ROMAI J., **1**, 2(2005),
1–2

THE CALCULATION OF THE FRONTOGENETIC FUNCTION

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Abstract The development of methods of diagnosis and forecasting of atmospheric fronts is one of priority problems of atmospheric dynamics. It is known that in overwhelming majority of cases the diagnosis of fronts is determined by the synoptic analysis. In this paper we develop some quantitative methods of diagnosis of fronts.

Development of methods of the diagnosis and the forecast of atmospheric fronts is one of prioritary problems of atmospheric dynamics. It is known, that in overwhelming majority of cases the diagnostics of fronts is determined by the synoptic analysis. In the present work we develop quantitative methods of the diagnosis of fronts. One parameter on which it is possible to judge the presence of the atmospheric front is the frontogenetic function, representing the total derivative with respect to time from a horizontal gradient of temperature. Using the thermodynamics equation, the frontogenetic function can be written as

F = $\frac{d}{dt}\,|\nabla\theta|$ = Convergence + deformation + a bend + inflows of heat

where convergence = - $\frac{1}{2|\nabla\theta|}(\theta_x^2+\theta_y^2)(U_x+V_y)$

deformation = - $-\frac{1}{2|\nabla\theta|}[(\theta_x^2-\theta_y^2)(U_x-V_y)\ +\ 2\theta_x\,\theta_y(U_y+V_x)]$

bend = - $\frac{\theta_z}{|\nabla\theta|}(\theta_x\,W_x+\theta_y\,W_y)$, Inflows of heat = $\frac{1}{|\nabla\theta|}\nabla\theta\nabla H$

where ∇ −horizontal gradient, H - inflows of heat owing to evaporation and condensation, θ - temperature, u, v, w - components of speed of a wind.

To positive values F the process of amplification of horizontal gradients of temperature, i.e. frontogenesis, while to negative values, a negative - frontolise correspond.

As at calculation of the frontogenetic function it is necessary to calculate derivatives from nonlinear terms, vertical speeds, sources and drains high smoothness requirements on the initial data must be imposed. Indeed, a series of calculations which have been carried out on GARP data, has shown high sensitivity of the frontogenetic function on the possible noise in given measurements (fig. 1a,b). In addition the noise level quite often appears to be equal in intensity to the level of a useful signal. In order to suppress the noise the median filtration, which is one of methods of nonlinear processing signal

as used. Median filtration keeps sharp differences in fields whereas the usual linear filter smoothes these differences.

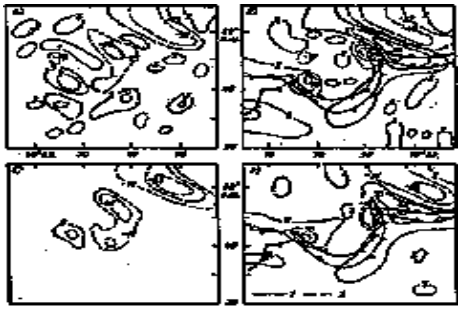

Fig. 1 Frontogenetic function without using median filter (a,b). Below after filtration. GARP dataset H500 and H1000 for 00h. 5 January 1979. --- for $F > 0$, —- $F < 0$. Fronts position were obtained by using the synoptic analysis.

A series of experiments with GARP data by using the median filter has shown that the best results are obtained with the filter with the aperture 5x5, in case when the field of temperature is exposed to a filtration only. Median filter substantially suppresses noise, allocates a useful signal, leaving constant its site.

Position of areas of positive values of F and change of their intensity can be connected with certain sites of fronts rather easily. The greatest positive values of F are marked along a zone of the Arctic front while at the top of an internal wave begun the occlusion of the cyclone. Concurrence of maxima of positive values of F to the position of frontal sections and tops of waves on them on 12 has forecasts value. Areas of negative values F coincide either with areas of divergence of anticyclones on a surface 500GPa, or with the position of the anticyclones crosspieces - saddles on the ground.

In order to study the structure of the front, the vertical cuts of an atmosphere along two parallels $42{,}5^0$ and $47{,}5^0$n.l., for 00h. January 5, 1979, GARP data were executed. They cross the cold fronts under an ungle of almost 90^0, but on different distances from the top of a wave. On vertical cuts the following parameters were analyzed: temperature, horizontal speed of the wind, relative humidity, vertical speed, convergence, deformation, a bend and total frontogenetic function.

Making separate comparison on the frontogenetic function shows that a bend isentropic surface is sometimes more than its other components. The contribution of heat inflows of in the total frontogenetic function appeared insignificant.

ROMAI J., **1**, 2(2005),
3–6

SINGULAR PERTURBATION FOR A PROBLEM FROM HYDRAULICS

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Abstract Of concern is the comparison of the solution (u, v) to the nonlinear b.v.p. (P) below with the solution $(u^\varepsilon, v^\varepsilon)$ of its elliptic regularization (P_ε). An asymptotic approximation for $(u^\varepsilon, v^\varepsilon)$ is constructed using the boundary layer method of Vishik-Lyusternik. The order of this approximation is found.

2000MSC: 35J25, 35L20, 41A60

Keywords: boundary function method, regularly perturbed problem, singularly perturbed problem.

1. INTRODUCTION

Consider problem (P) (arising in hydraulics) given by

$$\begin{cases} u_t + v_x + \alpha u(x,t) = f(x,t), \ (x,t) \in Q = (0,1) \times (0,T) \\ v_t + u_x + \beta v(x,t) = g(x,t), \ (x,t) \in Q = (0,1) \times (0,T), \end{cases} \tag{S}$$

$$u(0,t) = u(1,t) = 0, \ 0 < t < T, \tag{BC}$$

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ 0 < x < 1 \tag{C}$$

and its elliptic regularization (P_ε)

$$\begin{cases} \varepsilon u_{tt}^\varepsilon - u_t^\varepsilon = v_x^\varepsilon + \alpha u^\varepsilon(x,t) - f(x,t), \ (x,t) \in Q \\ \varepsilon v_{tt}^\varepsilon - v_t^\varepsilon = u_x^\varepsilon + \beta v^\varepsilon(x,t) - g(x,t), \ (x,t) \in Q, \end{cases} \tag{S$_\varepsilon$}$$

$$u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0, \ 0 < t < T, \tag{BC$_\varepsilon$}$$

$$\begin{cases} u^\varepsilon(x,0) = u_0(x), \ v^\varepsilon(x,0) = v_0(x), \ 0 < x < 1 \\ u^\varepsilon(x,T) = u_1(x), \ v^\varepsilon(x,T) = v_1(x), \ 0 < x < 1. \end{cases} \tag{C$_\varepsilon$}$$

Problem (P) governs the unsteady fluid flow (water-hammer) with nonlinear pipe friction. It is also a model for the fluid flow through a tree-structured system of transmission pipelines [4]. In problem (P), u denotes the instantaneous discharge at a point and v is the elevation of hydraulic gradeline.

We construct an asymptotic approximation $(Y^\varepsilon, Z^\varepsilon)$ of $(u^\varepsilon, v^\varepsilon)$ with the aid of the solution of problem (P) and find the order of this approximation in the spaces $L^2(Q)$ and $C\left([0,T]; L^2(0,1)\right)$.

This approximation of problem (P) with its elliptic regularization is stronger than those from [3] and from other related papers. In [2], an elliptic regularization is also used to approximate the solution of the heat equation.

2. THE ASYMPTOTIC APPROXIMATION

Denote $K = L^2(0,1)$ and $H = L^2(0,1)^2$. It is known that, under some specific hypotheses, problems (P) and (P_ε) have unique strong solutions (u,v) in $W^{1,\infty}(0,T;H)$ ([4]) and $(u^\varepsilon, v^\varepsilon) \in W^{2,2}(0,T;H)$, respectively [1].

Our purpose is to investigate the behavior of the solution $(u^\varepsilon, v^\varepsilon)$ of (P_ε) as $\varepsilon \to 0$. Remark that (u,v) does not generally satisfy the boundary condition (C_ε), therefore at least in some neighborhood $(\rho, T]$ of the final point $t = T$, functions $(u^\varepsilon, v^\varepsilon)$ and (u,v) are not close to each other. Thus (P_ε) is a singularly perturbed problem and the set $L = (0,1) \times (\rho, T]$ is a boundary layer. We shall prove that $u^\varepsilon - u$ and $v^\varepsilon - v$ converge to zero in $C([0,\rho];K)$ and construct an asymptotic approximation for $(u^\varepsilon, v^\varepsilon)$ which is valid in the whole domain Q. Using the boundary function method of Vishik -Lyusternik [5], [6], we are looking for an asymptotic approximation $(Y^\varepsilon, Z^\varepsilon)$ of $(u^\varepsilon, v^\varepsilon)$ of the form

$$Y^\varepsilon(x,t) = u(x,t) + \Pi(x,\tau), \ Z^\varepsilon(x,t) = v(x,t) + \Phi(x,\tau), \tag{1}$$

where $\tau = (T-t)/\varepsilon$ is the boundary layer variable and Π, Φ are boundary layer functions. They satisfy $\Pi(x,\infty) = \Phi(x,\infty) = 0$, $0 < x < 1$. We find

$$\Pi(x,\tau) = e^{-\tau}[u_1(x) - u(x,T)], \ \Phi(x,\tau) = e^{-\tau}[v_1(x) - v(x,T)]. \tag{2}$$

Since in the equations which define the remainder, the second derivatives u_{tt}, v_{tt} arise, we need high order regularity results for (P). Suppose that:

$(H1)$ $\alpha, \beta \geq 0$;

$(H2)$ $f, g \in W^{2,\infty}(0,T;K)$;

$(H3)$ $f(.,0), g(.,0) \in H^1(0,1)$, $u_0, v_0 \in H^2(0,1)$, $f(0,0) = f(1,0) = 0$, $u_0(0) = u_0(1) = 0$, $v_0'(0) = v_0'(1) = 0$.

Theorem 2.1. *If $(H1)-(H3)$ hold, then $(u,v) \in W^{2,\infty}(0,T;H)$ and $u_x, v_x \in L^\infty(0,T;K)$.*

Proof. One differentiates formally problem (P) with respect to t. Denoting $U = u_t$, $V = v_t$, one obtains the equation

$$\begin{cases} U_t + V_x + \alpha U(x,t) = f_t(x,t), \ (x,t) \in Q = (0,1) \times (0,T) \\ V_t + U_x + \beta V(x,t) = g_t(x,t), \ (x,t) \in Q = (0,1) \times (0,T), \end{cases}$$

with the boundary conditions

$$U(0,t) = U(1,t) = 0, \ 0 < t < T,$$

and

$$U(x,0) = U_0(x), \ V(x,0) = V_0(x), \ 0 < x < 1,$$

where $U_0(x) = f(x,0) - \alpha u_0(x) - v_0'(x)$ and $V_0(x) = g(x,0) - \beta v_0(x) - u_0'(x)$, for $x \in (0,1)$.

One shows that this problem admits a unique strong solution $(U,V) \in W^{1,\infty}(0,T;H)$. Let $S(t)$, $t \geq 0$ be the nonlinear semigroup generated by the linear operator $A : D(A) \subseteq H \times H \to H \times H$,

$$D(A) = \{(U,V) \in H, \ U(0,t) = U(1,t) = 0, \ 0 < t < T\},$$

$$A(U,V) = (\alpha U, \beta V).$$

With the aid of this semigroup and of the constant variation formula, we express (U,V) and show that it coincides with the derivatives (u_t, v_t) of the solution of (P). Thus we conclude the proof.

Now we are interested to find the order of $u^\varepsilon - Y^\varepsilon$, $v^\varepsilon - Z^\varepsilon$, where $(Y^\varepsilon, Z^\varepsilon)$ is given in (1). The main result is the following.

Theorem 2.2. *If $u_1 \in H_0^1(0,1)$, $v_1 \in H^1(0,1)$, then under assumptions $(H1) - (H3)$, for sufficiently small $\varepsilon > 0$, the pair $(Y^\varepsilon, Z^\varepsilon)$ is an asymptotic approximation of $(u^\varepsilon, v^\varepsilon)$ in the entire domain Q and we get the estimates:*

$$|u^\varepsilon - u|_{L^2(Q)} = o\left(\varepsilon^{1/2}\right), \ |v^\varepsilon - v|_{L^2(Q)} = o\left(\varepsilon^{1/2}\right), \tag{3}$$

$$|u^\varepsilon - u - \Pi|_{C([0,T];K)} = o\left(\varepsilon^{1/4}\right), \ |v^\varepsilon - v - \Phi|_{C([0,T];K)} = o\left(\varepsilon^{1/4}\right). \tag{4}$$

Conclusions. *a) If $u(.,T) = u_1$, $v(.,T) = v_1$, then (u,v) is an asymptotic approximation of $(u^\varepsilon, v^\varepsilon)$ in the entire domain Q. Problem (P_ε) is regularly perturbed and estimates (4) hold with $\Pi = \Phi = 0$.*

b) Otherwise, for every $\rho \in (0,T)$, (u,v) is an asymptotic approximation of $(u^\varepsilon, v^\varepsilon)$ in $Q_1 = (0,1) \times (0,\rho]$, (P_ε) is singularly perturbed and we have

$$|u^\varepsilon - u|_{C([0,\rho];K)} = o\left(\varepsilon^{1/4}\right), \ |v^\varepsilon - v|_{C([0,\rho];K)} = o\left(\varepsilon^{1/4}\right). \tag{5}$$

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ROMAI J., **1**, 2(2005),
7–10

GEOMETRIC DYNAMICS OF CUTTING MACHINES

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The kinematic scheme of the teeth-wheel cutting machine, Romanian production FD-320A is represented in fig. 1.

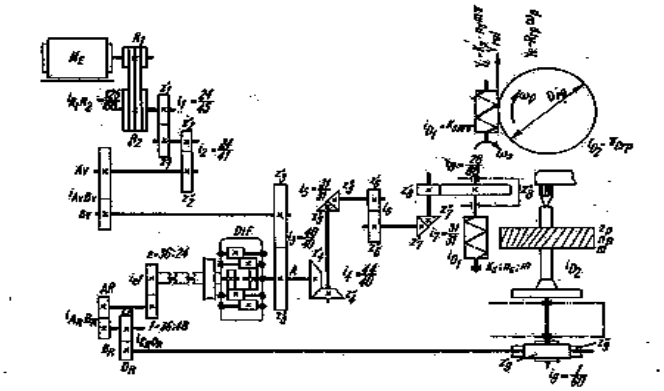

Fig. 1.

In it three branches can be distinguished: the branch from the point M_E to the point A; the branch from the point A to the tool screw milling; the branch the point A to the piece. Shortly, these branches will be named: uncommon branch, common branch (of the tool) and working surface branch.

In this paper we apply the Udrişte method [6] in order to obtain further information regarding the description of the cutting machine behavior during the threading. In order to increase the threading capacity of the teeth wheel cutting machine FD-320A it is necessary to diminish as much as possible the vibrations that appear during the cutting process. The increase of the cutting capacity and of the quality of the working surfaces determines the increasing of the production capacity. In order, to study the vibrational movement of the elastic system presented in fig. 2d from [5] we use the Riemann manifold $(\mathrm{R}^6, \delta_{ij})$.

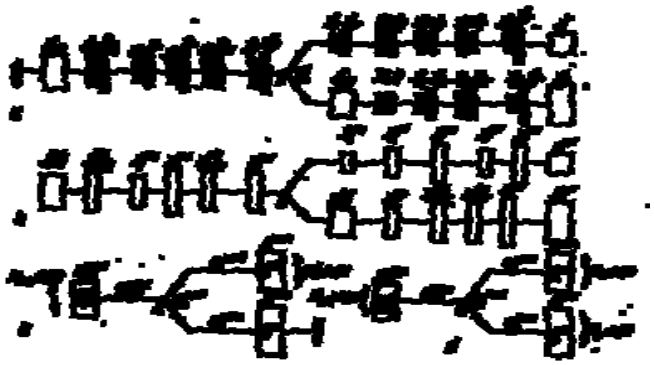

Fig. 2.

The vibrations of the elastic system are modeled by the solutions of the differential kinematic system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{C_{me}}{J_{me}} \dfrac{C_S(x_3 - x_1) + C_P(x_5 - x_1)}{C_{me} + C_S + C_P}, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \dfrac{C_S}{J_S} \dfrac{C_{me}(x_1 - x_3) + C_P(x_5 - x_3)}{C_{me} + C_S + C_P}, \\ \dot{x}_5 = x_6, \\ \dot{x}_6 = \dfrac{C_P}{J_P} \dfrac{C_{me}(x_1 - x_5) + C_S(x_3 - x_5)}{C_{me} + C_S + C_P}, \end{cases} \tag{1}$$

where the constants $J_{me},\ C_S,\ C_P$ represent the inertia moment and two constants of elasticity respectively. The index selection took into account the electrical engine, tool and piece. Let us build the Lagrange extension by Udrişte method. To this aim we introduce the „cutting machine" vector field which has the following six components

$$\begin{cases} X_1(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = x_2, \\ X_2(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = \dfrac{C_{me}}{J_{me}} \dfrac{C_S(x_3 - x_1) + C_P(x_5 - x_1)}{C_{me} + C_S + C_P}, \\ X_3(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = x_4, \\ X_4(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = \dfrac{C_S}{J_S} \dfrac{C_{me}(x_1 - x_3) + C_P(x_5 - x_3)}{C_{me} + C_S + C_P}, \\ X_5(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = x_6, \\ X_6(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6) = \dfrac{C_P}{J_P} \dfrac{C_{me}(x_1 - x_5) + C_S(x_3 - x_5)}{C_{me} + C_S + C_P}, \end{cases} \tag{2}$$

where $\mathbf{X} = (X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6)$ and $\mathbf{x} = (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)$.

The autonomous system (1) has the form

$$\frac{dx_i}{dt} = X_i(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6),\ i = \overline{1,\ 6}. \tag{3}$$

The equilibria of (1) are $x = (a, 0, a, 0, a, 0)$, where a is a constant. The function

$$f = \frac{1}{2} \sum_{i=1}^{6} X_i^2 \tag{4}$$

represents the energy density associated with the vector field „cutting machine" and the Euclidian structure δ_{ij}). The geometric dynamics associated with the elastic system is described by the second order differential system

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_j \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{dx_j}{dt}, \ i, \ j = \overline{1, \ 6}, \tag{5}$$

which proves to be an Euler-Lagrange extension. In other words, the Lagrangian $L = \frac{1}{2} \sum\limits_{i=1}^{6} \left(\frac{dx_i}{dt} - X_i \right)^2$ or $L = \frac{1}{2} \sum\limits_{i=1}^{6} \left(\frac{dx_i}{dt} \right)^2 - \sum\limits_{i=1}^{6} X_i \frac{dx_i}{dt} + f$ determines the system (5) (which has 6 degrees of freedom. The trajectories of (5) contain also the solutions of the system (1). This system describes a pregeodesic movement in a gyroscopical field forces. More precisely, $\sum\limits_{i=1}^{6} \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{dx_j}{dt}$ is the gyroscopical force and $\frac{\partial f}{\partial x_i}$ represents a conservative force. In order to see if the trajectories are pregeodesic we build the associated Hamiltonian $H = \frac{1}{2} \sum\limits_{i=1}^{6} \left(\frac{dx_i}{dt} \right)^2 - f$, and the geometric structure formed of Riemann metrics $g_{ij} = (H + f)\delta_{ij}$, and of the nonlinear connection $N_j^i = \Gamma_{jk}^i - F_j^i$, where Γ_{jk}^i is the Riemann connection induced by g_{ij} metrics.

The skqewsymmetric matrix of F_j^i elements

$$F_j^i = \delta^{ih} F_{jh}, \tag{6}$$

$$F_{ij} = \frac{\partial X_j}{\partial x_i} - \frac{\partial X_i}{\partial x_j} \tag{7}$$

corresponds to the rotor in the case of three dimmensional space.

The solutions of the system (5) (computations using (1),(2),(4) were made) are horizontal pregeodesics on the Riemann-Jacobi-Lagrange manifold (R^6 $Omega, \ g_{ij}, \ N_j^i$), where Ω is the set of equilibrium points, and N_j^i is a nonlinear connection.

Theorem 0.3. *Any nonconstant trajectory of the dynamic system associated with (5), which has the total energy H (constant), is a reparameterized horizontal pregeodesic of the Riemann-Jacobi-Lagrange manifold* (R^6 $\Omega, \ g_{ij} = (H + f)\delta ij, \ N_j^i = \Gamma_{jk}^i + F_j^i, \ i, \ j = \overline{1, \ 6}$).

Because the new detaching of the splint maintains the vibration process, the phenomenon is called a regenerative vibration. It is known that the delay is the central idea of the regenerative effect. In analyzing the nonlinear model of the vibrating tool machine, mathematically described by differential equations with a delay [2] the following stages are followed: 1) the analysis of the linear part of the system

$$\dot{X}(t) = L(p)X(t) + R(p)X(t-\tau) + F(X(t),\ X(t-\tau),\ p), \tag{8}$$

where $L(p) = \begin{pmatrix} 0 & 1 \\ -(1+p) & -2\xi \end{pmatrix}$, $R(p) = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$,

$$F(X(t),\ X(t-\tau),\ p) = \frac{3p}{10}\begin{pmatrix} 0 \\ (x_1(t)-x_1(t-\tau))^2 - (x_1(t)-x_1(t-\tau))^3 \end{pmatrix};$$

2) the analysis of the roots of the characteristic equation and Hopf bifurcation; 3) the determination of the generalized eigensubspaces associated with the system (6) at the Hopf bifurcation point; 4) the determination of the centre manifold at the bifurcation point and the associated limit cycle; 5) the study of the orbit of the system (6) and the specification of the invariants of the limit cycle. Starting with the simplified scheme of the dynamical system machine-tool-piece-contrivance-tool during the cutting process, the stability of the cutting machine functioning is studied [1]. Actually, the aim is the lifting of the stability charts. They are graphical representations which have on the abscissa the n revolutions of the machine and on the ordinal the w value of the threading depth. In order to calculate the revolution of the cutting and of the threading depth a computation algorithm was formulated too.

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ROMAI J., **1**, 2(2005),
11–16

CONTINUITY OF CHARACTERISTICS OF A THIN LAYER FLOW DRIVEN BY A SURFACE TENSION GRADIENT

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Abstract The flow of a thin layer down an inclined rigid impermeable plane driven simultaneously by the gravity and a surface tension gradient is considered. Geometric and mechanical reasons suggested us the continuity of the depth of the layer, the pressure and the volume flux of the fluid. By using suitable asymptotic expansions it was shown that, indeed, this is the case.

2000 MSC: 76D08
Keywords: viscous flow, thin-layer approximation, surface tension gradient

1. PROBLEM FORMULATION

Consider the flow of a thin liquid layer and use a lubrication (thin-layer) approximation. The liquid is a Newtonian viscous fluid, of constant density ρ and coefficient of dynamic viscosity μ. The thin layer has a steady flow, down a plane inclined at an angle α, $0 \leq \alpha \leq \pi$ to the horizontal. The flow is driven simultaneously by gravity and a surface tension gradient, $\partial\gamma/\partial x > 0$.

With respect to the Cartesian coordinates system Oxyz as indicated in fig.1,

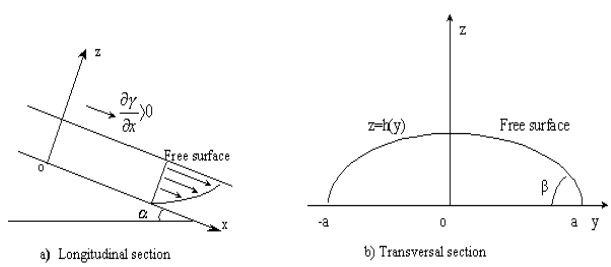

Fig.1. The geometry of the thin layer.

the fluid velocity has the form $\mathbf{u} = u(y,z)\mathbf{i}$. Introduce the notation: p is the pressure in the fluid , p_a is the atmospheric pressure, $\mathbf{g}$ is the gravitational acceleration, $\overline{\gamma}$ is the reference value of surface tension, β is the angle of the three-phase contact line (it is assumed constant, such that $\beta < \pi/2$), $2a$ is the

width of the layer, h_m is the maximum depth of the liquid ($h_m = h(0)$). A constant value of β means that any contact angle hysteresis is ignored. We neglected the viscosity of the air above the liquid and we took the surface curbature to be $h^{''} = d^2h/dy^2$, because $h^2 << h$. Denote by $z = h(y)$ the free-surface profile. Then, the Navier-Stokes equations are

$$0 = -p_x + \rho g \sin\alpha + \mu(u_{yy} + u_{zz}),\ 0 = -p_y,\ 0 = -p_z - \rho g \cos\alpha. \tag{1}$$

In the thin-layer theory, [1], [5] these equations reduce to

$$0 = -p_x + \rho g \sin\alpha + \mu u_{zz},\ 0 = -p_y,\ 0 = -p_z - \rho g \cos\alpha. \tag{2}$$

and are to be integrated subject to the boundary conditions

$$u = 0,\ \text{on } z = 0; \tag{3}$$

$$p - p_a = -\overline{\gamma} h^{''}, \gamma u_z = \partial\gamma/\partial x \text{ on } z = h(y). \tag{4}$$

The origin of the stress τ could be very diverse. We are interested in the case when τ comes from a local variation of surface, i.e $\tau = \partial\gamma/\partial x$.

Impose the following contact conditions

$$h = 0,\ h = \pm tan\beta,\ \text{at } y = \pm a. \tag{5}$$

The angle of the layer and the horizontal straightline belongs to the following three cases : i) $\alpha \in (0, \pi/2)$, ii)$\alpha = \pi/2$, iii)$\alpha \in (\pi/2, \pi)$. Taking into account the closed form expressions (Section 2) for the velocity profile, free surface profile, pressure and nondimensional volum flux, in all these cases, in Section 3 their continuity at $\alpha = \pi/2$ is shown.

2. CLOSED FORM SOLUTIONS

In [2] we obtained the following results. The velocity of the fluid $u(y, z)$ is

$$u(y, z) = \frac{\rho g sin\alpha}{2\gamma}(-z^2 + 2zh) + \frac{1}{\mu} \cdot \frac{\partial\gamma}{\partial x} \cdot z. \tag{6}$$

The free surface profide $z = h(y)$ is given by

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot \frac{coshB - coshB\xi}{sinhB},\ \alpha \in (0, \pi/2) \tag{7}$$

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot B \cdot \frac{1 - \xi^2}{2},\ \alpha = \pi/2 \tag{8}$$

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot \frac{cosB\xi - cosB}{sinB}, \ \alpha \in (\pi/2, \pi) \tag{9}$$

where $B = a(\frac{\rho g|cos\alpha|}{\gamma})^{1/2}$, for $\alpha \neq \pi/2$; $B = 0$, for $\alpha = \pi/2$, $\xi = y/a \in [-1, 1]$.

The pressure of the fluid $p = p(z)$ has the expression

$$p(z) = p_a - \rho g z cos\alpha + tan\beta\sqrt{\rho g\gamma\,|cos\alpha|} \begin{cases} cothB, & \alpha \in (0, \pi/2) \\ B^{-1}, & \alpha = \pi/2 \\ cotB, & \alpha \in (\pi/2, \pi) \end{cases} \tag{10}$$

The nondimensional volume flux reads

$$\overline{Q} = F(B) + \frac{9}{2}\frac{\partial\gamma}{\partial x}\frac{1}{B}\frac{1}{tan\beta\,|tan\alpha|}G(B), \tag{11}$$

where

$$F(B) = 15Bcoth^3B - 15coth^2B - 9BcothB + 4, \ \alpha \in (0, \pi/2) \tag{12}$$

$$F(B) = \frac{12}{35}B^4, \ \alpha = \pi/2 \tag{13}$$

$$F(B) = -15Bcot^3B + 15cot^2B - 9BcotB + 4, \ \alpha \in (\pi/2, \pi) \tag{14}$$

and

$$G(B) = 3Bcoth^2B - 3cothB - B, \ \alpha \in (0, \pi/2) \tag{15}$$

$$G(B) = \frac{4}{15}B^3, \ \alpha = \pi/2 \tag{16}$$

$$G(B) = 3Bcot^2B - 3cotB + B, \ \alpha \in (\pi/2, \pi). \tag{17}$$

Remark 2.1. *Rigorously speaking, for the case $\alpha = \pi/2$, these formulae cotain nondeterminations of the form $0/0$. It is easy to see that the elimination of this nondetermination proceeds immediately. However, we kept the above forms for the sake of symmetry. On the other hand, this kind of expressions are currently used in the litterature on thin layers.*

3. CONTINUITY OF H, P AND Q AT $\alpha = \pi/2$

Some of the above expressions are sufficiently complicated to allow us to decide if h, p and Q are continuous or not at $\alpha = \pi/2$. The asymptotic study carried out in this section shows that they are continuous, indeed.

3.1. CONTINUITY OF H(Y)

For the sake of simplity, we consider the following functions $f_1(x) = \frac{coshx - coshx\xi}{xsinhx}$, $f_2(x) = \frac{cosx\xi - cosx}{xsinx}$, where B:=x. The asymptotic expansion of $f_1(x)$ and $f_2(x)$, as x→0, are

$$f_1(x) \sim \frac{\left(1 + \frac{x^2}{2!} + ...\right) - \left(1 + \frac{x^2\xi^2}{2!} + ...\right)}{x\left(\frac{x}{1!} + \frac{x^3}{3!} + ...\right)} = \frac{x^2\left(\frac{1}{2!} - \frac{\xi^2}{2!}\right) + ...}{x^2 + \frac{x^4}{3!} + ...} = \frac{\frac{1-\xi^2}{2!}}{1 + \frac{x^2}{3!} + ...} \sim \frac{1-\xi^2}{2};$$

$$f_2(x) \sim \frac{\left(1 - \frac{x^2\xi^2}{2!} + \frac{x^4\xi^4}{4!} + ...\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}...\right)}{x\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)} = \frac{x^2\left(\frac{1}{2!} - \frac{\xi^2}{2!}\right) + ...}{x^2 - \frac{x^4}{3!} + ...} \sim \frac{1-\xi^2}{2}.$$

Here we used [6] and the asymptotic expansions $cosx \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!}...$ and $sinx \sim \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$, as x→ 0. In fact, $\frac{1-\xi^2}{2}$ is the asymptotic representation of $f_1(x)$ and $f_2(x)$, as x→0 and it was obtained by performing the asymptotic expansion of the involved sines and cosines. The same results can by obtained by using the l$'$*Hospital* rule, namely

$$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} \frac{sinhx - \xi sinhx\xi}{sinhx + xcoshx} = \lim_{x\to 0} \frac{coshx - \xi^2 coshx\xi}{xsinhx + 2coshx} = \frac{1-\xi^2}{2};$$

$$\lim_{x\to 0} f_2(x) = \lim_{x\to 0} \frac{sinx - \xi sinx\xi}{sinx + xcosx} = \lim_{x\to 0} \frac{cosx - \xi^2 cosx\xi}{-xsinx + 2cosx} = \frac{1-\xi^2}{2}.$$

Therefore

$$\lim_{\alpha\nearrow\pi/2} h(y,\, \alpha) = h(y, \pi/2) \tag{18}$$

and

$$\lim_{\alpha\searrow\pi/2} h(y,\, \alpha) = h(y, \pi/2). \tag{19}$$

Here α is the asymptotic variable. The relations (18), (19) and (8) show that h(y,α) is continuous at $\alpha = \pi/2$.

3.2. CONTINUITY OF P

Consider the following functions $g_1(x) = xcothx$ and $g_2(x) = xcotx$, where B:=x. Using the asymptotic expansions [5] $cothx \sim \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...$ and $cotx \sim \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...$, as x→ 0, we have $g_1(x) \sim x(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...) = 1 + ...$, $g_2(x) \sim x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) = 1 + ...$, and, by means of the l$'$ Hospital rule, we get $\lim\limits_{x\to 0} g_1(x) = \lim\limits_{x\to 0} \frac{coshx + xsinhx}{coshx} = 1$;

$\lim\limits_{x\to 0} g_2(x) = \lim\limits_{x\to 0} \frac{x}{tanx} = \lim\limits_{x\to 0} cos^2 x = 1$, whence

$$\lim_{\alpha\nearrow\pi/2} p(y,\,\alpha) = p(y,\pi/2), \tag{20}$$

$$\lim_{\alpha\searrow\pi/2} p(y,\,\alpha) = p(y,\pi/2). \tag{21}$$

The relation (20), (21) and (10) show that p(y,α) is continuous at $\alpha = \pi/2$.

3.3. CONTINUITY OF Q

The following asymptotic representations $F(B) \sim \frac{12}{35}B^4$, $G(B) \sim \frac{4}{15}B^3$, as $B \to 0$ (i.e. as $\alpha \to \pi/2$) hold. Indeed, consider the functions $F_1(x) = \frac{15xcoth^3x - 15xcoth^2x - 9xcothx + 4}{x^4}$, $F_2(x) = \frac{-15xcot^3x + 15cot^2x - 9xcotx + 4}{x^4}$, $G_1(x) = \frac{3xcoth^2x - 3cothx - x}{x^3}$, $G_2(x) = \frac{3cot^2x - 3cotx + x}{x^3}$, and take into account that $coth^2x \sim \frac{1}{x^2} + \frac{x^2}{9} + ... + \frac{2}{3} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...$, $cot^2x \sim \frac{1}{x^2} + \frac{x^2}{9} + ... - \frac{2}{3} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...$

Remark that the first nonvanishing term of the asymptotic expansion of F_1, F_2, G_1, G_2 is obtained if the first four terms in the asymptotic expansion of $coth^2x$ and cot^2x are considered. We have

$F_1(x) \sim 5 \cdot \frac{1}{x}(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) \cdot [\frac{3x(\frac{1}{x^2} + \frac{x^2}{9} + ... + \frac{2}{3} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...)}{x^3} - \frac{3(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...) - x}{x^3}] + \frac{-4x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) + 4}{x^4} =$

$= (\frac{5}{x^2} + \frac{5}{3} - \frac{5x^2}{45} + \frac{10x^4}{945} + ...)(\frac{4}{15} - \frac{36}{945}x^2 + ...) + \frac{4}{45} - \frac{8x^2}{945} + ... = \frac{4}{9} + \frac{4}{45} - \frac{36}{189} = \frac{12}{35}$,

$F_2(x) \sim 5 \cdot \frac{1}{x}(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) \cdot [\frac{-3x(\frac{1}{x^2} + \frac{x^2}{9} + ... - \frac{2}{3} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...)}{x^3} + \frac{3(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) - x}{x^3}] + \frac{-4x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + ...) + 4}{x^4} =$

$= (\frac{5}{x^2} - \frac{5}{3} - \frac{5x^2}{45} - \frac{10x^4}{945} - ...)(-\frac{9}{45} - \frac{30}{945}x^2 - \frac{3}{45} - \frac{6}{945}x^2 ...) + \frac{4}{45} + \frac{8x^2}{945} + ... = \frac{12}{35}$,

$G_1(x) \sim \frac{3}{x^2}(\frac{x^2}{9} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...) - \frac{3}{x^3}(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) - \frac{1}{x^2} = \frac{1}{3} - \frac{6}{45} + \frac{12x^2}{945} - \frac{6x^2}{135} + \frac{3}{45} - \frac{6x^2}{945} + ... = \frac{1}{3} - \frac{3}{45} = \frac{4}{15}$,

$G_2(x) \sim \frac{3}{x^2}(\frac{x^2}{9} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...) - \frac{3}{x^3}(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) - \frac{1}{x^2} = \frac{1}{3} - \frac{6}{45} - \frac{12x^2}{945} - \frac{6x^2}{135} - \frac{3}{x^4} + \frac{1}{x^2} + \frac{3}{45} + \frac{6x^2}{945} + ... + \frac{1}{x^2} = \frac{1}{3} - \frac{3}{45} = \frac{4}{15}$

and , by the l$'$Hopital rules, we get $\lim\limits_{x\to 0} F_1(x) =$

$$\lim_{x\to 0} \frac{15xcoshx(1 + sinh^2x) - 15sinhx(1 + sinh^2x) - 9xcoshxsinh^2x + 4sinh^3x}{x^4sinh^3x} =$$

$$= \lim_{x\to 0} \frac{15xsinhx - 27coshxsinh^2x + 6xsinh^3x + 12cosh^2xsinhx}{4x^3sinh^3x + 3x^4sinh^2xcoshx} =$$

$$= \lim_{x\to 0} \frac{-36sinh^2x + 36xcoshxsinhx}{12x^2sinh^2x + 20x^3sinhxcoshx + 3x^4 + 6x^4sinh^2x} =$$

$$= \lim_{x\to 0} \frac{36(x+2xsh^2x - shxchx)}{24xsinh^2x + 84x^2sinhxcoshx + 32x^3 + 64x^3sinh^2x + 12x^4sinhxcoshx} =$$

$$= \lim_{x\to 0} \frac{144xsinhxcoshx}{180x^2 + 24sinh^2x + 216xsinhxcoshx + 360x^2sinh^2x + 12x^4 + 176x^3sinhxcoshx + 24x^4cosh^2x} =$$

$$= \lim_{x\to 0} \frac{144xcoshx}{180\frac{x}{sinhx} + 24\frac{sinhx}{x} + 216coshx + 360xsinhx + 12x^3\frac{x}{sinhx} + 176x^2sinhxcoshx + 24x^3coshx} =$$

$= \frac{12}{35}$;

$\lim_{x\to 0} F_2(x) = \lim_{x\to 0} \frac{-15xcos^3x+15sinxcos^2x-9xcosxsin^2x+4sin^3x}{x^4sin^3x} = \lim_{x\to 0} \frac{27xcos^2x-27sinxcosx+9xsin^2x}{7x^5} = \lim_{x\to 0} \frac{-36xcosx+36sinx}{35x^3} = \lim_{x\to 0} \frac{36xsinx}{35\cdot 3x^2} = \frac{12}{35}$;

$\lim_{x\to 0} G_1(x) = \lim_{x\to 0} \frac{4xsinhxcoshx-4sinh^2x}{3x^2sinh^2x+2x^3sinhxcoshx} = \lim_{x\to 0} \frac{4xsinhx}{6xsinhx+3xcosh^2x+6xcosh^2x+2x^3sinhx} = \lim_{x\to 0} \frac{4}{6+3coshx\cdot\frac{x}{sinhx}+6coshx\cdot\frac{x}{sinhx}+2x^2} = \frac{4}{15}$,

$\lim_{x\to 0} G_2(x) = \lim_{x\to 0} \frac{3xcos^2x-3sinxcosx+xsin^2x}{x^3\cdot sin^2x} = \lim_{x\to 0} \frac{-4xcosx+4sinx}{5x^3} = \lim_{x\to 0} \frac{4xsinx}{15x^2} = \frac{4}{15}$;

whence

$$\lim_{\alpha\nearrow\pi/2} Q(y,\,\alpha) = Q(y,\pi/2) \tag{22}$$

and

$$\lim_{\alpha\searrow\pi/2} Q(y,\,\alpha) = Q(y,\pi/2). \tag{23}$$

The relations (22), (23) and (11)-(17) show that $Q(y,\,\alpha)$ is continuous at $\alpha = \pi/2$.

Consequently, by direct asymptotic expansions and by the l$'$Hopital rule, we proved that the following characteristics $h(y,\,\alpha)$, $p(y,\,\alpha)$ and $Q(y,\,\alpha)$ of the flow of the thin layer are continuous at $\alpha = \pi/2$.

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DISTRIBUTION OF DYNAMIC PERTURBATION TO THE TWO-COMPONENT ELASTIC MEDIUM

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Abstract The present paper is devoted to the solution of the non-stationary self-similar problems a linearly-elastic waves in the quarter-space, filled with a two-component medium, under the action of impulse external force of the boundary. We explain the idea of Smirnov-Sobolev method of functionally invariant solutions, using as an example the problem of the shock of quarter of elastic space on rigid fixed barrier.

Keywords: linearly-elastic waves, two-component medium, Smirnov-Sobolev method, functionally invariant solutions, self-similar solution.

Herein we explain the idea of the Smirnov-Sobolev method of functionally invariant solutions, using as an example the problem of shock of a quarter of elastic space on a rigid fixed barrier. The material is considered to be homogeneous and isotropic, composed of two rigid media. It is supposed that the medium moves uniformly with velocity v_0, parallel to the xOy plane, and at an instant t=0 it strikes at a rigid fixed barrier in the plane xOz. The xOz plane is considered to be free of strains. When $t = 0$, on the plane xOz, shearing stresses are equal to zero, i.e. there is the possibility of sliding.

It is necessary to study the flat ondular movement that satisfies, for $x \geq 0$, $y \geq 0$, the equations of motion. In U_i, V_i $(i = 1, 2)$ these equations read

$$\begin{aligned}
&A_{i1}\frac{\partial^2 U_i}{\partial x^2} + A_{i12}\frac{\partial^2 U_i}{\partial y^2} + (A_{i1} - A_{i2})\frac{\partial^2 V_i}{\partial x \partial y} + B_{i1}\frac{\partial^2 U_{3-i}}{\partial x^2} + B_{i2}\frac{\partial^2 U_{3-i}}{\partial y^2} + \\
&+ (B_{i1} - B_{i2})\frac{\partial^2 V_{3-i}}{\partial x \partial y} == \rho_{1i}\frac{\partial^2 U_1}{\partial t^2} + \rho_{i2}\frac{\partial^2 U_2}{\partial t^2} - (-1)^i b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right), \\
&A_{i1}\frac{\partial^2 V_i}{\partial x^2} + A_{i12}\frac{\partial^2 V_i}{\partial y^2} + (A_{i1} - A_{i2})\frac{\partial^2 U_i}{\partial x \partial y} + B_{i1}\frac{\partial^2 V_{3-i}}{\partial x^2} + B_{i2}\frac{\partial^2 V_{3-i}}{\partial y^2} + \\
&+ (B_{i1} - B_{i2})\frac{\partial^2 U_{3-i}}{\partial x \partial y} == \rho_{1i}\frac{\partial^2 V_1}{\partial t^2} + \rho_{i2}\frac{\partial^2 V_2}{\partial t^2} - (-1)^i b\left(\frac{\partial V_1}{\partial t} - \frac{\partial V_2}{\partial t}\right),
\end{aligned} \tag{1}$$

where $A_{i1} = \lambda_i + 2\mu_i + (-1)^i \frac{\rho_{3-i}\alpha_2}{\rho}$; $A_{i2} = \mu_i - \lambda_5$; $B_{i1} = \lambda_{2+i} + 2\mu_3 + \frac{\rho_i \alpha_2}{\rho}(-1)^i$; $B_{i2} = \lambda_5 + \mu_3$; $(i = 1,\ 2)$.

The solution of the system (1) should satisfy the initial conditions

$$U_1 = 0, \quad \frac{\partial U_1}{\partial t} = -V_0, \quad U_2 = 0, \quad \frac{\partial U_2}{\partial t} = -V_0,$$

$$V_1 = 0, \quad \frac{\partial V_1}{\partial t} = 0, \quad V_2 = 0, \quad \frac{\partial V_2}{\partial t} = 0 \tag{2}$$

and the following boundary conditions on the Oy axis

$$
\begin{aligned}
U_1|_{x=0} &= 0,\\
U_2|_{x=0} &= 0,\\
\left[(2\mu_1+\lambda_5)\frac{\partial U_1}{\partial y}+(2\mu_3-\lambda_5)\frac{\partial U_2}{\partial y}+(2\mu_1-\lambda_5)\frac{\partial V_1}{\partial x}+(2\mu_3+\lambda_5)\frac{\partial V_2}{\partial x}\right]\Bigg|_{x=0} &= 0,\\
\left[(2\mu_3+\lambda_5)\frac{\partial U_1}{\partial y}+(2\mu_2-\lambda_5)\frac{\partial U_2}{\partial y}+(2\mu_3-\lambda_5)\frac{\partial V_1}{\partial x}+(2\mu_2+\lambda_5)\frac{\partial V_2}{\partial x}\right]\Bigg|_{x=0} &= 0;
\end{aligned}
\tag{3}
$$

on the Ox axis

$$
\begin{aligned}
&\left[\lambda_1\tfrac{\partial U_1}{\partial x}+(\lambda_1+2\mu_1)\tfrac{\partial V_1}{\partial y}+\lambda_3\tfrac{\partial U_2}{\partial x}+(\lambda_3+2\mu_3)\tfrac{\partial V_2}{\partial y}\right]\Bigg|_{y=0}=0,\\
&\left[\lambda_2\tfrac{\partial U_2}{\partial x}+(\lambda_2+2\mu_2)\tfrac{\partial V_2}{\partial y}+\lambda_4\tfrac{\partial U_1}{\partial x}+(\lambda_4+2\mu_3)\tfrac{\partial V_1}{\partial y}\right]\Bigg|_{y=0}=0,\\
&\left[(2\mu_1+\lambda_5)\tfrac{\partial U_1}{\partial y}+(2\mu_3-\lambda_5)\tfrac{\partial U_2}{\partial y}+(2\mu_1-\lambda_5)\tfrac{\partial V_1}{\partial x}+(2\mu_3+\lambda_5)\tfrac{\partial V_2}{\partial x}\right]\Bigg|_{y=0}=0,\\
&\left[(2\mu_3+\lambda_5)\tfrac{\partial U_1}{\partial y}+(2\mu_2-\lambda_5)\tfrac{\partial U_2}{\partial y}+(2\mu_3-\lambda_5)\tfrac{\partial V_1}{\partial x}+(2\mu_2+\lambda_5)\tfrac{\partial V_2}{\partial x}\right]\Bigg|_{y=0}=0.
\end{aligned}
\tag{4}
$$

The conditions (3) correspond to the possibility of the free sliding of the medium along a rigid barrier, and (4), to the free surface.

The quantities U_1, U_2, V_1, V_2 are the components of the displacement vector of the rigid phases; ρ_{11}, ρ_{22} are the effective masses of components at the relative driving; $\rho_{11}+\rho_{12}=\rho_1$, $\rho_{22}+\rho_{12}=\rho_2$, ρ_{12} is the „linking parameter" between the components of the mixture. It has the mass dimension or an additional apparent mass in relation to moving components; $a_2=\lambda_3-\lambda_4$ is a constant that has a dimensional stress; λ_j, μ_j, $\left(j=\overline{1,5}\right)$ are the Lamé constants; ρ_1, ρ_2 are the densities of phases; b is the diffusivity coefficient.

Remark, that the functions $\partial U_i/\partial t$ on the straightline $x=0$ at $t=0$ undergo a rupture, reflected by the physical content of the shock.

The decay of this rupture for $t>0$ leads to the appearance of some propellant lines of the rupture. Two of them, the straightline $x=at$ and the arc of circle $x^2+y^2=a^2t^2$ divide the plane xOy into three sub regions: I – zone of inertial movement; II – the zone of one-dimensional movement from the barrier Oy; III – zone of reflection from the free edge Ox.

By a we denoted the maximum velocity of a longitudinal wave in a material.

Some ideas concerning the nature of gaps distributing in the quarter of a plane at impact with a rigid obstacle, can be stated on the basis of the elementary laws of wave propagation. The gap of the initial data leads to the fact that in the medium two waves of a strong gap with their speeds are diffused, and thus the area (II) is formed, where the problem is one-dimensional. The free boundary conditions produce secondary (diffracted) expansion waves and shift (III).

The study of the solution of this problem in the first two zones does not imply special difficulties. In the zone last the solution is considerably easy to

found, if it is looked for in the form of functions of volumetric strains Δ_i and shear strains Ω_i, given trough transitions by the relations

$$\Delta_i = \partial U_i/\partial x + \partial V_i/\partial y; \quad \Omega_i = \Delta_i = \partial V_i/\partial x - \partial U_i/\partial y, \quad (i = 1, 2). \quad (5)$$

Then the system (1) becomes

$$\begin{aligned} &A_{i1}\frac{\partial \Delta_i}{\partial x} - A_{i2}\frac{\partial \Omega_i}{\partial y} + B_{i1}\frac{\partial \Delta_{3-i}}{\partial x} - B_{i2}\frac{\partial \Omega_{3-1}}{\partial y} = \\ &= \rho_{1i}\frac{\partial^2 U_1}{\partial t^2} + \rho_{i2}\frac{\partial^2 U_2}{\partial t^2} - (-1)^i b\left(\frac{\partial U_1}{\partial t} - \frac{\partial U_2}{\partial t}\right) \\ &A_{i1}\frac{\partial \Delta_i}{\partial y} + A_{i2}\frac{\partial \Omega_i}{\partial x} + B_{i1}\frac{\partial \Delta_{3-i}}{\partial y} + B_{i2}\frac{\partial \Omega_{3-1}}{\partial x} = \\ &= \rho_{1i}\frac{\partial^2 V_1}{\partial t^2} + \rho_{i2}\frac{\partial^2 V_2}{\partial t^2} - (-1)^i b\left(\frac{\partial V_1}{\partial t} - \frac{\partial V_2}{\partial t}\right) \end{aligned} \quad (6)$$

After some transformation we obtain

$$\begin{aligned} &A_{i1}\Delta\Delta_i + B_{i1}\Delta\Delta_{3-i} = \rho_{1i}\frac{\partial^2 \Delta_1}{\partial t^2} + \rho_{i2}\frac{\partial^2 \Delta_2}{\partial t^2} - (-1)^i b\left(\frac{\partial \Delta_1}{\partial t} - \frac{\partial \Delta_2}{\partial t}\right), \\ &A_{i2}\Delta\Omega_i + B_{i2}\Delta\Omega_{3-i} = \rho_{1i}\frac{\partial^2 \Omega_1}{\partial t^2} + \rho_{i2}\frac{\partial \Omega_1}{\partial t^2} - (-1)^i b\left(\frac{\partial \Omega_1}{\partial t} - \frac{\partial \Omega_2}{\partial t}\right). \end{aligned} \quad (7)$$

For $\Delta_1 = \delta_1 + \delta_2$, $\Delta_2 = \beta_1\delta_1 + \beta_2\delta_2$, $\Omega_1 = \omega_1 + \omega_2$, $\Omega_2 = \gamma_1\omega_1 + \gamma_2\omega_2$, the equations (7) can be reduced to the wave equations (for zero diffusivity (b=0))

$$a_j^2\Delta\delta_j = \partial^2\delta_j/\partial t^2; \quad b_j^2\Delta\omega_j = \partial^2\omega_j/\partial t^2, \quad (8)$$

where $a_j^2 = \frac{A_{11}+\beta_j B_{11}}{\rho_{11}+\beta_j\rho_{12}} = \frac{A_{21}+\beta_j A_{21}}{\rho_{12}+\beta_j\rho_{22}}$, $b_j^2 = \frac{A_{12}+\gamma_j B_{12}}{\rho_{11}+\gamma_j\rho_{12}} = \frac{B_{22}+\gamma_j A_{22}}{\rho_{12}+\gamma_j\rho_{22}}$, and the parameters β, γ are determined from the following algebraic equations

$$a_1^*\beta^2 + b_1^*\beta + c_1^* = 0, \quad a_2^*\gamma^2 + b_2^*\gamma + c_2^* = 0, \quad (9)$$

where $a_1^* = B_{11}\rho_{22} - A_{21}\rho_{12}$; $b_1^* = \rho_{12}(B_{11} - A_{21}) + \rho_{22}(A_{11} - B_{21})$; $c_1^* = A_{11}\rho_{12} - A_{21}\rho_{11}$; $a_2^* = B_{12}\rho_{22} - A_{22}\rho_{12}$; $b_2^* = \rho_{12}(B_{12} - A_{22}) + \rho_{22}(A_{21} - B_{22})$; $c_2^* = A_{21}\rho_{12} - A_{22}\rho_{11}$,

$$\frac{\rho_{11} + \rho_{12}\beta}{A_{11} + B_{11}\beta} = \frac{\rho_{12} + \rho_{22}\beta}{A_{21}\beta + B_{21}}; \quad \frac{\rho_{11} + \rho_{12}\gamma}{A_{12} + B_{12}\gamma} = \frac{\rho_{12} + \rho_{22}\gamma}{A_{22}\gamma + B_{22}}.$$

By a_j we denote the longitudinal velocities, and by b_j the velocities of the transverse waves.

Since the initial system is hyperbolic and the expressions of the velocities contain the elastic constants λ_k, μ_k, we must have the additional constraints $A_{11}B_{21} - A_{21}B_{11} \neq 0$; $\mu_1\mu_2 - \mu_3^2 \neq 0$; $p_1p_2 - p_3p_4 \neq 0$; $\vartheta \neq 1$, where $p_j = \lambda_j + \mu_j$; $\vartheta = \vartheta_1 - \vartheta_2$; $\vartheta_1 = \frac{\alpha_2(\rho_2p_2 - \rho_1p_4)}{\rho(p_1p_2 - p_3p_4)}$; $\vartheta_2 = \frac{\alpha_2(\rho_1p_1 - \rho_2p_3)}{\rho(p_1p_2 - p_3p_4)}$.

Let us look for a self-similar solution in the third zone by using the Smirnov-Sobolev method of invariant relations [4] . The variables are written as $\xi = x/(at)$, $\eta = y/(at)$, and the solution as

$$U_i(x, y, t) = at\bar{U}_i(\xi, \eta), V_i(x, y, t) = at\bar{V}_i(\xi, \eta),$$

$$\delta_i\,(x,\,y,\,t) = \bar{\delta}_i\,(\xi,\ \ \eta)\,, \omega_i\,(x,\,y,\ t) = \bar{\omega}_i\,(\xi,\ \eta)\,. \tag{10}$$

Thus, the waves equations (8) become the equations of a mixed type: elliptic inside a circle of radii a_i, b_i $(i = 1, 2)$, and hyperbolic outside it

$$\begin{aligned}
&\left(\xi^2 - \alpha_i^2\right) \tfrac{\partial^2 \bar{\delta}_i}{\partial \xi^2} + 2\,\xi\,\eta \tfrac{\partial^2 \bar{\delta}_i}{\partial \xi\,\partial \eta} + \left(\eta^2 - \alpha_i^2\right) \tfrac{\partial^2 \bar{\delta}_i}{\partial \eta^2} + 2\,\xi \tfrac{\partial \bar{\delta}_i}{\partial \xi} + 2\,\eta \tfrac{\partial \bar{\delta}_i}{\partial \eta} = 0,\\
&\left(\xi^2 - \varepsilon_i^2\right) \tfrac{\partial^2 \bar{\omega}_i}{\partial \xi^2} + 2\,\xi\,\eta \tfrac{\partial^2 \bar{\omega}_i}{\partial \xi\,\partial \eta} + \left(\eta^2 - \varepsilon_i^2\right) \tfrac{\partial^2 \bar{\omega}_i}{\partial \eta^2} + 2\,\xi \tfrac{\partial \bar{\omega}_i}{\partial \xi} + 2\,\eta \tfrac{\partial \bar{\omega}_i}{\partial \eta} = 0,
\end{aligned} \tag{11}$$

where $\alpha_i = a_i/a$, $\varepsilon_i = b_i/a$, $\bar{A}_{i\,j} = A_{i\,j}/a$, $\bar{B}_{i\,j} = B_{j\,i}/a$, $(i = 1,\ 2)$.

In order to completely formulate the problem in δ_i, ω_i, we must add the boundary conditions. Thus, the boundary conditions (0.3) at $\xi = 0$ read

$$\overline{\Omega}_1\big|_{\xi=0} = \overline{\Omega}_2\big|_{\xi=0} = \left.\frac{\partial \overline{\Delta}_1}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \overline{\Delta}_2}{\partial \xi}\right|_{\xi=0} = 0. \tag{12}$$

The boundary conditions (4) at $\eta = 0$ have the form

$$\begin{aligned}
&a_{11} \tfrac{\partial \overline{\delta_1}}{\partial \xi} + a_{12} \tfrac{\partial \overline{\delta_2}}{\partial \xi} + a_{13} \tfrac{\partial \overline{\omega_1}}{\partial \eta} + a_{14} \tfrac{\partial \overline{\omega_2}}{\partial \eta} = 0,\\
&a_{21} \tfrac{\partial \overline{\delta_1}}{\partial \xi} + a_{22} \tfrac{\partial \overline{\delta_2}}{\partial \xi} + a_{23} \tfrac{\partial \overline{\omega_1}}{\partial \eta} + a_{24} \tfrac{\partial \overline{\omega_2}}{\partial \eta} = 0,\\
&a_{31} \tfrac{\partial \overline{\delta_1}}{\partial \eta} + a_{32} \tfrac{\partial \overline{\delta_2}}{\partial \eta} + a_{33} \tfrac{\partial \overline{\omega_1}}{\partial \xi} + a_{34} \tfrac{\partial \overline{\omega_2}}{\partial \xi} = 0,\\
&a_{31} \tfrac{\partial \overline{\delta_1}}{\partial \eta} + a_{32} \tfrac{\partial \overline{\delta_2}}{\partial \eta} + a_{33} \tfrac{\partial \overline{\omega_1}}{\partial \xi} + a_{34} \tfrac{\partial \overline{\omega_2}}{\partial \xi} = 0,
\end{aligned} \tag{13}$$

where a_{ij} $(i, j = \overline{1, 4})$ are constants depending on the parameters of a material and on the parameters β, γ. Thus, the problem is reduced to solve the system of differential equations (11) under the boundary conditions (1), (2).

The characteristic equations for (11) are

$$\left(\xi^2 - \alpha_i^2\right)\ d\eta^2 - 2\xi\,\eta\ d\xi\ d\eta + \left(\eta^2 - \alpha_i^2\right)\ d\,\xi^2 = 0,$$

$$\left(\xi^2 - \varepsilon_i^2\right)\ d\eta^2 - 2\xi\,\eta\ d\xi\ d\eta - \left(\eta^2 - \varepsilon_i^2\right)\ d\,\xi^2 = 0.$$

The common solution for these equations are the straightlines

$$c_i\xi \pm \sqrt{1 - \alpha_i^2 c_i^2}/\alpha_i\eta - 1 = 0, \quad d_i\xi \pm \sqrt{1 - \varepsilon_i^2 d_i^2}/\varepsilon_i\eta - 1 = 0. \tag{14}$$

Special solutions of these equations are the circles of radii a_i and b_i $(i = 1, 2)$ respectively. Relations (14) describe the two sets of straightlines in the hyperbolic zones tangent to a circle of the relevant radius. Obviously it is possible to present an ondular pattern of the perturbed area.

Let us reduce the equations (11) to a canonical form. For this purpose from (14) we derive c_i and d_i

$$c_i = \frac{\xi}{\xi^2 + \eta^2} \pm \frac{\eta\sqrt{\xi^2 + \eta^2 - \alpha_i^2}}{\alpha_i\left(\xi^2 + \eta^2\right)}; \quad d_i = \frac{\xi}{\xi^2 + \eta^2} \pm \frac{\eta\sqrt{\xi^2 + \eta^2 - \varepsilon_i^2}}{\varepsilon_i\left(\xi^2 + \eta^2\right)}. \tag{15}$$

Thus, the reduction of the initial set of equations (11) in variables ξ, η to a canonical form, reduces to the problem of searching a solution of the Laplace equations in the given quarter of the plane. The general solution of the Laplace equations is looked for as the real part of some arbitrary analytical functions in the relevant quarters of the plane

$\bar{\delta}_j = Ref_j(Q_j)$,(j=1,2) where $Q_j = \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{\alpha_j^2-(\xi^2+\eta^2)}}{\alpha_j(\xi^2+\eta^2)}$, (j=1,2),

$\bar{\omega}_j = ReF_j(Q_{2+j})$, (j=1,2) where $Q_{2+j} = \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{\varepsilon_j^2-(\xi^2+\eta^2)}}{\varepsilon_j(\xi^2+\eta^2)}$, (j=1,2).

The boundary conditions (1) at $\xi = 0$ in the variables Q become

$$Re\left[f_j'(Q_j)\right] = 0,\ Re\left[iF_j'(Q_{2+j})\right] = 0. \tag{16}$$

Since $\eta = 0$, we have $Q_j = Q_{2+j} = 1/\xi = Q$, the conditions (2) imply

$$\begin{aligned}
&a_{11}f_1'(Q)\left(-Q^2\right) + a_{12}f_2'(Q)\left(-Q^2\right) + a_{13}F_1'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\varepsilon_1^2}}\right) + \\
&a_{14}F_2'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\varepsilon_2^2}}\right) = i\varphi_1(Q), a_{21}f_1'(Q)\left(-Q^2\right) + a_{22}f_2'(Q)\left(-Q^2\right) + \\
&a_{23}F_1'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\varepsilon_1^2}}\right) + a_{24}F_2'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\varepsilon_2^2}}\right) = i\varphi_2(Q), \\
&a_{31}f_1'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\alpha_1^2}}\right) + a_{32}f_2'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\alpha_2^2}}\right) + a_{33}F_1'(Q)\left(-Q^2\right) + \\
&a_{34}F_2'(Q)\left(-Q^2\right) = i\varphi_3(Q), a_{41}f_1'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\alpha_1^2}}\right) + a_{42}f_2'(Q)\left(iQ\sqrt{Q^2-\tfrac{1}{\alpha_2^2}}\right) + \\
&a_{43}F_1'(Q)\left(-Q^2\right) + a_{44}F_2'(Q)\left(-Q^2\right) = i\varphi_4(Q),
\end{aligned} \tag{17}$$

where $\varphi_m(Q)$, $\left(m = \overline{1,\ 4}\right)$ are arbitrary real-valued functions of a real argument.

Solving this system of the algebraic equations in $f_j^{`}(Q)$ and $F_j^{`}(Q)$, we find

$$f_j^{`}(Q) = \frac{D_j}{D}, \quad F_j^{`}(Q) = \frac{D_{2+j}}{D}, \tag{18}$$

where D is the determinant of the system, and D_j are the Cramer determinants. Equating D to zero, we obtain the Rayleigh equations, defining the velocities of the Rayleigh waves along the free boundary of a two-component medium.

On the basis of formulas (18) the functions $f_j^{`}(Q)$ and $F_j^{`}(Q)$ may be analytically continuated to all right quarter of the complex plane variable Q.

In order to find the functions $f_j^{`}(Q)$ and $F_j^{`}(Q)$ it is necessary to determine the auxiliary functions $\varphi_m(Q)$. For this purpose it is necessary to use the boundary conditions and the fact that the functions $\bar{\delta}_j$ are bounded at the Rayleigh points.

Before determining the functions $\varphi_m(Q)$ let us mention some of their proprieties. On the basis of the equalities (16), (17) it follows that functions $\varphi_1(Q)$ and $\varphi_2(Q)$ on a conjugate axis are real-valued functions, and $\varphi_3(Q)$ and $\varphi_4(Q)$ are imaginary-valued functions.

By the reflection principle, these functions can be analytically continuated in the second quadrant, and all of them are real on the entire real axis. Hence, they may be analytically continuated in the lower half-plane too.

The formulas of analytic continuation of these functions show that $\varphi_1(Q)$, $\varphi_2(Q)$ are even, and $\varphi_3(Q)$, $\varphi_4(Q)$ are odd. Let us mention that the singular points of the functions $\varphi_m(Q)$ may be situated only on the boundaries of the appropriate area and only at the points of a break. At the passage to a plane Q these points went on the real axis. Therefore all singular points of these functions in the planes Q are situated on the real axis. These points are 0; $\pm 1/\alpha_i$; $\pm 1/\varepsilon_i$; ∞ and the Rayleigh point, and they may be only poles of the first order. It follows that in the plane Q, $\varphi_m(Q)$ are rational functions. Investigating them at each of the indicated points, we conclude that $\varphi_1(Q)$ and $\varphi_2(Q)$ have a pole of the first order at the points $Q = \pm 1/\alpha_1$, and $Q = \pm 1/\alpha_2$. Similarly, $\varphi_3(Q)$ and $\varphi_4(Q)$ have poles at the points $Q = 0$ and $Q = \pm 1/\varepsilon_1$, $Q = \pm 1/\varepsilon_2$.

As a consequence these functions may be written as

$$\varphi_1(Q) = \frac{C_1^*}{\left(Q^2 - 1/\alpha_1^2\right)\left(Q^2 - 1/\alpha_2^2\right)}, \varphi_2(Q) = \frac{C_2^*}{\left(Q^2 - 1/\alpha_1^2\right)\left(Q^2 - 1/\alpha_2^2\right)},$$
$$\varphi_3(Q) = \frac{C_3^*}{Q\left(Q^2 - 1/\varepsilon_1^2\right)\left(Q^2 - 1/\varepsilon_2^2\right)}, \quad \varphi_4(Q) = \frac{C_4^*}{Q\left(Q^2 - 1/\varepsilon_1^2\right)\left(Q^2 - 1/\varepsilon_2^2\right)}, \tag{19}$$

where C_m^* - are arbitrary constants.

From the condition that the functions δ_i are bounded at the Rayleigh point we obtain that $C_1^* = C_2^* = 0$, and from the boundary conditions at $\eta = 0$ we find C_3^* and C_4^*.

Substituting (0.21) in formulas (0.20) we obtain explicit expressions for functions $f_j^{'}(Q)$ and $F_j^{'}(Q)$.

By a direct check it is possible to show that the obtained solution, defined by these functions, satisfies the initial equations, the Laplace equations, and the boundary conditions formulated earlier. From them it is possible to determine the functions $\bar{\delta}_j(\xi,\ \eta)$, $\bar{\omega}_j(\xi,\ \eta)$, hence, the components of the displacement vector and the stress tensor in the entire quarter of a circle of radius equal to unity.

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ROMAI J., **1**, 2(2005),
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EXTENSIONS AND MAPPINGS OF TOPOLOGICAL SPACES

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Abstract In the present paper the class of extensions of topological spaces and the methods of constructing of special extensions are investigated. The notions of quasi-compactness, compactness and double compactness are considered. Various problems of the theory of extensions are stated.

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Key words: compactness, extension, compactification, remainder, uniform space, singular mapping, superperfect mappings.

Introduction

Compactness is one of the most important notions. A quasi-compactness is a class of spaces which is multiplicative, hereditary with respect to closed subspaces and contains an infinite T_0-space.

The concept of a compact space was introduced by L. Vietoris [117], P.S. Alexandroff and P.S. Urysohn [2] and is due to the works of E. Borel, H. Lebesgue, K. Kuratowski, W. Serpinski, S. Saks (see [34, 44, 90]).

The general notion of compactness is due to the works of P. S. Alexandroff and P. S. Urysohn [2], E. Hewitt [64], R. Arens and J. Dugundji [7], L. Nachbin [83], S. Mrowka and R. Engelking [43,81], H. Herrlich [62], H. Herrlich and J. Vander Slot [63], M. Hušek and J. de Vries [67], Z. Frolik [51], R. N. Bhanmik and D. N. Misra [19], G. Viglino [118], A. P. Shostak [140] (see [44]).

For every space E there exists the minimal quasi-compactness P such that $E \in P$ (see [43, 44, 81]).

Theory of compactifications is a wide and vast branch of topology and its applications.

One-point compactification of the plane was studied by G. Riemann and compactifications of open subsets of the plane were studied by C. Caratheodory in connection with some problems of analytic functions. The notion of the extension was used by R. Dedekind and G. Cantor in the theory of real numbers and by F. Hausdorff in the theory of metric spaces (see [30, 34, 44, 90, 121]).

Let P be a quasi-compactness. A generalized P-extension of a space X is a pair (eX, f), where $eX \in P$, $f : X \to eX$ is a continuous mappings and the

set $f(X)$ is dense in eX. If f is an embedding, then eX is called a P-extension or a P-compactification of the space X.

The general problems of the theory of P-extensions are the following.

First General Problem: To find the methods to construct and study the P-extensions and special P-extensions of a given space X.

Second General Problem: To study the class $GE(X)$ of all generalized P-extensions of a given space X.

Third General Problem. Under which conditions the class $GE(X)$ is a complete lattice?

Fourth General Problem. Let GE(X) be a lattice and let $\beta_P X$ be the maximal element in $GE(X)$. To study the properties of spaces $\beta_P X$ and $\beta_P X \setminus X$.

Fifth General Problem. Let X and Y be spaces. Under which conditions there exists a P-extension eX of X such that Y and $eX \setminus X$ are homeomorphic?

Various important problems of the theory of extensions were formulated in [3, 12, 17, 34, 49, 59, 90, 103, 119, 121, 129].

The purpose of the present paper is to investigate the class of P-extensions of topological spaces and the methods of constructing of new P-extensions of topological spaces.

In Section 1 we discuss the general notions and problems. We introduce the notion of double compactness. In the final part of the section we give examples and concrete problems of the theory of extensions.

Section 2 is devoted to investigation of the methods of construction of extensions.

The method of perfect mappings was used by M. C. Raybom [89] in the constructions of Hausdorff compactifications for locally compact spaces. We introduce the method of superperfect mappings for arbitrary spaces. These methods are used for investigation of the lattice of compactifications (see [1, 5, 27, 32, 55, 58, 68, 71,
72, 74, 76, 82, 95, 106, 114, 116, 119, 124]).

The method of singular mappings was introduced in [32] for construction of the Hausdorff compactifications of locally compact spaces.

The Wallman-Shanin method was introduced by W. H. Wallman [122] and N. A. Shanin [96, 97, 98, 99]. The notion of the base-ring was introduced by O. Frink [50], E. F. Steiner [106, 109], V. I. Zaitsev [128]. In [50] O. Frink formulated the problem: Is every Hausdorff compactification of a completely regular space of the Wallman-Shanin type? The problem of O. Frink was studied by many authors (see [49, 55, 79, 85, 90, 105, 106, 109, 113]) and it was negatively solved by V. M. Uljanov [115].

The spectrum of rings (see [15, 29, 52, 53, 58, 65, 66, 84, 90, 110, 111, 119, 124]) was used by L. I. Calmutskii [24, 28, 131, 132, 133] to introduce the notion of spectral compactifications.

In Section 3 we study the uniform extensions of completely regular spaces. The construction of maximal uniform extension μX of a space X is due to J. Diendonné and to F. Hausdorff [44]. The concept of a uniform space and the notion of a complete uniform space were introduced by A. Weil (see [44]). The completions of separable metric spaces were studied by J. M. Aarts and P. V. van Emde Boas [1]. The completions of arbitrary metric spaces were studied by V. K. Bel'nov [20,127]. An important part of the methods of construction of extensions of a space is to present the "new points" of the extension as a space with concrete properties. We simplify and extend the "Bel'nov's gluing method" to theory of uniform completions of arbitrary completely regular spaces.

In this article we shall use the following notation:

We denote by $cl_X A$ or clA the closure of a set A in a space X.

We denote by $|A|$ the cardinality of a set A.

We denote by $w(X)$ the weight of a space X.

The interval $[0, 1]$ is denoted by I.

On the set $N = \{1, 2, \dots\}$ we consider only the discrete topology.

We use the terminology from [44,34,90].

General notions and problems

Let L be a partially ordered set. Fix a non-empty subset A of L. We consider that $a = \vee A$ if $a \geq x$ for every $x \in A$ and if $b \geq x$ for each $x \in A$, then $b \geq a$. We consider that $c = \wedge A$ if $c \leq x$ for every $x \in A$ and if $b \leq x$ for each $x \in A$, then $b \leq c$.

The set L is called:

- an upper semi-lattice if there exists the element $\vee L$ and for every two elements $x, y \in L$ there exists the element $x \vee y = \vee\{x, y\}$;

- a lower semi-lattice if there exists the element $\wedge L$ and for every two elements $x, y \in L$ there exists the element $x \wedge y = \wedge\{x, y\}$;

- a complete upper semi-lattice if for every non-empty subset $A \subseteq L$ there exists the element $\vee A$;

- a lattice if L is an upper semi-lattice and a lower semi-lattice;

- a complete lattice if L is a lower semi-lattice and a complete upper semi-lattice.

We mention that in the complete lattice L for every non-empty subset $A \subseteq L$ there exists the element $\wedge A$.

Let L be a complete upper semi-lattice and M be a non-empty subset of L. If for every two elements $x, y \in M$ we have $x \vee y \in M$, then M is called an upper subsemi-lattice of L. In the similar way there are defined the notions of a lower subsemi-lattice and of a sublattice.

1.1. Extensions of spaces

1.1.1. Definition. A g-extension of a space X is called a pair $(Y,\ f)$, where Y is a non-empty T_0-space, $f : X \to Y$ is a continuous mapping and $\{cl_Y f(A) : A \subseteq X\}$ is a closed base of the space Y.

1.1.2. Definition. A g-extension $(Y,\ f)$ of a space X is called an extension of X if f is an embedding of X in Y.

1.1.3. Remark. If $(Y,\ f)$ is a g-extension of a space X, then the set $f(X)$ is dense in Y.

Denote by $E(X)$ the family of all extensions of a space X and by $GE(X)$ the family of all g-extensions of the space X. The family $GE(X)$ is partially ordered in the standard way: $(Y_1,\ f_1) \leq (Y_2,\ f_2)$ if there exists a continuous mapping $\varphi : Y_2 \to Y_1$ such that $f_1(x) = \varphi(f_2(x))$ for every $x \in X$, i. e. $f_1 = \varphi \circ f_2$.

If $(Y_1,\ f_1),\ (Y_2,\ f_2) \in GE(X),\ \varphi : Y_2 \to Y_1$ and $\psi : Y_1 \to Y_2$ are continuous mappings, $f_1 = \varphi \circ f_2$ and $f_2 = \psi \circ f_1$, then $\psi = \varphi^{-1}$ and φ and ψ are homeomorphisms. Thus $(Y_1,\ f_1) = (Y_2,\ f_2)$ provided $(Y_1,\ f_1) \leq (Y_2,\ f_2)$ and $(Y_2,\ f_2) \leq (Y_1,\ f_1)$.

If $i \in \{0,\ 1,\ 2,\ 3,\ 3\frac{1}{2}\}$, then $GE(X) = \{(Y,\ f) \in GE(X) : Y$ is a T_i-space$\}$ and $E_i(X) = E(X) \cap GE_i(X) = \{(Y,\ f) \in E(X) : Y$ is a T_i-space$\}$.

1.1.4. Proposition. Let $f : X \to Y$ be a continuous mapping of a space X into a T_i-space Y, the set $f(X)$ is dense in Y and $i \geq 3$. Then $(Y,\ f) \in GE_i(X)$.

Proof. Let F be a closed non-empty subset of Y and $y \in Y \setminus F$. There exist two open subsets U and V of Y such that $F \subseteq U$, $y \in V$ and $U \cap V = \emptyset$. We put $\Phi = cl_Y(f(X) \cap U)$. Then $F \subseteq \Phi$ and $y \notin \Phi$. Hence $\{cl_Y A : A \subseteq f(X)\}$ is a closed base of the space Y. The proof is complete.

1.1.6. Definition. A pair $(Y,\ f)$ is called a weak g-extension (wg-extension) of a space X if $f : X \to Y$ is a continuous mapping, Y is a T_0-space and the set $f(X)$ is dense in Y.

We denote by $WGE(X)$ the family of all wg-extensions of a space X,
$WE(X) = \{(Y,\ f) \in WGE(X) : f$ is an embedding$\}$,
$WGE_i(X) = \{(Y,\ f) \in WGE(X) : Y$ is a T_i-space$\}$ and
$WGE_i(X) = WE(X) \cap WGE_i(X)$.

1.1.7. Proposition. Let X be a non-empty T_0-space. Then:

1. $WE(X)$ is n*ot* a set.
2. $WGE(E)$ is not a set.
3. If $i \geq 2$, then $WGE_i(X)$ is a set.
4. $GE(X)$is a set.

Proof. Let Z be a non-empty T_0-space and $Z \cap X = \emptyset$. We put $Y = X \cup Z$, $f(x) = x$ for every $x \in X$,- $\mathrm{Im}_0 = \{H \subseteq X : H$ is open in $X\} \cup \{X \cup V : V \subseteq Z$ and V is open in $Z\}$. Then Im_0 is a T_0-topology on Y and

$(Y, f) \in WE(X)$. Thus $WE(X)$ is not a set. Hence $WGE(X)$ is not a set, too.

Let $m = |X|$ and $\tau = 2^{2^\tau}$. If $(Y, f) \in GE(X) \cup WGE_2(X)$, then $|Y| \leq \tau$. Therefore $WGE_2(X) \cup GE(X)$ is a set. The proof is complete.

1.1.8. Proposition. Let X be an infinite T_1-space. Then $WE_1(X)$ is not a set. In particular, $WE_1(X)$ is not a set.

Proof. Let Z be a non-empty T_1-space and $Z \cap X = \emptyset$. We put $Y = X \cup Z$, $f(x) = x$ for every $x \in X$ and $\Gamma_1 = \{H \subseteq X : H$ is open in $X\} \cup \{V \subseteq Y : V \cap Z$ is open in Z and the set $X \backslash V$ is finite$\}$. Then Γ_1 is a T_1-topology on Y, $\Gamma_0 \subseteq \Gamma_1$ and $((Y, \Gamma_1), f) \in WE(X)$. The proof is complete.

1.1.9. Remark. If in the proof of Proposition 1.1.7 or of Proposition 1.1.8 the space Z is compact, then the space (Y, Γ_0) or (Y, Γ_1) is compact, too.

If (Y_1, f_1), $(Y_2, f_2) \in WGE(X)$, then $(Y_1, f_1) \leq (Y_2, f_2)$ if there exists a continuous mapping $\varphi : Y_2 \to Y_1$ such that $f_1 = \varphi \circ f_2$.

1.1.10. Proposition. The relation $\leq$ is an ordering on $WGE_2(X)$.

Proof. Is obvious.

1.1.11. Example. Let X be a non-empty space. Then $\leq$ is not an ordering on $WE(X)$.

Let Z be a non-empty T_0-space, $Z \cap X = \emptyset$ and $b \in Z$. Consider the space $Y_1 = Z \cup X$ with the topology $\Gamma_0 = \{U \subseteq X : U$ is open in $X\} \cup \{V \cup X : V$ is open in $Z\}$ and subspace $Y_2 = \{b\} \cup X$ of Y_1. Let $f(x) = x$ for each $x \in X$. Then (Y_1, f), $(Y_2, f) \in WE(X)$. We put $\varphi(y) = y$ for every $y \in Y_2$, $f = \psi \,|\, X$ and $\psi(y) = b$ for every $y \in Z$. Then the mappings $\varphi : Y_2 \to Y_1$ and $\psi : Y_1 \to Y_2$ are continuous and $\varphi(x) = \psi(x) = x$ for each $x \in X$. Thus $(Y_1, f) \leq (Y_2, f)$, $(Y_2, f) \leq (Y, f)$ and $(Y, f) \neq (Y_2, f)$ provided $|Z| \geq 2$.

1.1.12. Example. Let X be an infinite T_1-space. Then $\leq$ is not an ordering on $WE_1(X)$.

Let Z be a T_1-space, $|Z| \geq 2$, $b \in Z$, $Y_1 = Z \cup X$ be a space with the topology $\Gamma_1 = \{U \subseteq X : U$ is open in $X\} \cup \{V \subseteq Y_1 : V \cap Z$ is open in Z and the set $X \backslash V$ infinite$\}$, $Y_2 = \{b\} \cup X$ be a subspace of Y_1 and $f(x) = x$ for each $x \in X$. Then (Y_1, f), $(Y_2, f) \in WE_1(X)$, $(Y_1, f) \leq (Y_2, f), (Y_2, f) \leq (Y_1, f)$ and $(Y_1, f) \neq (Y_2, f)$.

Let X be a space. On the class $WGE(X)$ we consider the relation $\sim$:$(Y_1, f_1) \sim (Y_2, f_2)$ iff $(Y_1, f_1) \leq (Y_2, f_2)$ and $(Y_2, f_2) \leq (Y_1, f_1)$. Obviously, $\sim$ is a relation of equivalence. Denote by $WGE^0(X)$ the classes of equivalence on $WGE(X)$ and by $WE^0(X)$ the classes of equivalence on $WE(X)$.

Obviously $\leq$ is an ordering on the a class $WGE^0(X)$.

1.1.13. Proposition. Let $H = \{(Y_\alpha, f_\alpha) \in WGE(X) : \alpha \in A\}$ be a set, $f(x) = (f_\alpha(x) : \alpha \in A)$ for every $x \in X$ and Y be the closure of the set $f(X)$ in the space $\Pi\{Y_\alpha : \alpha \in A\}$. Then:

1. $(Y, f) \in WGE(X)$ and we put $(Y, f) = \vee H$.
2. $(Y_\alpha, f_\alpha) \leq (Y, f)$ for each $\alpha \in A$.

3. If $(Z, g) \in WGE(X)$ and $(Y_\alpha, f_\alpha) \leq (Z, g)$ for each $\alpha \in A$, then $(Y, f) \leq (Z, g)$.

4. If $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$ and Y_α is a T_i-space, then Y is a T_i-space.

5. If $H \cap WE(X) \neq \emptyset$, then $(Y, f) \in WE(X)$.

Proof. For every $\beta \in A$ we consider the projection $\varphi_\beta : Y \to Y_\beta$, where $\varphi_\beta(y_\alpha : \alpha \in A) = y_\beta$ for any $(y_\alpha : \alpha \in A) \in Y$. Then $f_\alpha = \varphi_\alpha \circ f$ for each $\alpha \in A$. The assertions 1, 2, 4 and 5 are proved. Let $(Z, g) \in WGE(X)$ and $(Y_\alpha, f_\alpha) \leq (Z, g)$ for each $\alpha \in A$. For any $\alpha \in A$ we fix a continuous mapping $\psi_\alpha : Z \to Y_\alpha$ such that $f_\alpha = \psi_\alpha \circ g$. Consider the mapping $\psi : Z \to \Pi\{Y_\alpha : \alpha \in A\}$, where $\psi(z) = (\psi_\alpha(z) : \alpha \in A)$. The mapping ψ is continuous, $\psi(g(X)) = f(X)$ and $\psi(Z) \subseteq Y$. The assertion 3 and Proposition are proved.

1.1.14. Question. Is it true that $WE^0(X)$ is a set for each topological space X?

Obviously, $WE^0(X)$ is a set for every space X iff $WGE^0(X)$ is a set for every space X.

1.1.15. Remark. Let X be a non-empty space, D_0 be a singleton space, $f_m : X \to D_0$ be the unique mapping of X into D_0, $f_M(x) = x$ for each $x \in X$. Then (X, f_M) is the maximal element in $WGE(X)$ and (D_0, f_M) is the minimal element in $WGE(X)$. Obviously, $(X, f_M), (D_0, f_M) \in GE(X)$.

1.1.16. Question. Let X be a space and H be a non-empty subset of the set $GE(X)$. Is it true that $\vee H \in GE(X)$?

1.1.17. Corollary. Let $i \geq 2$. Then $WGE_i(X)$ is a complete lattice.

1.1.18. Corollary. Let $i \geq 2$. Then $WE_i(X)$ is a complete upper semi-lattice.

1.1.19. Corollary. Let $i \geq 3$. Then $GE_i(X)$ is a complete lattice.

1.1.20. Corollary. Let $i \geq 3$. Then $E_i(X)$ is a complete upper semi-lattice.

1.2. The canonical functor $m : WGE(X) \to GE(X)$

Consider a topological space X. Fix a wg-extension (Y, f) of the space X. Let Γ_Y be the topology of the space Y. On Y consider a new topology Γ_{Yf} generated by the closed base $\{cl_Y H : H \subseteq f(X)\}$. There exist a set Y_f and a mapping $P_Y : Y \to Y_f$ such that $P_Y^{-1}(P_Y(H)) = H$ for every $H \in \Gamma_{Yf}$ and $\Gamma^0_{Yf} = \{P_Y(H) : H \in \Gamma_{Yf}\}$ is a T_0-topology on a set Y_f.

If $y \in Y$, then$P_Y^{-1}(P_Y(y)) = (\cap\{Y \in \Gamma_{Yf} : y \in U\}) \cap (\cap\{Y \setminus U : U \in \Gamma_{Yf}, y \notin U\})$. Consider the mapping $Pf : X \to Y_f$, where $Pf = P_Y \circ f$. By construction, $(Y_f, Pf) \in GE(X)$. We put $(Y_f, Pf) = m(Y, f)$, $Y_f = m(Y)$ and Pf=$m(f)$.

The canonical functor $m : WGE(X) \to GE(X)$ is constructed.

From the construction it follows.

1.2.1. Proposition. If $(Y, f) \in GE(X)$, then $m(Y, f)$=(Y, f).

1.2.2. Question. Is it true that the functor m is covariant?

1.3. The canonical functors $m_i : WGE(X) \to WGE_i(X)$

Fix $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$. For every space Y there exist a unique T_i-space Y/i and a unique projection $i_Y : Y \to Y/i$ with the properties:

1. i_Y is a continuous mapping onto Y/i;
2. for every continuous mapping $\varphi : Y \to Z$ in a T_i-space Z there exists a unique continuous mapping $\bar{\varphi} : Y/i \to Z$ such that $\varphi = \bar{\varphi} \circ i_Y$.
3. if $\psi : Y \to Z$ is a continuous mapping, then there exists a unique continuous mapping $\bar{\psi} : Y/i \to Z/i$ such that $\bar{\psi} \circ i_Y = i_Z \circ \psi$.

The space Y/i with the projection i_Y is called the i-replic of the space Y.

Fix a space X. If $(Y, f) \in WGE(X)$, then we put $f_i = i_Y \circ f$ and $m_i(Y, f) = (Y/i, f_i)$. From the construction it follows.

1.3.1. Proposition. $m_i : WGE(X) \to WGE_i(X)$ is a covariant functor. If $(Y, f) \leq (Z, g)$, then $m_i(Y, f) \leq m_i(Z, g)$. If $(Y, f) \in WGE(X)$, then $m_i(Y, f) = (Y, f)$.

1.4. Compactness

The notion of compactness is due to E. Mrowka [81,43], E. Hewit [64], R. Arens and S. Dugundji [7].

A class P of topological T_0-spaces is called a strongly compactness if the following conditions are fulfilled:

C_1. the class P is non-empty;

C_2. there exists a space $X \in P$ such that $|X| > 2$;

C_3. the class P is multiplicative, i. e. if $\{X_\alpha \in P : \alpha \in A\}$ is a non-empty set of spaces from P, then $\Pi\{X_\alpha : \alpha \in A\} \in P$;

C_4. the class P is closed hereditary, i. e. if Y is a closed subspace of a space $X \in P$, then $Y \in P$;

C_5. if Y is a dense subspace of a space $X \in P$, then $\{cl_X A : A \subseteq Y\}$ is a closed base of the space X.

A class of spaces P with properties $C_1 - C_4$ is called a quasi-compactness. A quasi-compactness P of Hausdorff spaces is called a compactness.

Fix a quasi-compactness P. For every space X we put $WPGE(X) = \{(Y, f) \in WGE(X) : Y \in P\}, WPE(X) = WPGE(X) \cap WE(X), PGE(X) = WPGE(X) \cap GE(X)$ and $PE(X) = PGE(X) \cap E(X)$.

If P is a compactness, then $WPGE(X) = PGE(X)$ and $WPE(X) = PE(X)$. From Proposition 1.1.7. it follows that $PGE(X)$ and $PE(X)$ are the sets for each space X.

1.4.1. Theorem. Let P be a compactness and X be a space. Then $\vee H \in WPGE(X)$ for every non-empty set $H \subseteq WPGE(X)$.

Proof. Follows immediately from the conditions C_3, C_4 and properties of Hausdorff spaces.

1.4.2. Corollary. Let P be a compactness. Then $WPGE(X)$ is a complete lattice for every space X.

Denote by $(\beta_P X,\, \beta_P)$ the maximal element of the lattice $WPGE(X)$, where P is a compactness.

1.4.3. Corollary. Let P be a compactness, X be a space and $WPE(X) \neq \emptyset$. Then:

1. $WPE(X)$ is a complete upper semi-lattice;
2. $(\beta_P X,\, \beta_P) \in WPE(X)$.

1.4.4. Theorem. Let P be a compactness. For every continuous mapping $f : X \to Y$ of a space X into a space Y there exists a continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$ such that $\beta_P \circ f = \beta_P f \circ \beta_P$. If every space $Z \in P$ is a T_2-space, then the mapping $\beta_P f$ is unique.

Proof. We may consider that $f(X)$ is dense in Y. Then $g = \beta_P \circ f : X \to \beta_P Y$ is a continuous mapping, the set $g(X)$ is dense in $\beta_P Y$ and $(\beta_P Y,\, g) \in WPGE(X)$. Thus there exists a continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$ such that $g = \beta_P \circ f \circ \beta_P$. The proof is complete.

1.4.5a. Corollary. Let P be a compactness and $f : X \to Y$ be a continuous mapping of a space X into a space $Y \in P$. Then $Y = \beta_P Y$ and there exists a unique continuous mapping $\beta_P f : \beta_P X \to Y$ such that $f = \beta_P f \circ \beta_P$.

1.4.5b. Remark. Let P be a quasi-compactness. Then there exists $\beta_P X \in WPGE(X)$ such that $\beta_P X \in \vee PGE(X)$.

1.4.6. Proposition. Let P be a strongly compactness. Then every space $x \in P$ is a Hausdorff space, i. e. P is a compactness.

Proof. Let $d(X) = \min\{|H| : H \subseteq X, cl_X H = H\}$ be the density of a space X. Consider the space F=$\{0,1\}$ with the topology $\mathrm{Im} = \{\emptyset, \{1\}, \{1,0\}\}$. Suppose that $X \in P$ and X is not a T_1-space. Then the space F is embeddable in X. Suppose that $F \subseteq X$. Denote by b, c the cardinality larger than 2^c (see Proposition 1.1.7). For some cardinal m the space Y is embeddable in $F^m \subseteq X^m$([44],Theorem 2.3.26). Let Z be the closure of Y in X^m. Then the space Z is separable and $|Z| \geq |Y| > 2^c$. If $S \in P$, then $|S| \leq \exp(\exp(d(S)))$. Thus $|S| \leq 2^c$ for every separable space $S \in P$. Therefore every space $S \in P$ is a T_1-space.

Suppose that $X \in P$ and X is not a T_2-space. There exist two distinct points $a, b \in X$ such that $V \cap W \neq \emptyset$ provided V and W are open subsets of X, $a \in V$ and $b \in W$. Fix a cardinal number $\tau > \exp(\exp(|X|))$. We put $\Phi = \{a, b\}$. In X^τ we consider the diagonal $\Delta(X)$ (see [44], p.110). Let Y be the closure of the set $\Delta(X)$ in X^m. Then $\Phi^\tau \subseteq Y$, $|\Delta(X)| = |X|$, $d(Y) \leq |X|$, $|Y| \leq \exp(\exp(|X)) < \tau|$ and $|\Phi^\tau| = 2^\tau = \exp(\tau)$, a contradiction. The proof is complete.

1.5. Double compactness

A class P of topological T_0-spaces is called a double compactness if the following conditions are fulfiled:

D_1. the class P is non-empty;

D_2. there exists a space $X \in D$ such that $|X| \geq 2$;

D_3. if Γ is a topology of the space $X \in P$ then there is determined the completely regular topology $d\Gamma$ on X such that $(X, d\Gamma) \in P$, $\Gamma \subseteq d\Gamma$ and $dd\Gamma = d\Gamma$;

D_4. if $f : X \to Y$ is a continuous mapping of a space (X, Γ) into a space (Y, Γ') and $X, Y \in P$, then f is a continuous mapping of the space $(X, d\Gamma)$ into a the space $(Y, d\Gamma')$;

D_5. if $\{(X_\alpha, \Gamma_\alpha) : \alpha \in A\}$ is a non-empty set of spaces, $(X_\alpha, \Gamma_\alpha) \in P$ for each $\alpha \in A$, $X = \Pi\{X_\alpha : \alpha \in A\}$, Γ is the product of topologies Γ_α on X and Γ' is the product of topologies $d\Gamma_\alpha$ on X, then $\Gamma' \subseteq d\Gamma$;

D_6. if $(X, \Gamma) \in P$, $Y \subseteq X$ and Y is a closed subset of the space $(X, d\Gamma)$, then $(Y, \Gamma \,|\, Y) \in P$ and $d(\Gamma \,|\, Y) \supseteq d\Gamma \,|\, Y$, where $\Gamma \,|\, Y = \{U \cap Y : U \in \Gamma\}$ for the topology Γ on X.

1.5.1. Proposition. Let P be a class of spaces, X be a space, $\{Y_\alpha : \alpha \in A\}$ be a non-empty family of subspaces of the space X, $Y = \cap\{Y_\alpha : \alpha \in A\}$ and $Y_\alpha \in P$ for each $\alpha \in A$. Then:

1. if P is a double compactness, then $Y \in P$;
2. if P is a compactness, then $Y \in P$.

Proof. We may consider that $X = Y_\alpha$ for some $\alpha \in A$. If X is a T_2-space, then Y is a closed subspace of the space $\Pi\{Y_\alpha : \alpha \in A\}$. The assertion 2 is proved. If P is a double compactness, then Y is a closed subspace of the space $\Pi\{Y_\alpha : \alpha \in A\}$ in the topology $d\Gamma$. The assertion 1 and Proposition are proved.

Fix a double compactness P. For every space X we put $PGE(X) = \{(Y, f) \in WGE(X) : Y \in P$ and $f(X)$ is a dense subset of the space $(Y, d\Gamma)$ and $PE(X) = WE(X) \cap PGE(X)$.

From the condition D_6 it follows that $PGE(X)$ and $PE(X)$ are sets.

1.5.2. Theorem. Let P be a double compactness. Then $PGE(X)$ is a complete lattice for every space X.

Proof. Let $\{(Y_\alpha, f_\alpha) : \alpha \in A\}$ be a non-empty subset of the set $PGE(X)$. Denote by Γ_α the topology of the space Y_α and by Γ the topology of the space $\Pi\{Y_\alpha : \alpha \in A\}$. Consider the mapping $f : X \to \Pi\{Y_\alpha : \alpha \in A\}$, where $f(x) = (f_\alpha(x) : \alpha \in A)$ for each $x \in X$. Let Y be the closure of the set $f(X)$ in the space $(\Pi\{Y_\alpha : \alpha \in A\}, d\Gamma)$. Then $(Y, f) \geq (Y_2, f_2)$ for each $\alpha \in A$. From the condition D_4 it follows that if $(Z, g) \in PGE(X)$ and $(Z, g) \geq (Y_\alpha, f_\alpha)$ for each $\alpha \in A$, then $(Z, g) \geq (Y, f)$. Thus $(Y, f) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in A\}$. The proof is complete.

1.5.3. Corollary. Let P be a double compactness, let X be a space and $PE(X) \neq \emptyset$. Then $PE(X)$ is a complete upper semi-lattice.

1.5.4. Theorem. Let P be a double compactness. Then:

1. for every continuous mapping $f : X \to Y$ of a space X into a space Y there exists a unique continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$, $\beta_P \circ f = \beta_P f \circ \beta_P$;

2. for every continuous mapping $f : X \to Y$ of a space X into a space $Y \in P$ there exists a unique continuous mapping $\beta_P f : \beta_P X \to Y$ such that $f = \beta_P f \circ \beta_\Gamma$.

Proof. Let Z be the closure of the set $\beta_P(f(X))$ in the space $(\beta_P Y, d\Gamma)$. Then $(Z, \beta_P \circ f) \in PGE(X)$ and the assertion 1 is proved. If $Y \in P$, then $\beta_P Y = Y$. The proof is complete.

1.6. Examples

1.6.1. Example. Let C be the class of compact Haussdorff spaces. Then C is a strongly compactness. If $(Y, f) \in CGE(X)$, then we say that (Y, f) is a g-compactification of X. If $(Y, f) \in CE(X)$, then (Y, f) is called a compactification of X. For every space X the g-compactification $\beta X = \beta_P X$ is the Stone-Čech g-compactification of X. If X is a completely regular space, then βX is the Stone-Čech compactification of X.

1.6.2. Example. Let C_0 be the class of zero-dimensional compact spaces. Then C_0 is a strongly compactness. If $indX > 0$, then $C_0E(X)=\emptyset$. If $indX=0$, then $mfX = \beta_{C_0}X$ is the Morita-Freudenthal compactification of X. We put $mfX = \beta_{C_0}X$ and $(mfX, mf) = (\beta_{G_0}X, \beta_{G_0})$. The g-compactification mfX is called the maximal zero-dimensional g-compactification of the space X.

1.6.3. Example. Let X be a completely regular space. A subset L of X is called bounded in X if the set $f(L)$ is bounded in the space of reals R for every continuous function $f : X \to R$. A space X is called μ-complete if the closure clL of every bounded subset L is compact. Let C_μ be the class of all μ-complete spaces. Then C_μ is a strongly compactness. The g-extension $(\beta_{G_\mu}X, \beta_{C_\mu}) = (\mu^*X, \mu^*)$ is called the maximal μ-completion of the space X. If X is a completely regular space, then $(\mu^*X, \mu^*) \in E(X)$ and μ^*X is the μ-completion of X.

1.6.4. Example. Let R be the space of reals. A space Z is called a realcompact space if it is homeomorphic to a closed subspace of some space R^A. The class R of all realcompact spaces is a strongly compactness. The g-extension $(\nu X, \nu) = (\beta_R X, \beta_R)$ is the maximal g-realcompactification of the space X. If X is a completely regular space, then νX is the realcompactification of X and $(\nu X, \nu) \in E(X)$. Every realcompact space is μ-complete. Therefore $\nu X \leq \mu^*X$ and $\mu^*X \subseteq \nu X$.

1.6.5. Example. Let U be the class of all complete uniform spaces. If X is a completely regular space, then by U_X we denote the universal uniformity on X (see [44]). Every uniform space is considered and a topological space too. Thus for every space X in $UGE(X)$ the maximal element $(\mu X, \mu)$ is determined, where μX is a complete uniform space, $\mu : X \to \mu X$ is a contin-

uous mapping and the set $\mu(X)$ is dense in μX. The space μX is called the Diendonné completion of the space X. If X is completely regular, then μX is the completion of the uniform space (X, U_X). If $X = \mu X$, then the space X is called a Deudonné complete space. Every Deudonné complete space is μ-complete. For every space X we may consider that $\mu^* X \subseteq \mu X \subseteq \nu X \subseteq \beta X$.

1.6.6. Example. Let P be a compactness such that every space $(Y, \Gamma) \in P$ be completely regular. For every space $(X, \Gamma) \in P$ we put $d\Gamma = \Gamma$. Then P is a double compactness. Therefore every compactness of completely regular spaces may be considered as a double compactness.

1.6.7. Example. For every space (X, Γ) we put $c\Gamma = \{U \in \Gamma : U$ is a compact subset$\}$ and $d\Gamma$ is the topology generated by the open base $\{U_1 \cap U_2 \cap \ldots \cap U_n : n \in N,\ U_1,\ U_2, \ldots, U_n \in \Gamma\} \cup \{X \setminus U : U \in c\Gamma\}$.

A space (X, Γ) is called a spectral space if $c\Gamma$ is an open base of the space X, $U \cap V \in c\Gamma$ is an open base of the space X, $U \cap V \in c\Gamma$ for all $U,\ V \in c\Gamma$ and $(X, d\Gamma)$ is a compact Hausdorff space. Let S be the class of all spectral spaces. Then S is a double compactness. For every T_0-space X we have $SE(X) \neq \emptyset$, i. e. $\beta_S : X \to \beta_S X$ is an embedding.

1.6.8. Proposition. If (X, Γ) is a spectral space, then:

1. (X, Γ) is a compact T_0-space;
2. $(X, d\Gamma)$ is a zero-dimensional compact space;
3. $d\Gamma = \Gamma$ if (X, Γ) is a T_1-space.

Proof. Is obvious (see [132]).

1.6.9. Remark. The class of spectral compactifications of a space X was studied in [24,28,131,132,133].

1.6.10. Example. Let E=[0,1], $F = \{2^{-n} : n \in N\}$ and Im be the topology generated by the base $\{\{t \in E : a < t < b\}$: a, b are real numbers$\} \cup \{V_n = \{t \in E : t < 2^{-n}\} \setminus F : n \in N\}$(see [2] or [44], Example 1.5.7). Then E is a T_2-space and E is not regular. If X is the subspace of irrational numbers of E or $X = E \setminus F$, then $\{cl_E H : H \subseteq X\}$ is not a closed base of E. Thus $E \notin P$ for every compactness P. There exists a minimal quasi-compactness P of Hausdorff spaces such that $E \in P$. Therefore P is a compactness and P is not a strongly compactness.

1.6.11. Example. Let P be the class of all compact T_0-spaces. Then P is a quasi-compactness and P is not a compactness. Obviously $OPE(X) \neq \emptyset$ for every T_0-space X.

1.6.12. Example. Let P be the class of all compact T_1-spaces. Then P is a quasi-compactness and P is not a compactness. It is well–known that $\omega X \in PE(X)$ for every T_1-space X.

1.7. Problems

1.7.1. Problem. Let P be a compactness or a double compactness.

1. Under which conditions the lattices $PGE(X)$ and $PGE(Y)$ are isomorphic?

2. Under which conditions the upper semi-lattices $PE(X)$ and $PE(Y)$ are isomorphic?

3. Which topological properties of a space X are characterized in terms of the objects $PGE(X)$ and $PE(X)$?

4. Which properties of the lattice $PGE(X)$ are characterized by the properties of the space X?

5. Let X be a space and $PE(X) \neq \emptyset$. Which properties of the upper semi-lattice $PE(X)$ are characterized be the properties of the space X?

The program of matching "interesting" topological properties of a completely regular space X with "interesting" properties of the complete upper semi-lattice $PE(X)$ is very important in the theory of extensions. N. Boboc and G. Siretchi [22] has proved that $CE(X)$ is a lattice iff the space X is locally compact. In [76] K. D. Magil has proved that for two locally compact spaces X and Y the semi-lattices $CE(X)$ and $CE(Y)$ are isomorphic iff the spaces $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic.

Another program of investigation is to find the "interesting" compactness and double compactness.

1.7.2. Problem. Let P be a compactness or a double compactness, let X be a space and $PE(X) \neq \emptyset$.

1. Find the methods of constructions the extension $\beta_P X$, some extensions from $PE(X)$ or all extensions $PE(X)$.

2. Let Z be a space. Under which conditions there exists an extension $(Y, f) \in PE(X)$ such that $Y \setminus f(X)$ and Z are homeomorphic?

3. Under which conditions there exists an extension $(Y, f) \in PE(X)$ such that $\dim(Y \setminus f(X)) \geq m$, where $m \in N$?

4. Let $Z \in P$. Under which conditions there exist an extension $(Y, f) \in PE(X)$ and a closed subspace $Z' \subseteq Y \setminus f(X)$ such that Z and Z' are homeomorphic?

2. Some methods of construction of extensions

A mapping $f : X \to Y$ of a space X into a space Y is called:

- a perfect mapping if $f(X) = Y, f$ is continuous, closed and the fibers $f^{-1}(y),\ y \in Y$, are compact;

- a superperfect mapping if $f(X) = Y, f$ is continuous, perfect and there exists a compact set $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for each $x \in X \setminus \Phi$;

- a singular mapping if f is continuous and the set $f^{-1}(V)$ is non-compact for any non-empty open subset V of Y;

- an almost perfect mapping if $f(X) = Y, f$ is continuous, closed and there exists a closed compact set $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for each $x \in X \setminus \Phi$.

2.1. Method of perfect mappings

Let X be an open dense subspace of a space eX, $Y{=}eX\backslash X$ and $h : Y \to Z$ be a perfect mapping onto a space Z. We put $e_hX = Z \cup X$ and consider the mapping $f : eX \to e_hX$, where $f(x) = x$ for every $x \in X$ and $h = f\,|\,Y$. On a space e_hX we consider the quotient topology $\{W \subseteq e_hX : f^{-1}(W)$ is open in $eX\}$.

2.1.1. Property. The mapping f is perfect.

Proof. By construction, the mapping f is continuous and the fibers $f^{-1}(y)$, $y \in e_hX$, are compact. Let F be a closed subset of the space eX. Then $F_1 = h^{-1}(h(F \cap Y))$ is a closed subset of Y, $\Phi = F_1 \cup F$ is closed subset of eX and $\Phi = f^{-1}(f(F))$. Thus $f(F)$ is closed in e_hX. The proof is complete.

2.1.2. Property. If $i \in \{1, 2, 3, 4\}$ and eX is a T_i-space, then e_hX is a T_i-space. Moreover, if eX is a normal space, then e_hX is a normal space.

Proof. The property to be a T_i-space, $i \in \{1, 2, 3, 4\}$, is preserved by the perfect mappings.

2.1.3. Property. If eX and Z are T_0-spaces, then e_hX is a T_0-space.

Proof. Obvious.

2.1.4. Property. X is an open dense subspace of the space e_hX.

Proof. Obvious.

We put $LC(X) = \cup\{U : U$ is an open subset of X and cl_XU is compact$\}$ – the set of locally compactness of a space X. Let $RC(X){=}X\backslash LC(X)$. A space X is almost locally compact if the set $LC(X)$ is dense in X. If $RC(X){=}\emptyset$, then the space X is locally compact.

2.1.5. Theorem. Let eX be an extension of the almost locally compact space X, the set $LC(X)$ is open in eX, $Y{=}eX\backslash LC(X)$, $h : Y \to Z$ is a perfect mapping onto a space Z and $h^{-1}(h(x)) = \{x\}$ for every $x \in RC(X)$. Then there exist an extension e_hX of a space X and a perfect mapping $f : eX \to e_hX$ such that the set $LC(X)$ is open in e_hX.

Proof. Let $X_1 = LC(X)$. Then eX is an extension of the space X_1 and the set X_1 is open in eX. Properties 2.1.1 – 2.1.4 complete the proof.

2.2. Method of superperfect mappings

2.2.1. Theorem. If $f : X \to Y$ is an almost perfect mapping onto a T_1-space Y, then f is superperfect.

Proof. There exists a closed compact subset $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for every $x \in X \setminus \Phi$. Let $F = f(\Phi)$. If $y \in Y\backslash F$, then $f^{-1}(y)$ is a singleton. If $y \in F$, then $f^{-1}(y)$ is a compact set as a closed subset of the subspace Φ. Thus the fibers $f^{-1}(y)$ are compact. The proof is complete.

We say that a subset H of a space X is compact in X if the set cl_XH is compact.

A set $N(f) = \{x \in X : f^{-1}(f(x)) \neq \{x\}\}$ is called the kernel of a mapping $f : X \to Y$.

A mapping $f : X \to Y$ is almost perfect iff $f(X) = Y$, f is closed, continuous and the kernel $N(f)$ is compact in X.

2.2.2. Theorem. Let X be a subspace of a space X_1, $Y = X_1 \setminus X$, $h : Y \to Z$ be an almost perfect mapping onto a space Z and the set $cl_Y N(h)$ is closed in X. Then there exist a unique space S and an unique almost perfect mapping $f : X_1 \to S$ such that $N(f) = N(h)$ and $h = f \,|\, Y$.

Proof. We put $S = X \cup Z$, $Y_1 = cl_Y N(h)$, $X_2 = X_1 \setminus cl_Y N(h)$, $f(x) = h(x)$ for every $x \in Y$ and $f(x) = x$ for every $x \in X$. The space X_2 is open in X_1 and $g = h \,|\, Y_1 : Y_1 \to Z_1 = h(Y_1)$ is a continuous closed mapping. On S we consider the quotient topology. Obviously, $N(f) = N(h)$. By construction, f is a closed continuous mapping. The proof is complete.

2.2.3. Corollary. Let eX be an extension of a space X, $Y{=}eX\setminus X$, $h : Y \to Z$ be an almost perfect mapping onto a space Z and the set $cl_Y N(f)$ be closed in eX. Then there exist an extension e_hX of the space X and an almost perfect mapping $f : eX \to e_hX$ such that:

1. Z is a subspace of the space e_hX and $Z = e_hX \setminus X$;
2. $h = f \,|\, Y$;
3. $N(f) = N(h)$.

2.3. Method of singular mappings

Let P be a quasi-compactness.

A space X is called locally P-compact if for every point $x \in X$ there exists an open subset $U \subseteq X$ such that $x \in U$ and $cl_X U \in P$.

We say that a mapping $f : X \to Y$ is a P-singular mapping if f is continuous and $cl_X f^{-1}(V) \notin P$ for every non-empty open subset $V \subseteq X$.

Consider that the compactness P fulfills the following conditions:

S_1. If Y and Z are closed subspaces of a space X and Y, $Z \in P$, then $Y \cup Z \in P$.

S_2. If Y is a closed subspace of the space X, $Y \in P$, $Z \in P$ provided $Z \subseteq X \setminus Y$ is a closed subset of X and $X\setminus Y{=}\cup\{V : V$ is open in X and $cl_X V \subseteq X \setminus V\}$, then $X \in P$.

In the class of regular spaces Condition S1 follows from Condition S2.

2.3.1. Construction. Let $f : X \to Y$ be a P-singular mapping of a locally P-space X into a compact space $Y \in P$. Obviously that the set $f(X)$ is dense in Y. We put $eX = X \cup Y$, with the topology generated by the open base $\{U \subseteq X : U$ is open in $X\} \cup \{V \cup (f^{-1}(V) \setminus U) : V$ is open in Y, U is open in X and $cl_X \in UP\}$.

Property 1. $eX \in P$.

By construction, $Y \in P$ and $X = eX \setminus Y = \cup\{U \subseteq X : U$ is open in eX and $cl_X U \in P\}$. If U is open in X and $cl_X U$, then $cl_{eX} U = cl_X U$. Let Z be a closed subspace of eX and $Z \cap Y = \emptyset$. For every point $y \in Y$ there exist an open subset V_y of Y and an open subset U_y of X such that $cl_X U_y \in P$, $y \in V_y$

and $Z \cap (f^{-1}V \setminus cl_X U_y) = \emptyset$. Since Y is compact, there exists a finite set F such that $Y = \cup\{V_y : y \in F\}$. Then $Z \subseteq \cup\{cl_X U_y : y \in F\}$. By virtue of Condition S_1, $\cup\{cl_X U_y : y \in F\} \in P$ and $Z \in P$. Condition S_2 completes the proof.

Property 2. X is an open dense subspace of the space eX.

Obviously, X is open in eX. Let $y \in Y$, V be an open subset of Y, U be an open subset of X, $y \in V$ and $cl_X U \in P$. Then the set $W = V \cup (f^{-1}(V) \setminus cl_X U)$ is open in eX and $y \in W$. Since $cl_X f^{-1}(V) \notin P$, then $W \cap X = f^{-1}(V) \setminus cl_X U \neq \emptyset$. Thus the set X is dense in eX.

Property 3. Let $i \in \{0, 1, 2\}$ and X, Y be T_i-spaces. Then eX is a T_i-space.

Let $x, y \in eX$ and $x \neq y$.

Case 1. $x, y \in Y$ and $i \leq 1$.

If V is open in Y, $x \in V$ and $y \notin V$, then $W = V \cup f^{-1}(V)$ is open in eX, $x \in W$ and $y \notin W$.

Case 2. $x, y \in Y$ and $i = 2$.

There exist two open subsets V_1 and V_2 of Y such that $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2 = \emptyset$. The sets $W_i = V_i \cup f^{-1}(V_i)$ are open in $eX, x \in W_1, y \in W_2$ and $W_1 \cap W_2 = \emptyset$.

Case 3. $x \in X$ and $y \in Y$.

There exists an open subset U of X such that $x \in U$ and $cl_X U \in P$. We put $W = eX \setminus cl_X U = Y \cup (f^{-1}(Y) \setminus cl_X U)$. The set W is open in eX, $y \in W$ and $U \cap W = \emptyset$.

Case 4. $x, y \in X$.

Since X is an open subspace of the space eX and X is a T_i-space, the proof is complete.

Property 4. If every closed subset Z of X is compact provided $Z \in P$, then eX is a compact space.

Proof. Obvious.

Property 5. Let $\varphi : eX \to Y$ be the mapping for which $f = \varphi \,|\, X$ and $\varphi(y) = y$ for all $y \in Y$. Then φ is a continuous mapping.

Proof. If V is open in Y, then $\varphi(V \cup (f^{-1}(V) \setminus clU)) = V$. The proof is complete.

Property 6. Let X be a T_2-space and for every open subset U of X with $cl_X U \in P$ there exists an open subset W of X such that $cl_X U \subseteq W$, $cl_X W \in P$ and $cl_X W$ is a normal subspace of X. Then eX is a normal space.

Proof. Let F and Φ be two closed subsets of eX and $F \cap \Phi = \emptyset$.

Case 1. $F \subseteq Y$ and $\Phi \subseteq Y$.

There exists a continuous function $h : Y \to [0, 1]$ such that $F \subseteq h^{-1}(0)$ and $\Phi \subseteq h^{-1}(1)$. We put $g(x) = h(\varphi(x))$ for every $x \in eX$.

The function $g : eX \to [0, 1]$ is continuous $F \subseteq g^{-1}(0)$ and $\Phi \subseteq g^{-1}(1)$.

Case 2. $\Phi \cap Y = \emptyset$.

There exist the open subsets U and W of X such that $\Phi \subseteq cl_X U \subseteq W$ and $cl_X W \in P$. Then $cl_X W$ is a normal subspace of X and the set $cl_X W$ is closed in eX. There exists a continuous function $h : X \to [0, 1]$ such that $\Phi \subseteq h^{-1}(1)$ and $(F \cap X) \cup (X \setminus W) \subseteq h^{-1}(0)$. We put $g(y)$=0 for every $y \in Y$ and $g(x) = h(x)$ for every $x \in X$. The function $g : eX \to [0, 1]$ is continuous, $F \subseteq Y \subseteq g^{-1}(0)$ and $\Phi \subseteq g^{-1}(1)$.

Case 3. $F \subseteq Y$.

Let $\Phi_1 = Y \cap \Phi \neq \emptyset$. There exists a continuous function $g_1 : eX \to [-1, 1]$ such that $F \subseteq g_1^{-1}(1)$ and $\Phi_1 \subseteq g_1^{-1}(-1)$. The set $U = \{x \in eX : g_1(x) < 0\}$ is open in eX. We put $g_2(x) = \sup\{g_1(x),\ 0\}$. The function $g_2 : eX \to [0, 1]$ is continuous, $F \subseteq g_2^{-1}(1)$ and $\Phi_1 \subseteq U \subseteq g_2^{-1}(0)$. The set $\Phi_2 = \Phi \setminus U$ is closed in eX and $\Phi_2 \cap Y = \emptyset$. There exists a continuous function $g_3 : eX \to [0, 1]$ such that $F \subseteq g_3^{-1}(1)$ and $\Phi_2 \subseteq g_3^{-1}(0)$. Now we put $g(x) = g_3(x) \cdot g_2(x)$ for every $x \in eX$. The function $g : eX \to [0, 1]$ is continuous, $F \subseteq g^{-1}(1)$ and $\Phi \subseteq g^{-1}(0)$.

Case 4. $F_1 = F \cap Y \neq \emptyset$ and $\Phi_1 = \Phi \cap Y \neq \emptyset$.

There exists a continuous function $g_1 : eX \to [0, 2]$ such that $\Phi \subseteq g_1^{-1}(0)$ and $F_1 \subseteq g_1^{-1}(2)$. The set $U = \{x \in eX : g_1(x) > 1\}$ is open in eX. Let $F_2 = F \setminus U$. The set F_2 is closed in eX and $F_2 \cap Y = \emptyset$. There exists a continuous function $g_2 : eX \to [0, 1]$ such that $\Phi \subseteq g_2^{-1}(0)$ and $F_2 \subseteq g^{-1}(1)$. Now we put $g(x) = \min\{1, g_1(x) + g_2(x)\}$ for every $x \in eX$. The function $g : eX \to [0, 1]$ is continuous, $\Phi \subseteq g^{-1}(0)$ and $F \subseteq g^{-1}(1)$. The proof is complete.

2.3.2. Remark. In [31,32] the method of singular mappings was applied for the construction of Hausdorff compactifications of locally compact spaces.

2.4. Wallman-Shanin method

A family L of subsets of a space X is called an l-base on a space X if L is a closed base and $F \cup H,\ F \cap H \in L$ for all $F,\ H \in L$.

Let L be an l-base on the space X. An L-filter in the space X is a non-empty family ξ of subsets of X which satisfies the following conditions:

F_1. $\xi \subseteq L$ and $\emptyset \notin \xi$.

F_2. If $F,\ H \in L$, $F \subseteq H$ and $F \in \xi$, then $H \in \xi$.

F_3. If $F,\ H \in \xi$, then $F \cap H \in \xi$.

A maximal L-filter is called an L-ultrafilter. A filter ξ is called a free L-filter if $\cap\xi = \emptyset$.

A family L of subsets of the space X is called a net in the space X at a point $x \in X$ if for every neighbourhood U of x there exists $H \in L$ such that $x \in H \subseteq U$. A family L of subsets of X is a net in the space X if L is a net of X at each point $x \in X$ (see [9,10,11]).

For every point $x \in X$ we put $\xi_L(x) = \{F \in L : x \in F\}$.

2.4.1. Lemma. Let L be an l-base and $x \in X$. The following assertions are equivalent:

1. L is a net of the space X at the point x;
2. $\xi_L(x)$ is an L-ultrafilter.

Proof. Suppose that $\xi_L(x)$ is an L-ultrafilter. If $H \in L$ and $x \notin H$, then $H \notin \xi_L(x)$. Then $H \cap F = \emptyset$ for some $F \in \xi_L(x)$. Thus L is a net at the point $x \in X$. Consider that L is a net at the point $x \in X$, $H \in L$ and $H \notin \xi_L(x)$. Then there exists $F \in L$ such that $x \in F \subseteq X \setminus H$. Thus $\xi_L(x)$ is an L-ultrafilter. The proof is complete.

Denote $\omega_L X = \{\xi_L(x) : x \in X\} \cup \{\xi : \xi$ is a free L-ultrafilter$\}$. We identify the point $x \in X$ with the filter $\xi_L(x)$ and obtain $X \subseteq \omega_L X$. For every $F \in L$ we put $\langle F \rangle = \{\xi \in \omega_L X : F \in \xi\}$. Let $< L >= \{< F >: F \in L\}$.

2.4.2. Lemma. For every $H, F \in L$ we have $< H \cup F >=< H > \cup < F >$ and $< H \cap F >=< H > \cap < F >$

Proof. If $H \cup F \in \xi \in \omega_L X$, then $\xi \cap \{H, F\} \neq \emptyset$. Thus $< H \cup F >=< H > \cup < F >$. If $H \cap F = \emptyset$, then $< H > \cap < F >=< \emptyset >= \emptyset$. Let $\Phi = H \cap F \neq \emptyset$. If $\Phi \in \xi \in \omega_L X$, then $H, F \in \xi$ and $\xi \in< H > \cap < F >$. If $\xi \in< H > \cap < F >$, then $H, F \in \xi$ and $H \cap F \in \xi$. The proof is complete.

On $\omega_L X$ we consider the topology generated by a closed base $< L >$.

We say that the extension Y of a space X is an end – T_1-extension if the set $\{y\}$ is closed in Y for every point $y \in Y \setminus X$.

2.4.3. Theorem. If L is an L-base of a space X, then:

1. $\omega_L X$ is a compactification of the space X.
2. $\omega_L X \in E(X)$.
3. $\omega_L X$ is an end – T_1-extension of X.

Proof. For every $F \in L$ we have $< F > \cap X = F$ and $< F >$ is the closure of F in $\omega_L X$. By construction, $\omega_L X$ is a compact space. If $\xi \in \omega_L X$ is an L-ultrafilter, then $\{\xi\}$ is a closed subset of $\omega_L X$. The proof is complete.

2.4.4. Corollary. $\omega_L X$ is a T_1-space iff X is a T_1-space and L is a net of the space X.

2.4.5. Definition. If L is the family of all closed subsets of a space X, then $\omega X = \omega_L X$ is called the Wallman compactification of the space X.

The compactification ωX is a T_1-space iff X is a T_1-space. The compactification ωX for a T_1-space X was constructed by H. Wallman (see [122]). The T_1-compactifications of the type $\omega_L X$ were constructed by N. A. Shanin [96,98].The general case was examined in [29,133].

A compactification bX of a space X is called the compactification of the Wallman-Shanin type if there exists an l-base of X such that $bX = \omega_L X$.

In [18,100,115] it was proved that there exists a Hausdorff compactification bX of some discrete space X which is not of the Wallman-Shanin type.The papers [15,18,24,29,49,50,70,79,85,100,109,113,132,133] contained sufficient conditions provided the compactification to be of the Wallman-Shanin type.

2.5. $\omega\alpha$–compactification

Fix a space X and an l-base L of X.

2.5.1. Definition. A compactification bX of a space X is called an $\omega\alpha_L$-compactification if there exists a continuous closed mapping $f : \omega_L X \to bX$ such that $f(x) = x$ for each $x \in X$.

If L is the family of all closed subsets X, then an $\omega\alpha_L$-compactification is called an $\omega\alpha$-compactification. The $\omega\alpha$-compactifications of T_1-space were introduced and examined by P. C. Osmatescu [87].

If bX is an $\omega\alpha_L$-compactification of a space X, then the mapping $f : \omega_L X \to bX$ is a natural projection if f is continuous closed and $f(x) = x$ for every $x \in X$.

2.5.2. Proposition. Let bX be an $\omega\alpha_L$-compactification of a space X and $f : \omega_L X \to bX$ be the natural projection. Then:

1. $f(\omega_L X) = bX$;
2. $f(\omega_L X \backslash X) = bX \backslash X$;
3. bX is an end-T_1-extension of the space X;
4. $f^{-1}(x) = \{x\}$ for each $x \in X$;
5. $bX \in E(X)$;
6. the natural projection $f : \omega_L X \to bX$ is unique.

Proof. Let $(Y_1, f_1) \in GE(X)$, $(Y_2, f_2) \in WGE(X)$, $\varphi : Y_1 \to Y_2$ be a closed mapping and $f_2 = \varphi \circ f_1$. Then $(Y_2, f_2) \in GE(X)$. Thus the assertion 5 is proved.

Since f is a closed mapping and the set $f(\omega_L X)$ is dense in bX, then $bX = f(\omega_L X)$. The assertion 1 is proved.

Obviously, $bX \backslash X \subseteq f(\omega_L X \backslash X)$.

If $x \in \omega_L X \setminus X$, then the set $\{x\}$ is closed in $\omega_L X$ and the set $\{f(x)\}$ is closed in bX. Therefore the assertion 3 is proved.

Let $x \in X$, $y \in \omega_L X \backslash X$ and $f(y) = x$. There exists an L-ultrafilter ξ such that $y = \xi$ and $y \in cl_{\omega_L X} F$ for every $F = \xi$. Since f is continuous, then $x \in cl_{bX} F$ for every $F \in \xi$. There exists $H \in \xi$ such that $x \notin H$. Then $f(< H >) = cl_{bX} H$ and $cl_{bX} H \cap bX = H$, a contradiction. The assertion 4 is proved.

Let $f, g : \omega_L X \to bX$ be two continuous mappings and $f(x) = g(x)$ for all $x \in X$. Then $f(< H >) = cl_{bX} H = g(< H >)$ for each $H \in L$, $f(y) = \cap\{cl_{bX} H : H \in L, y \in < H >\}$ and $g(y) = \cap\{cl_{bX} H : H \in L, y \in < H >\}$ for every $y \in \omega_L X \setminus X$. Thus f(y)=$g(y)$ for every $y \in \omega_L X$. The proof is complete.

2.5.3. Theorem. The set $\Omega L(X)$ of all $\omega\alpha_L$-compactifications of the space X is a complete upper semi-lattice and $\omega_L X$ is the maximal element in $\Omega L(X)$.

Proof. Let $\{Y_\alpha : \alpha \in A\}$ be a non-empty subset of the set $\Omega L(X)$ and $f_\alpha : \omega_L X \to Y_\alpha$ be the natural projection of $\omega_L X$ onto Y_α. Consider the

mapping $f : \omega_L X \to \Pi\{Y_\alpha : \alpha \in A\}$, where $f(y) = (f_\alpha(y) : \alpha \in A)$ for every $y \in \omega_L X$. We put $Y = f(\omega_L X)$. Then f is a continuous mapping, $f \,|\, X$ is an embedding of X into Y, Y is a compactification of X, $f(\mathrm{x})= x$ for each $x \in X$. For every $\alpha \in A$ there exists a projection $g_\alpha : Y \to Y_\alpha$, where $f_\alpha = g_\alpha \circ f$. Since $g_\alpha(A) = f_\alpha(f^{-1}(A))$ for each $A \subseteq Y$, the mapping g_α is closed. If $F \subseteq \omega_L X$, then $f(F) = Y \cap \Pi\{f_\alpha(F) : \alpha \in A\}$ and the mapping f is closed. Therefore $Y = \vee\{Y_\alpha : \alpha \in A\}$ and $\Omega L(X)$ is a complete upper semi-lattice. The proof is complete.

We put $a_L X = X \cup \{a\}$, where $a \notin X$ and $\{U \subseteq X : U$ is open in $X\} \cup \{a_L X \setminus F : F \subseteq X,\ F$ is closed in $\omega_L X\}$ is the open base of the space $a_L X$. The mapping $p : \omega_L X \to a_L X$, where $p^{-1}(a) = \omega_L X \setminus X$ and $f(x) = x$ for each $x \in X$, is continuous. Thus $a_L X \in WE(X)$ and $a_L X$ is a compactification of X.

2.5.4. Theorem. The following assertions are equivalent:

1. $\Omega L(X)$ is a complete lattice;
2. $a_L X$ is a minimal element of the lattice $\Omega L(X)$;
3. the set X is open in $\omega_L X$;
4. $a_L X$ is an end-T_1-extension of X.

Proof. Let $Y \in \Omega L(X)$, $y_1, y_2 \in Y \setminus X$ and $y_1 \neq y_2$. We put $Z = Y \setminus \{y_2\}$, $\varphi(y) = y$ for every $y \in Z, \varphi(y_2) = \varphi(y_1) = y_1$ and on Z consider the quotient topology. Then $\varphi : Y \to Z$ is a closed mapping, Z is a compactification of X, $Z \in \Omega L(X)$ and $Z \leq Y$. Thus the compactification $Y \in \Omega L(X)$ is not a minimal element in $\Omega L(X)$ provided $|\, Y \setminus X \,| \geq 2$.

Let Y be the minimal element in $\Omega L(X)$ and $f : \omega_L X \to Y$ be the projection. Then $Y \setminus X$ is a singleton, X is open in Y, $X = f^{-1}(X)$ is open in $\omega_L X$ and $Y = a_L X$.

If X is open in $\omega_L X$, then the mapping $p : \omega_L X \to a_L X$ is closed. The proof is complete.

2.5.5. Theorem. Let X be a locally compact space and the l-base L be a net in the space X. Then:

1. X is an open subset of $\omega_L X$.
2. $a_L X$ is an $\omega\alpha_L$-compactification of X.
3. $a_L X$ is the minimal element of the complete lattice $\Omega L(X)$.

Proof. For every point $x \in X$ there exists an open subset U_x such that $x \in U_x$ and the set $\Phi_x = cl_X U_x$ is compact. Every filter $\xi \in \omega_L X$ is an L-ultrafilter. If F is a closed subset of X, then $cl_{\omega_L X} F = \cap\{< H >: H \in L,\ F \subseteq H\}$. Fix $x \in X$. There exists $H_x \in L$ such that $x \notin H_x$ and $X \setminus U_x \subseteq H_x$. Since L is a net of X, there exists $F_x \in L$ such that $x \in F_x \subseteq U_x \cap (X \setminus H_x)$. Thus $\xi(x) \notin < H_x >$. Therefore $x \in \omega_L X \setminus < H_x >$. If $\xi \in \omega_L X \setminus X$, then there exists $H \in \xi$ such that $H \cap \Phi_x = \emptyset$. Then $H \subseteq X \setminus U_x \subseteq H_x$ and $\xi \in < H_x >$. Therefore $V_x = \omega_L X \setminus < H_x >$ is open in $\omega_L X$ and $x \in V_x \subseteq X$. The assertion 1 is proved. The Theorem 2.5.4. completes the proof.

2.5.6. Example. Let X be a non compact T_1-space. Denote by L_1 the family of all closed subsets of X. Fix $\xi \in \omega X \setminus X$. We put $L = \{F \cup H : F \in \xi,\ H \in L_1\}$. Then L is an l-base of X and $\mid \omega_L X \setminus X \mid = 1$. Thus $\Omega L(X)$ is a complete lattice and a singleton set. In this case $\omega_L X = a_L X$.

2.5.7. Corollary. The set $\Omega(X)$ of all $\omega\alpha$-compactifications of the space X is a complete upper semi-lattice with the maximal element ωX.

2.5.8. Corollary. If the space X is locally compact, then $\Omega(X)$ is a complete lattice with the maximal element ωX and minimal element aX, where $aX = a_L X$ for the l-base L of all closed subsets of X.

2.5.9. Corollary. For a T_3-space X the following assertions are equivalent:

1. X is locally compact;
2. $\Omega(X)$ is a complete lattice.

If X is a complete regular space X, then we denote by $SC(X)$ the family of all Hausdorff compactifications. In this case the Stone-Čech compactification βX is the maximal element in $SC(X)$ and $SC(X)$ is a complete subsemi-lattice of the upper semi-lattice $\Omega(X)$.

2.5.10. Corollary (N. Boboc and G. Siretchi [22]**).** For a complete regular space X the following assertions are equivalent:

1. X is locally compact;
2. $SC(X)$ is a complete lattice and sublattice of $\Omega(X)$.

If is well–know that the Stone-Čech compactification βX of a completely regular space X is a $\omega\alpha$-compactification [3] of the Wallman-Shanin type (see [3, 50, 86, 96, 109]).

From Theorem 2.1.5 it follows.

2.5.11. Corollary. Let X be an almost locally compact space, L be an l-base of X and $f : \omega_L X \setminus LC(X) \to Y$ be a continuous perfect mapping onto a T_1-space X such that $f^{-1}(x) = x$ for every $x \in X \setminus LC(X)$. Then there exists a unique $\omega\alpha_L$-compactification bX of the space X such that $bX \setminus LC(X)$ is homeomorphic to Y.

2.5.12. Corollary. Let X be a locally compact space, L be an l-base of X and Y be a T_1-space.

1. If there exists a closed mapping $f : \omega_L X \setminus X \to Y$ onto Y, then there exists a unique $\omega\alpha_L$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.
2. If there exists a closed mapping $f : \omega X \setminus X \to Y$ onto Y, then there exists an $\omega\alpha$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

From Theorem 2.2.2 it follows.

2.5.13. Corollary. Let L be an l-base of a space X and Y be a T_1-space.

1. If $f : \omega_L X \setminus X \to Y$ is an almost perfect mapping onto Y and $cl_{\omega_L X} N(f) \subseteq \omega_L X \setminus X$, then there exists a unique $\omega\alpha_L$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

2. If $f : \omega X \setminus X \to Y$ is an almost perfect mapping onto Y and $cl_{\omega_L X} N(f) \subseteq \omega_L X \setminus X$, then there exists an $\omega\alpha$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

2.6. Spectral compactifications

Let $SE(X)$ be the set of all spectral compactifications of a space X.

A mapping $g : X \to Y$ of a space X into a space Y is a spectral mapping if g is continuous and a set $g^{-1}(U)$ is compact provided the set U is open and compact in Y.

If Y, $Z \in SE(X)$, then we consider that $Z \leq Y$ if there exists a spectral mapping $g : Y \to Z$ such that $g(x) = x$ for every $x \in X$. In this conditions $SE(X)$ is a complete upper semi-lattice with the maximal element $\beta_S X$ (see Example 1.6.7).

Let L be an l-base of a space X. The filter $\xi \subseteq L$ is a simple L-filter if $\xi \cap \{F, H\} \neq \emptyset$ provided $F \cup H \in \xi$ and F, $H \in L$. Every maximal L-filter is simple. The filter $\xi(x)$ is simple for every $x \in X$.

Denote by $s_L X$ the set of all simple L-filters. For every $H \in L$ we put $<< H >>= \{\xi \in s_L X : H \in \xi\}$. Then $<< L >>= \{<< H >>: H \in L\}$ is a closed base of the space $s_L X$. We identify $x \in X$ with $\xi(x)$. Then X is a subspace of $s_L X$, X is dense in $s_L X$ and the set $s_L X \setminus << H >>$ is open and compact in $s_L X$ for every $H \in L$. Thus $s_L X$ is a spectral compactification of X. We mention that $\omega_L X \subseteq s_L X$.

If bX is a spectral compactification of X, then $L = \{X \setminus U : U$ is an open and compact subset of bX $\}$ is an l-base of X and $bX = \omega_L X$ (see [24,25]).

In the papers [24,131,132] the class of all spectral compactifications was constructed and studied using the functional rings.

We mention that the spectrum of the simple ideals of a ring in the Zariski topology is a spectral space (see [132]).

3. Uniform extensions of topology spaces

In the present chapter every space is assumed to be a completely regular T_1-space.

A uniform space (X, U) is a set X and a family U of entourages of the diagonal $\Delta(X) = \{(x, x) : x \in X\}$ of X in $X \times X$ which satisfies the following conditions:

U_1. If $V \in U$ and $V \subseteq W$, then $W^{-1} = \{(x, y) : (y, x) \in W\} \in U$.

U_2. If V, $W \in U$, then $V \cap W \in U$.

U_3. For every $V \in U$ there exists $W \in U$ such that $2W \subseteq V$, where $2W = \{(x, y) :$ there exists $z \in X$ such that (x, z), $(z, y) \in W\}$.

U_4. $\cap U = \Delta(X)$.

Denote by $u-w(X, U)$ the weight of a uniform space (X,U). On a uniform space (X, U) we consider the topology $T(U)$, generated by the uniformity U.

Let X be a space with the topology T. We put $u-w(X) = \min\{u-w(X, U) : T(U) = T\} + \aleph_0$.

If X is discrete or metrizable, then $u - w(X) = \aleph_0$.

A pseudometric on a space X is a function $\rho : X \times X \to R$ into the reals such that $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. The pseudometric ρ is continuous if the sets $B(\rho, x, r) = \{y \in X : \rho(x, y) < r\}$, $x \in X$ and $r > 0$, are open in X.

Every uniformity is generated by a family of pseudometrics [44].

3.1. Lattice $UE(X)$

A uniform extension of a space X is a complete uniform space (eX, U) that contains X as a dense subspace.

Denote by $UE(X)$ the family of all uniform extensions of a space X.

If (eX, U), $(bX, V) \in UE(X)$, then we consider that $(e(X), U) \geq (bX, V)$ if there exists a uniformly continuous mapping $g : eX \to bX$ such that $g(x) = x$ for each $x \in X$.

3.1.1. Proposition. The set $UE(X)$ is a complete upper semi-lattice for every non-empty space X.

Proof. See Example 1.6.5

In the present chapter we consider the following two problems.

Problem 1. Let P be a property, X be a space and (Y, V) be a complete uniform space with the property P. Under which conditions there exists a uniform extension (Z, U) of X such that:

1. (Z, U) is a uniform space with the property P;
2. the uniform space (Y, V) is uniformly isomorphic to the subspace $Z \setminus X$ of (Z, U)?

Problem 2. Let X be a space and (Y, V) be a complete uniform space. Under which conditions there exists a uniform extension (Z, U) of X such that (Y, V) is uniformly isomorphic to some subspace $H \subseteq Z \setminus X$ of the space (Z, U)?

Concrete results related to the solution of the problems of this type play an important role in the study of classes of spaces and complete uniform spaces.

3.2. Discrete subspaces and uniform extensions

A subset L of a space X is strongly discrete if there exists a discrete family $\{H_x : x \in L\}$ of open subsets of X such that $L \cap H_x = \{x\}$ for every $x \in L$. For every space X we put $DS(X) = \{|L| : L$ is a strongly discrete infinite subset of $X\}$ and $d(X) = \min\{|H| : H$ is a dense subset of the space $X\}$.

If Y is a subspace of a space X, then we denote $DS(X, Y) = \{|H| : H \subseteq X \setminus Y$ and H is a strongly discrete infinite subset of $X\}$.

3.2.1. Proposition. Let Y be a subset of a space X, ρ and d be continuous pseudometrics on the space X, $r > 0$, $X_1 = \{x \in X : d(x, y) < 2r$ for some

$y \in Y\}$ and $X_2 = \{x \in X : d(x, y) < r$ for some $y \in Y\}$. Then there exists a continuous pseudometric ρ_1 on X such that $\rho_1(x, y) = \rho(x, y)$ and $\rho_1(z, u) = 0$ for all $x, y \in X \setminus X_1$ and $z, u \in X_2$. We say that ρ_1 is the (d, r)-modification of the pseudometric ρ.

Proof. Let $\rho_2 = d+\rho$. There exist a set Z, a metric ρ_3 on Z and a mapping $p : X \to Z$ such that $\rho_3(p(x), p(y)) = \rho_2(x, y)$ for every $x, y \in X$. We have $p^{-1}(p(x)) = \{y \in X : \rho_2(x, y) = 0\}$. There exists a continuous pseudometric ρ_4 on Z such that $\rho_4(p(x), p(y)) = \rho(x, y)$ for all $x, y \in X$.

Moreover, on Z there exists a continuous pseudometric d_1 such that $d_1(p(x), p(y)) = d(x, y)$ for all $x, y \in X$.

Now we put $Z_2 = \{z \in Z : d_1(z, y) < r$ for some $y \in p(Y)\}$ and $Z_1 = \{z \in Z : d_1(z, y) < 2r$ for some $y \in p(Y)\}$. By construction, $cl_Z Z_2 \subseteq Z_1$, Z_1 and Z_2 are open subsets of Z and $Z \setminus Z_1$ is closed subset of Z.

Since Z is a metric space, there exists a continuous pseudometrics ρ_5 on Z such that $\rho_5(x, y) = \rho_4(x, y)$ for all $x, y \in Z \setminus Z_1$ and $\rho_5(x, y) = 0$ for all $x, y \in Z_2$ (see [36, 101]). Obviously, $X_i = p^{-1}(Z_i)$, $i \in \{1, 2\}$. Thus, $\rho_1(x, y) = \rho_5(p(x), p(y))$ is the desired pseudometric.

3.2.2. Proposition. Let d be a continuous pseudometric on a space X, $Z \subseteq X$, ρ be a continuous pseudometric on a space Y, $r > 0$, Y_1 be a subset of Y, $f : Y_1 \to Z$ be an one-to-one mapping of Y_1 onto Z, $d(x, y) \geq 3r$ provided $x, y \in Z$ and $x \neq y$. We put $H_z = \{x \in X : d(x, z) \leq r\}$ for every $z \in Z$. Let $X_1 = X \cup Y$. Then:

1. $\{H_z : z \in Z\}$ is a discrete family of closed subsets of the space X;
2. there exists a pseudometric ρ_1 on X_1 such that:

– $\rho_1(x, y) = \rho(x, y)$ for all $x, y \in Y$;
– $\rho_1(y, f(y)) = 0$ for each $y \in Y$;
– $\rho_1(y, x) = \rho(y, f^{-1}(z)) + d(z, x)$ if $y \in Y$, $z \in Z$ and $x \in H_z$;
– $B(\rho_1, y, r) \cap X \subseteq \cup\{H_z : z \in Z\}$ for each $y \in Y$;
– for every $x \in X_1$ and $\varepsilon > 0$ the set $B(\rho_1, x, \varepsilon) \cap Y$ is open in Y and the set $B(\rho_1, x, \varepsilon) \cap X$ is open in X.

We say that ρ_1 is the (d, r)-extension of the pseudometric ρ.

Proof. There exist a metric space (Y_2, ρ_2) and a mapping $p : Y \to Y_2$ such that $\rho_2(p(y), p(z)) = \rho(y, z)$ for all $y, z \in Y$. We put $Y_3 = p(Y_1)$.

There exist a metric space (X_2, d_1) and a mapping $q : X \to X_2$ such that $d_1(q(x), q(y)) = d(x, y)$ for all $x, y \in X$. We put $Z_1 = q(Z)$. The mapping $q_1 = q \,|\, Z : Z \to Z_1$ is one-to-one. Let $g(z) = p(f^{-1}(q_1^{-1}(z)))$ for every $z \in Z_1$. By construction, $d_1(y, z) \geq 3r$ if $y, z \in Z_1$ and $y \neq z$.

The discrete sum $X_3 = Y_2 \oplus X_2$ is a metric space. Let $P_z = \{x \in X_2 : d_1(z, x) \leq r\}$ for every $z \in Z_1$. Then $H_z = q^{-1}(P_{q(z)})$ and $\{P_z : z \in Z_1\}$ is a discrete family of closed subsets of the space X_2.

We put $V_z = \{x \in X_2 : d_1(z, x) < r\}$, $V = \cup\{V_z : z \in Z_1\}$ and $P = \cup\{P_z : z \in Z_1\}$. On $Q = P \cup Y_2$ we consider the pseudometric ρ_3, where:

– $\rho_3(x, y) = \rho_2(x, y)$ if $x, y \in Y_2$;
– $\rho_3(z, g(z)) = \rho_3(g(z), z) = 0$ if $z \in Z_1$;
– $\rho_3(y, x) = \rho_3(x, y) = \rho_2(y, g(z)) + d_1(z, x)$ if $y \in Y_2,\ z \in Z_1$ and $x \in P_z$;
– $\rho_3(x, y) = d_1(x, y)$ if $x, y \in P_z$; for some $z \in Z_1$;
– $\rho_3(x, y) = d(x, z_1)+\rho_2(g(z_1), g(z_2))+d_1(z_2, y)$ if $z_1, z_2 \in Z_1$, $z_1 \neq z_2$, $x \in P_{z_1}$, $y \in P_{z_2}$.

By construction, the pseudometric ρ_3 is continuous on the closed subset Q of the metric space X_3. Thus, there exists a continuous pseudometric ρ_4 on X_3 such that $\rho_4(x, y) = \rho_3(x, y)$ for all $x, y \in Q$. Consider the mapping $\varphi : X \oplus Y \to X_3$, where $\varphi \,|\, X = q$ and $\varphi \,|\, Y = p$. Let $\rho_1(x, y) = \rho_4(\varphi(x), \varphi(y))$ for all $x, y \in X \oplus Y$. The proof is complete.

A space X is called a space of pointwise countable type if for every point $x \in X$ there exists a compact subset $\Phi(x) \ni x$ of countable character in X (see [10]).

A space X is called a space of countable type if for every compact subset $F \subseteq X$ there exists a compact subset $\Phi(F) \supseteq F$ of countable character in X (see [10,61]). M. Henriksen and J. R. Isbell [61] has proved that X is a space of countable type iff the remainder $\beta X \setminus X$ is a Lindelöf space.

3.2.3. Proposition. Let $X = Y \cup Z$, where Y is a closed subspace of the space X and every compact subset $F \subseteq Y$of a countable character in Y has a countable character in X. Then:

1. X is a first countable space iff Y and Z are the first countable spaces;
2. X is a space of pointwise countable type iff Y and Z are spaces of pointwise countable type;
3. X is a space of countable type iff Y and Z are spaces of countable type.

Proof. The assertions 1 and 2 are obvious.

Let X be a space of countable type. Then Y is a space of countable type as a closed subspace of X and Z is a space of countable type as an open subspace of space of countable type. Suppose that Y and Z are spaces of countable type. We put $Y_1 = cl_{\beta X} Y \setminus Y$ and $Z_1 = cl_{\beta X} Z \setminus Z$. By virtue of Theorem of M. Henriksen and J. R. Isbell [61], Y_1 and Z_1 are Lindelöf spaces. We affirm that $X_1 = \beta X \setminus X$ is a Lindelöf spaces. Let $\{U_\alpha : \alpha \in A\}$ be a cover of the set X_1 and U_α is open in βX for every $\alpha \in A$. Since Y_1 is a Lindelöf space and $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$, there exists a countable subset $A_1 \subseteq A$ such that $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$. Then $F = Y \setminus \cup\{U_\alpha : \alpha \in A_1\}$ is a compact subset of Y and there exists a compact subset $\Phi \subseteq Y$ such that $F \subseteq \Phi$ and Φ has countable character in X. Thus Φ is a G_δ-subset of βX and $Z_2 = Z \setminus \Phi$ is an F_δ-subset of Z_1. Thus Z_2 is a Lindelöf subspace of βX. Moreover, $Z_3 = Z_2 \setminus \cup\{U_\alpha : \alpha \in A_1\}$ is a Lindelöf subspace of βX. By construction $Z_3 \subseteq X_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$. Therefore there exists a countable subset $A_2 \subseteq A$ such that $A_1 \subseteq A_2$ and $Z_3 \subseteq \cup\{U_\alpha : \alpha \in A_2\}$. So $\{U_\alpha : \alpha \in A_2\}$ is a countable

cover of X_1. The M. Heriksen's and J. R. Isbell's theorem [61] completes the proof.

3.2.4. Proposition. Let $X = Y \cup Z$, Y be a closed subspace of the space X, Y_1 be a subset of Y and $\{V_n(y) : n \in N, y \in Y_1\}$ be a family of open subsets of the space X such that:
- if y, $z \in Y_1$ and $y \neq z$, then $cl_X V_1(y) \cap cl_Y V_1(z) = \emptyset$;
- if $y \in Y_1$ and $n \in N$, then $cl_X V_{n+1}(y) \subseteq \{y\} \cup V_n(y)$;
- the family $\{V_1(y) : y \in Y_1\}$ is discrete in Z and $\cup\{V_1(y) : y \in Y_1\} \subseteq Z$;
- if U is an open subset of Y, the set $U \cup \cup\{V_n(y) : y \in U \cap Y_1\}$ is open in X for every $n \in N$;
- if U is open in X and $y \in U \cap Y$, then $V_n(y) \subseteq U$ for some $n \in N$;
- $Y \cap Z = \emptyset$.

A. The following assertions are true:

1. If $V_n = \cup\{Y \cup V_n(y) : y \in Y_1\}$, then V_n is open in X and $cl_X V_{n+1} \subseteq V_n$ for every $n \in N$.

2. $Y = \cap\{V_n : n \in N\} = \cap\{cl_X V_n : n \in N\}$.

3. If the spaces Y and Z are paracompact spaces, then X is paracompact.

B. If for every open subset U of X and every point $x \in U$ there exist $n \in N$ and an open subset $V$of X such that $x \in V \subseteq V \cup \cup\{V_n(y) : y \in V \cap Y_1\} \subseteq U$, then the following assertions are true:

4. If the subspaces Y and Z are normal, then X is a normal space.

5. If Y and Z are perfectly normal spaces, then X is a perfectly normal space.

6. If a compact subset $F \subseteq Y$ has a countable character in Y, then F has a countable character in X.

7. If Y and Z are the first countable spaces, then X is first countable.

8. If Y and Z are spaces of countable (pointwise countable) types, then X is a space of countable (pointwise countable) type.

9. If Y and Z are metrizable spaces, then X is a metrizable space.

10. If Y and Z are complete metrizable spaces, then X is a complete metrizable space.

11. If Y and Z are paracompact p-spaces, then X is a paracompact p-space.

12. If Y and Z are Čech complete paracompact spaces, then X is a Čech complete paracompact space.

Proof. The assertions 1 and 2 are obvious.

Let Y and Z be paracompact spaces and ω be an open cover of the space X. There exists a sequence $\{\xi'_n = \{V'_\alpha : \alpha \in A_n\} : n \in N\}$ of open discrete families of the space Y such that for every $n \in N$ and every $\alpha \in A_n$ there is $W_\alpha \in \omega$ such that $V'_\alpha \subseteq W_\alpha$ and $Y = \cup\{V'_\alpha : \alpha \in \cup\{A_n : n \in N\}\}$. We put $V_\alpha = V'_\alpha \cup \cup\{W_\alpha \cap V_2(y) : y \in Y_1 \cap V'_\alpha\}$. Then $\xi_n = \{V_\alpha : \alpha \in A_n\}$ is a discrete family of open subsets of X.

There exists a sequence $\{\lambda_n = \{U_\beta : \beta \in B_n\} : n \in N\}$ of open discrete families of the space Z for which $Z = \cup\{U_\beta : \beta \in \cup\{B_n : n \in N\}\}$ and for every $\beta \in \cup\{B_n : n \in N\}$ there exists $W_\beta \in \omega$ such that $U_\beta \subseteq W_\beta$. For every $n,m \in N$ we put $\lambda_{nm} = \{U_\beta \setminus cl_X V_m : \beta \in B_n\}$. Then $\{\xi_n, \lambda_{nm} : n, m \in N\}$ is a σ-discrete refinement of the cover ω. Thus X is a paracompact space (see [44], Theorem 5.1.11).

Suppose that for every open subset U of X and point $x \in U$ there exist a natural number $n = n(x, U)$ and an open subset $V = V(x, U)$ of X such that $x \in V \subseteq \cup\{V \cup V_n(y) : y \in V \cap Y_1\} \subseteq U$.

Let Y and Z be normal spaces, F and Φ be closed subsets of X and $\Phi \cap F = \emptyset$.

Let $F \subseteq Y$ and $\Phi \cap Y = \emptyset$. For every $y \in F$ there exists $n = n(y) = n(y, X \setminus \Phi)$ and an open subset V_y of V such that $W_y \cap \Phi = \emptyset$, where $W_y = \cup\{V_y \cup V_n(y) : y \in V_y \cap Y_1\}$. Let $F_m = \{y \in F : n(y) \leq m\}$. Then $F = \cup\{F_m : m \in N\}$ and $F_n \subseteq F_{n+1}$ for every $n \in N$. We put $W_n = \cup\{V_y : y \in F_n\}$, $W_n^I = \cup\{W_n \cup V_n(y) : y \in W_n \cap Y_1\}$, $W_n'' = \cup\{W_n \cup V_{n+1}(y) : y \in W_n \cap Y_1\}$, $H_1 = \cup\{W_n' : n \in N\}$ and $H_2 = \cup\{W_n'' : n \in N\}$. By construction, $H_2 \subseteq H_1, cl_X H_2 \subseteq H_1 \cup Y$ and $H_1 \cap \Phi = \emptyset$. Thus $F \subseteq H_2$ and $\Phi \subseteq X \setminus cl_X H_2$.

Let $F \subseteq Y$ and $\Phi \cap Y \neq \emptyset$. There exist two open subsets H_1 and H_2 of Y such that $F \subseteq H_1, Y \cap \Phi \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_1' = \cup\{H_1 \cup V_2(y) : y \in H_1 \cap Y_1\}$, $H_2' = \cup\{H_2 \cup V_2(y) : y \in H_2 \cap Y_1\}$ and $\Phi_1 = \Phi \setminus H_2'$. Then Φ_1 is closed in X and $\Phi_1 \cap Y = \emptyset$. Thus there exist two open subsets $H_1'' \subseteq H_1'$ and H_2'' of X such that $F \subseteq H_1'', \Phi_1 \subseteq H_2''$ and $H_1'' \cap H_2'' = \emptyset$. Let $H_2''' = H_2' \cup H_2''$.Then $\Phi \subseteq H_2'''$ and $H_1'' \cap H_2''' = \emptyset$.

Suppose now that $F \cap Y = \emptyset$. Then there exist the open subsets H_1', H_1'', H_2' and H_2'' of X such that $H_1' \cap H_2' = H_1'' \cap H_2'' = \emptyset$, $F \cap Y \subseteq H_1', \Phi \subseteq H_2', F \setminus H_1' \subseteq H_1''$ and $\Phi \subseteq H_2''$. Let $H_1 = H_1' \cup H_2''$ and $H_2 = H_2' \cap H_2''$. Then $F \subseteq H_1, \Phi \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. The assertion 4 is proved. The assertion 5 follows from the assertions 1,2 and 4.

Let F be a compact subset of Y, $\{H_n : n \in N\}$ be a sequence of open subsets of Y and for every open subset $U \supseteq F$ there exists $n \in N$ such that $F \subseteq H_n \subseteq U$. We put $H_{nm} = \cup\{H_n \cup V_m(y) : y \in H_n \cap Y_1\}$. The sets H_{nm} are open in X. Let U be an open subset of X and $F \subseteq U$. There exists a finite set F' of F such that $F \subseteq \cup\{V(x, U) : x \in F'\}$. There exists n such that $F \subseteq H_n \subseteq \cup\{V(x, U) : x \in F'\}$ and $\max\{n(x, U) : x \in F'\} \leq n$. Then $F \subseteq H_{nm} \subseteq U$. Thus F has a countable character in X. The assertion 6 is proved. The assertions 7 and 8 follow from the assertion 6 and Proposition 3.2.3.

Let Y and Z be metric spaces. Fix a metric ρ_1 on a space Y. For every $y \in Y_1$ on a space $Z_y = clV_1(y)$ fix a metric d_y such that $V_n(y) \subseteq \{z \in Z_y : d_y(y, z) < 2^{-n}\}$. Now on $\cup\{Y \cup Z_y : y \in Y_1\}$ we construct the metric ρ_2 such that:

– $\rho_2(x,z) = \rho_1(x,z)$ if $x, z \in Y$;
– $\rho_2(x,z) = d_y(x,z)$ if $y \in Y_1$ and $x, z \in Z_y$;
– $\rho_2(x,z) = d_{y_1}(x,y_1) + \rho_1(y_1,y_2) + d_{y_2}(y_2,z)$ if $y_1, y_2 \in Y, y_1 \neq y_2, x \in Z_{y_1}$ and $z \in Z_{y_2}$.

Obviously, ρ_2 is a metric on a subspace $X_1 = \cup\{Y \cup Z_y : y \in Y_1\}$. Thus the paracompact space X is a union of two open metrizable subspaces $X_2 = \cup\{Y \cup V_1(y) : y \in Y_1\}$ and $X_3 = Z \setminus Y$.Thus X is a metrizable spaces. Suppose that Y and Z be complete metric spaces, In this case we consider that the metrics ρ_1 and d_y are complete. Then there exists a metric ρ on X such that:
– ρ is complete on $Z \setminus \cup\{V_2(y) : y \in Y_1\}$;
– $\rho(x,y) = \rho_2(x,y)$ if $x,y \in X_1$.

We affirm that the metric ρ is complete. Let $\{x_n : n \in N\}$ be a sequence of points and $\rho(x_n, x_m) < 2^{-n}$ provided $n \leq m$. Obviously, the sequence $\{x_n\}$ is convergent in X in the following cases:
– the set $\{n \in N : x_n \in Y\}$ is infinite;
– there exists $y \in Y_1$ such that the set $\{n \in N : x_n \in Zy\}$ is infinite;
– there exists $m \in N$ such that the set $\{n \in N : x_n \in Z \setminus \{V_m(y) : y \in Y_1\}\}$ is infinite.

Suppose that for every $n \in N$ there exists $y_n \in Y_1$ such that:
– $x_n \in Zy_n$;
– $y_n \neq y_m$ for $n \neq m$.
– In this case $\rho(y_n, y_m) < \rho(x_n, y_n) + \rho(y_n, y_m) + \rho(y_m, x_m) = \rho(x_n, x_m)$ and there exists $y \in Y$ such that $y = \lim y_n$. By construction, $y = \lim x_n$. The assertions 9 and 10 are proved.

Let Y and Z be two paracompact p-spaces. Fix a perfect mapping $\varphi : Y \to Y'$ onto a metric space Y'. Since Z is a paracompact space, we may consider that $V_n(y)$ is a co-zero set of Z for all $y \in Y_1$ and $n \in N$. Because the family $\{V_1(y) : y \in Y_1\}$ is discrete in Z, then there exists a perfect mapping $\Psi : Z \to Z'$ onto a metric space Z' such that $\Psi^{-1}(\Psi(V_n(y))) = V_n(y)$ for all $y \in Y_1$ and $n \in N$.

Let $X' = Y' \cup Z'$. Consider the mapping $g : X \to X'$, where $\varphi = g\,|Y$ and $\Psi = g\,|Z$. On X' we consider the quotient topology $\{U \subseteq X' : g^{-1}(U)$ is open in $X\}$. By construction, the mapping g is perfect. We put $Y_1' = g(Y_1) = \varphi(Y_1)$ and $V_n'(z) = \cup\{\Psi(V_n(y)) : y \in \Psi^{-1}(z)\}$ for all $z \in Y_1'$ and $n \in N$. By virtue of the assertion 9 the space X' is metrizable. If the spaces Y and Z are complete metrizable, then the space $X'$is complete metrizable. The assertions 11 and 12 are proved. The proof is complete.

3.2.5. Theorem. Let (e_1X, U_1) be a uniform extension of a space X, Y be an infinite discrete space and $|Y| \in DS(e_1X)$.Then there exists a uniform extension (eX, U) of X such that:
1. (eX, U) is a uniform extension of the space e_1X;
2. $eX = e_1X \cup Y$ and Y is a strongly discrete subset of the space eX;

3.$u-w(eX,U)$=$u-w(e_1X,U_1)$;

4. if e_1X is a paracompact space, then eX is a paracompact space;

5. if e_1X is a normal space, then eX is a normal space;

6. if e_1X is a collectionwise normal space, then eX is a collectionwise normal space;

7. if $e_1X$is a metacompact space, then eX is a metacompact space;

8. if e_1X is a perfectly normal space, then eX is a perfectly normal space;

9. if e_1X is Čech complete, then eX is a Čech complete space;

10. if e_1X is a p-space, then eX is a p-space;

11. the space eX has a countable base at every point $y \in Y$;

12. if e_1X is a first countable space, then eX is a first countable space;

13. if e_1X is a space of pointwise countable type, then eX is a space of pointwise countable type.

Proof. In the space e_1X we fix a discrete family $\{H_\alpha : \alpha \in A\}$ of non-empty open subsets, where $|A| = |Y|$. For every $\alpha \in A$ fix a point $b_\alpha \in X \cap H_\alpha$ and a continuous function $h_\alpha : e_1X \to [0,1]$, where $h_\alpha(b_\alpha) = 1$ and $e_1X \setminus H_\alpha \subseteq h_\alpha^{-1}(0)$. Let $h : A \to Y$ be a mapping such that the set $h^{-1}(y)$ is countable for each $y \in Y$. We consider that $h^{-1}(y) = \{\alpha(n,y) : n \in N\}$ and $b_{\alpha(n,y)} = x(n,y)$ for every $y \in Y$ and $n \in N$.

There exists a family P of continuous pseudometrics on e_1X such that P generates the uniformity U_1 on e_1X and $\rho_1 + \rho_2 \in P$ for all $\rho_1, \rho_2 \in P$. We consider that $\rho(b_\alpha, b\mu) \geq 1$ for all $\rho \in P, \alpha, \mu \in A$ and $\alpha \neq \mu$.

Let $U(n,y) = \cup\{H_{\alpha(m,y)} \cup \{y\} : m > n\}, g_{(n,y)}(y) = 1, g_{(n,y)}(z) = 0$ and $g_{(n,y)}(x) = \Sigma\{h_{\alpha(m,y)} : m > n\}$ for every $y, z \in Y, y \neq z, x \in e_1X$ and n∈N.

We put $d_n(x,z) = \sum\{|\ (g_{(n,y)}(x) - g_{(n,y)}(z)\ | : y \in Y\}$ and $d(x,z) = \sum\{2^{-n}d_n(x,z) : n \in N\}$ for all $x, z \in eX$ and $n \in N$.

By construction:

– $d(y,z)$=2 if $y, z \in Y$ and $y \neq z$;

– $d(y, x(n,y))$=2^{-n} for every $y \in Y$ and $n \in N$;

– $B(d, y, 2^{-n-1}) \subseteq U(n,y)$ for every $y \in Y$ and $n \in N$;

– $B(d,x,r) \cap e_1X$ is open in e_1X for every $x \in eX$ and r¿0.

On eX we consider the topology generated by the open base $\{B(d,y,2^{-n})$:$y \in Y, n \in N\}\cup\{U \subseteq e_1X$:$U$ is open in $e_1X\}$. At every point $y \in Y$ the space eX has a countable base and the set Y is strongly discrete in eX. From these properties of a space eX it follows the assertions 4-10 and 2. Obviously, $cl_{eX}\{b_\alpha : \alpha \in A\} \supseteq Y$. Therefore the sets X and e_1X are dense in eX.

Fix a pseudometric $\rho \in P$ and $n \in N$. Let $X_n = \cup\{B(d,y,2^{-n}) : y \in Y\}$ and $Y_n = Y \cup (\{\beta_\alpha : \alpha \in A\} \cap X_n)$. By virtue of Proposition 3.2.1, on eX , there exists a continuous pseudometric $e_n(\rho)$ such that $e_n(\rho)$(x,y)= ρ(x,y), if $x, y \in eX \setminus X_n$, and $e_n(\rho)$(x,y)=0, if $x, y \in X_{n+2}$. Then $P_1 = \{\alpha + e_n(\rho) : \rho \in P, n \in N\}$ is a family of continous pseudometrics on eX which generates the topology of the space eX and some uniformity U on eX.

We affirm that the uniform space (eX, U) is complete, i.e.(eX, U) is a uniform extension of the spaces e_1X and X.

Let ξ be a Cauchy filter of closed subsets of the space eX.

Case 1. $Y \in \xi$.

Fix $\rho_1 = d + e_1(\rho) \in P_1$. There exists $\Phi \in \xi$ such that $\rho(x, y) < 1$ for all $x, y \in \Phi$ and $\Phi \subseteq Y$. Let $z \in \Phi$. If y∈Y and $y \neq z$, then $\rho_1(z, y) \geq d(z, y) = 2$. Thus Φ is a singleton set and $\cap\xi = \Phi \neq \emptyset$.

Case 2. $Y \notin \xi$.

There exists $\Phi \in \xi$ such that $\Phi \cap Y = \emptyset$. We may consider that $d(x, y) < 2^{-4}$ for all $x, y \in \Phi$. Let $Y_0 = \{y \in Y : d(y, \Phi) < 2^{-4}\}$. If $Y_0 = \emptyset$, then $X_4 \cap \Phi = \emptyset$, $F = X \setminus X_4$ is a closed subset of eX, $\Phi \subseteq F$ and $F \in \xi$. In this case ξ is a Cauchy filter of the uniform space (e_1X, U_1) and $\cap\xi \neq \emptyset$.

Suppose that $Y_0 \neq \emptyset$. Then the set Y_0 is a singleton set. If $y_1, y_2 \in Y_0$, then there exists $x_1, x_2 \in \Phi$ such that $d(y_1, x_1) < 2^{-4}$ and $d(y_2, x_2) \in \Phi$. Then $d(y_1, y_2) \leq d(y_1, x_1) + d(x_1, x_2) + d(x_2, y_2) < 3 \cdot 2^{-4} < 2^{-2} < 2$ and $y_1 = y_2$. Suppose that Y_0=$\{y_0\}$. Since the set Φ is closed in eX and $y_0 \neq \Phi$, then there exists $n > 4$ such that $B(d, y_0, 2^{-n}) \cap \Phi = \emptyset$. In this case $X_n \cap \Phi = \emptyset$, $eX \setminus X_n \in \xi$ and ξ is a Cauchy filter of the uniform space (e_1X, U_1). Therefore $\cap\xi \neq \emptyset$. The proof is complete.

3.2.6. Theorem. Let (e_1X, U_1) be a uniform extension of a space X, (Y, ν) be a complete uniform space and $d(Y, \nu) \in DS(e_1X, U_1)$. Then there exists a uniform extension (eX, U) of X such that:

1.(eX, U) is a uniform extension of the space $(e_1X, T(U_1))$; 2. (Y, ν) is uniformly isomorphic to the subspace $eX \setminus e_1X$ of $(eX,\ U)$; 3. $u - w(eX, U) \leq u - w(e_1X, U_1) + u - w(Y, \nu)$; 4. if e_1X and Y are paracompact Čech complete spaces, then eX is a paracompact Čech complete space; 5. if e_1X and Y are paracompact p-spaces, then eX is a paracompact p-space; 6. if e_1X and Y are paracompact spaces, then eX is a paracompact space; 7. if e_1X and Y are normal spaces, then X is a normal space; 8.if e_1X and Y are perfectly normal spaces, then X is a perfectly normal space; 9. if e_1X and Y are first countable spaces, then X is a first countable space; 10. if e_1X and Y are spaces of pointwise countable type, then X is a space of pointwise countable type; 11. if e_1X and Y are spaces of countable type, then X is a space of countable type; 12. if e_1X and Y are metrizable spaces, then X is a metrizable space.

Proof. Let Y_1 be a dense subset of the space Y and $|Y_1| = d(Y)$. In the proof of Proposition 3.2.1 there were constructed a uniform extension (e_2X, U_2) of a space e_1X, a continuous pseudometric d on e_2X and a family P_1 of continuous pseudometrics on e_2X such that:

–$Z = e_2X \setminus e_1X$ is a strongly discrete subspace of the space e_2X ;

– $|Z| = |Y_1|$, i.e. there exists a one-to-one correspondence $h : Y_1 \to Z$;

– if $x, y \in Z$ and $x \neq y$, then $d(x, y) = 3$ and $\rho(x, y) = 0$ for every $\rho \in P_1$;

– the space e_2X has a countable base at every point $y \in Z$;

– $\{B(d, y, 2^{-n}) : n \in N\}$ is a base of the space e_2X at a point $y \in Z$;

– if $\rho \in P_1$, then there exists n∈N such that $\rho(x, z) = 0$ for all $x, z \in \cup\{B(d, y, 2^{-n}) : y \in Z\}$;

– the family of pseudometrics $P_2 = \{d+\rho : \rho \in P_1\}$ generates the uniformity U_2 on a space e_2X;

– for every $n \in N$ the family P_1 generates on $e_1X \setminus \cup\{B(d, y, 2^{-n}) : y \in Y_1\}$ the uniformity U_1.

Suppose that the uniformity V of the space Y is generated by the family P_3 of continuous pseudometrics.

We put $eX = e_1X \cup Y$.

For every $\rho \in P_3$ let $e(\rho)$ be the $(d, 1)$-extension of the pseudometrics ρ on eX. We identify the point $y \in Y_1$ with the point $h(y) \in Z$.

Fix $\rho \in P_1$ and $n \in N$. Let ρ' be the $(d, 2^{-n})$-modification of the pseudometric ρ on e_2X. We fix $z_0 \in Z$ and put $e_n(\rho)(x, y) = \rho'(x, y)$ if $x, y \in e_1X, e_n(\rho) = 0$ if $x, y \in Y$ and $e_n(\rho)(x, y) = e_n(\rho)(y, x) = \rho'(z_0, x)$ if $y \in Y$ and $x \in e_1X$. Now we put $P = \{e(\rho_1)+e_n(\rho_2) : \rho_1 \in P_3, \rho_2 \in P_1$ and $n \in N\}$. The pseudometrics P generates the uniformity U on eX.

Obviously, (Y, V) is a uniform subspace of the space (eX, U), e_1X is a dense subspace of the space eX.

Let ξ be a Cauchy filter of a space (X, U). The filter ξ is convergent in X in the following cases:

– $\Phi \subseteq Y$ for some $\Phi \in \xi$;

– $\Phi \subseteq e_1X \setminus \cup\{B(d, y, 2^{-n}) : y \in Y_1\}$ for some n∈N and $\Phi \in \xi$;

– there exist $n \in N$, $y \in Y$ and $\Phi \in \xi$ such that $\Phi \subseteq B(d, y, 2^{-n})$.

Suppose that for every $n \in N$ and $\Phi \in \xi$ the set $n(\Phi) = cl_X\{y \in Y_1 : \Phi \cap B(d, y, 2^{-n}) \neq \emptyset\}$ is non-empty. Then $\eta = \{n(\Phi) : \Phi \in \xi, n \in N\}$ is a Cauchy filter of the space (Y, V). If $y \in \cap\eta$, then $y \in \cap\xi$. Therefore (X, U) is a complete space and a complete extension of Y and e_1X. Proposition 3.2.4 completes the proof.

3.2.7. Problems. Let P be a topological property and Y and e_1X be two spaces with the property P. Is it true that eX has the property P in the following cases:

a) P is the property to be a metacompact space;

b) P is the property to be a p-space;

c) P is the property to be a Čech complete space;

d) P is the property to be a space with a G_δ-diagonal;

e) P is the property to be a symmetrizable space.

3.3. The gluing operation and σ-discretness

For every point x of a space X we put $DS(x, X) = \cap\{DS(cl_XH) : H$ is open in X and $x \in H\}$, $\tau - ds(x, X) = \{\sup A : A \subseteq DS(x, X)$ and $|A| \leq \tau\}$ and $ds(x, X) = \aleph_0 - ds(x, X)$.

If X is a metric space or a space with a σ-discrete net, then $\sup DS(x, X) \in ds(x, X)$.

For every subset A of a space X, by A^d we denote the derived set of A, i. e. the set of accumulation points of A.

3.3.1. Definition ((see [20]) for metric spaces**).** Let (X_1, U_1) be a uniform space, let A be a non-empty subset of the set X_1^d and $\{(Y_x, V_x) : x \in A\}$ be a family of complete uniform spaces. A uniform extension (Y, V) of the space $X = X_1 \setminus A$ is obtained by gluing the space (Y_x, V_x) at the point x for every $x \in A$ if the following conditions are satisfied:

G_1. $Y = X \cup \cup\{Y_x : x \in A\}$;

G_2. (Y_x, V_x) is a uniform subspace of the space (Y, V) for each $x \in A$;

G_3. the subspace X is dense in Y;

G_4. the natural mapping $f : Y \to X_1$, where $f(x) = x$ for every $x \in X$ and $f^{-1}(x) = Y_x$ for each $x \in A$, is continuous;

G_5. if $x, y \in A$ and $x \neq y$, then $Y_x \cap Y_y = \emptyset$.

3.3.2. Definition. Let (X_1, U_1) be a uniform space, A be a non-empty subset of X_1^d and a uniform extension (Y,V) the space $X = X_1 \setminus A$ is obtained by gluing of the space (Y_x, V_x) at the point x for every $x \in A$. The gluing is strongly at the point $x_0 \in A$ if for every open subset H of Y , that contains Y_{x_0}, there exists an open subset U of X_1 such that $x_0 \in U$ and $Y_x \subseteq H$ for each $x \in A \cap U$.

A mapping $f : X \to Y$ is called:

- a closed mapping at a point $y \in Y$ if $f^{-1}(y) \neq \emptyset$ and for every open subset $U \subseteq X$, that contains $f^{-1}(y)$, there exists an open subset V of Y such that $y \in V$ and $f^{-1}(V) \subseteq U$;

- a perfect mapping of a point $y \in Y$ if $f^{-1}(y)$ is a compact subset and f is closed at y.

Gluing is strongly at a point $x \in A$ iff the natural mapping $p : Y \to X_1$ is closed at a point x.

From the E. Michal's theorem [78] it follows.

3.3.3. Corollary. Let (X_1, U_1) be a space of pointwise countable type, $A \subseteq X_1^d$, $X = X_1 \setminus A$ and the uniform extension (Y, V) of X is obtained by gluing the spaces $\{(Y_x, V_x) : x \in A\}$ at the points of A.

The following assertions are equivalent:

1. the natural mapping $p : Y \to X_1$ is perfect; 2. gluing is strongly at each point $x \in A$.

Let ρ be a continuous pseudometric on a space X. There exist a metric space $(X/\rho, \hat{\rho})$ and a natural projection $\pi_\rho : X \to X/\rho$, where $\pi_\rho^{-1}(\pi_\rho(x)) = \{y \in X : \rho(x, y) = 0\}$ and $\rho(x, y) = \widehat{\rho}(\pi_\rho(x), \pi_\rho(y))$ for all $x, y \in X$. The natural projection is continuous. Denote by $\overline{X/\rho}$ the completion of a metric space $(X/\rho, \hat{\rho})$.

3.3.4. Definition. Let ρ be a continuous pseudometric on a space X. The pseudometric ρ is a metric on a subset $A \subseteq X$ if for every point $x \in A$ and every open subset $U \subseteq X$ that contains x there exists $\varepsilon > 0$ such that $B(x, \rho, \varepsilon) \subseteq U$.

3.3.5. Proposition. Let (X_1, U_1) be a complete uniform space, $A \subseteq X_1^d$, $X = X_1 \setminus A$, a continuous pseudometric d on X_1 is a metric on the set A, $\{(Y_x, V_x) : x \in A\}$ is a family of complete uniform spaces and the uniform extension (Z, W) of the space $Z_1 = \overline{X_1/d} \setminus \pi_d(A)$ is obtained by gluing the space (Y_x, V_x) at a point $\pi_\rho(x)$ for every $x \in A$. Then:

1.there exists a uniform extension (Y, V) of a space X which is a gluing of the spaces (Y_x, V_x) at the points $x \in A$; 2. if $x \in A$ and the gluing of (Y_x, V_x) is strongly in Z, then the gluing of (Y_x, V_x) is strongly in X_1, too; 3. (Y, V) is uniformly isomorphic to some closed subspace of the Cartesian product of the spaces (X_1, U_1) and (Z, W).

Proof. Consider the projection $\pi_d : X_1 \to X_1/d$ and the set $A_1 = \pi_d(A)$. Let (Z, W) be the uniform extension of the space $Z' = \overline{X_1/d} \setminus A_1$ obtained by gluing each space (Y_x, V_x) at a point $\pi_d(x)$, $x \in A$. By definition, $Z = Z' \cup \cup\{Y_x : x \in A\}$.

We put $Y = X \cup \cup\{Y_x : x \in A\}$. Consider the mappings $p : Y \to X_1$ and $q : Y \to Z$, where:

- $p(x) = x$ and $q(x) = \pi_d(x)$ for each $x \in X$;
- $p^{-1}(x) = Y_x$ for each $x \in A$;
- $q(y) = y$ for every $y \in Y_x$ and $x \in A$.

Now we consider the mapping $\varphi : Y \to X_1 \times Z$, where $\varphi(y) = (p(y), q(y))$ for every $y \in Y$. By construction, $\varphi(y) \neq \varphi(z)$ provided $y, z \in Y$ and $y \neq z$. We identify $y \in Y$ with $\varphi(y)$ and consider $Y = \varphi(Y)$ as a uniform subspace of the uniform space $X_1 \times Z$. Since $\varphi(Y_x) = Y_x$, for every $x \in A$, (Y_x, V_x) is a uniform subspace of the space Y.

Since $p \,|\, X : X \to X_1$ is an embedding and the mapping $q \,|\, X \to Z$ is continuous, the space $X = \varphi(X)$ is a subspace of the space Y.

Let $(x, z) \in X_1 \times Z$ and $(x, z) \notin Y = \varphi(Y)$.

<u>Case 1</u>. $x \in X$.

In this case $x \in Y$ and $\pi_d(x) = \varphi(x) \neq z$. There exist two open subsets H_1 and H_2 of Z such that $\pi_d(x) \in H_1$, $z \in H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_3 = \pi_d^{-1}(H_1)$ and $H = H_3 \times H_2$. Then $(x, z) \in H$ and $H \cap \varphi(Y) = \emptyset$.

<u>Case 2</u>. $x \in A$.

In this case $z \notin Y_x$ and $z \neq \pi_d(x)$. If $r : Z \to \overline{X_1/d}$ is the natural projection, then $r(z) \neq \pi_d(x)$ and there exist two open subsets H_1 and H_2 of Z such that $\pi_d(x) \in H_1$, $r(z) \in H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_3 = \pi_d^{-1}(H_1)$, $H_4 = r^{-1}(H_2)$ and $H = H_3 \times H_4$. Then $(x, z) \in H$ and $H \cap \varphi(Y) = \emptyset$. Therefore $\varphi(Y)$ is a closed subset of the space $X_1 \times Z$.

Obviously, that the set X is dense in Y. The proof is complete.

3.3.6. Proposition. Let (X_1, d) be a complete metric space, X be a dense subset of X, $X_1 \setminus X = L$, let $\{(Y_x, V_x) : x \in L\}$ be a family of complete uniform spaces, $L = \cup\{L_n : n \in N\}$, where L_n is a closed discrete subset of X_1 for each $n \in N$, $L_n \cap L_m = \emptyset$ for $n \neq m$, $\{H_x : x \in L\}$ be a family of open subsets of X_1, $\{L_n x : x \in L, n \in N\}$ be a family of closed discrete subsets of X_1 such that:

A_1. for every $n \in N$ the set $M_n = L_n \cup \cup\{L_n x : x \in L_n\}$ is closed in X_1;

A_2. for every $n \in N$ the family $\{H_x : x \in L\}$ is discrete in X_1;

A_3. if $m, n \in N$, $m \neq n$ and $x \in L$, then $x \in H_x \setminus L_m x$, $L_m x \cap L_n x = \emptyset$ and $L_m x \subseteq H_x$;

A_4. if $x \in L$ and $x_n \in L_n x$ for $n \in N$, then $d(x, x_n) < 2^{-n}$;

A_5. if $m, n \in N$, $n < m$ and $H_m = \cup\{H_x : x \in L_m\}$, then $M_n \cap clH_m = \emptyset$;

A_6. If $x \in L$, then $|L_n x| \leq |L_{n+1} x|$ for each $n \in N$, $\tau(x) = \sup\{L_m x : m \in N\}$ is an infinite cardinal and $d(Y_x) = \tau(x)$.

Then there exists a uniform extension (Y, V) of the space X such that:

1. $Y = X \cup \cup\{Y_x : x \in L\}$ and (Y, V) is a gluing of the spaces (Y_x, V_x) at the points $x \in L$; 2. $u - w(Y, V) = \sup\{u - w(Y_x, V_x) : x \in L\}$; 3. if $x \in L$ and $x_n \in L_n x$, then the sequence $\{x_n : n \in N\}$ is convergent to some point of Y_x; 4. $Y_x \subseteq cl_Y(\cup\{L_n x : n \in N\})$ for every $x \in L$; 5. if $x \in L$ and $y \in Y_x$, then $\chi(y, Y) = \chi(y, Y_x)$; 6. if $\{Y_x : x \in L\}$ are paracompact p-spaces, then Y is a paracompact p-space; 7. if $\{Y_x : x \in L\}$ are Čech complete paracompact spaces, then Y is a Čech complete paracompact space.

Proof. Fix $x \in L$. There exists a set A_x of cardinality $\tau(x)$. Assume that $A_x = \cup\{A_n x : n \in N\}$, where:

- if $n \leq m$, then $A_n x \subseteq A_m x$;
- $|A_n x| = |L_n x|$ for each $n \in N$.

We may suppose that $L_n x = \{x_{n\alpha} : \alpha \in A_n x\}$ and $A_x \cap A_y = \emptyset$ for $x \neq y$. Let $\{y_\alpha : \alpha \in Ax\}$ be a dense subset of the space Y_x.

Let $Y = X \cup \cup\{Y_x : x \in L\}$, $L'_n = \cup\{L_i : i \leq n\}$ and $Y_n = (X_1 \setminus L'_n) \cup \cup\{Y_x : x \in L'_n\}$ for every $n \in N$. If $n, m \in N$ and $n < m$, then consider the natural projection $p_{(m,n)} : Y_m \to Y_n$, where:

– $p_{(m,n)}(x) = x$ if $x \in X$;

– if $x \in L'_n$ and $y \in Y_x$, then $p_{(n,m)}(y) = y$;

– $p^{-1}_{(m,n)}(x) = Y_x$ if $x \in L'_m \setminus L'_n$.

For every $n \in N$ there exist the projections $p_{(\omega,n)} : Y \to Y_n$, $p : Y \to X_1$ and $p_n : Y_n \to X_1$ such that:

– $p_{(\omega,n)}(x) = p(x) = p_n(x) = x$ if $x \in X$;

– if $x \in L'_n$ and $y \in Y_x$, then $p_n(y) = p(y) = x$ and $p_{(\omega,n)}(y) = y$;

– if $x \in L \setminus L'_n$, then $p_n(x) = x$ and $p^{-1}_{(\omega,n)}(x) = Y_x$.

We assume that $L_1 = \emptyset$ and $Y_1 = X_1$.

Fix $n \in N$, where $n \geq 2$. Let $X'_n = X_1 \setminus L'_n \setminus$. On a set $X_n = X'_n \cup \cup\{A_x : x \in L'_n\}$ there exists a complete metric d_n such that:

– the metrizable space (X'_n, d) is a subspace of the space (X_n, d_n);

– if $x \in L'_n$ and $\alpha \in A_x$, then $\lim d_n(d, x_{m\alpha}) = 0$ and $d_n(d, x_{m\alpha}) < 1$ for every $m \in N$;

– if $\alpha, \beta \in \cup\{A_x : x \in L'_n\}$ and $\alpha \neq \beta$, then $d_n(\alpha, \beta) \geq 3$.

By virtue of Theorem 3.2.5 there exists a uniform structure V_n on Y_n such that:

1. (Y_n, V_n) is a uniform extension of the space X'_n and $Y_n \setminus X'_n$ is uniformly isomorphic to the discrete sum of the uniform spaces $\{(Y_x, V_x) : x \in L'_n\}$; 2. $u - w(Y_n, V_n) = \sup\{u - w(Y_x, V_x) : x \in L'_n\}$; 3. if $x \in L'_n$ and $\alpha \in A_x$, then $y_\alpha = \lim x_{m\alpha}$; 4. if $\{(Y_x, V_x) : x \in L\}$ are paracompact spaces, then Y_n is a paracompact space; 5. if $\{Y_x : x \in L\}$ are p-spaces, then Y_n is a p-space; 6. if $\{Y_x : x \in L\}$ are Čech complete spaces, then Y_n is a Čech complete space.

Consider the mapping $\varphi : Y \to \Pi\{Y_n : n \in N\}$, where $\varphi(y) = (p_{(\omega,n)}(y) : n \in N)$ for every $y \in Y$. The set $\varphi(Y)$ is closed in $\Pi\{Y_n : n \in N\}$. We identify Y with $\varphi(Y)$ and consider (Y, V) as a closed subspace of $\Pi\{(Y_n, V_n) : n \in N\}$. The proof is complete.

3.3.7. Theorem. Let (e_1X, U_1) be a first-countable uniform extension of a space X, let $L_n \subset eX \setminus X$ be a strongly discrete subset of the space e_1X, let $L = \cup\{L_n : n \in N\}$, $\{(Y_x, V_x) : x \in L\}$ be a family of complete uniform spaces and let $d\{(Y_x, V_x) \in ds(x, X)$ for every $x \in L$. If (e_1X, U_1) is a Baire space, then there exists a uniform extension (eX, U) and a uniformly continuous mapping $g : eX \to e_1X$ such that:

1. $g(x) = x$ for every $x \in e_1X \setminus L$; 2. for every $x \in L$ the space (Y_x, V_x) is uniformly isomorphic to the subspace $g^{-1}(x)$ of (eX, U); 3. $u - w(eX, U) \leq u - w(e_1X, U_1) + \sup\{u - w(Y_x, V_x) : x \in L\}$; 4. if $\{e_1X_x : x \in L\}$ are Čech complete spaces, then eX is a paracompact Čech complete space; 5. if $(e_1X_x, Y_x) : x \in L$ are paracompact p-spaces, then eX is a paracompact p-space; 6. $\chi(y, eX) = \chi(y, Y_x)$ for every $y \in Y_x$ and $x \in L$.

Proof. For every $x \in L$ we fix a sequence $\{\tau_n(x) \in DS(x, X) : n \in N\}$ such that:

– $\tau_n(x) \leq \tau_{n+1}(x)$ for every $n \in N$;

– $d(Y_x, V_x) = \sup\{\tau_n(x) : n \in N\}$.

Let $\tau(x) = \sup\{\tau_n(x) : n \in N\}$, A_x be a set of cardinality $\tau(x)$, A_nx be a subset of A_x of cardinality $\tau_n(x)$ and $A_nx \subseteq A_{n+1}x$ for every $n \in N$.

Since $\{L_n : n \in N\}$ are strongly discrete sets of the first countable Baire space e_1X, then there exist a family $\{H_x : x \in L\}$ of open subsets of e_1X and a family $\{L_nx : x \in L, n \in N\}$ of strongly discrete sets of the space e_1X such that:

– $\{H_x : x \in L_n\}$ is a discrete family of e_1X for every $n \in N$;

– $\{x\} \cup \cup\{L_nx : n \in N\} \subseteq H_x$ for every $x \in L$;

– the set $M_n = \cup\{\{x\} \cup L_m x : x \in \cup\{L_i : i \leq n\},\ m \in N\}$ is closed in e_1X for every $n \in N$;

– if $n,\ m \in N,\ n < m$ and $H_m = \cup\{H_x : x \in L_m\}$, then $M_n \cap clH_m = \emptyset$;

– if $n,\ m \in N,\ n < m$ and $x \in L$, then $L_n x \cap L_m x = \emptyset$;

– if $x \in L$ and $x_n \in L_n x$ for every $n \in N$, then $x = \lim x_n$;

– $|L_n x| = \tau_n(x)$ for every $x \in L$ and $n \in N$.

For every $n,\ m \in N$ we put $L_{nm} = \cup\{L_m x : x \in L_n\}$. Then L_{nm} is a strongly discrete subset of the space e_1X.

Since e_1X is a first countable space and $\{L_n,\ L_{nm} : n,\ m \in N\}$ are strongly discrete subsets of e_1X, then there exists a continuous pseudometric d on e_1X such that d is a metric on the set $A = \cup\{L_n \cup L_{nm} : n,\ m \in N\}$. Consider the projection $\pi_d : e_1X \to e_1X$. Let $(X_1,\ d_1)$ be the completion of the metric space $(e_1X/d,\ \hat{d})$. We put $L_n = \pi_d(L_n)$, $x = \pi_d(x)$ and $L_n x = \pi_d(L_n x)$ for all $n \in N$ and $x \in L$. The Propositions 3.3.6 and 3.3.5 complete the proof.

3.4. Ultrauniform spaces

An entourage U of the diagonal $\Delta(X)$ of a space X is called discrete if $U = U^{-1} = 2U$.

3.4.1. Definition. A uniform space $(X,\ U)$ is said to be ultrauniform if there exists a base B of uniformity U such that:

– every entourage $U \in B$ is discrete;

– the base B is linearly ordered, i. e. if $U,\ V \in B$, then $U \subseteq V$ or $V \subseteq U$.

The completion of an ultrauniform space is ultrauniform.

3.4.2. Proposition (S. Nedev and M. Choban [137]**).** Every ultrauniform space is hereditarily paracompact.

Proof. Let B be a linearly ordered base of the uniformity U on a space X and every entourage from B be discrete. For every $x \in X$ and $V \in B$ we put $a(x,\ V) = \{y \in X : (x,\ y) \in V\}$. Since V is a discrete entourage, the family $\xi(V) = \{a(x,\ V) : x \in X\}$ is a discrete cover of the space X. If $U,\ V \in B$ and $U \subseteq V$, then $a(x,\ U) \subseteq a(X,\ V)$ for every $x \in X$. Thus $\{a(x,\ V) : V \in B,\ x \in X\}$ is a base of rank one of the space X. Every space with a base of rank one is hereditarily paracompact [8]. A family L of subsets of X has rank one if for every two sets $A,\ B \in L$ we have $A \subseteq B$, or $B \subseteq A$, or $A \cap B = \emptyset$. The proof is complete.

3.4.3. Lemma. Let (X, U) be an ultrauniform space and $\tau = u{-}w(X,\ U)$. Then:

1. $\dim X = 0$; 2. if $\{H_\alpha : \alpha \in A\}$ is a family of open subsets of X and $|A| < \tau$, then $\cap\{H_\alpha : \alpha \in A\}$ is open in X, i. e. X is a P_τ-space; 3. τ is a regular cardinal.

Proof is obvious.

3.4.4. Theorem. Let (e_1X, U_1) be an ultrauniform extension of a space X, (Y, V) be a complete ultrauniform space and τ be an infinite regular cardinal which satisfies the following conditions:

– $d(Y, V) \in DS(e_1X, U)$ and $\tau \in DS(e_1X, U)$;

– Y and e_1X are P_τ-spaces;

– the uniform space (Y, V) is discrete or $u - w(Y, V) = \tau$.

Then there exists a uniform extension (eX, U) of the spaces X and e_1X such that:

1. (eX, U) is an ultrauniform space; 2. the space (Y, V) is uniformly isomorphic to the subspace $eX \setminus e_1X$ of (eX, U); 3. $u - w(eX, U) = \tau$.

Proof. Since $\tau + d(Y, V) \in DS(e_1X, U)$, then there exists a family $\{M_\alpha : \alpha \in A\}$ of subsets of the space X such that:

– $|A| = d(Y, V)$ and $|M_\alpha| = \tau$ for each $a \in A$;

– if $\alpha, \beta \in A$ and $\alpha \neq \beta$, then $M_\alpha \cap M_\beta = \emptyset$;

– the set $M = \cup\{M_\alpha : \alpha \in A\}$ is strongly discrete in e_1X.

We may assume that $M_\alpha = \{x_{\alpha\beta} : \beta < \tau\}$.

Since e_1X is a P_τ-space, then either e_1X is a discrete space, or $u{-}w(e_1X, U_1) = \tau$. Therefore there exists a family $\{\gamma_\beta = \{H_{\beta\lambda} : \lambda \in \Gamma_\beta\} : \beta < \tau\}$ of open discrete covers of the space e_1X such that:

– $\{H_\beta = \cup\{H_{\beta\lambda} \times H_{\beta\lambda} : \lambda \in \Gamma_\beta\} : \beta < \tau\}$ is a base of some complete uniformity U_2 on eX and $U_1 \subseteq U_2$;

– if $\beta < \xi < \tau$, then γ_ξ is a refinement of γ_β, i. e. $H_\xi \subseteq H_\beta$;

– (e_1X, U_2) is a complete ultrauniform space;

– if $\beta < \tau$ and $\lambda \in \Gamma_\beta$, then $|H_{\beta\lambda} \cap M| \leq 1$.

Since $|A| = d(Y, V)$, we may fix a dense subset $Y_1 = \{y_\alpha : \alpha \in A\}$ of the space Y. The uniform space (Y, V) is either discrete or $u - w(Y, V) = \tau$. Therefore there exists a family $\{\omega_\beta = \{V_{\beta\mu} : \mu \in Q_\beta\} : \beta < \tau\}$ of open discrete covers of the space Y such that:

– $B = \{V_\beta = \cup\{V_{\beta\mu} \times V_{\beta\mu} : \mu \in Q_\beta\} : \beta < \tau\}$ is a base of the uniformity V on Y;

– if $\beta < \xi < \tau$, then ω_ξ is a refinement of ω_β, i. e. $V_\xi \subseteq H_\beta$.

For every $\alpha \in A$ and $\beta < \tau$ we put $M_{\alpha\beta} = \{x_{\alpha\xi} : \beta \leq \xi < \tau\}$.

For every $\beta < \tau$ and $\mu \in Q_\beta$ we put $W_{\beta\mu} = V_{\beta\mu} \cup \cup\{H_{\beta\lambda} \in \gamma_\beta : y_\alpha \in V_{\beta\mu}$ and $M_{\alpha\beta} \cap H_{\beta\lambda} \neq \emptyset$ for some $\alpha \in A\}$, $\bar{\gamma}_\beta = \{H_{\beta\lambda} : \lambda \in \Gamma_\beta\} \cup \{W_{\beta\mu} : \mu \in Q_\beta\}$ and $W_\beta = \cup\{H \times H : H \in \bar{\gamma}_\beta\}$. Let U be the uniformity on $eX = e_1X \cup Y$ generated by the base $\{W_\beta : \beta < \tau\}$. The uniform extension (eX, U) is desired. The proof is complete.

3.5. Extensions of locally compact spaces

In this section every space is assumed to be a completely regular T_2-space.

3.5.1. Proposition. Let bY be a Hausdorff compactification of a non-empty space Y. Then there exists a pseudocompact space X such that:

1. $Y = \beta X \setminus X$ and $bY = cl_{bX}Y$; 2. the space $X_1 = \beta X \setminus bY = X \setminus bY$ is a countable compact locally compact dense subspace of the space X and $\beta X_1 = \beta X$; 3. $\dim X = \dim bY$; 4. if $g : bY \to Z$ is a continuous mapping onto a compact space Z and $g^{-1}(g(y)) = y$ for every $y \in bY \setminus Y$, then there exists a compactification bX of X such that $g(Y) = bX \setminus X$ and $Z = cl_{bX}g(Y)$.

Proof. Let ω_1 be the first uncountable ordinal number and W be the space of all ordinal numbers $\leq \omega_1$ in the topology generated by the linear order on W. We put $X = (W \times bY) \setminus (\{\omega_1\} \times Y)$ and $X_1 = (W \setminus \{\omega_1\}) \times bY$. Then $\beta X_1 = \beta X = W \times bY$.

Let $g : bY \to Z$ be a continuous mapping onto a compact space and $g^{-1}(g(y)) = y$ for every $y \in bY \setminus Y$. We put $bX = X_1 \cup Z$. Consider the mapping $\varphi : \beta X \to bX$, where $g = \varphi \,|\, bY$ and $\varphi(x) = x$ for every $x \in X_1$. Then $\varphi(x) = x$ for every $x \in X$. On bX consider the quotient topology. The proof is complete.

A space is called a continuum if it is a connected compact Hausdorff space. A space X is an arcwise connected or pathwise connected space if for every pair of points $a, b \in X$ there exists a continuous mapping $f : I \to X$ of the interval $I = [0, 1]$ into X such that $f(0) = a$ and $f(1) = b$. A space$X$ is locally arcwise connected if the family $\{U \subseteq X : U$ is an open arcwise connected subspace of $X\}$ is an open base of X (see [44]).

A space is called a Peano continuum if it is a locally arcwise connected continuum.

3.5.2. Definition. A space X is said to be a marginal arcwise connected space if there exist a cardinal τ, an embedding of X into I^m and a sequence of arcwise connected subspaces $\{X_n : n \in N\}$ of I^τ such that:

– $X = \cap\{X_n : n \in N\}$;

– for every open subset U of I^τ, which contains the closure clX of X in I^τ, there exists $n \in N$ such that $\cup\{X_i : i \geq n\} \subseteq U$.

3.5.3. Examples.

1. The Tychonoff cube I^m is a Peano continuum. 2. If a continuum X is a G_δ-subset of a Peano continuum Y, then X is a marginal arcwise connected space. 3. Every metrizable continuum is a marginal arcwise connected space.

The set $S(f) = Y \setminus \cup\{U : U$ is open in Y and the set $cl_X f^{-1}U$ is compact$\}$ is called the singularity set of the mapping $f : X \to Y$ of a space X into a space Y. If X is a locally compact space, then $S(\varphi) = \cap\{cl_Y f(X \setminus F) : F$ is a compact subset of $X\}$ (see [35]).

3.5.4. Proposition. Let $f : X \to Y$ be a continuous mapping of a locally compact non-compact space X into a compact space Y. Then:

1. $S(f) \neq \setminus$; 2. there exists a compactification bX of the space X such that the spaces $bX \setminus X$ and $S(f)$ are homeomorphic.

Proof. As in Section 2.3 we consider the compact space $Z = X \cup Y$ with the topology generated by the open base $\{U \subseteq X : U$ is open in $X\} \cup \{f^{-1}(V) \setminus F :$

V is open in Y and F is a compact subset of $X\}$. Then $S(f) = cl_Z X \setminus X$ and $bX = cl_Z X$. The proof is complete.

3.5.5.Theorem. Let X be a locally compact non-pseudocompact space and Y be a separable marginal arcwise connected space X. Then:

1. there exists a Hausdorff compactification bY of Y such that bY is a remainder of some Hausdorff compactification bX of X.

2. there exist a compact space Z, an embedding of Y in Z and a continuous mapping $\varphi : X \to Z$ such that $S(\varphi) = cl_Z Y$.

Proof. There exist a cardinal τ, an embedding of Y into I^τ and a sequence of arcwise connected subspaces $\{Y_n : n \in N\}$ of I^τ such that:

– $Y = \cap\{Y_n : n \in N\}$;

– for every open subset U of I^τ, which contains the closure clY of Y in I^τ, there exists $n \in N$ such that $\cup\{Y_i : i \geq n\} \subseteq U$.

We put $Z = I^\tau$ and $bY{=}cl_Z Y$. Fix a countable dense subset $B =\{b_n : n \in N\}$ of the space Y. Fix a point $b_0 \in Y$. For every $n \in N$ we fix a continuous mapping $g_n : I \to Y_n$ such that $g_n(0) = b_0$ and $g_n(1){=}b_n$.

Since X is non-pseudocompact, there exists a subset $A =\{a_n \in X{:}n \in N\}$ and a continuous function $f : X \to R$ such that $f(a_{n+1}) \geq 3 + f(a_n)$ for every $n \in N$. For every $n \in N$ we fix an open subset U_n of X and a continuous function $f_n : X \to I$ such that $a_n \in U_n, f(a_n) = 1, X \setminus U_n \subseteq f_n^{-1}$ and the set $cl_X U_n$ is compact.

Now we construct the mapping $\varphi : X \to Z$, where:

– $\varphi(x) = b_0$ if $x \in X \setminus \cup\{U_n : n \in N\}$;

– if $n \in N$ and $x \in U_n$, then $\varphi(x) = g_n(f_n(x))$.

Since the family $\{U_n : n \in N\}$ is discrete and $\varphi(x) = g_n(f_n(x))$ for every $n \in N$ and $x \in cl_X U_n$, the mapping φ is continuous.

Let $H_n = Z \setminus cl_Z Y_n$. For every $n \in N$ there exists $k = k(n) \in N$ such that $Y_i \cap H_n = \emptyset$ for every $i \geq k$.

Then $\varphi^{-1}(H_n) \subseteq \cup\{U_i : i \leq k\}$ and the set $cl_X \varphi^{-1}(H_n)$ is compact.

Since $bY = Z \setminus \cup\{H_n : n \in N\}$, we have $S(\varphi) \subseteq bY$. If U is open in Z and $U \cap bY \neq \emptyset$, then the set $N(U) = \{n \in N : b_n \in U\}$ is infinite. If $n \in N(U)$, then $a_n \in \varphi^{-1}(U)$. Therefore the set $\varphi^{-1}(U)$ is not compact and $bY = S(\varphi)$.

The assertion 2 is proved. The Construction 2.3.1 completes the proof.

3.5.6. Proposition. Let X be a locally compact non-pseudocompact space and bY be a compactification of a separable arcwise connected space Y. Then there exists a continuous mapping $\varphi : X \to bY$ such that $S(\varphi) = bY$, i.e. the mapping φ is singular.

Proof. As in the proof of Theorem 3.5.5. we consider that $Y \subseteq bY \subseteq I^\tau$ for some cardinal τ and put $Y_n = Y$ for each $n \in N$. The proof is complete.

3.5.7. Corollary. Let X be a locally compact non-pseudocompact space and K be a marginal arcwise connected compact space. Then K is a remainder of some Hausdorff compactification bX of X.

3.5.8. Corollary. (see [74], Theorem 3, when X is connected). Let X be a locally compact non-pseudocompact space and Y be a space which contains a dense separable and arcwise connected subspace. Then every compactification bY of $Y$is a remainder of X.

3.5.9. Corollary. (see [73] for Y metrizable). Let X be a locally compact non-pseudocompact space and Y be a separable Peano continuum. Then Y is a remainder of X.

3.5.10. Corollary. ([94], J.V.Rogers and [1], J.M.Aarts and P.van Emde Boas, for metrizable separable X). Let Y be metrizable continuum and let X be a locally compact non-pseudocompact space. Then Y is a remainder of X.

3.5.11. Theorem. Let X be a paracompact locally compact space. If the space X is not compact, then:

1. if $\tau \in DS(X)$, the cardinal τ is uncountable, Y is a compact space and $d(Y) \leq \tau$, then Y is a remainder of some compactification bX of the space X;
2. if *dimX=0* and Y is a remainder of some compactification of the discrete space D_m of the cardinality $m \leq \tau$,then Y is a remainder of some compactifion bX of the space X.

Proof. The space X can be represented as the union of a family $\{X_\alpha : \alpha \in A\}$ of disjoint closed-and-open subspaces of X each of which has the Lindelöf property ([44], Theorem 5.1.27).

There exist a locally compact metric space Z and a perfect mapping $\varphi : X \to Z$ of X onto Z such that $X_\alpha = \varphi^{-1}(\varphi(X_\alpha))$ for each $\alpha \in A$. Then $Z_\alpha = \varphi(X_\alpha)$ is an open-and-closed subset of Z for every $\alpha \in A$. If the set A is infinite and $\tau \in DS(X)$, then $\tau \leq |A|$. If the set A is countable, then the space X is Lindelöf and every closed discrete subspace of X is finite or countable .

<u>Case 1</u>. $\alpha_0 \in A$ and the subspace X_{α_0} is not compact.

In this case Z_{α_0} is a locally compact non compact space with a countable base.

Let Y be a metrizable connected compact space. By virtue of Aarts and Emde Boas theorem [1] (see Theorem 3.5.5) there exists a compactification bZ_{α_0} of the space Z_{α_0} such that $Y = bZ_{\alpha_0} \setminus Z_{\alpha_0}$. Then there exists a compactification $bX_{\alpha_0}$of the space X_{α_0} such that $Y = bX_{\alpha_0} \setminus X_{\alpha_0}$. Fix $y_0 \in Y$ and put $bX = X \cup Y$. On bX consider the following topology:

– the space X is an open subspace of the space bX;

– $bX_{\alpha_0} \setminus \{y_0\}$ is an open subspace of the space bX;

– if U is an open subset of bX and $y_0 \in U$, then $F = X \setminus (U \cup X_{\alpha_0})$ is a compact subset of X ;

– if V is an open subset of bX_{α_0}, $y_0 \in V$ and F is a compact subset of X, then $V \cup \cup\{X_\alpha \setminus F : \alpha \in A \setminus \{\alpha_0\}\}$ is open in bX.

In this conditions bX is a Hausdorff compactification of X and Y=$bX \setminus X$.

Case 2. $\tau = |A|$ is an infinite cardinal. For every $a \in A$ fix a point $a_\alpha \in X_\alpha$.

Let Y be a compact space and $d(Y) \leq \tau$. Let Y_1 be a dense subset of Y and $|Y_1| \leq \tau$. There exists a continuous mapping $\varphi : X \to Y$ into Y such that for every $y \in Y_1$ the set $\{\alpha \in A : \varphi(a_\alpha) = y\}$ is infinite. The continuous mapping φ is singular, i.e. the set $cl_X\varphi^{-1}(V)$ is not compact provided the set V is open and non-empty. By virtue of Construction 2.3.1 there exists a compactification bX of X such that $Y{=}bX\setminus X$.

Case 3. dimX=0.

We may assume that the set A is infinite and X_α is a compact subset of X for each $\alpha \in A$. Let Y be a compact space, D_m be a infinite discrete space, $m \leq \tau$, bD_m be a compactification of D_m and $Y{=}bD_m \setminus D_m$. We may assume that $m = \tau$. Consider that $D_m = D_\tau = \{d_\alpha : \alpha \in A\}$. Then there exists a mapping $\Psi : X \to D_\tau$ such that $\Psi^{-1}(d_\alpha) = X_\alpha$ for every $\alpha \in A$. The mapping Ψ is open and perfect. There exists a continuous extension $g : \beta X \to bD_\tau$ of the mapping Ψ. By construction, $g(\beta X \setminus X) = Y$. By virtue of Theorem 2.1.5 there exists a compactification bX of X such that $Y = bX \setminus X$. The proof is complete.

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ROMAI J., **1**, 2(2005),
69–77

THE OSCILLATORY LIFTING SURFACE EQUATION AND THE SELF-PROPULSION

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Abstract

Using appropriate quadrature formulas we discretize the oscillatory lifting surface integral equation in order to obtain the jump of the pressure across the surface. Integrating numerically we obtain the drag coefficient.

1. INTRODUCTION

In this paper we discretize the oscillatory lifting surface equation (for an inviscid incompressible fluid) for the delta wing in order to obtain the jump of the pressure coefficient field across the surface. We notice that if the frequency of the oscillation overpasses a certain value, the average drag coefficient is negative, i.e. a propulsive force appears.

2. THE INTEGRAL EQUATION

We consider a system of coordinates $Ox^{(1)}y^{(1)}z^{(1)}$ related to the wing and we introduce the dimensionless space coordinates $(x, z) = \left(\frac{x^{(1)}}{a}, \frac{z^{(1)}}{a}\right)$, taking the wing length (more precisely the maximum value of the wing chord) a as reference length along the vertical direction ($Oz^{(1)}$-axis direction) and along the direction of the unperturbed uniform flow ($Ox^{(1)}$-axis direction).

We introduce also the dimensionless space coordinate $y = \frac{y^{(1)}}{b}$, taking the wing half-span b as reference length along the $Oy^{(1)}$-axis direction. Denote by $D^{(1)}$ the wing projection on the $Ox^{(1)}y^{(1)}$ - plane. Let

$$0 = F(x^{(1)}, y^{(1)}, z^{(1)}, t) = z^{(1)} - h^{(1)}(x^{(1)}, y^{(1)}) \exp(i\omega t) \ ; \ (x^{(1)}, y^{(1)}) \in D^{(1)}, \tag{1}$$

be the equation of the oscillating wing (we neglect the thickness of the wing). Let $\rho_0 Re(f \exp(i\omega t))$ be the jump of the pressure over the oscillating wing (we denote by ρ_0 the density of the fluid at rest). ω is the frequency of the oscillation, V_0 is the translation velocity of the unperturbed flow with

respect to the $Ox^{(1)}y^{(1)}z^{(1)}$ frame of reference and $\varpi = \dfrac{b}{a}$ is the aspect ratio. Introducing the dimensionless functions and variables

$$h(x,y) = \frac{h^{(1)}(x^{(1)}, y^{(1)})}{a},\ \tilde{\omega} = \frac{\omega a}{V_0}, \tilde{d} = \frac{d}{V_0},\ \tilde{f}(x,y) = \frac{f(ax,by)}{V_0^2},$$

$$x_0 = x - \xi, \quad y_0 = y - \eta,$$

one demonstrates [3] that the integral equation of the oscillatory thin wing is

$$\begin{aligned} \frac{-\varpi}{4\pi} \int\int_D^* \tilde{f}(\xi,\eta) \exp(-i\tilde{\omega}x_0) \left(\int_{-\infty}^{x_0} \frac{\exp(i\tilde{\omega}u)}{(u^2+\varpi^2 y_0^2)^{3/2}} du \right) d\xi d\eta = \\ = \frac{\partial h(x,y)}{\partial x} + i\tilde{\omega} h(x,y). \end{aligned} \tag{2}$$

where $(x,y) \in D$ if and only if $(x^{(1)}, y^{(1)}) \in D^{(1)}$.

The star indicates the finite part in the Hadamard sense of the integral.

3. THE DISCRETIZATION OF THE INTEGRAL EQUATION

We consider the oscillating delta wing. The equations of the leading edge of $D^{(1)}$ are

$$y_{\pm}^{(1)}(x^{(1)}) = \pm \frac{b}{a}(x^{(1)}) \qquad ; \qquad x^{(1)} \in [0,a] \tag{3}$$

and the equations of the leading edge of D are

$$y_{\pm}(x) = \pm x \qquad ; \qquad x \in [0,1]. \tag{4}$$

In order to solve numerically the integral equation (2), we have to discretize the left-hand side in order to obtain a system of algebraic equations. To this aim we split the kernel $K(x,y;\xi,\eta) = \displaystyle\int_{-\infty}^{x_0} \frac{\exp(i\tilde{\omega}u)}{(u^2+\varpi^2 y_0^2)^{3/2}} du$ into several kernels in order to put into evidence the kind of singularities we are dealing with and to find afterwards the most convenable quadrature formulas.

We have step by step

$$\begin{aligned} \int_{-\infty}^{x_0} \frac{\exp(i\tilde{\omega}u)}{(u^2+\varpi^2 y_0^2)^{3/2}} du = \int_{-\infty}^{x_0} \frac{\exp(i\tilde{\omega}u) - 1}{(u^2+\varpi^2 y_0^2)^{3/2}} du + \\ + \frac{1}{\varpi^2 y_0^2} \left(1 + \frac{x_0}{|x_0|} \right) + \frac{1}{\varpi^2 y_0^2} \left(\frac{x_0}{\sqrt{x_0^2 + \varpi^2 y_0^2}} - \frac{x_0}{|x_0|} \right), \end{aligned} \tag{5}$$

$$\int_{-\infty}^{x_0} \frac{\exp(i\tilde{\omega}u)-1}{(u^2+\varpi^2 y_0^2)^{3/2}}du = \int_0^{x_0} \frac{\exp(i\tilde{\omega}u)-1}{(u^2+\varpi^2 y_0^2)^{3/2}}du + \int_0^{\infty} \frac{\exp(-i\tilde{\omega}u)-1}{(u^2+\varpi^2 y_0^2)^{3/2}}du, \tag{6}$$

$$\int_0^{\infty} \frac{\exp(-i\tilde{\omega}u)-1}{(u^2+\varpi^2 y_0^2)^{3/2}}du = -\frac{1}{\varpi^2 y_0^2} + \int_0^{\infty} \frac{\cos\tilde{\omega}u - i\sin\tilde{\omega}u}{(u^2+\varpi^2 y_0^2)^{3/2}}du, \tag{7}$$

The integral from the right-hand part of (7) represents the *sine* and *cosine Fourier transforms* of $(u^2+\varpi^2 y_0^2)^{-3/2}$ and it may be shown that

$$\int_0^{\infty} \frac{\cos\tilde{\omega}u}{(u^2+\varpi^2 y_0^2)^{3/2}}du = \frac{\widetilde{\omega}}{\varpi\,|y_0|} K_1\left(\widetilde{\omega}\varpi\,|y_0|\right), \tag{8}$$

$$\int_0^{\infty} \frac{\sin\tilde{\omega}u}{(u^2+\varpi^2 y_0^2)^{3/2}}du = \frac{\pi}{2}\frac{\widetilde{\omega}}{\varpi\,|y_0|}\left(L_{-1}\left(\widetilde{\omega}\varpi\,|y_0|\right) - I_1\left(\widetilde{\omega}\varpi\,|y_0|\right)\right), \tag{9}$$

where L_{-1} is a Strouve function and I_1, K_1 are Bessel functions and their series expansions are

$$I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!},\ \ L_{-1}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\frac{1}{2})} \tag{10}$$

$$K_1(x) = I_1(x)\ln\frac{x}{2} + \frac{1}{x} - \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!\,(k+1)!}(\psi(k+1)+\psi(k+2)), \tag{11}$$

We also have

$$\int_0^{x_0} \frac{\exp(i\tilde{\omega}u)-1}{(u^2+\varpi^2 y_0^2)^{3/2}}du = \int_0^{x_0} \frac{\exp(i\tilde{\omega}u)-1-i\tilde{\omega}u+\tilde{\omega}^2u^2/2}{(u^2+\varpi^2 y_0^2)^{3/2}}du + \frac{i\tilde{\omega}}{|y_0|} -$$

$$-\frac{i\tilde{\omega}}{({x_0}^2+\varpi^2 y_0^2)^{1/2}} + \frac{\tilde{\omega}^2 x_0}{2({x_0}^2+\varpi^2 y_0^2)^{1/2}} - -\frac{\widetilde{\omega}^2}{2}\ln\left(x_0 + \sqrt{({x_0}^2+\varpi^2 y_0^2)}\right). \tag{12}$$

Hence

$$K\left(x,y;\xi,\eta\right) = K_1\left(x,y;\xi,\eta\right) + ... + K_8\left(x,y;\xi,\eta\right)$$

and the integral equation (2) becomes

$$\frac{\varpi}{4\pi}\sum_{i=1}^{8}\int\int_D^{*} \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_i\left(x,y;\xi,\eta\right)d\xi d\eta$$

$$= -\left(\frac{\partial h(x,y)}{\partial x} + i\tilde{\omega}h(x,y)\right)\exp(i\tilde{\omega}x) \tag{13}$$

In the sequel we shall provide adequate quadrature formulas for the integrals from the right-hand side of the equation (13) in order to discretize it. Let

$$K_1(x,y;\xi,\eta) = \frac{1}{\varpi^2 y_0^2}\left(\frac{x_0}{\sqrt{x_0^2+\varpi^2 y_0^2}} - \frac{x_0}{|x_0|}\right),$$

$$K_2(x,y;\xi,\eta) = \frac{1}{\varpi^2 y_0^2}\left(1+\frac{x_0}{|x_0|}\right), K_3(x,y;\xi,\eta) = \frac{-i\widetilde{\omega}}{\sqrt{x_0^2+\varpi^2 y_0^2}},$$

$$K_4(x,y;\xi,\eta) = -\frac{\widetilde{\omega}^2}{2}\frac{x_0}{|x_0|}\ln\left(|x_0| + \sqrt{x_0^2+\varpi^2 y_0^2}\right),$$

$$K_5(x,y;\xi,\eta) = \frac{\widetilde{\omega}^2}{2}\ln(\varpi|y_0|)\left(1+\frac{x_0}{|x_0|}\right), K_6(x,y;\xi,\eta) = \frac{\widetilde{\omega}^2 x_0}{\sqrt{x_0^2+\varpi^2 y_0^2}},$$

$$K_7(x,y;\xi,\eta) = \frac{\widetilde{\omega}}{\varpi|y_0|}K_1(\widetilde{\omega}\varpi|y_0|) - \frac{1}{\varpi^2 y_0^2} - \frac{\widetilde{\omega}^2}{2}\ln\frac{\widetilde{\omega}\varpi|y_0|}{2} +$$

$$+\frac{i\pi\widetilde{\omega}^2}{2\varpi|y_0|}\left(I_1(\widetilde{\omega}\varpi|y_0|) - L_{-1}(\widetilde{\omega}\varpi|y_0|) + \frac{2}{\pi}\right) + \frac{\widetilde{\omega}^2}{2}\ln\frac{\varpi}{2},$$

$$K_8(x,y;\xi,\eta) = \int_0^{x_0}\frac{\exp(i\tilde{\omega}u) - 1 - i\tilde{\omega}u + \tilde{\omega}^2u^2/2}{(u^2+\varpi^2 y_0^2)^{3/2}}du.$$

Setting

$$\tilde{f}(\xi,\eta) = \frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}},$$

we have

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_1(x,y;\xi,\eta)\,d\xi d\eta =$$

$$= \frac{1}{\varpi^2}FP\int_{-1}^{1}\frac{1}{y_0^2}\left(\int_{|\eta|}^{1}\frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}}\exp(i\tilde{\omega}\xi)\left(\frac{x_0}{\sqrt{x_0^2+\varpi^2 y_0^2}} - \frac{x_0}{|x_0|}\right)d\xi\right)d\eta. \tag{14}$$

FP stands for the finite part in Charles Fox' sense [2]. Taking into account that $x(1) = x(-1) = 1$ we assume the behaviour

$$\int_{|\eta|}^{1} \frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}} \exp(i\tilde{\omega}\xi) \left(\frac{x_0}{\sqrt{x_0^2+\varpi^2 y_0^2}} - \frac{x_0}{|x_0|} \right) d\xi = \sqrt{1-\eta^2} G(x,y;\eta), \tag{15}$$

where $G(x,y;\eta)$ is finite in $\eta = \pm 1$. We consider on D a net consisting of the nodes (grid points, controll points) $(x_i, \overline{y}_j) = \left(\frac{i}{n}, \frac{2j+1}{2n}\right)$, $i = 1, ..., n$, $j = -i, -i+1, ..., i-1$. For the hypersingular integral occuring in (14) we may use the quadrature formula for equidistant controll points given by Dumitrescu [1]

$$FP \int_{-1}^{1} \frac{\sqrt{1-\eta^2} G(x_k, \overline{y}_l; \eta)}{(\overline{y}_l - \eta)^2} d\eta = \sum_{j=-n}^{n-1} G(x_k, \overline{y}_l; \overline{y}_j) A_{lj} \tag{16}$$

$$A_{lj} = -\arccos(y_j) + \arccos(y_{j+1}) + \frac{\sqrt{1-y_j^2}}{y_j - \overline{y}_l} -$$

$$-\frac{\sqrt{1-y_{j+1}^2}}{y_{j+1} - \overline{y}_l} - \frac{\overline{y}_l}{\sqrt{1-\overline{y}_l^2}} \ln \mid \frac{C_{l(j+1)}}{C_{lj}} \mid \tag{17}$$

with

$$C_{lj} = \frac{\sqrt{1-y_j} \cdot \sqrt{1+\overline{y}_l} - \sqrt{1+y_j} \cdot \sqrt{1-\overline{y}_l}}{\sqrt{1-y_j} \cdot \sqrt{1+\overline{y}_l} + \sqrt{1+y_j} \cdot \sqrt{1-\overline{y}_l}}. \tag{18}$$

We have the quadrature formula

$$G(x_k, \overline{y}_l; \overline{y}_j) = \sum_{i=j}^{n} g_{ij} B_{ijkl}, g_{ij} = g(\overline{x}_{ij}, \overline{y}_j), \tag{19}$$

$$\overline{x}_{ij} = \begin{cases} x_i - \frac{1}{2n}, -i < j < i-1, \\ x_i - \frac{1}{4n}, j \in \{-i, i-1\}, \end{cases}, \widetilde{x}_{ij} = \begin{cases} x_i - \frac{1}{n}, -i < j < i-1, \\ x_i - \frac{1}{2n}, j \in \{-i, i-1\}, \end{cases} \tag{20}$$

$$B_{ijk} = \frac{E_{ij} D_{ijkl}}{\sqrt{1-\overline{y}_j^2}}, E_{ij} = \exp(i\tilde{\omega}\overline{x}_{ij}) \ln \frac{x_i + \sqrt{x_i^2 - \overline{y}_j^2}}{\widetilde{x}_{ij} + \sqrt{\widetilde{x}_{ij}^2 - \overline{y}_j^2}}, \tag{21}$$

$$D_{ijkl} = \left(\frac{x_k - \overline{x}_{ij}}{\sqrt{(x_k - \overline{x}_{ij})^2 + \varpi^2 (\overline{y}_l - \overline{y}_j)^2}} - \frac{x_k - \overline{x}_{ij}}{|x_k - \overline{x}_{ij}|} \right), -i < j < i-1, \tag{22}$$

Finally, we deduce

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_1\left(x_k,\overline{y}_l;\xi,\eta\right)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=-i}^{i-1} g_{ij}K_{ijkl}^{(1)}, \tag{23}$$

$$K_{ijkl}^{(1)} = \frac{A_{lj}B_{ijkl}}{\varpi^2}. \tag{24}$$

We have

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_2\left(x,y;\xi,\eta\right)d\xi d\eta =$$

$$= \frac{2}{\varpi^2}FP\int_{-x}^{x}\frac{1}{y_0^2}\left(\int_{|\eta|}^{x}\frac{g(\xi,\eta)\exp(i\tilde{\omega}\xi)}{\sqrt{\xi^2-\eta^2}}d\xi\right)d\eta. \tag{25}$$

Assuming the behaviour

$$\int_{|\eta|}^{x_k}\frac{g(\xi,\eta)\exp(i\tilde{\omega}\xi)}{\sqrt{\xi^2-\eta^2}}d\xi = \sqrt{x_k^2-\eta^2}G^{(k)}\left(x_k;\eta\right), \tag{26}$$

we have

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_2\left(x_k,\overline{y}_l;\xi,\eta\right)d\xi d\eta =$$

$$= \frac{2}{\varpi^2}FP\int_{-x_k}^{x_k}\frac{\sqrt{x_k^2-\eta^2}G^{(k)}\left(x_k;\eta\right)}{\left(\overline{y}_l-\eta\right)^2}d\eta = \frac{2}{\varpi^2}\sum_{j=-k}^{k-1}G^{(k)}\left(x_k;\overline{y}_j\right)A_{lj}^{(k)}, \tag{27}$$

where

$$A_{lj}^{(k)} = -\arccos(\frac{y_j}{x_k}) + \arccos(\frac{y_{j+1}}{x_k}) +$$

$$+\frac{\sqrt{x_k^2-y_j^2}}{y_j-\overline{y}_l} - \frac{\sqrt{x_k^2-y_{j+1}^2}}{y_{j+1}-\overline{y}_l} - \frac{\overline{y}_l}{\sqrt{x_k^2-\overline{y}_l^2}}\ln\mid\frac{C_{l(j+1)}^{(k)}}{C_{lj}^{(k)}}\mid, \tag{28}$$

with

$$C_{lj}^{(k)} = \frac{\sqrt{x_k-y_j}\cdot\sqrt{x_k+\overline{y}_l}-\sqrt{x_k+y_j}\cdot\sqrt{x_k-\overline{y}_l}}{\sqrt{x_k-y_j}\cdot\sqrt{x_k+\overline{y}_l}+\sqrt{x_k+y_j}\cdot\sqrt{x_k-\overline{y}_l}}. \tag{29}$$

In order to compute $G^{(k)}\left(x_k;\overline{y}_j\right)$ we use the quadrature formula

$$G^{(k)}\left(x_k;\overline{y}_j\right) = \frac{1}{\sqrt{x_k^2-\overline{y}_j^2}}\int_{|\overline{y}_j|}^{x_k}\frac{g(\xi,\overline{y}_j)\exp(i\tilde{\omega}\xi)}{\sqrt{\xi^2-\overline{y}_j^2}}d\xi = \sum_{i=|j|}^{k}g_{ij}\frac{E_{ij}}{\sqrt{x_k^2-\overline{y}_l^2}}. \tag{30}$$

At last, we find

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_2\left(x_k,\overline{y}_l;\xi,\eta\right)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=-i}^{i-1} g_{ij}K_{ijkl}^{(2)} \tag{31}$$

with

$$K_{ijkl}^{(2)} = \begin{cases} \frac{2}{\varpi^2}A_{lj}^{(k)}\frac{E_{ij}}{\sqrt{x_k^2-\overline{y}_l^2}}; i \le k, \\ 0; i > k. \end{cases} \tag{32}$$

The kernels K_3 and K_4 are singular. We divide and multiply them by y_0^2 in order to obtaine quadrature formulas similar to the formulas for K_1. We get

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)\left[K_3\left(x_k,\overline{y}_l;\xi,\eta\right)+K_4\left(x_k,\overline{y}_l;\xi,\eta\right)\right]d\xi d\eta =$$

$$= \sum_{i=1}^{n}\sum_{j=-i}^{i-1} g_{ij}K_{ijkl}^{(3,4)} \tag{33}$$

with

$$K_{ijkl}^{(3,4)} = \frac{A_{lj}\left(\overline{y}_l-\overline{y}_j\right)^2}{\sqrt{1-\overline{y}_j^2}}\left[K_3\left(x_k,\overline{y}_l;\overline{x}_{ij},\overline{y}_j\right)+K_4\left(x_k,\overline{y}_l;\overline{x}_{ij},\overline{y}_j\right)\right]E_{ij}. \tag{34}$$

The kernels K_5, K_6, K_7 and K_8, have integrable singularities and we use the quadrature formulas

$$\int\int_D^* \tilde{f}(\xi,\eta)\exp(i\tilde{\omega}\xi)K_p\left(x_k,\overline{y}_l;\xi,\eta\right)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=-i}^{i-1} g_{ij}K_{ijkl}^{(p)},\ p=5,6,7,8$$

where

$$K_{ijkl}^{(5)} = \begin{cases} \widetilde{\omega}^2 B_{jl}^{(k)}E_{ij}, i \le k \\ 0, i > k, \end{cases}, \tag{35}$$

$$B_{jl}^{(k)} = \left(y_{j+1}-\overline{y}_l\right)\ln\left|y_{j+1}-\overline{y}_l\right| - \left(y_j-\overline{y}_l\right)\ln\left|y_j-\overline{y}_l\right|, \tag{36}$$

$$K_{ijkl}^{(p)} = E_{ij}K_p\left(x_k,\overline{y}_l;\overline{x}_{ij},\overline{y}_j\right)/n,\ p=6,7,8. \tag{37}$$

In order to compute $K_7\left(x_k,\overline{y}_l;\overline{x}_{ij},\overline{y}_j\right)$ we use the series expansions of the Bessel and Strouve functions and we take into account that

$$K_7\left(x_k,\overline{y}_l;\overline{x}_{ij},\overline{y}_l\right) = -\frac{\widetilde{\omega}^2\left(\psi(1)+\psi(2)\right)}{4}+\frac{\pi i\widetilde{\omega}}{4},$$

$$\psi(1) = -0.5772, \psi(2) = 0.4228. \tag{38}$$

The kernel $K_8\left(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j\right)$ is an integral which is evaluated numerically with the trapezoidal rule.

Denoting

$$K_{ijkl} = K_{ijkl}^{(1)} + K_{ijkl}^{(2)} + K_{ijkl}^{(3,4)} + K_{ijkl}^{(5)} + K_{ijkl}^{(6)} + K_{ijkl}^{(7)} + K_{ijkl}^{(8)},$$

we obtain, discretizing the $2d$ integral ecuation (2),

$$\frac{\varpi}{4\pi} \sum_{i=1}^{n} \sum_{j=-i}^{i-1} g_{ij} K_{ijkl} = -\left(\frac{\partial h(x_k, \overline{y}_l)}{\partial x} + i\tilde{\omega} h(x_k, \overline{y}_l) \right) \exp(i\tilde{\omega} x_k) \tag{39}$$

4. THE AERODYNAMIC COEFFICIENTS

In the sequel we shall deal with the jump of the pressure coefficient

$$C_p(x^{(1)}, y^{(1)}, t) = Re[\tilde{f}(x, y) \exp(i\omega t)]. \tag{40}$$

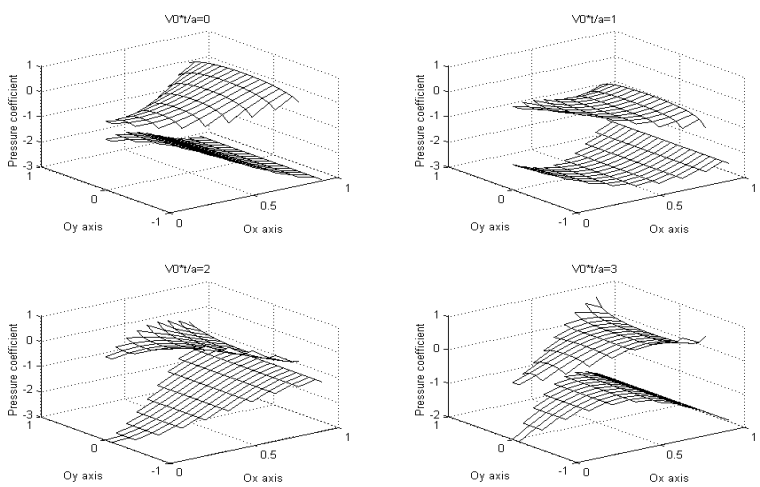

Fig. 1. Pressure coefficient field computed numerically.

Among the aerodynamic characteristics of the wing, in this paper we are interested in the drag coefficient

$$C_D(t) = -2 \int \int_D n_x C_p(ax, by, t) dx dy. \tag{41}$$

We consider the oscillating delta wing whose equation is

$$h(x,y) = \alpha \exp(i\widetilde{\omega}_1 x), \ \widetilde{\omega}_1 = a\omega_1; \qquad (x,y) \in D. \tag{42}$$

In order to compute the drag coefficient numerically we use the quadrature formulas

$$C_D(t) = \frac{2\alpha i\tilde{\omega}_1 \exp(i\omega t)}{n^2} \cdot \sum_{k=1}^{N-1} \exp\left(i\tilde{\omega}_1 \frac{k}{n}\right) \cdot \left[\frac{1}{2} C_p\left(\frac{ka}{n}, \frac{(2k-1)b}{2n}, t\right) + \right.$$

$$\left. \sum_{j=-k}^{k-1} C_p\left(\frac{ka}{n}, \frac{(2j+1)b}{2n}, t\right) + \frac{1}{2} C_p\left(\frac{ka}{n}, \frac{(-2k+1)b}{2n}, t\right)\right], \tag{43}$$

5. THE PROPULSIVE FORCE

In order to have an intuitive view of the temporal variation of the pressure coefficient distribution over the wing, in fig.1 we present the pressure coefficient fields (divided to α) for the values of the nondimensional time $\frac{V_0}{a}t \in \{0,1,2,3\}$ and for the nondimensional frequencies $\tilde{\omega} = \pi/4$, $\widetilde{\omega}_1 = -\pi$. Above there is the pressure coefficient field and below there is the wing. We considered $N = 15$ and $\varpi = 1/8$. Since the drag coefficient varies periodically, we may calculate the *average drag coefficient*

$$\widetilde{C}_D = \frac{1}{T}\int_0^T C_D(t)\, dt \simeq \frac{1}{k}\sum_{l=1}^{k} C_D\left(\frac{lT}{k}\right),$$

where T is the period of the oscillation.

In fig.2, for $\widetilde{\omega}_1 = \pi/4$, we present the average drag coefficient (divided by α^2) versus the reduced frequency $\widetilde{\omega}$. We notice that the average drag coefficient is negative, i.e. a propulsive force appears and the absolute value of this propulsive force increases when $\widetilde{\omega}$ increases.

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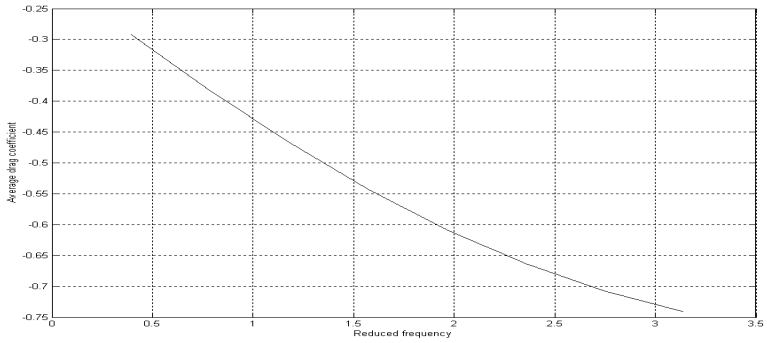

Fig. 2. Average drag coefficient versus reduced frequency

ROMAI J., **1**, 2(2005),
79–82

EXAMPLES OF L-PARTITIONS OF L-FUZZY SETS

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Abstract If X is a set and L a lattice, then a function $\varphi : X \to L$ is called a fuzzy L-subset of X. In [1], by generalizing similar results about partitions of fuzzy subsets, L-partitions of L- fuzzy subsets are presented. In this paper we give examples of L- partitions of L-fuzzy subsets.

2000 MSC 94D05, 03B52.
Key words: L- fuzzy sets L- fuzzy partition, fuzzy logic.

1. INTRODUCTION

Because the conventional computer logic was not capable to treat the data representing vague values of some properties and parameters, L. A. Zadeh introduced the fuzzy sets theory in his work "Fuzzy sets" published in 1965. While the classic sets can be represented by their characteristic function (membership function) which takes the values 0 or 1, the elements in fuzzy sets have different degrees of membership from 0 to 1.
Fuzzy sets and fuzzy logic opened ways of research in may areas such as: artificial inteligence, multicriteria decision making, design of controlled systems, databases query, image processing, data analysis.
Fuzzy logic is very useful for Case Based Reasoning Systems because the analogical reasoning can operate with linguistic expressions and the fuzzy logic is created to operate with linguistic expressions.

Similarity measurement and the cases classifications are based on fuzzy sets partitions which allowed to develop many fuzzy models.

In [1], the author generalized fuzzy partitions to fuzzy L-partitions and presented some results regarding these notions. These results are presented in Section 2 without proofs. In Section 3 some examples of fuzzy L-partitions, which is the aim of this paper, are described.

2. *L*- FUZZY SETS AND *L*- FUZZY PARTITIONS

In this section we introduce some notions and results from [1] on L-fuzzy sets and L- fuzzy partitions.

Definition 2.1. *Let X be a set and let L be a lattice. A function $\varphi : X \to L$ is called fuzzy L-subset of the set X. Consider $(L, \wedge, \vee, 0, 1)$ a complete distributive lattice and $(-)' : (L', \wedge, \vee, 0, 1) \to (L, \vee, \wedge, 1, 0)$ an involutive antiisomorphism such that $(x')' = x$, $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$ for any $x, y \in L$. Suppose that there exists an element x_0 in L such that $x_0' = x_0$ and for any $x \in L$, $x \leq x_0$ or $x \geq x_0$. This kind of lattice exists. Indeed, for example we may consider the lattice $L = ([0, \frac{1}{2}], \vee, \wedge 0, \frac{1}{2})$ and $L' = ([\frac{1}{2}], \wedge, \vee 1, \frac{1}{2})$ and $x' = 1 - x$ for any $x \in L$. We can see that $x_0 = \frac{1}{2}$ satisfies the required conditions and $(-)'$ is an involutive antiisomoprhism satisfying DeMorgan's laws.*

Proposition 2.1. *If the lattice L and the element x_0 are above defined, then the following properties are satisfied:*
1) $x \leq x'$ iff $x \leq x_0$;
2) $x \wedge x' \leq x_0 \leq x \vee x'$ for any x in L.

Remark 2.1. $(\sup_k x_k)' = \inf_k x_k'$

Definition 2.2. *Let X be a set and let φ be an L-fuzzy subset of X. A family of L-fuzzy subsets (ν_n) that satisfies the following conditions:*
1) $\nu_i \leq \nu_j'$, $\forall i \neq j$
2) $\varphi \wedge (\sup_n \nu_n)' \leq (\varphi \wedge (\sup_n \nu_n)')'$
3) $\sup_n \nu_n \leq \varphi$
is called a partition of φ.
In case $\varphi = 1_L$ the family (ν_n) is called a complete partition.

Theorem 2.1. *The family (ν_n) is a complete partition iff it satisfies the following conditions:*
1) $\nu_i \leq \nu_j'$, $\forall i \neq j$,
2) $\sup_n \nu_n \geq (\sup_n \nu_n)'$.

Theorem 2.2. *Assuming $\sup_n \varphi_n \geq \sup_n (\varphi_n)'$, the sequence (ν_n) defined bellow is a complete partition*

$$\nu_n = \begin{cases} \varphi_1, & \text{if } n = 1 \\ \varphi_n \wedge (\sup\limits_{k<n} \nu_k)', & \text{if } n > 1 \end{cases}$$

Theorem 2.3. *If the family (ν_n) is a complete partition, then the family $(\varphi \wedge \nu_n)$ is a partition of φ for any L- fuzzy subset φ.*

3. EXAMPLES OF FUZZY *L*- PARTITIONS

Consider the set $X = [0,1]$, the lattice $L = ([0,1], \wedge, \vee, 0, 1)$, $x' = 1 - x$ and a family of three L-fuzzy subsets of X $\nu_1, \nu_2, \nu_3 : X \to L$ defined bellow

$$\nu_1 = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{4}] \\ -6 \cdot x + \frac{5}{2}, & \text{if } x \in [\frac{1}{4}, \frac{5}{12}] \\ 0, & \text{if}, x \in (\frac{5}{12}, 1] \end{cases}$$

$$\nu_2 = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}] \\ 6 \cdot x - \frac{3}{2}, & \text{if } x \in [\frac{1}{4}, \frac{5}{12}] \\ -6 \cdot x + \frac{7}{2}, & \text{if}, x \in [\frac{5}{12}, \frac{7}{12}] \\ 0, & \text{if}, x \in (\frac{7}{12}, 1] \end{cases}$$

$$\nu_3 = \begin{cases} 0, & \text{if } x \in [0, \frac{5}{12}] \\ 6 \cdot x - \frac{5}{2}, & \text{if } x \in [\frac{5}{12}, \frac{7}{12}] \\ 1, & \text{if}, x \in (\frac{7}{12}, 1] \end{cases}$$

It follows that

$$\nu_1'(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}] \\ -6 \cdot x + \frac{7}{2}, & \text{if } x \in [\frac{1}{4}, \frac{5}{12}] \\ 0, & \text{if}, x \in (\frac{5}{12}, 1] \end{cases}$$

$$\nu_2'(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{4}] \\ -6 \cdot x + \frac{5}{2}, & \text{if } x \in [\frac{1}{4}, \frac{5}{12}] \\ 6 \cdot x - \frac{5}{2}, & \text{if}, x \in [\frac{5}{12}, \frac{7}{12}] \\ 0, & \text{if}, x \in (\frac{7}{12}, 1] \end{cases}$$

$$\nu_3'(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{5}{12}] \\ -6 \cdot x + \frac{7}{2}, & \text{if } x \in [\frac{5}{12}, \frac{7}{12}] \\ 1, & \text{if}, x \in (\frac{7}{12}, 1] \end{cases}$$

With the graphs of these functions and by a simple computation we can easily check that the following conditions are satisfied:
1) $\nu_i \leq \nu_j'$, $\imath, j \in \{1,2,3\}, i \neq j$,
2) $\sup\{\nu_1, \nu_2, \nu_3\} \geq \sup(\{\nu_1, \nu_2, \nu_3\})'$.
Then, according to Theorem 2.1, it follows that the family $\{\nu_1, \nu_2, \nu_3\}$ is a complete fuzzy L- partition.
With ν_1, ν_2, ν_3 above defined let us consider the following fuzzy L- subsets μ_1, μ_2, μ_3
$\mu_1 = \nu_1$
$\mu_2 = \nu_2 \wedge \mu_1' = \nu_2 \wedge \nu_1'$
$\mu_3 = \nu_3 \wedge (\sup\{\nu_1, \nu_2\})' = \nu_3 \wedge \inf\{\nu_1', \nu_2'\}$
According to Theorem 2.2, the family $\{\mu_1, \mu_2, \mu_3\}$ is a complete fuzzy L-partition.

If L is an L- fuzzy subset, then the families $\{\varphi \wedge \nu_1, \varphi \wedge \nu_2, \varphi \wedge \nu_3\}$ and $\{\varphi \wedge \mu_1, \varphi \wedge \mu_2, \varphi \wedge \mu_3\}$ are fuzzy L- partitions of φ according to Theorem 2.3.

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ROMAI J., **1**, 2(2005),
83–86

HOPF BIFURCATION INTO A MODEL OF EVOLUTION OF THE CAPITAL OF A FIRM

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Abstract A Cauchy problem for a system of two ordinary differential equations of first order depending on three parameters and having third order nonlinearities is considered. It governs a microeconomical dynamics [1]. It is proved that for some region in the parameter space a Hopf bifurcation occurs [2] and, by applying a theorem from [3], it is shown that this bifurcation is degenerated. This shows a time-periodic asymptotic behaviour of the capital and working force of the firm. A graphical representation for a phase portrait of the associated dynamical system supports this theoretical result.

Problem setting. The evolution of the capital of a firm can be modelled by the Cauchy problem for the system of ordinary differential equations (s.e.d.o.)

$$\begin{cases} \dot{x}(t) = cxy^2 + bx, \\ \dot{y}(t) = x + \lambda y - 1. \end{cases} \tag{1}$$

where $x, y : \mathbb{R} \to \mathbb{R}$ are the state functions and b, c, λ are the parameters. They have a well defined economic meaning. For b, c , $\lambda \neq 0$, $1 - \lambda\sqrt{-b/c} \neq 0$ and $-bc > 0$ the s.e.d.o. (1) has three equilibrium points: $P_1(0, 1/\lambda)$; $P_2\big(1 - \lambda\sqrt{-\frac{b}{c}}, \sqrt{-\frac{b}{c}}\big)$ and $P_3\big(1 + \lambda\sqrt{-\frac{b}{c}}, -\sqrt{-\frac{b}{c}}\big)$, while for $\lambda = 0$, P_1 goes to the infinity. We focus our attention on P_2 by fixing b and c and letting λ be the bifurcation parameter. In addition, we assume that $-bc > 0$ and $\lambda \in \mathbf{R}$

Analyzing the dynamics generated by the of system (1) about the point P_2, as $\lambda \to 0$, we show that as λ tends to 0, a Hopf bifurcation sets.

Manifold $H_{\mathbb{C}}$. Let $x_0 = 1 - \lambda\sqrt{-\frac{b}{c}}$ and $y_0 = \sqrt{-\frac{b}{c}}$, be the coordinates of P_2 and perform the change of unknown functions $u = x - x_0$, $v = y - y_0$, translating P_2 at the origin. Since $x = u + x_0,\ y = v + y_0$, (1) becomes

$$\begin{cases} \dot{u} = 2vcx_0y_0 + 2cy_0uv + cx_0v^2 + cuv^2, \\ \dot{v} = u + \lambda v. \end{cases} \tag{2}$$

The matrix of the system linearized about the origin is $A(\lambda) = \begin{pmatrix} 0 & 2cx_0y_0 \\ 1 & \lambda \end{pmatrix}$, and $trA = \lambda$, $detA = -2cx_0y_0$. The manifold $\mathbf{H}_{\mathbb{C}}$, the set of the Hopf bifurcation values, is defined as $\mathbf{H}_{\mathbb{C}} = \{(b, c, \lambda) \in \mathbb{R}^3 \mid trA = 0 \text{ and } detA > 0\}$.

Hence, at the points of $\mathbf{H}_{\mathbb{C}}$ we have $x_0 = 1$, and, consequently, $y_0 > 0$, $\lambda = 0$, implying

$$\mathbf{H}_{\mathbb{C}} = \{(b, c, 0) \mid \ c < 0, \ b > 0\}.$$

Let us prove that the non hyperbolic singularities of (2), coresponding to the parameters in $\mathbf{H}_{\mathbb{C}}$, are degenerated Hopf singularities of order greater than or equal to one.

The eigenvalues, of the matrix A are the roots $t_{1,2}$ of the equation

$$t^2 - \lambda t - 2cx_0y_0 = 0$$

and they have the expresion

$$t_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 + 8cx_0y_0}}{2}.$$

We have $t_{1,2}(\lambda) = \eta(\lambda) \ \pm \ i\omega(\lambda)$, where

$$\eta(\lambda) = \lambda/2 \ \text{ and } \omega(\lambda) = \sqrt{-\lambda^2 - 8\lambda b + 8\sqrt{-bc}}/2.$$

For $\lambda = 0$, $x_0 = 1$, $c < 0$, $b > 0$ they become purely imaginary, namely $t_{1,2}(0) = \pm i\sqrt{2}\sqrt[4]{-bc}$. Therefore, for $c < 0$ and $b > 0$ the characteristic equation has the complex roots $t_{1,2}$, which for $\lambda = 0$ become purely imaginary.

Remark that about $\lambda = 0$, the discriminant in the expression of $t_{1,2}$ is negative (indeed, $\lambda^2 < -8c\sqrt{-b/c}\,[1-\lambda\sqrt{-b/c}]$, since the solutions of the equation $\lambda^2 + 8b\lambda - 8\sqrt{-bc} = 0$ are real and of opposite sign). In addition, $Ret_i(0) = 0$, $Im \ \mid t_i(0) \mid = \sqrt{2}\sqrt[4]{-bc} \neq 0$, $\frac{\partial Re\ t_i}{\partial \lambda}(0) = \frac{1}{2} \neq 0$, $i = 1, 2$. Therefore, for small λ, P_2 is a focus.

Consequently, the Hopf theorem applies, showing that at $\lambda = 0$ a Hopf bifurcation occurs. Let us see if it is degenerated or non degenerated. To this aim it is necessary to determine the first Lyapunov coefficient.

First Lyapunov coefficient. We have the matrix

$$A(0) = \begin{pmatrix} 0 & -2\sqrt{-bc} \\ 1 & 0 \end{pmatrix}$$

which has the above-written eigenvalues $t_{1,2}(\lambda) = \eta(\lambda) + i\omega(\lambda)$, where $t_1(0) = \eta(0) + i\omega(0)$, with $\eta(0) = 0$, $\omega^2(0) = 2\sqrt{-bc}$, i.e. $t_{1,2}(0) = \pm i\omega(0)$, where $\omega(0) > 0$.

The eigenvector $\mathbf{v}$ of $A(0)$ corresponding to the eigenvalue $t_1(0)$ is

$$\mathbf{v} = \begin{pmatrix} i\omega \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} \omega \\ 0 \end{pmatrix}.$$

Thus, the matrix which ensures the passage to the Jordan form of matrix A is $P = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}$. We have $P^{-1} = \frac{1}{\omega}\begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$. By means of the the change $\begin{bmatrix} u \\ v \end{bmatrix} = P\begin{bmatrix} x \\ y \end{bmatrix}$, i.e. $\begin{cases} u = x\omega, \\ v = y. \end{cases}$ the system (2) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P^{-1}AP\begin{pmatrix} x \\ y \end{pmatrix} + (2cy_0\omega xy + cx_0y^2 + c\omega xy^2)\begin{pmatrix} \frac{1}{\omega} \\ 0 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2cy_0xy + \frac{c}{\omega}x_0y^2 + cxy^2 \\ 0 \end{pmatrix}.$$

In our case, the expresion from [3], of the first Lyapunov coeficient reads

$$l_1 = \frac{1}{16}[f_{xyy}(0,0)] + \frac{1}{16\omega}\big[f_{xy}(0,0)f_{xx}(0,0) + f_y(0,0)\big],$$

where

$$f(x,y) = 2cy_0xy + \frac{c}{\omega}x_0y^2 + cxy^2,$$

implying

$$l_1(0) = \frac{1}{16}\cdot 2c + \frac{1}{16\omega}\big(2cy_0 \cdot \frac{2cx_0}{\omega}\big) = 0.$$

Consequently, at $\lambda = 0$, for fixed $c < 0$ and $b > 0$, a Hopf bifurcation sets in. The Hopf bifurcation point is degenerated of order greater than or equal to one. This agrees with the phase-plane portrait about the origin(fig.1), suggesting a nonlinear center.

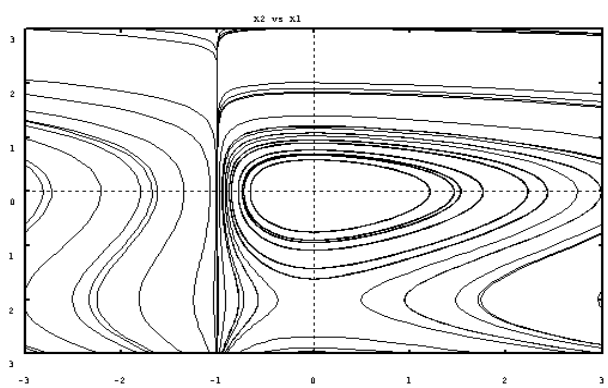

Fig.1. Phase portrait of the dynamical system generated by (2) for $c = -1$, $b = 1$, $\lambda = 0$.

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ROMAI J., **1**, 2(2005),
87–95

STABILITY SURFACES IN A CONVECTION PROBLEM FOR A MICROPOLAR FLUID. I. THEORETICAL RESULTS

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Abstract The simplest direct method and two direct and one variational methods based upon Fourier series expansions to solve eigenvalue problems involving ordinary differential equations of high order and the coefficients of which depend on several parameter are presented. The two main steps, namely the determination of the characteristic and secular equation, are shown. A concrete example is worked out.

1. INTRODUCTION

The linear stability theory problems involve mostly high order ordinary differential equations with constant coefficients and some homogeneous boundary conditions. To solve these problems several methods have been developed; the most known are the direct method and the methods based on Fourier series expansions [1]. An example of application of the Fourier series expansions methods is worked out. In Section 2 we describe the simplest direct method, in Section 3 we give details for the application of the methods based on series. For a concrete example, in Section 4 we give theoretical results obtained by applying three methods.

2. THE DIRECT METHOD

The direct method is one of the most frequently used method to solve eigenvalue problems and the simplest. The general solution of the equations of the problems is written as a linear combination (with unknown coefficients) of a basis consisting of the eigenvectors and, in the case of multiple characteristic values, generalized eigenvectors. The method consists in substituting the general solution in the boundary conditions and then imposing the condition that the Cramer determinant of the obtained linear algebraic system in the unknown coefficients to be zero. This condition generates the secular equation. If the determinant is zero, then the system has nontrivial solutions. The general solution of the equations depends on the multiplicity of the roots λ_i of

the characteristic equation associated with the eigenvalue problem. Whence the importance of discussing the multiplicity of λ_i. If the λ_i's are distinct, then the general solution is written as a linear combination of hyperbolic sinus and cosinus functions. If λ_i are multiple, of m_{λ_i} multiplicity respectively, then the general solution is a product of a polynomial in z by $\cosh\lambda_i z$ and $\sinh\lambda_i z$, of degree $m_{\lambda_i} - 1$.

3. METHODS BASED ON FOURIER SERIES EXPANSION

Frequently, the linear problems in hydrodynamic stability theory are eigenvalue problems consisting of high order differential equations with constant coefficients depending on several parameters which make the discussion of multiplicity of the roots of the characteristic equation very difficult. In this way, the application of the direct method becomes very difficult and alternative methods must be used. Some of these methods are based on Fourier series: the direct methods based on series and the variational methods based on series too. The direct methods based on series are: of Chandrasekar - Galerkin type and of Budiansky- DiPrima type.

In the first one, the unknown functions are written in the form of expansions in Fourier series upon a complete set of function (in $L^2[a, b]$) which satisfy all boundary conditions. The characteristic equation is the equality to zero of an infinite determinant.

The second one is a direct expansion approach too, however the functions are chosen such that they satisfy only part of the boundary conditions of the problem. In this case the characteristic equation is the equality to zero of an infinite series.

The variational methods can also be of the Chandrasekar - Galerkin type or Budiansky-DiPrima type depending of whether all the boundary conditions are satisfied or only part of them. A functional J is derived whose stationary points are the eigenfunctions of the considered problem and conversely. To solve the eigenvalue problem, for example, $Lf = 0$, is equivalent to solving the associated variational problem $\delta J = 0$. We say that a variational principle is established. Then the unknown functions of $\delta J = 0$ are written in form of expansion in Fourier series upon complete sets that satisfies or not all the boundary conditions. Then, they are substituted in the variational problem.

4. THE SECULAR EQUATION OF A STABILITY PROBLEM

Consider a particular stability problem [2] and write its secular equation using the direct Chandrasekar - Galerkin method, the direct Budianski-DiPrima

method and the variational Budiansky-DiPrima method. In [2] the problem was investigated only by the direct method.

If the exchange of stability principle is valid, the linear neutral stability in the presence of normal modes perturbations of a conduction state of a layer of a thermally conducting micropolar fluid, situated between two horizontal rigid walls maintained at constant temperature and subject to an external magnetic field, is governed by the following two-point problem

$$\begin{cases} (1+R)\Big[(D^2-a^2)^2-QD^2\Big]W+R(D^2-a^2)Z-R_a a^2\Theta=0, \\ \Big[A(D^2-a^2)-2R\Big]Z-R(D^2-a^2)W=0, \\ (D^2-a^2)\Theta+W-\overline{\delta}Z=0, \end{cases} \tag{1}$$

$$W=DW=Z=\Theta=0 \text{ at } z=\pm 0.5. \tag{2}$$

The number $R_a>0$ stands for the Rayleigh number, $a>0$ is the wave number, A, R, $\overline{\delta}$ are micropolar parameters, Q is the intensity of the magnetic field and the functions W, Θ, $Z:[-0.5,0.5]\to\mathbb{R}$ characterize the amplitude of the perturbation of the vertical component of the velocity, temperature and the vertical component of the spin vorticity, respectively.

The Chandrasekar-Galerkin method

Let us apply the Chandrasekar method to (1)-(2). To this aim, we expand the unknown functions W, Θ, Z upon sets of orthonormal functions that satisfy all the boundary conditions. Assume that W, Θ and Z are even functions of z.

Taking into account (2) develop W upon the total set (in $L^2(-0.5;0.5)$) $\{C_n\}_{n\geq 1}$, $C_n(z)=\dfrac{\cosh(\lambda_n z)}{\cosh(\lambda_n/2)}-\dfrac{\cos(\lambda_n z)}{\cos(\lambda_n/2)}$ where λ_n are the roots of the equation $\tanh(\lambda/2)+\mathrm{tg}(\lambda/2)=0$.

This means that $C_n=DC_n=0$ at $z=\pm 0.5$. Choosing $W=\sum\limits_{n=1}^{\infty}W_{2n-1}C_n$, it follows that $Z=0$ at $z=\pm 0.5$ too.

The unknown functions Θ, Z are developed upon the orthonormal set $\{E_{2n-1}\}_{n\geq 1}$, where $E_{2n-1}(z)=\sqrt{2}\cos(2n-1)\pi z$, i.e.

$$\Theta=\sqrt{2}\sum_{n=1}^{\infty}\Theta_{2n-1}\cos(2n-1)\pi z, Z=\sqrt{2}\sum_{n=1}^{\infty}Z_{2n-1}\cos(2n-1)\pi z.$$

It is immediate that $\Theta=Z=0$ at $z=\pm 0.5$.

The Chandrasekar - Galerkin procedure imposes that the left-hand side of the equation obtained by introducing all these expansions in (1) is orthogonal

to E_{2m-1}, $m = 1, 2, ...$ Denoting $l_{nm} = (C_n, E_{2m-1})$, $q_{nm} = (D^2C_n, E_{2m-1})$, $r_{nm} = (D^4C_n, E_{2m-1})$ we obtain

$$\begin{cases} (1+R)\sum\limits_{n=1}^{\infty}[r_{nm} - (2a^2+Q)q_{nm} + a^4 l_{nm}]W_{2n-1} - \\ -R[(2m-1)^2\pi^2 + a^2]Z_{2m-1} - R_a a^2 \Theta_{2m-1} = 0, \\ \\ -R\sum\limits_{n=1}^{\infty}(q_{nm} - a^2 l_{nm})W_{2n-1} - [A((2m-1)^2\pi^2 + a^2) + 2R]Z_{2m-1} = 0, \\ \\ \sum\limits_{n=1}^{\infty} l_{nm} W_{2n-1} - \overline{\delta} Z_{2m-1} - \Big[(2m-1)^2\pi^2 + a^2\Big]\Theta_{2m-1} = 0. \end{cases} \tag{3}$$

where $(.,.)$ is the inner product in $L^2(-0.5; 0.5)$.

Eliminating Θ_{2m-1} and Z_{2m-1} between $(3)_{1,2,3}$ we obtain the following infinite system of linear algebraic equations in the constant coefficients W_1, W_3, W_5,...

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big\{(1+R)[r_{nm} - (2a^2+Q)q_{nm} + a^4 l_{nm}] + \frac{R^2 A_m (q_{nm} - a^2 l_{nm})}{AA_m + 2R} - \\ &-R_a a^2\Big[\frac{l_{nm}}{A_m} + \frac{\overline{\delta} R(q_{nm} - a^2 l_{nm})}{A_m(A_m + 2R)}\Big]\Big\}W_{2n-1} = 0, \end{aligned} \tag{4}$$

where $A_m = (2m-1)^2\pi^2 + a^2$.

This leads to the secular equation

$$\lim_{n\to\infty}\begin{vmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ ... & ... & ... & ... \\ d_{n1} & d_{n2} & ... & d_{nn} \end{vmatrix} = 0$$

where d_{nn} is the coefficient of W_{2n-1} in (4). From the secular equation we obtain a hypersurface S of equation $R_a = R_a(a, R, A, \overline{\delta}, Q)$. It becomes a surface for three fixed parameters and a curve for four fixed parameters.

If we take some fixed values of three of the parameters a, R, A, then we can represent the secular surface in various two-dimensional planes: (a, R_a), (A, R_a), (R, R_a) or for two-fixed parameters, in the three-dimensional spaces: (A, a, R_a), (A, R, R_a), (a, R, R_a).

The Budiansky-DiPrima method

To apply the Budiansky - DiPrima method we expand the unknown functions W, Θ, Z upon total sets of orthonormal functions that do not satisfy all

boundary conditions. We develop the functions W, Θ, Z upon the total set $\{E_{2n-1}\}_{n\in\mathbb{N}}$, where E_{2n-1} are defined before. We have

$$\begin{cases} W = \sum\limits_{n=1}^{\infty} \sqrt{2} W_{2n-1} \cos(2n-1)\pi z, \\ Z = \sum\limits_{n=1}^{\infty} \sqrt{2} Z_{2n-1} \cos(2n-1)\pi z, \\ \Theta = \sum\limits_{n=1}^{\infty} \sqrt{2} \Theta_{2n-1} \cos(2n-1)\pi z. \end{cases} \tag{5}$$

Thus, the functions W, Z, Θ satisfy the conditions $W = Z = \Theta = 0$ at $z = \pm 0.5$, while the condition $DW = 0$ at $z = \pm 0.5$ introduces a constraint for the problem (1)-(2).

Using the backward integration technique, we obtain the expressions of all derivatives that appear in (1). We replace these expressions in (1), impose the condition that the obtained equations be orthogonal to E_{2m-1}, $m = 1, 2, ...$ and we have

$$\begin{cases} (1+R)\{2\sqrt{2}(-1)^{n+1} D^2 W(0.5)(2n-1)\pi + [A_n^2 + Q(2n-1)^2\pi^2] W_{2n-1}\} - \\ -RA_n Z_{2n-1} - R_a a^2 \Theta_{2n-1} = 0, \\ \\ RA_n W_{2n-1} - (AA_n + 2R) Z_{2n-1} = 0, \\ \\ W_{2n-1} - \overline{\delta} Z_{2n-1} - A_n \Theta_{2n-1} = 0. \end{cases} \tag{6}$$

We considered only the limit case $Q = 0, \overline{\delta} = 0$; the case Q, $\overline{\delta} \neq 0$ will be treated elsewhere.

In this case, the system (6) becomes

$$\begin{cases} (1+R) A_n^2 W_{2n-1} - RA_n Z_{2n-1} - R_a a^2 \Theta_{2n-1} = \\ = 2\sqrt{2}(-1)^n (1+R)(2n-1)\pi\alpha \\ \\ RA_n W_{2n-1} - (AA_n + 2R) Z_{2n-1} = 0, \\ \\ W_{2n-1} - A_n \Theta_{2n-1} = 0, \end{cases} \tag{7}$$

where $\alpha = D^2 W(0.5) \neq 0$.

The constraint has the form

$$\sum_{n=1}^{\infty} (-1)^n \sqrt{2}(2n-1)\pi W_{2n-1} = 0 \tag{8}$$

The expressions of the unknown constants W_{2n-1} are obtained by solving the system (7). Replacing these expressions in (8), we obtain the secular equation

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2 A_n (AA_n + 2R)}{(AA_n + 2R)[A_n^3 + RA_n^3 - R_a a^2] - R^2 A_n^3} = 0. \tag{9}$$

<u>The variational method</u>

The problem (1) can be written in the form $Lf = 0$,

$$\begin{cases} (1+R)(D^2-a^2)^2 W + R(D^2-a^2)Z - R_a a^2 \Theta = 0, \\ -R(D^2-a^2)W + [A(D^2-a^2) - 2R]Z = 0, \\ W + (D^2-a^2)\Theta = 0. \end{cases} \tag{11}$$

where $L : \mathcal{A} \to [\mathcal{C}^0(-0.5, 0.5)]^3$ is the matricial linear differential operator

$$L = \begin{pmatrix} (1+R)(D^2-a^2I)^2 & R(D^2-a^2I) & -R_a a^2 I \\ -R(D^2-a^2I) & A(D^2-a^2I) - 2RI & \mathcal{O} \\ I & \mathcal{O} & D^2 - a^2 I \end{pmatrix},$$

I is the identity operator on $\mathcal{A}$, $\mathcal{O}$ is the null operator on $\mathcal{A}$, the set $\mathcal{A}$ consists of all vector functions $f = (W, Z, \Theta)$ satisfying (2) and

$$f \in [\mathcal{C}^\infty(-0.5, 0.5)]^4 \cap \mathcal{C}^0(-0.5, 0.5) \times \mathcal{C}^0(-0.5, 0.5) \times \mathcal{C}^0(-0.5, 0.5).$$

The domain of definition $\mathcal{A}$ is embedded in $[L^2(-0.5, 0.5)]^3$.

We say that an operator L is symmetric if the equality

$$(Lf, g) = (f, Lg)$$

holds for every $f, g \in \mathcal{A}$. It is selfadjoint if it is symmetric and $L = L^*$. In our case direct (and quite easy) computations show that L is not selfadjoint.

Remark. Sometimes [6] a nonselfadjoint matricial differential linear operator can become selfadjoint by multiplying one or more equations in its corresponding system of equations by some constants.

If we multiplying the equation $(10)_1$ by (-1) and the equation $(10)_3$ by $(R_a a^2)$, the equation (10) becomes $L_1 f = 0$, where L_1 is the selfadjoint operator

$$L_1 = \begin{pmatrix} -(1+R)(D^2-a^2I)^2 & -R(D^2-a^2I) & R_a a^2 I \\ -R(D^2-a^2I) & A(D^2-a^2I) - 2RI & \mathcal{O} \\ R_a a^2 I & \mathcal{O} & R_a a^2 (D^2 - a^2 I) \end{pmatrix}.$$

The inner product $(L_1 f, g)$ is obtained by multiplying the equations $(10)_{1,2,3}$ by $g = (W^*, Z^*, \Theta^*)$, respectively, by adding the results and then integrating the obtained sum over $[-0.5, 0.5]$. Integrating by parts in the expression $(L_1 f, g)$ and taking into account the boundary conditions (2) we are lead to

$$(L_1 f, g) = (f, L_1^* g).$$

Since the operator L_1 is selfadjoint, we check that $L_1 = L_1^*$.

The eigenvalue problem for the selfadjoint operator consist in the system of ordinary differential equations

$$\begin{cases} -(1+R)(D^2-a^2)^2 W^* - R(D^2-a^2)Z^* + R_a a^2 \Theta^* = 0, \\ -R(D^*-a^2)W^* + [A(D^2-a^2) - 2R]Z^* = 0, \\ R_a a^2 W^* + R_a a^2 (D^2-a^2)\Theta^* = 0, \end{cases} \tag{11}$$

and the boundary conditions

$$W^* = DW^* = Z^* = \Theta^* = 0 \text{ at } z = \pm 0.5.$$

Define the functional $J : \mathcal{A}_1 \to \mathbb{R}$ by $J(f) = (L_1 f, f)$, where the set $\mathcal{A}_1$ is defined as follows

$$\mathcal{A}_1 = \{f = (W, Z, \Theta) \in [\mathcal{C}^\infty(-0.5, 0.5)]^3 | W, Z, \Theta \text{ satisfy } W = Z = \Theta = 0\}.$$

We have

$$J(f) = (1+R)D^2 W DW\Big|_{-0.5}^{0.5} + \int_{-0.5}^{0.5} \Big\{ -(1+R)[(D^2W)^2 + 2a^2(DW)^2 + a^4W^2] + \\ +2R[DZDW + a^2WZ] + 2Ra^2W\Theta - [A(DZ)^2 + Aa^2Z^2 + 2RZ^2] - \\ -R_a a^2 (D\Theta)^2 - R_a a^4 \Theta^2 \Big\} \tag{12}$$

Theorem 1. $L_1(f) = 0$ *iff* $\delta J(f) = 0$.

If $\overline{f} \in \mathcal{A}_1$ is a solution of the equation $L_1 \overline{f} = 0$ it follows that the variation of the functional J at f in the $\mathcal{A}_1$ vanishes, i.e. $\delta J(\overline{f}) = 0$, in other words f makes the functional J stationary. We have (for simplicity we removed the bar)

$$\delta J(f) = 2\int_{-0.5}^{0.5} \Big\{ -(1+R)(D^4W - 2a^2D^2W + a^4W) + R(D^2Z - a^2Z - \\ -R_a a^2)\Theta \Big\}\delta W + \Big\{ -R(D^2W - a^2W) + [A(D^2Z - a^2Z) - 2RZ] \Big\}\delta Z + \\ + \Big\{ W + (D^2\Theta - a^2\Theta) \Big\}\delta\Theta = 0. \tag{13}$$

Since δW, δZ and $\delta\Theta$ are arbitrary,$\delta J(f) = 0$ implies that (11) hold. Conversely, if (11) hold, it follows that $\delta J(f) = 0$, whence the Theorem 1.

Replacing in (12) the series expansions for W, DW, D^2W, Z, DZ, Θ and $D\Theta$ we obtain a function in the coefficients W_{2n-1}, Z_{2n-1} and Θ_{2n-1}. Imposing to this function to be stationary we obtain an infinite linear system in these coefficients. Eliminating Z_{2n-1} and Θ_{2n-1} between the equations of this algebraic system it is obtained W_{2n-1}. Substituting the obtained expression for W_{2n-1} in the constraint (8) we are lead to (9). Therefore, as it was expected, we obtained the same secular equation.

The use of the functional J diminishes the computations to the half because J contains only two derivatives of W and one derivative for Z and Θ, i.e. half of the derivatives occurring in the system (1).

5. CONCLUSIONS

After describing the simplest direct method, and three direct and variational methods based on Fourier series, we treated a concrete example by the direct and variational Budiansky and DiPrima method, leading to the secular equation defining the manifolds separating the stability domain from the instability domain in some parameter space. The numerical study of these manifolds will be done elsewhere.

The method that we choose to apply depends on the form of the eigenvalue problem and must avoid complicate Fourier coefficients.

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ON LOCAL CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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Abstract In this paper, we introduce the concept of probabilistic (ε, λ)-local g-contraction which is a generalization of probabilistic contraction of Sehgal' type. We prove fixed point theorems for mappings defined on a non-empty set X with values in a probabilistic metric space. The set X is endowed with a probabilistic structure induced by a mapping g.

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Keywords: probabilistic metric space, fixed points.

1. INTRODUCTION

The study of contraction mappings defined on probabilistic metric spaces was initiated by H. Sherhood [18], V.M. Sehgal [15], A.T. Bharucha-Reid [16].Since many papers and books dealing with numerous generalizations and applications have appeared. different classes of probabilistic contractions have been defined and probabilistic versions of Banach theorem were stated. Probabilistic fixed point theorems are important because they generalize deterministic results and offer a important tool to solve random equations. This paper state some fixed point results for some local contractions with values in probabilistic metric spaces.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{I} = [0, 1]$. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non decreasing and left-continuous with $\inf F = 0$ and $\sup F = 1$.

In what follows we always denote by $\mathcal{D}$ the set of all distribution functions. $\mathcal{D}^+ = \{F : F \in \mathcal{D}, F(0) = 0\}$ is the set of all distribution functions associated to non-negative, one-dimensional random variables.

For every $a \in \mathbb{R}^+$, a specific distribution function is defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$.

A mapping $T : I \times I \to I$ is called a t-norm if it satisfies the following conditions:

$(T1)$ $T(a, 1) = a$,

$(T2)$ $T(a, b) = T(b, a)$,

$(T3)$ $T(c,d) \geq T(a,b)$ if $c \geq d$ and $d \geq b$,
$(T4)$ $T(T(a,b),c) = T(a,(T(b,c))$.

The most utilized t-norms in probabilistic metric spaces theory are $T = Min$, $T = Prod$ and $T = T_m$, where $Min(a,b) = \min\{a,b\}$, $Prod(a,b) = a \cdot b$ and $T_m(a,b) = \max\{a+b-1,0\}$.

Definition 1. A probabilistic metric space of Menger type (briefly Menger space) is an ordered triple $(S,\mathcal{F},T)$, where S is a nonempty set, $\mathcal{F}$ is a mapping from $S \times S$ into $\mathcal{D}^+$, T is a t-norm and the conditions :

(M_1) $F_{x,y}(t) = H_0(t)$ if and only if $x = y$.
(M_2) $F_{x,y}(t) = F_{y,x}(t)$, for all $t \in \mathbb{R}$.
(M_3) $F_{x,z}(t_1+t_2) \geq T(F_{x,y}(t_1), F_{y,z}(t_2))$, for all $x,y,z \in S$ and $t_1,t_2 \geq 0$.

are satisfied. $\mathcal{F}(x,y)$ is denoted by $F_{x,y}$, the mappings $\mathcal{F}$ is called a probabilistic metric and (M_3) is a probabilistic version of triangle inequality.

The study of probabilistic metric spaces was introduced by K. Menger in [10]. Since, a lot of results were obtained in this area [1], [13].

A important class of metric spaces is given of random normed spaces.

Definition 2. Let L be a linear space and let $\mathcal{F}$ be a mapping defined on L with values in $\mathcal{D}^+$. ($\mathcal{F}(x)$ is denoted by F_x.) If the following conditions are satisfied :

(N_1) $F_x = H_0$, if only if $x = \theta$ (the null vector).
(N_2) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$, for $t \in \mathbb{R}$, $\alpha \in \mathbb{K}^*$, where $\mathbb{K}$ is the field of scalars.
(N_3) $F_{x+y}(t_1+t_2) \geq T(F_x(t_1), F_y(t_2))$, for all $x,y \in L$ and $t_1,t_2 \in \mathbb{R}^+$.

then the ordered triple $(L,\mathcal{F},T)$ is called a random normed space. This notion was firstly studied in [17],[9],[11]. For more details we refer [1],[13].

A uniformity on a Menger space $(S,\mathcal{F},T)$ is defined by the family :

$$\mathcal{V} = \{V(\varepsilon,\lambda) = \{(x,y) \in S \times S : F_{u,v}(\epsilon) > 1-\lambda\}, \epsilon > 0, \lambda \in (0,1)\}.$$

In what follows we will consider a Menger space $(S,\mathcal{F},T)$ under a t-norm T, that satisfies the weakest condition $\sup\{T(t,t) : t < 1\} = 1$, which ensures the existence of the above uniformity on S.

2. LOCAL G-CONTRACTION MAPPINGS

Theorem 1. Let g be a injective mapping defined on a non empty set X with values into a Menger space $(S,\mathcal{F},T)$. Then the following statements are true :

(a) The mapping $\mathcal{F}^g$ defined on $X \times X$ with values in $\mathcal{D}^+$, by $\mathcal{F}^g(x,y) = F_{g(x),g(y)}$ is a probabilistic metric on X, that is, $(X,\mathcal{F}^g,T)$ is a Menger space under the same t-norm T.

(b) If $S_1 = g(X)$ and $(S_1, \mathcal{F}, T)$ is a complete Menger space, then $(X, \mathcal{F}^g, T)$ is also a complete Menger space.

(c) If $(S_1, \mathcal{F}, T)$ is compact then $(X, \mathcal{F}^g, T)$ is also compact.

Proof. We will prove only the statement (c). Let $(x_n)_{n\geq 1}$ be a sequence in X. Then $(u_n)_{n\geq 1}$ with $u_n = g(x_n)$ is a sequence in S_1, that is a compact Menger space. Now, we can find a subsequence $\{v_n : n \geq 1\} \subset \{u_n : n \geq 1\}$ convergent to an element $v \in S_1$. This is equivalent to $F_{v_n,v}(t) \to H_0(t)$, $(n \to \infty)$, for every $t > 0$. If we set $y_n = g^{-1}(v_n)$ and $y = g^{-1}(v)$ then we have

$$F^g_{y_n,y}(t) = F_{g(y_n),g(y)}(t) = F_{v_n,v}(t) \to H_0(t),$$

when $\to \infty$, for every $t > 0$. This show us that $(X, \mathcal{F}^g, T)$ is a compact Menger space.

Definition 3. Let f, g be two mappings defined on a non empty set X with values into a Menger space $(S, \mathcal{F}, T)$, let us suppose that g is bijective and $\varepsilon > 0$, $\lambda \in (0, 1)$. The mapping f is called an (ε, λ)-local g-contraction with a constant $k \in (0, 1)$ if

$$(C) \qquad F_{g(x),g(y)}(\varepsilon) > 1 - \lambda \quad \text{implies} \quad F_{f(x),f(y)}(kt) \geqslant F_{g(x),g(y)}(t),$$

for every $t > 0$.

The notion of (ε, λ)-local g-contraction is justified because the imagines of two points x, y by the function f are probabilistic more near than imagines of the same points by the function g, whenever $g(x)$ and $g(y)$ are in a neighborhood.

Definition 4. Under the conditions of Definition 3 the set X will be called (ε, λ)-g-chainable if for every $x, y \in X$ there exists a finite sequence $x = x_0, x_1, ..., x_n = y$ such that $F_{g(x_{i+1}),g(x_i)}(\varepsilon) > 1 - \lambda$, for every $i = 0, 1.....n - 1$. The finite sequence $x = x_0, x_1, ..., x_n = y$ is called $(\varepsilon, \lambda) - g-$chain joining x and y.

Theorem 2. If $f : X \to S$ is a (ϵ, λ)-local g-contraction, then we have :
(a) f is a continuous mapping on $(X, \mathcal{F}^g, T)$ with values in $(S, \mathcal{F}, T)$.
(b) $g^{-1} \circ f$ is a continuous mapping on $(X, \mathcal{F}^g, T)$ with values into itself.

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in X such that $x_n \to x \in X$, under the probabilistic metric $\mathcal{F}^g$. This implies that $F^g_{x_n,x}(t) \to H_0(t)$, for every $t > 0$. From the (ε, λ)-local g-contraction condition (C) it follows $F_{f(x_n),f(x)}(t) \to H_0(t)$, for every $t > 0$. This show us that the mapping f is continuous. The above convergence implies $F_{g\circ g^{-1}\circ fx_n,g\circ g^{-1}\circ fp}(t) \to H_0(t)$, for every $t > 0$. Now, we can write $F^g_{g^{-1}\circ fx_n,g^{-1}\circ fp}(t) \to H_0(t)$, for every $t > 0$. So, the mapping $g^{-1} \circ f$ defined on the Menger space $(X, \mathcal{F}^g, T)$ with values in itself is continuous.

Remark 1. The above concept of (ε, λ)-g-contraction is a generalization of Sehgal' type (ε, λ)-local contraction [16] which can be obtained when $X = S$

and g is the identity on the Menger space $(S, \mathcal{F}, T)$.

Let T be a t-norm and let us define the family $\mathcal{T} = \{T^m\}_{m\geq 1}$ by $T^1(u) = T(u,u), ..., T^{m+1}(u) = T(T^m(u), u), \quad m \geq 1, u \in [0,1]$. The t-norm T is of h-type if the family $\mathcal{T}$ is equicontinuous at $t = 1$ [6-7] .

Theorem 3. If f and g are two mappings defined on a non empty set X with values into a complete Menger space $(S, \mathcal{F}, T)$, under a continuous t-norm T of H-type, g is bijective, f is an (ε, λ)-local g-contraction and the set X is (ε, λ)-g-chainable, then there exists a unique point $x^* \in X$ such that $f(x^*) = g(x^*)$. We say that x^* is a fixed point of the (ε, λ)-local g-contraction f. Moreover, $x^* = \lim\limits_{n\to\infty} x_n$, where $x_0 \in X$ and $g(x_{n+1}) = f(x_n)$.

Proof. First, we will prove that the mapping $h = g^{-1} \circ f$ defined on Menger space $(X, \mathcal{F}^g, T)$ with values into itself is an (ε, λ)-local contraction. Let us suppose that $y \in U_x(\varepsilon, \lambda)$ under the probabilistic metric $\mathcal{F}^g$ on X, then $F^g_{x,y}(\varepsilon) > 1 - \lambda$, that is, $F_{g(x),g(y)}(\varepsilon) > 1 - \lambda$. By the contraction condition (C) we have $F_{f(x),f(y)}(kt) \geqslant F_{g(x),g(y)}(t)$. This implies $F_{g\circ g^{-1}\circ f(x), g\circ g^{-1}\circ f(y)}(kt) \geqslant F_{g(x),g(y)}(t)$ or, equivalently $F^g_{g^{-1}\circ f(x), g^{-1}\circ f(y)}(kt) \geqslant F^g_{x,y}(t)$, So, $h = g^{-1} \circ f$ is an (ε, λ)-local contraction. If X is (ε, λ)-g-chainable set and $x, y \in X$, then there exists a finite sequence $x = x_0, x_1, ..., x_n = y$ such that $F_{g(x_{i+1}),g(x_i)}(\varepsilon) > 1-\lambda$, for every $i = 0, 1.....n-1$. The above inequality implies $F^g_{x_{i+1},x_i}(\varepsilon) > 1-\lambda$, that is, $x_{i+1} \in V_{x_i}(\varepsilon, \lambda)$ under the probabilistic metric $\mathcal{F}^g$ and it follows that Menger space $(X, \mathcal{F}^g, T)$ is (ε, λ)-chainable.

Now, we will prove that the mapping h has a unique fixed point. Let $x = x_0, x_1, ..., x_n = h(x)$ be a finite sequence such that $F^g_{x_{i+1},x_i}(\varepsilon) > 1 - \lambda$, $i = 1, 2, ..., n-1$. The contraction condition (C) implies that $F_{f(x_{i+1}),f(x_i)}(\varepsilon) \geqslant F^g_{x_{i+1},x_i}(\frac{\varepsilon}{k}) > 1 - \lambda$. So, we can write $F_{g\circ g^{-1}\circ f(x_{i+1}), g\circ g^{-1}\circ f(x_i)}(\varepsilon) > 1 - \lambda$, or equivalently, $F^g_{h(x_{i+1}),h(x_i)}(\varepsilon) > 1 - \lambda$. This means that $h(x) = h(x_0), h(x_1), ...,$ $h(x_n) = h^2(x)$ is an (ε, λ)-chain for $h(x)$ and $h^2(x)$. By induction it follows that $F^g_{h^m(x),h^{m+1}(\varepsilon)}(\varepsilon) \geqslant 1 - \lambda$.

So, $h^m(x) = h^m(x_0), h^m(x_1), ..., h^m(x_n) = h^{m+1}(x)$ is an (ε, λ)-chain for $h^m(x)$ and $h^{m+1}(x)$.

Reasoning in a similar way to that of [16] or [6-7] it follows that there exists a unique fixed point x^* of the mapping h and $\lim\limits_{n\to\infty} h^n(x_0) = x^*$ for each $x_0 \in X$. Moreover, because $x_{n+1} = h(x_n)$ we have $g(x_{n+1}) = f(x_n)$ and $g(x^*) = f(x^*)$.

Remark 3. When $X = S$ and g is the identity on S we obtain some known results from [16],[2].

If we take in account that every metric space (S, d) can be made, in a natural way, a Menger space $(S, \mathcal{F}, T)$ by setting $F_{x,y}(t) = H_0(t - d(x, y)))$ and $T = Min$ then, by Theorem 3 one obtains fixed point theorems for mappings with values into metric spaces.

Definition 5. Let f, g be two mappings defined on a non empty set X with

values into a metric space (S, d), let us suppose that g is bijective and let ε be a positive real. The mapping f will be called an ε-local g-contraction with a constant $k \in (0, 1)$ if

$$(C) \qquad d(g(x), g(y)) < \varepsilon \quad \text{implies} \quad d(f(x), f(y)) \leqslant kd(g(x), g(y)).$$

Definition 6. Under the conditions of Definition 5 the set X will be called $(\varepsilon - local)$-g-chainable if for every $x, y \in X$ there exists a finite sequence $x = x_0, x_1, ..., x_n = y$ such that $d(g(x_{i+1}), g(x_i) < \varepsilon$, for every $i = 0, 1.....n-1$.

Theorem 4. If f and g are two mappings defined on a non empty set X with values into a complete metric space (S, d), g is bijective, X is $\varepsilon - g$-chainable and f is a ε-local g-contraction then, there exists a unique point $x^* \in S$ such that $f(x^*) = g(x^*)$.

Proof. Let us suppose, without lose the generality, that $d(x, y) \in [0, 1)$. Otherwise, we define the mapping $d_1(x, y) = 1 - e^{-d(x,y)}$. The pair (S, d_1) is also a metric space and the systems of neighborhoods defined by the metrics d and d_1 are equivalent.

Let f be a ε-local g-contraction on (S, d), that is, $d(gx, gy) < \varepsilon$ implies $d(fx, fy) \leqslant k(gx, gy)$. If $F_{g(x),g(y)}(\varepsilon) > 1-\lambda$ it follows that $H_0(\varepsilon - d(gx, gy) > 1 - \lambda$, what implies $d(gx, gy) < \varepsilon$. As f is an ε-local g-contraction we have $df(x), f(y) \leqslant kd(gx, gy)$ and $F_{fx,fy}(\varepsilon) = H_0(\varepsilon - d(fx, fy)) \geqslant H_0(\epsilon - kd(gx, gy)) = H_0[k(\frac{\epsilon}{k} - d(gx, gy)) = F_{gx,gy}(\frac{\varepsilon}{k})$. Similarly, it follows that X is an ε-g-chainable set. The conclusion yields by Theorem 3.

Definition 7. Under the conditions of Definition 3 the mapping f will be called an g-contraction with a constant $k \in (0, 1)$ if

$$(C_1) \qquad F_{f(x),f(y)}(kt) \geqslant F_{g(x),g(y)}(t),$$

for every $t > 0$.

Theorem 5. If f and g are two mappings defined on a non empty set X with values into a complete Menger space $(S, \mathcal{F}, T)$, under a continuous t-norm T of H-type, g is bijective, f is a g-contraction, then there exists a unique point $x^* \in X$ such that $f(x^*) = g(x^*)$. Moreover, $x^* = \lim\limits_{n\to\infty} x_n$, where $x_0 \in X$ and $g(x_{n+1}) = f(x_n)$.

Proof. The contraction condition (C_1) can be write in the form $F^g_{g^(-1)\circ f(x),g^{-1}\circ f(y)}(kt) \geqslant F^g_{x,y}(t)$. So, the mapping $g_{-1} \circ f$ is a probabilistic contraction of Menger $(x, \mathcal{F}^g, T)$, space into itself. Taking into account Theorem 2 and Theorem 1 and the fact that a t-norm T of H-type has the fixed point property [2], [7] the conclusion results.

Now, we consider a more particular situation, namely we suppose that $(S, \mathcal{F}^1, T)$ is a random normed space. If we define the mapping $\mathcal{F} : S \times S \to D^+$ by $\mathcal{F}(x, y) = F^1_{x-y}$, then $(S, \mathcal{F}, T)$ becomes a probabilistic metric space with

probabilistic metric $\mathcal{F}$ induced by random norm $\mathcal{F}^1$. So, in this situation the previous results are valid.

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A SOLUTION OF THE BOUNDARY INTEGRAL EQUATION OF THE THREE-DIMENSIONAL AIRFOIL IN SUBSONIC FLOW USING LINEAR BOUNDARY ELEMENTS

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Abstract A boundary singular integral equation, formulated in velocity vector terms, is obtained in [1] using an indirect integral method for the three- dimensional potential subsonic flow around bodies . In order to solve this equation a boundary element method using triangular linear elements is developed here. The integral representation of the velocity is reduced by discretization to an algebaric system. We test the method solving this algebaric system in a particular case, in which we know an exact solution for the singular integral equation, and comparing the values obtained for the exact solution and the calculated one we remark a high degree of accuracy.

1. INTRODUCTION

The boundary integral method (BEM) is a modern theory of great efficienty used to solve boundary value problems for systems of partial differential equatios. The principal advantage of the BEM over other numerical methods is the ability to reduce the problem dimension by one. This property reduces the size of the solution system leading to improved computational efficiency. To achieve this reduction of dimension it is necessary to formulate the governing equation as a boundary integral equation, which is usually a singular one. In order to solve the integral equation we construct a boundary mesh using different types of boundary elements. If the body is three-dimensional, the boundary elements have the nodes on the external surface of the body, and they are usually of two types: quadrilateral and triangular elements. In this paper we use triangular liniar elements for solving the singular integral equation resulting as an application of the integral boundary equations to the three-dimensional problem of incompressible potential fluids flow past bodies.

2. THE BOUNDARY INTEGRAL EQUATION

In the problem we want to solve, we consider that the uniform, steady, potential motion of an ideal inviscid fluid of subsonic velocity $U_\infty \bar{i}$, pressure $p_\infty$and density ρ_∞ is perturbed by the presence of a fixed body of a known boundary , denoted by Σ, assumed to be smooth and closed. We want to find out the perturbed motion, and the fluid action on the body. Denoting by $\overline{v}$ the perturbation velocity and by $\overline{V}$ the velocity field for the perturbed motion, we have $\overline{V} = U_\infty \bar{i} + \overline{v}$.

The boundary condition on Σ is $\overline{v} \cdot \overline{n} = -U_\infty \bar{i} \cdot \overline{n}$, where $\overline{n}$ is the normal unit vector outward the fluid (inward the body). It is also required that the perturbation velocity vanishes at infinity, i.e. $\lim\limits_{\infty} \overline{v} = 0$.

Assimilating the body with a continuous distribution of sources on the boundary Σ , having an unknown intensity (persumed to satisfay Hölder condition), for $\overline{v}$ we have the integral representation

$$\overline{v}\left(\overline{\xi}\right) = -\frac{1}{4\pi} \iint\limits_{\Sigma} f\left(\overline{x}\right) \frac{\overline{x} - \overline{\xi}}{\left\|\overline{x} - \overline{\xi}\right\|^3} da \tag{1}$$

As $\overline{\xi} \to x_0 \in \Sigma$ we get the perturbation velocity at any point of the boundary

$$\overline{v}\left(\overline{x}_0\right) = -\frac{1}{2} f\left(\overline{x}_0\right) \overline{n}_0 - \frac{1}{4\pi} \iint\limits_{\Sigma} f\left(\overline{x}\right) \frac{\overline{x} - \overline{x}_0}{\left\|\overline{x} - \overline{x}_0\right\|^3} da, \tag{2}$$

where $\overline{n}_0 = \overline{n}\left(\overline{x}_0\right)$

Using the boundary condition we obtain a singular integral equation for the unknown f

$$f\left(\overline{x}_0\right) + \frac{1}{2\pi}{}' \iint\limits_{\Sigma} f\left(\overline{x}\right) \frac{\left(\overline{x} - \overline{x}_0\right) \overline{n}_0}{\left\|\overline{x} - \overline{x}_0\right\|^3} da = 2U_\infty \bar{i} \cdot \overline{n}_0, \tag{3}$$

where the sign " ' " denotes the principal value in the Cauchy sense of the integral (for more information about obtaining this singular integral equation see [1], [2], [3]).

3. THE DISCRETE EQUATION

A collocation method is used, for example in [2], to solve this integral equation.

In the boundary element approach used herein, in order to solve the integral equation (3), the body surface, Σ, is divided into M triangles, denoted by $T_j,\ j = \overline{1, M}$;the extremes $\overline{x}_i, i = \overline{1, N}$ of the panels being situated on Σ. In order to describe the behavior of the unknown f, locally, on a boundary element, we use a linear model. Then the integral equation becomes

$$f\left(\overline{x}_0\right) + \frac{1}{2\pi} \sum_{j=1}^{M} \iint\limits_{T_j} f\left(\overline{x}\right) \frac{\left(\overline{x} - \overline{x}_0\right) \overline{n}_0}{\left\|\overline{x} - \overline{x}_0\right\|^3} da = 2U_\infty n_0^x. \tag{4}$$

For $\overline{x}_0 = \overline{x}_i, i = \overline{1, N}$ we have, for a fixed i, the integral equation

$$f(\overline{x}_i) + \frac{1}{2\pi} \sum_{j=1}^{M} \iint_{T_j} f(\overline{x}) \frac{(\overline{x} - \overline{x}_i)\, \overline{n}_i}{\|\overline{x} - \overline{x}_i\|^3} da = 2U_\infty n_i^x. \tag{5}$$

If x_i is one of the triangle T_j corner points, then the integral calculated on T_j has a singularity, otherwise it does not. Thus, we have to calculate two types of integrals, with and without singularities. They can be computed analytically.

We calculate the integrals using a local system of coordinates . Denoting by $\overline{x}_1, \overline{x}_2, \overline{x}_3$ the corner points (nodes) of a triangle, by $\lambda_1, \lambda_2, \lambda_3$ the intrinsic triangular coordinates $\left(\lambda_i \in [0,1], i = \overline{1,3}, \lambda_1 + \lambda_2 + \lambda_3 = 1\right)$, and using the parametric representation

$$\lambda_2 = r\cos\theta \ , \ \lambda_3 = r\sin\theta \ , \ \theta \in \left[0, \frac{\pi}{2}\right] \ , \ r \in [0, \rho] \ , \ \rho(\cos\theta + \sin\theta) = 1, \tag{6}$$

for an interior point of the triangle we have the relation

$$\overline{x} = \overline{x}_1 + (\overline{x}_2 - \overline{x}_1)\, r\cos\theta + (\overline{x}_3 - \overline{x}_1)\, r\sin\theta. \tag{7}$$

The transformation of the surface element from the global Cartesian system of coordinates to the intrinsec system of coordinates in a triangular element gives $da = 2Sd\lambda_1 d\lambda_2 = 2Srdrd\theta$, where S is the area of the triangle.

For f we assume the linear interpolation

$$f = f_1\lambda_1 + f_2\lambda_2 + f_3\lambda_3 \Longleftrightarrow f = f_1 + (f_2 - f_1)\,\lambda_2 + (f_3 - f_1)\,\lambda_3 \tag{8}$$

First we consider that T_j has nodes that are different for $\overline{x}_i$. Denoting by $\overline{x}_1^j, \overline{x}_2^j, \overline{x}_3^j$ the corner points of the panel T_j, and by f_{j1}^1, f_j^2, f_j^3 the values of the unknown function at these nodes, and using the formulas below we have

$$\overline{x} = \overline{x}_j^1 + \left(\overline{x}_j^2 - \overline{x}_j^1\right) r\cos\theta + \left(\overline{x}_j^3 - \overline{x}_j^1\right) r\sin\theta, \tag{9}$$

$$f = f_j^1 + \left(f_j^2 - f_j^1\right) r\cos\theta + \left(f_j^3 - f_j^1\right) r\sin\theta. \tag{10}$$

We can write

$$\frac{1}{2\pi} \iint_{T_j} f(\overline{x}) \frac{(\overline{x} - \overline{x}_i)\, \overline{n}_i}{\|\overline{x} - \overline{x}_i\|^3} da = \frac{S_j \overline{n}_i}{\pi} \left[f_j^1 \bar{A}_{ij}^1 + \left(f_j^2 - f_j^1\right) \overline{B}_{ij}^1 + \left(f_j^3 - f_j^1\right) \overline{C}_{ij}^1 \right]$$

where S_j is the area of T_j and

$$\overline{A}_{ij}^1 = \left(\overline{x}_j^1 - \overline{x}_i\right) \int_0^{\frac{\pi}{2}} I_1(\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \overline{e}_j(\theta)\, I_2(\theta)\, d\theta, \tag{11}$$

$$\overline{B}_{ij}^{1} = \left(\overline{x}_j^1 - \overline{x}_i\right) \int_0^{\frac{\pi}{2}} \cos\theta \; I_2(\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \cos\theta \; \overline{e}_j(\theta)\, I_3(\theta)\, d\theta, \tag{12}$$

$$\overline{C}_{ij}^{1} = \left(\overline{x}_j^1 - \overline{x}_i\right) \int_0^{\frac{\pi}{2}} \sin\theta \; I_2(\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \sin\theta \; \overline{e}_j(\theta)\, I_3(\theta)\, d\theta, \tag{13}$$

$$\overline{e}_j(\theta) = \left(\overline{x}_j^2 - \overline{x}_j^1\right)\cos\theta \; + \left(\overline{x}_j^3 - \overline{x}_j^1\right)\sin\theta, \tag{14}$$

$$I_n(\theta) = \int_0^{\rho} \frac{r^n}{\left(ar^2 + 2br + c\right)^{\frac{3}{2}}} dr \;,\; n = 1, 2, 3, \tag{15}$$

$$a = \left\|\overline{e}_j(\theta)\right\|^2 \;,\; b = \left(\overline{x}_j^1 - \overline{x}_i\right)\overline{e}_j(\theta) \;,\; c = \left\|\overline{x}_j^1 - \overline{x}_i\right\|^2. \tag{16}$$

Computing the integrals in (15) we find the expressions

$$I_1(\theta) = \frac{\sqrt{c}}{\Delta} - \frac{b\rho + c}{\Delta\sqrt{a\rho^2 + 2b\rho + c}}, \tag{17}$$

$$I_2(\theta) = \frac{\left(b^2 - \Delta\right)\rho + bc}{a\Delta\sqrt{a\rho^2 + 2b\rho + c}} - \frac{b\sqrt{c}}{a\Delta} + \frac{1}{\sqrt{a^3}} \ln \frac{\sqrt{a}\sqrt{a\rho^2 + 2b\rho + c} + a\rho + b}{b + \sqrt{ac}}, \tag{18}$$

$$\begin{aligned} I_3(\theta) \;=\; & \frac{\sqrt{a\rho^2 + 2b\rho + c}}{a^2} - \frac{3b}{\sqrt{a^5}} \ln \frac{\left|a\rho + b + \sqrt{a}\sqrt{a\rho^2 + 2b\rho + c}\right|}{b + \sqrt{ac}} + \\ & + \frac{\sqrt{c}\left(b^2 - 2\Delta\right)}{a^2\Delta} + \frac{c\left(\Delta - b^2\right) + \rho b\left(3\Delta - b^2\right)}{a^2\Delta\sqrt{a\rho^2 + 2b\rho + c}}. \end{aligned} \tag{19}$$

Introducing the notation $\overline{A}_{ij} = \overline{A}_{ij}^1 - \overline{B}_{ij}^1 - \overline{C}_{ij}^1$, the integral equation becomes

$$\frac{1}{2\pi} \iint_{T_j} f(\overline{x}) \frac{\left(\overline{x} - \overline{x}_i\right)\overline{n}_i}{\left\|\overline{x} - \overline{x}_i\right\|^3} da = \frac{S_j \overline{n}_i}{\pi} \left[f_j^1 \bar{A}_{ij} + f_j^2 \overline{B}_{ij}^1 + f_j^3 \overline{C}_{ij}^1\right]. \tag{20}$$

Consider that the triangle , denoted by T_k, has a corner point at $\overline{x}_i$. We calculate the singular integrals occurring in (11) by using the relations

$$\overline{x} = \overline{x}_i + \left(\overline{x}_k^2 - \overline{x}_i\right) r\cos\theta + \left(\overline{x}_k^3 - \overline{x}_i\right) r\sin\theta, \tag{21}$$

$$f = f_i + \left(f_k^2 - f_i\right) r\cos\theta + \left(f_k^3 - f_i\right) r\sin\theta, \tag{22}$$

where $\overline{x}_k^2$, $\overline{x}_k^3$ are the other two nodes of T_k, and f_k^2, f_k^3 are the values of the unknown function f at these nodes. If S_k is the area of T_k , denoting

$$\overline{e}_k(\theta) = \left(\overline{x}_k^2 - \overline{x}_i\right)\cos\theta \; + \left(\overline{x}_k^3 - \overline{x}_i\right)\sin\theta \tag{23}$$

we obtain

$$\begin{aligned} \frac{1}{2\pi} \iint_{T_k} f(\overline{x}) \frac{\left(\overline{x} - \overline{x}_i\right)\overline{n}_i}{\left\|\overline{x} - \overline{x}_i\right\|^3} da = \;=\; & \frac{\overline{n}_i}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\rho} \left[f_i + \left(f_k^2 - f_i\right) r\cos\theta\right] \frac{r\overline{e}_k(\theta)}{r^3 \left\|\overline{e}_k(\theta)\right\|^3} 2S_k r dr d\theta + \\ & + \frac{\overline{n}_i}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\rho} \left[\left(f_k^3 - f_i\right) r\sin\theta\right] \frac{r\overline{e}_k(\theta)}{r^3 \left\|\overline{e}_k(\theta)\right\|^3} 2S_k r dr d\theta. \end{aligned} \tag{24}$$

Using the fact that the finite part of $\int\limits_0^{\rho} \frac{1}{r} dr$ is equal to $\ln(\rho)$, we get the equivalent form of the right-hand side term

$$\frac{\overline{n}_i S_k}{\pi} f_i \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)}{\|\overline{e}_k(\theta)\|^3} \ln \rho(\theta)\, d\theta + \frac{\overline{n}_i S_k}{\pi} \left(f_k^2 - f_i\right) \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)\cos\theta}{\|\overline{e}_k(\theta)\|^3} \rho(\theta)\, d\theta +$$

$$+ \frac{\overline{n}_i S_k}{\pi} \left(f_k^3 - f_i\right) \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)\sin\theta}{\|\overline{e}_k(\theta)\|^3} \rho(\theta)\, d\theta. \tag{25}$$

Finally, the singular integral has the representation

$$\frac{1}{2\pi} \iint\limits_{T_k} f(\overline{x}) \frac{(\overline{x} - \overline{x}_i)\,\overline{n}_i}{\|\overline{x} - \overline{x}_i\|^3} da = \frac{S_k \overline{n}_i}{\pi} \left[f_i \bar{A}_{ik}^2 + \left(f_k^2 - f_i\right) \overline{B}_{ik}^2 + \left(f_k^3 - f_i\right) \overline{C}_{ik}^2 \right] \tag{26}$$

where

$$\overline{A}_{ik}^2 = \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)}{\|\overline{e}_k(\theta)\|^3} \ln \rho(\theta)\, d\theta \ , \ \overline{B}_{ik}^2 = \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)\cos\theta}{\|\overline{e}_k(\theta)\|^3} \rho(\theta)\, d\theta \ , \tag{27}$$

$$\overline{C}_{ik}^2 = \int\limits_0^{\frac{\pi}{2}} \frac{\overline{e}_k(\theta)\sin\theta}{\|\overline{e}_k(\theta)\|^3} \rho(\theta)\, d\theta. \tag{28}$$

Denoting $\overline{A}_{ik} = \overline{A}_{ik}^2 - \overline{B}_{ik}^2 - \overline{C}_{ik}^2$, it becomes

$$\frac{1}{2\pi} \iint\limits_{T_k} f(\overline{x}) \frac{(\overline{x} - \overline{x}_i)\,\overline{n}_i}{\|\overline{x} - \overline{x}_i\|^3} da = \frac{S_k \overline{n}_i}{\pi} \left[f_i \bar{A}_{ik} + f_k^2 \overline{B}_{ik}^2 + f_k^3 \overline{C}_{ik}^2 \right] \tag{29}$$

As shown in the previous section, all the integrals occurring in (29) can be computed analytically.

Using the relations (27) and (28) the singular integral equation has the form

$$f_i + \sum_{j \in A_1} \frac{S_j \overline{n}_i}{\pi} \left[f_j^1 \bar{A}_{ij} + f_j^2 \overline{B}_{ij}^1 + f_j^3 \overline{C}_{ij}^1 \right] + + \sum_{k \in A_2} \frac{S_k \overline{n}_i}{\pi} \left[f_i \bar{A}_{ik} + f_k^2 \overline{B}_{ik}^2 + f_k^3 \overline{C}_{ik}^2 \right] = 2U_\infty n_i^x \tag{30}$$

where A_1 is the set of the triangles that have no corner point at $\overline{x}_i$, and A_2 is the set of the triangles that have a corner point at $\overline{x}_i$.

Further we return to the global system of coordinates, and we obtain the equation in terms of the nodal unknowns $f_j = f(\overline{x}_j)$, $j = \overline{1, N}$. Denoting by $A_{i,j}$ the coefficient of f_j, $j = \overline{1, N}$ we get

$$\sum_{j=1}^{N} A_{i,j} f_j = 2U_\infty n_i^x \tag{31}$$

This algebraic system can be written as $[A]\{f\} = \{B\}$.

After solving the system (31) we may compute the velocity for the N nodes chosen for the discretization of the boundary. We deduce the formula

$$\begin{aligned}\overline{v}\left(\overline{x}_i\right) &= -\frac{1}{2}f_i\left(\overline{n}_i+\frac{1}{\pi}\sum_{k\in A_2}S_k\overline{A}_{ik}\right)-\sum_{j\in A_1}\frac{S_j}{2\pi}\left[f_j^1\bar{A}_{ij}+f_j^2\overline{B}_{ij}^1+f_j^3\overline{C}_{ij}^1\right]-\\ &\quad-\sum_{k\in A_2}\frac{S_k}{2\pi}\left[f_k^2\overline{B}_{ik}^2+f_k^3\overline{C}_{ik}^2\right].\end{aligned}\tag{32}$$

4. TESTING THE METHOD

In order to test the method we consider the uniform motion in the presence of a sphere of radius R, centred at the origin of the system of coordinates. In this case the integral equation (3) can be solved analytically. A solution of this equation can be found in [6].

Using the spherical coordinates for the nodal points

$$\overline{x}=r\left(\sin q_1\cos q_2\overline{i}+\sin q_1\sin q_2\overline{j}+\cos q_1\overline{k}\right)\tag{33}$$

and the method of successive approximations to integrate (3), for the case when the velocity at infinity is $-U_\infty\overline{k}$, we obtain the exact solution

$$f\left(q_1,q_2\right)=\frac{3}{2}U_\infty\cos q_1.\tag{34}$$

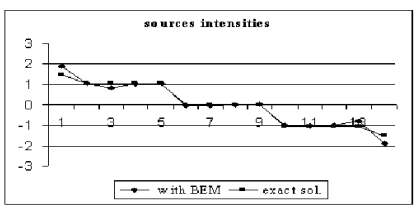

Fig.1. The sources intensities for the 14 control points.

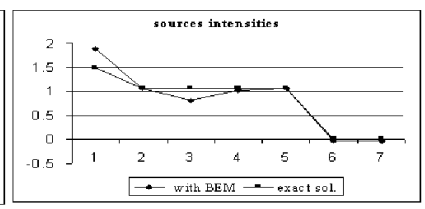

Fig.2. The sources intensities for the first 7 control points.

Comparisons between the analytical values of the intensity on the sphere and the values calculated by means of the boundary element method (with a computer code in MATHCAD) are performed in figs.1, 2, 3. The boundary is represented by 24 planar triangles and has 14 control points .

Remark that the calculated and analytical values of the intensity are very close.

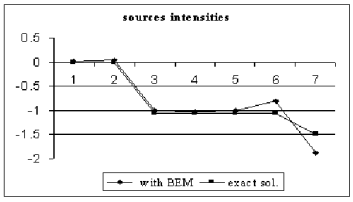

Fig.3. The sources intensities for the last 7 control points.

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ROMAI J., **1**, 2(2005),
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THE DISCRETE TIME ANALYSIS OF THE TURBULENCE IN EXCITABLE MEDIA

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Abstract The importance of turbulence models in excitable media is pointed out among the recent important contributions in fluid mechanics, since turbulence represents a feature of far from equilibrium systems. This paper exhibits some interesting features for such a model. It concerns the analysis of the deformations efficiency in length and surface for a 3D mathematical model associated with a vortex phenomenon. The experiments were realized on filaments of the aquatic algae Spirulina Platensis, based on a special vortex tube created by S. Savulescu and with a large applications area. The conditions in which the chaotic behavior issues are pointed out.

Keywords: turbulent mixing, deformations, vortex, chaotic behavior.

1. INTRODUCTION

The turbulence term is mostly associated with fluid dynamics, but it also represents an important feature of the systems with very few freedom degrees. We can define turbulence as *chaotic behavior of far from equilibrium systems, with infinity of freedom degrees.* In this area there are two important theories: a) the transition theory from smooth laminar flows to chaotic flows, characteristic to turbulence; b) statistic studies of the complete turbulent systems. The statistical idea of flow is represented by the map

$$x = \Phi_t\left(X\right) \text{ with } X = \Phi_t\left(t=0\right)\left(X\right) \tag{1}$$

We say that X is mapped in x after a time t.

In the continuum mechanics the relation (1.1) is called the *flow*, and it must be of class C^k. From the dynamic standpoint we have a map $\Phi_t\left(X\right) \longrightarrow x$ which is a diffeomorphism of class C^k. Moreover, (1.1) must satisfy the relation

$$0 < J < \infty,\ J = \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{2}$$

or, alternately, $J = det(D\Phi_t(X))$, where D denotes the derivation with respect to the reference configuration, in this case X. The relation (1.2) involves two particles, X_1 and X_2, which occupy the same position x at a moment. Nontopological behavior (like break up, for example) *is not allowed.*

The basic measure for the deformation with respect to X is the deformation gradient $\mathbf{F} = \left(\nabla_X \Phi_t\left(\mathbf{X}\right)\right)^T$ or, equivalently, $F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right)$, *or* $\mathbf{F} = D\Phi_t\left(\mathbf{X}\right)$,

where ∇_X denotes differentiation with respect to X. According to (1.2), $\mathbf{F}$ is nonsingular. The basic measure for the deformation with respect to $\mathbf{x}$ is the velocity gradient. By differentiation of x with respect to X we obtain the relation $dx_i = \frac{\partial x_i}{\partial X_j} \cdot dX_j$, which gives the deformation of an infinitesimal filament of length $|d\mathbf{X}|$ and orientation $\mathbf{M}(= d\mathbf{X}/|d\mathbf{X}|)$ from its reference position, to the present position dx , with the length $|dx|$ and the orientation $\mathbf{m}(= d\mathbf{x}/|d\mathbf{x}|)$, namely $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$. This relation represents the basic deformation of a material filament. The corresponding relation for the area of an infinitesimal material surface is $d\mathbf{a} = (\det \mathbf{F}) \cdot \left(\mathbf{F}^{-1}\right)^T \cdot d\mathbf{A}$. In this case in the present configuration we have the area $d\mathbf{a} = |d\mathbf{a}|$ and the orientation $\mathbf{n} = d\mathbf{a}/|d\mathbf{a}|$, and in the reference configuration, the area $d\mathbf{A} = |d\mathbf{A}|$, and the orientation $\mathbf{N} = d\mathbf{A}/|d\mathbf{A}|$.

The basic deformation measures used in applications are the length deformation λ and surface deformation η, with the relations [1], [2] $\lambda = \lim\limits_{|d\mathbf{X}|\to 0} \frac{|d\mathbf{x}|}{|d\mathbf{X}|}$, $\eta = \lim\limits_{|d\mathbf{A}|\to 0} \frac{|d\mathbf{a}|}{|d\mathbf{A}|}$, obtained from

$$\lambda = (C : MM)^{\frac{1}{2}}\ ,\ \eta = (\det F) \cdot \left(C^{-1} : NN\right)^{\frac{1}{2}}\ , \tag{3}$$

where $\mathbf{C}(= \mathbf{F}^T\ U2022\mathbf{F})$ is the Cauchy-Green deformation tensor, and $\mathbf{M} = d\mathbf{X}/\left|d\mathbf{X}\right|$, $\mathbf{N} = d\mathbf{A}/\left|d\mathbf{A}\right|$. The scalar form of (1.3) is $\lambda = C_{ij} \cdot M_i \cdot N_j$, $\eta = (\det F) \cdot \left(C_{ij}^{-1} \cdot N_i \cdot N_j\right)$, *with* $\sum M_i^2 = 1$, $\sum N_j^2 = 1$. The deformation tensor $\mathbf{F}$ and the associated tensors $\mathbf{C}, \mathbf{C}^{-1}$ represent the basic quantities in the deformation analysis for the infinitesimal elements. In most of cases, the flow $\Phi_t(X)$ is unknown and it must be obtained by integrating the velocity field. In this framework the mixing concept implies the *stretching* and *folding* of the material elements. If in an initial location P there is a material filament $d\mathbf{X}$ and an area element $d\mathbf{A}$, the specific length and surface deformations are given by the relations $\frac{D(\ln \lambda)}{Dt} = \mathbf{D} : \mathbf{mm}$, $\frac{D(\ln \eta)}{Dt} = \nabla \mathbf{v} - \mathbf{D} : \mathbf{nn}$, where $\mathbf{D}$ is the deformation tensor, obtained by decomposing the velocity gradient in its symmetric and non-symmetric part, i.e. $\nabla \mathbf{v} = \mathbf{D} + \mathbf{\Omega}$, where $\mathbf{D} = \frac{\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)}{2}$ is the *symmetric tensor*, $\mathbf{\Omega} = \frac{\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right)}{2}$ is the *antisimetric tensor*.

The flow $\mathbf{x} = \Phi_t(\mathbf{X})$ has *a good mixing* if the mean values $D(ln\lambda)/Dt$ and $D(ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations $\mathbf{M}$ and $\mathbf{N}$. As these two quantities are bounded, the deformation efficiency can be naturally quantified. Thus, it is defined the deformation efficiency in length, $e_\lambda = e_\lambda(X, M, t)$ of the material element dX, as $e_\lambda = \frac{D(ln\lambda)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \leq 1$, and similarly, the deformation efficiency in surface, $e_\eta = e_\eta(X, N, t)$ of the area element dA. In the case of an isochoric flow

(the Jacobian equals 1), we have $e_\eta = \frac{D(ln\eta)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \leq 1$, while in the case of a non-isochoric flow, we have $e_\eta = \frac{D(ln\eta)/Dt}{2^{1/2}\cdot(\mathbf{D}:\mathbf{D})^{1/2}} \leq 1$.

2. THE ANALYSIS OF THE MATHEMATICAL MODEL

The analysis of the deformation efficiency in length and surface for a three-dimensional mathematical model associated with a vortex phenomena for the aquatic algae Spirulina Platensis, has pointed out interesting analytical properties [5]. The mathematical model was associated to experiments, realized for a particular flow in a special vortex tube closed at one end. Locally, there exists an annular vorticity zone of high intensity, which has a whirlpool behavior. Since the turbulence occurs at small scales, phenomena like solid particles retaining, the mixture of textile fibers or break-up of cellular filaments of aquatic algae, are allowed. The special vortex tube is a modified version (aprox. 0.1 bar) of the Ranque-Hilsch tube [4]. If the tube is completely closed at one end, in this region it is obtained a very high intensity whirlpool. The mechanism has two versions: a small version (with a 15-20mm diameter), and a large version (100-300mm diameter), depending on two categories of processing particles – at small and at large scale. From physical standpoint, the generated whirlpool has four mechanisms [3], [4]: a) *convection*, realized by the streamlines which are oriented to the tube cover, the pressure source being near the middle axis of the tube; b) *turbulent diffusion*, realized by the fluctuating velocity and pressure; c) *stratification effects*, realized by the temperature and pressure gradients; d) *turbulent mixing*, realized by the concentrating velocity in the annular structures. In order to process the algae it was used a non-dimensional parameter $\tau_a = \frac{t\cdot Q}{D^3}$, where t represents the time (sec), D the diameter (m^3), and Q the mass flow (m^3/sec). By fragmenting the long chains of cellular filaments, there where obtained isolated cell units, or, as rare events, the break –up of the cell membrane (having less than 100 angstrom). The initial and final microscopic observations pointed out [4], [5] that the fragmentation degree increases with τ_a. The analytical study of the discrete mathematical model associated with the above phenomena has confirmed the experimental study. The following three-dimensional model was considered

$$\dot{x_1} = G\cdot x_2,\ \dot{x_2} = K\cdot G\cdot x_1,\ \dot{x_3} = c,\ \ -1 < K < 1,\ \ c = \ const. \tag{4}$$

This is a generalization to three dimensions of the two dimensional version used in [2], a widespread model for isochoric flows. In (2.1) the third component (corresponding to axis z) represents the rotation velocity, supposed to be constant. The initial condition attached to (2.4) is

$$x_1(0) = X_1,\ x_2(0) = X_2,\ x_3(0) = X_3. \tag{5}$$

Solving the Cauchy problem (2.4)-(2.5), the solution $x_i = x_i(X_j),\ i,\ j = 1,2,3$ was founded as

$$x_1 = \frac{1}{2}\cdot\left(X_1 + \frac{1}{\sqrt{K}}\cdot X_2\right)\cdot\exp\left(\dot{\gamma}\cdot t\right) - \frac{1}{2}\cdot\left(\frac{X_2}{\sqrt{K}} - X_1\right)\cdot\exp\left(-\dot{\gamma}\cdot t\right),$$

$$x_2 = \frac{1}{2} \cdot \left(\sqrt{K} \cdot X_1 + X_2\right) \cdot \exp\left(\dot{\gamma} \cdot t\right) + \frac{1}{2}\left(X_2 - \sqrt{K} \cdot X_1\right) \cdot \exp\left(-\dot{\gamma} \cdot t\right), \tag{6}$$
$$x_3 = c \cdot t + X_3,$$

where $\dot{\gamma} = G \cdot \sqrt{K}$.

From physical standpoint, (2.6) represents the state x_i of the system, at the moment t, with respect to the reference state $X_j, j = 1, 2, 3$. In our case, it represents the state of the aquatic algae after the vortex experiment. The deformations in length and surface of the material filaments, with the vortex conditions imposed were studied. The deformation tensor $\mathbf{F}$, and then the tensors $\mathbf{C}, \mathbf{C}^{-1}$, were founded, with quite complicated expressions [5]. The computations are quite complex, such that in order to analyze the deformations e_λ and e_η at *successive moments*, a discrete graphical analysis was carried out [5]. It was searched the issue of special phenomena appearing at *random values* of the unit vectors in length $M = (M_1, M_2, M_3)$ and in surface $N = (N_1, N_2, N_3)$.

3. RESULTS

The studied cases, and the events corresponding to different values of the orientations in length (M_1, M_2, M_3) and in surface (N_1, N_2, N_3) are very few, at least 60 cases. Their statistical interpretation is realized in [5], including the two-dimensional case. It was defined as rare event the event of breaking-up the material filaments, with a corresponding mathematical standpoint in the sudden failing of the running program, or failing the required accuracy. The breaking up of the filaments of Spirulina Platensis is produced by the *local alternative phenomena of stretching and folding*, in a favorable context of random distributed events. There are few cases with linear behavior, but there are also cases with alternative stretching /folding , or with breakup. In fact there are four types of events [5]: a) events with a *relative linear* behavior; b) events with a *linear-negative* behavior, corresponding to alternative stretching /folding of the filaments; statistically, these are very few; c) *mixing phenomena*, when large or small deviations, or strong discontinuities occur. In this case small pieces of algae are taking off, and the rest of algae undergo a new vortex phenomena; d) *rare events*, corresponding to *turbulent mixing*; in this case the filaments of Spirulina are broken up. A very important remark is that the mixing, and especially the turbulent mixing, occurs at *irrational values*, $\sqrt{2}$, $\sqrt{3}$, of $2 \cdot \dot{\gamma}$, and also of M_i and N_j $(i, j = 1, 2, 3)$. This is not surprising, taking into account that they can be considered as random values. Also, the turbulence occurs at small values of the time units, being in agreement with experiments. Depending to the context, τ_a can be measured in seconds, hours or even larger units.

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A TESTING PROCEDURE FOR DETERMINISTIC COVER FINITE STATE MACHINES

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Abstract The paper deals with the testing procedures of software system specified by *finite* sequential functions. It is shown that using the cover finite state machines, the W and Wp testing methods may be adapted to cope with such systems.

1. INTRODUCTION

The testing methodologies for software systems modeled by finite state machines are mainly aimed to the automatic generation of a test set such that a correct behaviour of the system on all the cases of the test guarantees the correctness in the general case. Obviously, that goal, contradicting the famous statement by E. W. Dijkstra *program testing can be used to show the presence of bugs, but never to show their absence* may be accomplished only in some restricted classes of software systems. Such a class comprises the communication protocols specified by *finite state machines* but implemented by some other finite state machines that meet some additional requirements concerning their form and size.

A special case is the test generating procedure for protocols modeled by *deterministic* finite state machines using the so called $W - method$ [2]. Given two deterministic finite state machines S and I, the former representing the specification and the latter the implementation, a test set is a set of sequences that, when applied to the two machines with identical results, guarantees their identical behaviour (if the two machines are not equivalent then at least one of these sequences will produce different results on the two machines). The test set will be generated from the specification S and, in principle, no information is available about the implementation I, except that the difference between the number of states of the implementation and that of the specification has to be at most k, a positive integer estimated by the tester. A variant of this method, called the Wp-*method*, that, in certain circumstances may slightly reduce the size of the test set at the expense of the complexity of the generation algorithm, has also been developed [7]. The method was also generalized to *non-deterministic finite state machines* [17]. The case of partially specified de-

terministic finite state machines is also considered in [1]. Furthermore, the W method has been used as a basis for test set generation for more complex models such as a type of extended finite state machine called *stream X-machine (Eilenberg machine)* [9]; it has been shown that, in certain circumstances, generating test sets for stream X-machines (Eilenberg machines) can be reduced to generating test sets for deterministic finite state machines that are partially specified [10], [9], [12], [11], [14], [13].

Due to the fact that many applications of regular languages use actually only finite languages, some recent investigations on regular languages and finite automata [4], [20] are focused on the specification of finite languages.As the number of states of the automaton that accepts a finite language is at least one more than the length of the longest word in the language, and can even be in the order of exponential to that number, these papers deal with the so called *cover finite automata*. Informally, a cover finite automaton A of a finite language L is a finite automaton that accepts all words in L and possibly other words that are strictly longer than any word in L. In many cases, a minimal deterministic cover automaton of L has a much smaller size than a minimal deterministic automaton that accepts L. Thus, in practice cover automata can be used to reduce the size of automata for finite languages.

In this paper we adapt the W and Wp methods to work with *cover finite state machines* taking advantage of the reduced complexity of the specifications based on these tools.

2. FINITE STATE MACHINES

This section introduces the finite state machine and related concepts and results that will be used later in the paper.

For an alphabet A, and a set $U \subseteq A^*$, we define U^n by $U^0 = \{\epsilon\}$ and $U^n = U^{n-1}U$ for $n \geq 1$. Also, $U[n] = \cup_{0 \leq k \leq n} U^k$. For a sequence $a \in A^*$, $length(a)$ denotes the number of elements of a; in particular $length(\epsilon) = 0$. For a finite language $L \subseteq A^*$, $length(L)$ denotes the length of the longest word(s) in L, i.e. $length(L) = max\{length(s) \mid s \in L\}$. For a (partial) function $f : A \longrightarrow B$, $dom\ f$ denotes the domain of f, i.e. the subset of A for which f is defined.

Definition 2.1. *A* deterministic finite state machine *(*DFSM *for short) M is a tuple* $(\Sigma, \Gamma, Q, h, q_0)$*, as follows:*

- Σ *is the finite* input alphabet*;*
- Γ *is the finite* output alphabet*;*
- Q *is the finite* set of states*;*
- h *is the (partial)* next state and output function*,* $h : Q \times \Sigma \longrightarrow Q \times \Gamma$*;*

- q_0 *is the* initial state $q_0 \in Q$.

A DFSM is usually described by a state transition diagram. M is said to be *completely specified* if h is a total function. Otherwise M is said to be *partially specified*. Moreover, the (partial) function $h : Q \times \Sigma \longrightarrow Q \times \Gamma$ may be seen as broken up into two (partial) functions:

- $h_1 : Q \times \Sigma \longrightarrow Q$, the next state function;
- $h_2 : Q \times \Sigma \longrightarrow \Gamma$, the output function

such that $h(q, \sigma) = (h_1(q, \sigma), h_2(q, \sigma)$.

Definition 2.2. *The next state function h_1 can be extended to a (partial) function $h_1^* : Q \times \Sigma^* \longrightarrow Q$ defined by*

- $h_1^*(q, \epsilon) = q,\ q \in Q$;
- $h_1^*(q, s\sigma) = h_1(h_1^*(q, s), \sigma),\ q \in Q, s \in \Sigma^*, \sigma \in \Sigma$.

The output function h_2 can be extended to a (partial) function $h_2^* : Q \times \Sigma^* \longrightarrow \Gamma^*$ defined by

- $h_2^*(q, \epsilon) = \epsilon,\ q \in Q$;
- $h_2^*(q, s\sigma) = h_2^*(q, s)h_2(h_1^*(q, s), \sigma),\ q \in Q, s \in \Sigma^*, \sigma \in \Sigma$.

Definition 2.3. *For $q \in Q$, the* function computed by M in q, *denoted by $\{_M^q$, is defined by*

$$\{_M^q(s) = h_2^*(q_0, s), s \in \Sigma^*.$$

The function computed by M in q_0 is simply called the function computed by M *and is denoted by $\{_M$.*

Definition 2.4. *A state $q \in Q$ is called* accessible *if $\exists s \in \Sigma^*$ such that $h_1^*(q_0, s) = q$. A is called* accessible *if $\forall q \in Q$, q is accessible.*

Definition 2.5. *Two states $q_1, q_2 \in Q$ are called Y-*equivalent*, $Y \subseteq \Sigma^*$, if $h_2(q_1, s) = h_2(q_1, s)\ \forall\ s \in Y$. Otherwise q_1 and q_2 are called Y-*distinguishable*. If $Y = \Sigma^*$ then q and q' are simply called* equivalent *or* distinguishable*. Two DFSMs are called (Y-)equivalent or (Y-)distinguishable if their initial states are (Y-)equivalent or (Y-)distinguishable.*

Definition 2.6. *M is called* reduced *if $\forall q_1, q_2 \in Q$ with $q_1 \neq q_2$, q_1 and q_2 are distinguishable.*

Definition 2.7. *M is called minimal if any other DFSM that computes $\{_M$ has at least the same number of states as M.*

Theorem 2.1. *M is minimal if and only if M is accessible and reduced.*

This is a well known result, for a proof see for example [6].

Definition 2.8. *Let $M = (\Sigma, \Gamma, Q, h, q_0)$ and $A' = (\Sigma, \Gamma, Q', h', q'_0)$ be two DFSMs over the same input alphabet. Then a function $g : Q \longrightarrow Q'$ is called an* isomorphism *if*

- *g is bijective;*
- $g(q_0) = q'_0$;
- $g(h_1(q, \sigma)) = h'_1(g(q), \sigma)$, $\forall\ q \in Q, \sigma \in \Sigma$;
- $h_2(q, \sigma) = h'_2(g(q), \sigma)$, $\forall\ q \in Q, \sigma \in \Sigma$.

Theorem 2.2. *For two minimal DFSMs M and M', $\{_M = \{_{M'}$ if and only if M and M' are isomorphic.*

This is a well known result, for a proof see for example [6]. Techniques for constructing the minimal DFSM that computes the same function as a given DFSM also exist, for more detail see for example [6] or [3].

Definition 2.9. *A (partial) function $F : \Sigma^* \longrightarrow \Gamma^*$ is called a sequential function (*S-function *for short) if $F = \{_M$ for some DFSM M.*

Definition 2.10. *Given an S-function $F : \Sigma^* \longrightarrow \Gamma^*$ and $s \in$ dom F, we denote by $F^s : \Sigma^* \longrightarrow \Gamma^*$ the (partial) function defined by: $F^s(x) = F(s)^{-1}F(sx)$, $x \in \Sigma^*$ such that $sx \in$ dom F.*

Naturally, F^s is also an S-function. Furthermore, it is easy to verify that if $M = (\Sigma, \Gamma, Q, h, q_0)$ is such that $F = \{_M$, then $F^s = \{^q_M$, where $q = h^*_1(q, s)$.

Definition 2.11. *A sequential function $F : \Sigma^* \longrightarrow \Gamma^*$ is called a finite sequential function (*FS-function *for short)of length l if dom F is finite and l is the length of the longest word(s) in dom F.*

Obviosuly, if F is an FS-function, then F^s is also an FS-function.

3. THE *W* AND *WP* METHODS FOR DFSMS

here we present the basis of the methodology used for automatic generation of test sequences from a DFSM specification.

3.1. THE *W*-METHOD

The W- method involves the selection of two sets of input sequences, a state cover and a characterisation set, as defined next.

Definition 3.1. *$S \subseteq \Sigma^*$ is called a* state cover *of $M = (\Sigma, \Gamma, Q, h, q_0)$ if $\epsilon \in S$ and $\forall q \in Q \setminus \{q_0\}$, $\exists s \in S$ such that $h_1^*(q_0, s) = q$.*

Definition 3.2. *$W \subseteq \Sigma^*$ is called a* characterisation set *of $M = (\Sigma, \Gamma, Q, h, q_0)$ if any two distinct states of M, $q, q' \in Q$, $q \neq q'$, are W-distinguishable.*

Note that a state cover and a characterisation set exist if M is minimal.

The test set generated by the W-method in the context of completely specified DFSMs is

$$U_k = S\Sigma[k+1]W,$$

where

- S is a state cover of the specification M
- W is a characterisation set of the specification M

The idea is that the set $S\Sigma[1]$ (usually called a *transition cover* of M) ensures that all the states and all the transitions of M are also present in M' and $\Sigma[k]W$ ensures that M' is in the same state as M after each transition is used. Notice that the latter set contains W and also all sets $\Sigma[i]W$, $1 \leq i \leq k$. This ensures that M' does not contain extra states. If there were up to k extra states, then each of them would be reached by some input sequence of up to length k from the existing states.

Theorem 3.1. *[2] Let $M = (\Sigma, \Gamma, Q, h, q_0)$ and $M' = (\Sigma, \Gamma, Q', h', q_0')$ be completely specified DFSMs, M minimal, such that $card(Q') - card(Q) \leq k$, $k \geq 0$. Then M and M' are equivalent if and only if M and M' are U_k-equivalent.*

However, the W-method in the above form does not work for partially specified finite state machines. Intuitively, this happens because, if M or M' are not completely specified, M and M' may be $\{s\}$-equivalent for a word s but $\{t\}$-distinguishable for some prefix t of s. Therefore a solution to the problem would be to take the set of all prefixes of U_k instead of just U_k. In [1] it is shown that only a subset of this set is required, this is

$$U_k' = U_k \cup S\Sigma[k]\Sigma = S\Sigma[k+1]W \cup S\Sigma[k]\Sigma.$$

Theorem 3.2. *[1] Let $M = (\Sigma, \Gamma, Q, h, q_0)$ and $M' = (\Sigma, \Gamma, Q', h', q_0')$ be (possibly partially specified DFSMs), M minimal, such that $card(Q') - card(Q) \leq k$, $k \geq 0$. Then M and M' are equivalent if and only if M and M' are U_k'-equivalent.*

3.2. THE WP-METHOD

A variant of the W-method is the partial W method (Wp-method) [7]. This reduces the size of the test set at the expense of a slightly more complex generation algorithm. Instead of using the whole set W to check each state q, only a subset of this set can be used in certain cases. This subset W_q depends on the reached state q and is called an identification set of q.

Definition 3.3. *For $q \in Q$, $W_q \subseteq \Sigma^*$ is called an* identification set of q *if* $\forall$ $q' \in Q \setminus \{q\}$, *q and q' are W_q-distinguishable.*

Definition 3.4. *A set $\mathcal{W} \subset 2^{\Sigma^*}$ that contains an identification set W_q of q for each state q of M is called an identification set of M.*

Naturally, the union of the identification sets $W_q \in \mathcal{W}$ is a characaterization set.

For completely specified DFSMs, the test set genarated by the Wp-method is

$$V_k = S\Sigma[k]W \cup R\Sigma[k] \otimes \mathcal{W},$$

where

- S is a state cover of the specification M
- $R = S\Sigma \setminus S$
- W is a characterisation set of the specification M
- $\mathcal{W}$ is an identification set of M such that for each identification sets $W_q \in \mathcal{W}$, $W_q \subseteq W$.

and for a set $A \subseteq \Sigma$, $A \otimes \mathcal{W}$ consists of the sequences s in A concatenated with the corresponding W_q such that q is reached by s, i.e.

$A \otimes \mathcal{W} = \{st \mid s \in A \wedge t \in W_q \wedge h_1^*(q_0, s) = q$ for some $q \in Q\}$.

Intuitively, the first component $V_k^1 = S\Sigma[k]W$ checks that all the states defined by the specification are identifiable in the implementation. At the same time, the transitions leading from the initial state to these states are checked for correct output and state transfer. The second component $V_k^2 = R\Sigma[k] \otimes \mathcal{W}$ checks the implementation for all the transitions that are not checked by V_k^1.

Naturally, since in general the sets W_q may be proper subsets of W, the W_p may yield shorter test sets that the W method.

Theorem 3.3. *[7]*

Analogously to the W-method, the Wp-method can also be extended to cope with partially-specified DFSMs [1]. In this case, the test set generated is

$$V_k' = V_k \cup S\Sigma[k]\Sigma = S\Sigma[k]W \cup R\Sigma[k] \otimes \mathcal{W} \cup S\Sigma[k]\Sigma.$$

4. DETERMINISTIC COVER FINITE STATE MACHINES OF FS FUNCTIONS

This section introduces the concept of (minimal) deterministic cover finite state machine of a finite sequential function, similar to the (minimal) deterministic finite cover automaton of a finite language [4]. The concepts and results in this section are natural extensions of the concepts and results in [4]. The proofs are similar to those for the coresponding results given in [4] and consequently will be omitted.

A deterministic cover finite state machine of a finite sequential function F is a DFSM that provides the computation specified by F but may also process other words that are longer than any word in *dom F*.

Definition 4.1. *Let $M = (\Sigma, \Gamma, Q, h, q_0)$ be a DFSM, $F : \Sigma \longrightarrow \Gamma$ an FS-function and $l = length(dom\ f)$. Then M is called a* deterministic cover finite state machine *(*DCFSM *for short) of F if $\{_M \mid \Sigma[l] = F$.*

Example 4.1. *Consider $F_n : \{a, b\}^* \longrightarrow \{0, 1\}^*$, $n \geq 1$, with $F_n(s) = 1^{length(s)}$, $s \in \Sigma[n]$, $F_n(s) = 1^n 0$, $s \in \Sigma^{n+1}$ and undefined elsewhere. Then the 3 DFSM3 represented in Figure 1 (a), (b) and (c) are DCFSMs of F.*

Definition 4.2. *A DCFSM M of an FS-function F is called* minimal *if for any DCFSM M' of F, the number of states of M is less or equal to the number of states of M'.*

The DFSM in Figure 1 (a) is not a minimal DCFSM of F_n defined in Example 4.1 since there are DCFSMs of F (represented in Figure 1 (b) and (c)) with less states than it.

Two similarity relations are defined in [4] and used to characterize and construct a minimal deterministic finite cover automaton of a finite language. These are naturally extended to DFSMs as follows:

- a similarity relation between input words w.r.t. a FS-function F;
- an l-similarity relation between the states of a DFSM, where l is the length of the longest word(s) in *dom F*.

Their formal definitions are now presented.

Definition 4.3. *Let $F : \Sigma^* \longrightarrow \Gamma^*$ be a FS-function of length l. Then $\sim_F$ is a relation on Σ^* defined by: $s \sim_F t$ if $F^s \mid \Sigma[n] = F^t \mid \Sigma[n]$, where $n = l - max\{length(s), length(t)\}$.*
We say that s is similar *to t w.r.t. F. The relation $\sim_F$ is called the similarity relation on Σ^* w.r.t. F. When $s \sim_F t$ does not hold we write $s \nsim_F t$.*

Remark 4.1. *The similarity relation w.r.t. F is reflexive and symmetric but not transitive. For instance, for F_n as defined in Example 4.1 with $n \geq 1$, $\epsilon \sim_F a^{n+1}$, $a \sim_F a^{n+1}$, but $\epsilon \nsim_F a$.*

Definition 4.4. *Let $M = (\Sigma, \Gamma, Q, h, q_0)$ be an accessible DFSM. For each state $q \in Q$ we define $level_M(q)$ as the length of the shortest path(s) from q_0 to q, i.e.*
$level_M(q) = min\{length(s) \mid s \in \Sigma^*, h_1^*(q_0, s) = q\}$.

For M as represented in Figure 1 (c) we have $level_M(i) = i$, $0 \leq i \leq n$.

Definition 4.5. *Let $M = (\Sigma, \Gamma, Q, h, q_0)$ be an accessible DFSM. For each state $q \in Q$ we define $x_M(q)$ as the minimum path from q_0 to q, i.e.*
$x_M(q) = min\{s \mid s \in \Sigma^*, h_1^*(q_0, s) = q\}$,
where the minimum is taken according to the quasi-lexicographical order on Σ^*.

Remark 4.2. *If $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is an ordered set, $n > 0$, then the quasi-lexicographical order on Σ^*, denoted $\prec$, is defined by: $x \prec y$ if $length(x) < length(y)$ or $length(x) = length(y)$ and $x = z\sigma_i v$, $y = z\sigma_j u$, $i < j$, for some $z, u, v \in \Sigma^*$ and $1 \leq i, j \leq n$.*

For M as represented in Figure 1 (c) we have $x_M(i) = a^i$, $0 \leq i \leq n$.

Definition 4.6. *Let $M = (\Sigma, \Gamma, Q, h, q_0)$ be an accessible DFSM and $l \geq 0$. Then $\sim_M^l$ is a relation on Q defined by*
$p \sim_M^l q$ *if* $F^q \mid \Sigma[n] = F^p\Sigma[n]$, *where* $n = l - max\{level_M(p), level_M(q)\}$.
*We say that p is l-*similar *to q w.r.t. M. The relation $\sim_M^l$ is called the l-*similarity relation on Q w.r.t. M. *When $p \sim_M^l q$ does not hold we write $p \nsim_M^l q$.*

Remark 4.3. *Analogously to the similarity relation on Σ^* w.r.t. F, the l-similarity relation on Q w.r.t. M is reflexive and symmetric but not transitive. For M_n as represented in Figure 1 (a) and $l = n+1$ with $n \geq 1$, $0 \sim_{M_n}^l n+1$, $1 \sim_{M_n}^l n+1$, but $0 \nsim_{M_n}^l 1$.*

The following lemma establishes the link between the two similarity relations.

Lemma 4.1. *Let $F : \Sigma^* \longrightarrow \Gamma^*$ be an FS-function, $l = length(dom\ F)$ and $M = (\Sigma, \Gamma, Q, h, q_0)$ an accessible DCFSM of F. Then $\forall\ p, q \in Q$, $p \sim_M^l q$ if and only if $x_M(p) \sim_{dom\ F} x_M(q)$.*

Definition 4.7. *An accessible DFSM $M = (\Sigma, \Gamma, Q, h, q_0)$ is called l-*reduced *if $\forall\ p, q \in Q$ with $p \neq q$, $p \nsim_M^l q$.*

That is, an accessible DFSM is l-reduced if any two distinct states are not l-similar w.r.t. M.

The following theorem identifies necessary conditions for a DCFSM to be minimal.

Theorem 4.1. *If $M = (\Sigma, \Gamma, Q, h, q_0)$ is a minimal DCFSM of an FS-function $F : \Sigma^* \longrightarrow \Gamma^*$ then M is accessible and l-reduced, where $l = length(dom\ F)$.*

Corollary 4.1. *A minimal DCFSM of an FS-function $F : \Sigma^* \longrightarrow \Gamma^*$ is also a minimal DFSM.*

The converse is, however, false as illustrated by Example 4.1: the DFSM represented in Figure 1 (a) is a minimal DFSM but not a minimal DCFSM.

5. THE W AND WP METHODS FOR BOUNDED WORDS

Now we will extend the W and Wp methods to DCFSMs. Given a specification, in the form of a DCFSM M of an FS-function F, we need to construct a set of input sequences whose length does not exceed l, the length of the longest sequence in *dom* F that, when applied to any implementation M' will detect any response to input sequences of length at most l that does not conform to the response specified by M, provided that the difference between the number of states of the implementation and that of the specification is at most a nonnegative integer k.

As shown by the next example, this extension is not straightforward, since it is not sufficient to extract the sequences of length at most l from the test sets that establish equivalence for unbounded sequences.

Example 5.1. *For M represented in Figure 1 (c), $S = \{\epsilon, a, \ldots, a^n\}$ is a state cover of M and $W = \{a^{n+1}\}$ is a characterisation set, so $U_0 = S\Sigma[1]W = \{\epsilon, a, \ldots, a^n\}\{\epsilon, a, b\}\{a^{n+1}\}$ and $U_0 \cap \Sigma^{n+1} = \{a^{n+1}\}$. Consider M' as represented in Figure 1 (d). It is easy to see that M and M' are only distingusihed by sequences of the form bx, $x \in \Sigma^n$, so M and M' are $(U_0 \cap \Sigma^{n+1})-$equivalent even though M' is not a DCFSM of F_n. Furthermore, $U_0' = U_0 \cup S\Sigma$, and $U_0' \cap \Sigma^{n+1} = \{\epsilon, a, \ldots, a^n\}\{a, b\}$, so M and M' are also $(U_0 \cap \Sigma^{n+1})-$equivalent.*

We show now how the definitions of the test set generated by W and Wp methods can be revised so that these methods can be extended to DCFSMs. These will be called the *W and Wp methods for bounded words* or, for short, the *bounded W and Wp methods.*

5.1. THE W-METHOD FOR BOUNDED WORDS

The W-method for bounded words will only select state covers and characterisation sets with special properties, as defined next.

Definition 5.1. *$S \subseteq \Sigma^*$ is called a* proper state cover *of $M = (\Sigma, \Gamma, Q, h, q_0)$ if $\forall q \in Q$, $\exists s \in S$ such that $h_1^*(q_0, s) = q$ and $length(s) = level_M(q)$.*

That is, a proper state cover contains sequences of minimum length that reach the states of M. In particular, $S = \{x_M(q) \mid q \in Q\}$ is a proper state cover of M.

Definition 5.2. *$W \subseteq \Sigma^*$ is called a* strong characterisation set *of $M = (\Sigma, \Gamma, Q, h, q_0)$ if any for two states of M, $q, q' \in Q$ and $k > 0$, q and q' are $\Sigma[k-1]$-equivalent and Σ^k-distinguishable $\Longrightarrow$ q and q' are $(W \cap \Sigma^k)$-distinguishable.*

That is, a strong characterization set contains sequences of minimum length that distinguish between the states of M.

Then the test set generated by the bounded W-method in the case where M and M' are completely specified is

$$Y_k = S\Sigma[k+1]W \cap \Sigma[l],$$

where

- S is a proper state cover of M
- W is a strong characterization set of M

Example 5.2. *For M represented in Figure 1 (c), $S = \{\epsilon, a, \ldots, a^n\}$ is a proper state cover of M. For $0 \leq i < j \leq n$, i and j are Σ^{n-j}-equivalent and Σ^{n-j+1}-distinguishable, so $W = \{a, \ldots, a^n\}$ is a strong characterization set of M. Thus $Y_0 = S\Sigma[1]W \cap \Sigma[n+1] = \{\epsilon, a, \ldots, a^n\}\{\epsilon, a, b\}\{a, \ldots, a^n\} \cap \Sigma[n+1] = = \{a^i b^j a^k \mid 0 \leq i \leq n, 0 \leq j \leq 1, 0 \leq k \leq n, i+j+k \leq n+1\}$. For M' as represented in Figure 1 (d), since $ba^n \in Y_0$, Y_0 distinguishes between M and M', these are Y_0-distinguishable.*

As for unbounded sequences, the above test set may not distinguish between partially specified machines. Similarly, the test set can be extended to the general case where the machines may be partially specified. The extended test set is

$$Y'_k = Y_k \cup S\Sigma[k]\Sigma \cap \Sigma[l].$$

5.2. THE *WP*-METHOD FOR BOUNDED WORDS

Similarly, the Wp-method for bounded words selects only those identification sets that contain sequences of minum length that distinguish the states of the machine. These are called strong identifications sets.

Definition 5.3. *For $q \in Q$, $W_q \subseteq \Sigma^*$ is called a* strong identification set *of q, if for any state $q' \in Q$ and $k > 0$, q and q' are $\Sigma[k-1]$-equivalent and Σ^k-distinguishable $\Longrightarrow$ q and q' are $(W \cap \Sigma^k)$-distinguishable.*

A set $\mathcal{W}$ that contains a strong identification set W_q of q for each state of M is called a strong identification set of M.

Then the bounded Wp-method generates the following test set for the case where the specification and the implementation are completely specified

$$Z_k = S\Sigma[k]W \cap \Sigma[l] \cup R\Sigma[k] \otimes \mathcal{W} \cap \Sigma[l],$$

where

- S is a proper state cover of the specification M
- $R = S\Sigma \setminus S$
- W is a strong characterisation set of the specification M
- $\mathcal{W}$ is a strong identification set of M such that for each identification set $W_q \in \mathcal{W}$, $W_q \subseteq W$.

Example 5.3. *Consider M, M', S and S' as in Example 5.2. Then $R = S\Sigma \setminus S = \{b, \ldots, a^n b\} \cup a^{n+1}\}$. It is easy to verify that $W_0 = \{a, a^n\}$ is a strong identification set of 0 and for $1 \leq i \leq n$, $W_i = \{a, a^{n-i+1}\}$ is a strong identification set of i, so $\mathcal{W} = \{\{a\}, \ldots, \{a, \ldots, a^n\}\}$ is a strong identification set of M. Thus $Z_0 = SW \cap \Sigma[n+1] \cup R \otimes \mathcal{W} \cap \Sigma[n+1] = \{a, \ldots, a^{n+1}\} \cup \{b\}\{a, \ldots, a^n\} \cup \ldots \{a^{n-1}b\}\{a\}$. Since $ba^n \in Z_0$, Z_0 distinguishes between M and M'.*

In the case where the machines may be partially specified, the test set will be extended to

$$Z'_k = Z_k \cup S\Sigma[k]\Sigma \cap \Sigma[l].$$

6. CONCLUSIONS

The use of deterministic cover finite state machines instead of finite state machines in case of finite sequential function is a matter of practical interest because it leads to small sized specification of software systems. Apart from the topics concerning the applicability of W and Wp methods in such environments, the theoretical support presented in the paper may be used as a tool for the proof of the error detection power of these methodologies and wiil be the subject of a forthcoming paper.

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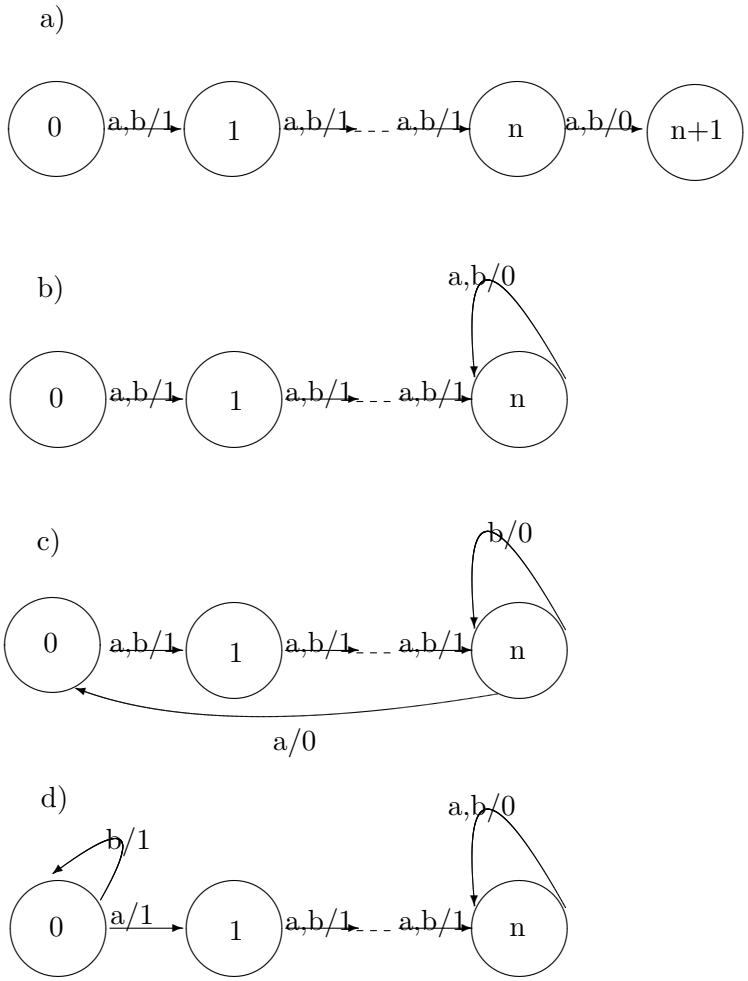

Figure 1 DCFSMs of F

ROMAI J., **1**, 2(2005),
129–132

AN ANT SYSTEM FOR A ROAD NET CONNECTING THE MAIN TOWNS OF ROMÂNIA

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Abstract In this paper we use an *Ant System* for *Travelling Salesman Problem* (*TSP*) in order to solve a practical problem. The idea is to consider the main towns in the România and find the best route passing exactly once through each of them, given the existing road network.

Keywords: *Travelling Salesman Problem, Ant System, Ant Colony Optimization, Meta-heuristic*

Introduction. Due to its strategic position, our country became an important place for investors all over the world. Even some regions are more preferred than others (such as Bucharest, the western part including Timişoara and Arad, or the coast of the Black Sea with Constanţa), the problem of passing through all towns in a tour is obviously important. This can be of special interest for supply or distribution companies, in service area and for many other activities, too. In this paper we propose the use of a modern algorithm (an Ant System, reffered in the following as AS) for the Travelling Salesman Problem (reffered in the following as TSP) in which we take into consideration the most important towns of Romania, as vertices. These towns and the associated numbers (that have been assigned to the towns within the AS) are: 0 - Bucharest, 1 - Alba Iulia, 2 - Arad, 3 - Bacău, 4 - Baia Mare, 5 - Bistriţa, 6 - Botoşani, 7 - Braşov, 8 - Brăila, 9 - Buzău, 10 - Cluj Napoca, 11 - Constanţa, 12 - Craiova, 13 - Deva, 14 - Drobeta Turnu-Severin, 15 - Focşani, 16 - Galaţi, 17 - Iaşi, 18 - Oradea, 19 - Piatra Neamţ, 20 - Piteşti, 21 - Ploieşti, 22 - Rîmnicu Vîlcea, 23 - Reşita, 24 - Satu Mare, 25 - Sibiu, 26 - Suceava, 27 - Tîrgovişte, 28 - Tîrgu Jiu, 29 - Tîrgu Mureş, 30 - Timişoara, 31 - Tulcea.

Ant Algorithms. Ant Algorithms have been proposed in 1991 by M. Dorigo and his coleagues [5] as a multi-agent approach for the difficult combinatorial optimization problemes. Actually, there is an intense research concerning the extension and the application of the algorithms based on "Ant Model" in very many discrete optimization problems. Recent applications, very important ones for practical area, include verhicle routing, sequential ordering, graph colouring, resources allocation and information routing in com-

munication networks. Ant Algorithms have been inspired by observing the real ant colonies. Ants are social insects whose behaviour concerns mostly the survival of the colony, to the detriment of the individuals. These aspects have drawn specialists' attention, due to the high degree of structuration of the colonies. A special aspect refferes to the fact that ants always find the shortest paths between the food sources and their nests. Walking on these paths, the ants deposit on the ground a substance called pheromones and so, they leave a track. Ants detect these tracks and choose the paths with the biggest concentrations of pheromones. The tracks also allow them to come back to their nests.

Observing that this behaviour could be used to project minimum-path algorithms, informaticians developed a meta-heuristic (*Ant Colony Optimization* - ACO), in which a colony of artificial agents (called "ants") cooperate, in order to find good solutions for discrete optimization problem. Cooperation is a key of the ACO algorithms. The resources are assigned to these simple agents that communicate in an indirect way - by means of the pheromone tracks mechanism. Starting from an initial state, each ant builds a solution. For this, the ant gather information about the characteristics of the problem and on the performances of the other ants and uses them in order to modify the state (or even the representation) of the problem. Ants can action concurrently and independently, but always having - by means of the permanent communication - a collective behaviour.

Ant System for a TSP having as vertices the main towns in România. The first applications of the ACO algorithms have been realized on the TSP problem, due to the fact that in this case the modelling process is simple and natural.

A general definition of TSP is the following. Consider a set N of vertices representing towns and a set E of edges fully connecting the vertices in N. Let d_{ij} be the length on the edge (i, j) in E (the distance between the towns i and j from N). TSP is the problem of finding a Hamiltonian circuit of minimal length for the graph $G = (N, E)$, namely a closed circuit in which every town is visited once and only once and whose length is the sum of the lengths of the edges that compose the circuit.

Ant Systems (AS) are the first ACO algorithms [6]. In an *AS*, the artificial ants build solutions (circuits) for TSP moving, on the graph of the problem, from a town to another. The algorithm performs *NC* (Number of Cycles) iterations. During each iteration, m ants (placed into the m towns) build a circuit executing m steps in which a probabilistic decision is applied - they choose the next town to be visited. When, being in the node i, the ant choose to move to node j (whose choosing probability has been maximum), the edge (i, j) is added to the current circuit and this step repeats until the full circuit

is built. After the ants build the circuits (a circuit for each of them) they "lay" pheromones, represented by variables associated to the visited edges. These quantities of pheromones are proportional with the quality of the generated solutions and represent the way the ants indicate, one to each other, the "desirability" in choosing the edge (i, j): the smaller the length of the circuit is, the bigger the quantity of pheromones is. In this way, the search is directed to better solutions. There also appears the necessity of modelling the "evaporation" of the pheromones and this helps to avoid the stagnation phenomena (the situations in which all ants would find the same circuit).

At the end of an iteration, the algorithm keeps the best circuit. This is (eventually) improved in the next iterations.

The papers [1,4,5,7] present different results obtained with different AS, distinguishing by: the way to compute the probability of choosing the new town, the way of keeping the tracks of pheromones and/or different values for the parameters of the AS. The test problems are also variated, with m (the number of the cities) starting from 30 and up to 500. For example, in [4] the authors studied TSP for 500 towns in SUA. We proposed to apply the AS algorithm that we built for a problem with the above-mentioned 32 towns in Romania. The distances between towns have been taken from www.sosele.com.

The algorithm have been coded in Java and run on an IBM computer, 1,7 GHz and 256 MB RAM. The values of the parameters are: $\alpha = 1$ (controls the euristic, that depends on the distances between towns), $\beta = 5$ (controls the pheromones level), $Q = 100$ (represents a scale factor), $\rho = 2/3$ (controls the degree of the evaporation of the pheromones) and $NC = 1000$ (number of iterations).

In 100 executions, the best solution furnished by the AS for the proposed problem has 3304 km and corresponds to the following tour: Bucureşti, Ploieşti, Buzău, Braşov, Tîrgovişte, Piteşti, Craiova, Drobeta Turnu-Severin, Tîrgu Jiu, Rîmnicu Vîlcea, Sibiu, Alba Iulia, Deva, Reşiţa, Timişoara, Arad, Oradea, Satu Mare, Baia Mare, Cluj Napoca, Tîrgu Mureş, Bistriţa, Suceava, Botoşani, Iaşi, Piatra Neamţ, Bacău, Focşani, Galaţi, Brăila, Tulcea, Constanţa (and back to Bucureşti). The mean of the lengths of the solutions is 3365.42 (km) and the maximum length is 3396 (km).

In each run, the program offers a new solution, so that, if other criteria are considered too, the user can pick the solution that fits his interests. The authors are interested in solving some concrete problems and will answer to all those who require the program they designed.

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ROMAI J., **1**, 2(2005),
133–142

INTERNET WORMS: PROPAGATION MODELING AND ANALYSIS

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Abstract Several Internet-scale incidents from the recent years have demonstrated the ability of self-propagating code, also known as "network worms", to infect large numbers of hosts, exploiting vulnerabilities in the largely deployed operating systems and applications. Capable of infecting a substantial portion of the hosts within several minutes, and impacting the world-wide network operations by generating a distributed denial of service (DDoS) attack on the whole Internet, network worms are considered a major security threat. So, a better understanding of the worms' propagation means will help to implement a more efficient detection and defense.

1. INTRODUCTION

Recent worm incidents have indicated a lot of interest in implementing a variety of scanning strategies to increase the worms' spreading speed and defeat security defense measures. In this paper, we present some mathematical models to analyze various scanning strategies that attackers have already used or might use in the future. Mathematical analysis provides a better understanding of how multiple factors affect a worm's propagation, and it can help us to build a better defense against future worms. The scanning strategies presented in the paper include idealized scan, uniform scan, divide-et-impera scan, local preference scan, sequential scan.

2. EPIDEMIC MODELS

Computer worms look similar to biological viruses in their propagation behaviors and self-replication. Thus, the mathematical models developed for the study of biological infectious diseases can be adapted to the study of computer worm propagation. We briefly introduce two classical deterministic epidemic models: simple epidemic model in homogeneous system and in interacting groups, respectively. The models and analyses presented in this paper are based primarily on these two models and their underlying principles.

A. *Simple epidemic model for homogeneous systems*

The simple epidemic model is characterized by the following assumptions:

- each and every host occupies one of two states: susceptible or infectious (this model is also named in the literature as „SI model”);
- once a host is infected, it stays permanently in the infectious state;
- each host is assumed to have equal probability to contact any other host.

The equation of the model is:

$$\frac{dI(t)}{dt} = \beta I(t)[N - I(t)], \tag{1}$$

where N is the total number of hosts under consideration, $I(t)$ is the total number of infectious hosts at time t, η is the average worm scan rate, Ω is the size of the worm's scanning space, β is the pairwise rate of infection in worm propagation model, $\beta = \eta/\Omega$.

At $t = 0$, there are $I(0)$ infected hosts, and $[N - I(0)]$ susceptible. The epidemic model SI has the following analytical solution

$$I(t) = \frac{I(0)N}{I(0) + [N - I(0)]e^{-\beta N t}}, \tag{2}$$

The spreading of an Internet worm or an epidemic disease is, in fact, a stochastic process, but when considering a large-scale system consisting of a large population N, which is the case of an Internet worm, it can be used a mean value analysis based on the law of large numbers. Also, in the Internet context, an infected host has equal probability of contacting any other host when the worm uniformly scans the Internet. Thus, a uniform scan worm can be modeled the same way as an epidemic disease in a homogeneous system.

B. *Simple epidemic model for interacting groups*

It is an extension of SI model, defined by equation (1), for non-homogeneous systems. In this model, the system consists of K groups; each group has population N_1, N_2, ..., N_k. Interactions between groups are different from interactions within a group. The equation of the model becomes

$$\frac{dI_k(t)}{dt} = [\beta_{kk} I_k(t) + \Sigma_{j \neq k} \beta_{jk} I_j(t)][N_k - I_k(t)], \tag{3}$$

where $k \in \{1, ..., K\}$, β_{jk} is the infection rate, per interacting pair, of the susceptible hosts in the k-th group by the infectious hosts in the j-th group.

3. THE MODELS AND THE ANALYSIS OF WORM SCANNING STRATEGIES

3.1. *Idealized worm*

This kind of worms would posses the IP addresses of all vulnerable hosts in the Internet. Even if it is virtually impossible to implement them on the global scale of the Internet, this type of scanning strategy is valuable as a research model.

3.1.1. *Perfect worm*

Considered to be the fastest propagation worm, it knows the addresses of all vulnerable hosts in the Internet, and all infected hosts fully cooperate with each other such that they will not try to scan and infect an already infected host. For this type of worm, any scan is successful in infecting another vulnerable host.

The propagation model for the perfect worm is

$$\frac{dI(t)}{dt} = \begin{cases} \eta I(t), \ I(t) < N, \\ 0, \ I(t) = N. \end{cases} \tag{4}$$

Suppose the perfect worm starts with $I(0)$ infected hosts, the solution for (4) is

$$I(t) = min[I(0)e^{\eta t}, \ N]. \tag{5}$$

To illustrate the propagation speed of the perfect worm, we assume it has some of the parameters identified for Code Red ($\eta = 358$ hosts/minute, $N = 360000$ total vulnerable hosts, and $I(0) = 10$).

From (5) it follows that all vulnerable population will be infected in $T = [lnN - lnI(0)]/\eta = 1.76sec$. However, the above scenario did not consider various time delays in the worm's propagation like: the time to transfer the worm code to the vulnerable host, the time to execute the worm infectious code. If we consider the delay (e) from the moment when a worm scan is sent out, to the moment when the vulnerable host just infected begins to infect others, then the propagation model for the perfect worm becomes

$$\frac{dI(t)}{dt} = \begin{cases} \eta I(t-\varepsilon), \ I(t) < N, \\ 0, \ I(t) = N, \end{cases} \tag{6}$$

where $I(t-\varepsilon) = 0, \ \forall t < \varepsilon$.

3.1.2. *Flash worm*

Staniford et al. defined the „flash worm", as one which knows the IP addresses of all vulnerable hosts in the Internet ($N = \Omega$) and uniformly scans the vulnerable population. The propagation of a flash worm satisfies the epidemic

spreading assumptions in a homogeneous system and can be modeled by the simple epidemic model (1). The equation for the flash worm becomes

$$\frac{dI(t)}{dt} = \eta I(t)[N - I(t)]/N. \tag{7}$$

Assuming some of the parameters identified for Code Red ($\eta = 358$ hosts/minute, $N = 360000$ vulnerable hosts, and $I(0) = 10$), it can be determined that a flash worm can infect almost all the vulnerable host in $T = 2.5$ seconds. Based on these results, the delays due to code propagation are significant and should be factored in the worm's propagation model. Assuming a propagation delay ε, (7) becomes

$$\frac{dI(t)}{dt} = \eta I(t-\varepsilon)[N - I(t)]/N,\ textwhere\ I(t-\varepsilon) = 0,\ \forall t < \varepsilon. \tag{8}$$

3.2. *Uniform scan worms*

When a worm does not have the IP addresses of the vulnerable hosts, the simplest solution is to scan randomly the entire IPv4 space ($\Omega = 232$) in order to identify potential victims. This scanning strategy was used by Code Red (07/2001) and Slammer (01/2003). Based on (1) the propagation model for Code Red type worms becomes

$$\frac{dI(t)}{dt} = 2^{-32}\eta I(t)[N - I(t)]/N. \tag{9}$$

An analysis between (7) and (9) shows that the scanning space of a flash worm is much smaller than IPv4 (used by a Code Red type worm). Also, because the propagation delays (e) are negligible in comparison with worm's spreading speed, we can ignore them.

3.2.1. *"Hit List" worm*

This type of worm was defined by Staniford, as a way to improve the spreading speed of a uniform scan worm. It has a list with IP addresses of some vulnerable host in Internet. First, the worm behaves like a „flash worm" by scanning and infecting the hosts from hit-list, and then it scans randomly the entire IPv4 space to infect other vulnerable hosts.

3.2.2. *"Routing Worm"*

Based on the BGP routing table, it has been established that only 28.6% of the IPv4 space is routable. With no change in the scanning strategy, any worm can improve its performances by scanning a smaller IP address space. Many of the recent worms have already taken of this information. For example, the scanning space for Win32.Doomjuice.a (2004) consisted of 160 class A networks ($\Omega = 160 * 224$)

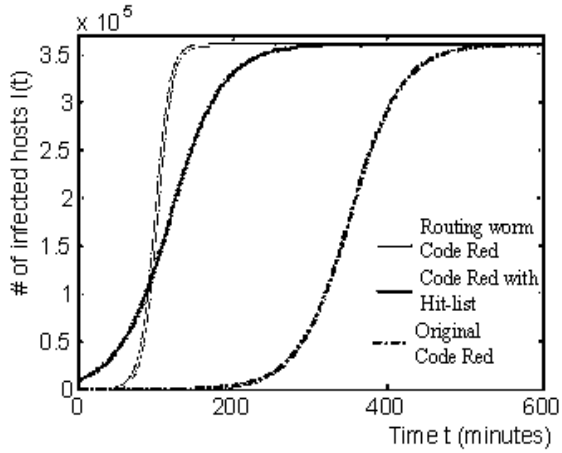

Fig 1. The Propagation Performances for possible Code Red versions taking advantage of "Hit List" and "Routing Worm" improvements.

It can be noticed that „Hit List" worm with $I(0) = 10.000$ can infect more hosts in a short time due to its hit-list, but it has a slower speed than a „Routing worm" with $\Omega = 0.286 * 2^{32}$.

3.2.3. *"Divide-et-impera" scan worm*

Another possibility to enhance the performances of a uniform scan worm would be to use a „divide-et-impera" approach to allow different infected hosts to scan and infect vulnerable hosts on distinct IP spaces. In the propagation of this kind of worm, there will be no case when two infected hosts would try to probe the same target.

The model assumes there is only one infected host initially in the system, and that vulnerable hosts are uniformly distributed in the entire scanning space O. Each infected host uniformly scans IP addresses in its scanning space. Once a target is infected, half of the scanning space allocated to the host that infected the target is transferred to the target (the space passed to the target includes the target host), while the infecting host remains with half of its original scanning space.

Based on these assumptions, none of the infected host will be subject to be probed, and the scanning space will be $\Omega' = \Omega - I(t)$. So, the equation for a uniform scan worm that uses a „Divide-et-impera" strategy is:

$$\frac{dI(t)}{dt} = \eta I(t)[N - I(t)]/(\Omega - I(t)), \tag{10}$$

For Internet worms, the number of vulnerable hosts N is much smaller than Ω ($I(t) < N << \Omega$). Therefore, $\Omega - I(t) \approx \Omega$ which means that when vulnerable hosts are uniformly distributed, a „divide-et-impera" scan worm

propagates in the same way as a uniform scan worm, and can be modeled by the simple epidemic model (1).

3.3. *Subnet scan worm*

The uniform scan is the simplest scanning strategy that a worm may use. However, it is not the optimal one because the vulnerable hosts are not uniformly distributed in Internet. A worm could increase its spreading speed when it scans with a higher probability in the IP spaces that have a higher density of vulnerable hosts.

In the following, we model and analyze a subnet scanning worm that has probability p to uniformly scan IP addresses in its own "/n" prefix subnetwork and probability (1-p) to uniformly scan other IP addresses. A „/n" prefix subnetwork represents the IP space where the addresses have the same first n bits. Thus, in the current IPv4 Internet, a „/n" prefix subnetwork contains 2^{32-n} IP addresses.

This class of worms can be modeled based on the simple epidemic model for interacting groups (3), where the worm scanning space Ω consists of K „/n" prefix subnetworks ($\Omega = K * 2^{32-n}$), $\beta' = \beta_{kk}$ - represents the pairwise rate of infection in local scan and $\beta'' = \beta_{jk} = \beta_{kj}$ is the pairwise rate of infection in remote scan. The model's equation becomes

$$\frac{dI(t)}{dt} = [\beta' I_k(t) + \Sigma_{j \neq k} \beta'' I_j(t)][N_k - I_k(t)], \tag{11}$$

where $k \in \{1, \ ..., \ K\}$ and

$$\beta'' = \frac{p\eta}{2^{32-n}}, \ \beta'' = \frac{1-p)\eta}{(K-1)2^{32-n}}. \tag{12}$$

The type of analysis can be easily extended to other kinds of local preference scan strategies, such as local preference scanning with several levels of locality. For example, Code Red II (08/2001) had two-level locality in its local preference scan - the worm scanned the local class A network with a probability $p_A = 0.5$ and the local Class B network with a probability $p_B = 0.375$. Another example is Win32/Sasser.worm (05/2004) that probed the local class A network with a probability $p_A = 0.25$ and the local Class B network with a probability $p_B = 0.25$.

When vulnerable hosts are uniformly distributed in a worm's scanning space, and so the worm propagation in each subnetwork is identical, subnet scanning does not help a worm in its propagation speed.

Assuming the vulnerable hosts are uniformly distributed in m out of the K subnetworks ($N_1 = ... = N_m = N/m$) and all other $(K - m)$ subnetworks are not allocated ($N_{m+1} = ... = N_K = 0$), the equation for the each subnetwork

becomes

$$\frac{dI(t)}{dt} = [\beta' + (m-1)\beta'']I_k(t)][N_k - I_k(t)], \text{ where } k = 1, ..., m. \tag{13}$$

The equation for the propagation in Internet (for all subnetworks) is

$$\frac{dI(t)}{dt} = m\frac{dI_1(t)}{dt} = [\beta' + (m-1)\beta'']I(t)][N - I(t)]/m. \tag{14}$$

If a subnet scan worm wants to propagate as fast as possible, the worm should select a preference probability p that maximize the pairwise rate of infection $[\beta' + (m-1)\beta"]/m$ in (14). Thus, the optimal preference probability should be $p = 1$. Such a conclusion may seem unexpected, but it is reasonable based on the assumption used that all those m „/n" subnetworks are considered identical. If $p = 1$, which means a worm only scans its own subnetwork, then no worm scans will be wasted in those $(K - m)$ empty subnetworks. In this way, the worm achieves its fastest spreading speed. In reality, no subnetwork is exactly the same as the others, and the worm has to scan remote networks in order to propagate to every part of the entire Internet.

The next chart shows the comparative simulation results between a Class A routing worm, subnet scan worm ($K = 256,\ m = 116$) for various local preference probabilities, and the original Code Red worm.

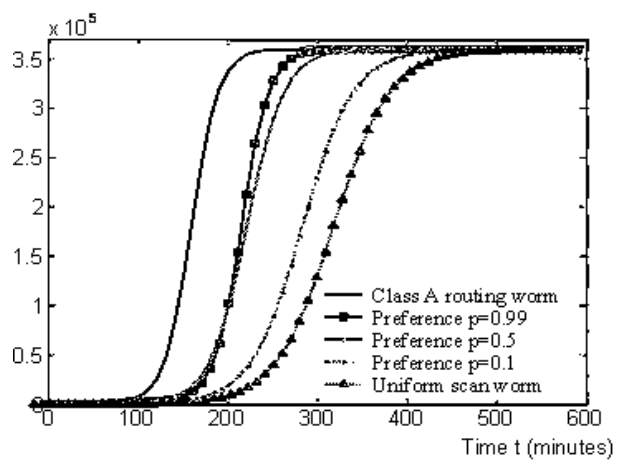

Fig. 2. Comparison of a Class A routing worm, subnet scan worm ($K = 256,\ m = 116$) and the original Code Red worm.

When vulnerable hosts are not uniformly distributed in a worm's scanning space, subnet scan increases the worm's propagation speed comparing with uniform scan. The optimal local preference scan probability p increases when the local scan is on larger subnetworks.

3.4. *Sequential scan worm*

This scanning strategy assumes that once a vulnerable host is infected, it selects a starting address (x) from where it scans IP addresses sequentially (x+1, x+2,...).

A worm can select the start IP address (x) randomly, or close to its own address with higher probability. For example, for its starting point, the Blaster (08/2003) used the first address of the host's Class C subnetwork with probability $p = 0.4$, and chose a random IP address with a probability 0.6.

If vulnerable hosts are uniformly distributed in a worm's scanning space, a random sequential scan worm has the same propagation speed as a uniform scan worm and can be modeled by the SI epidemic model (1).

If a worm uses a local preference in selecting the start address, the worm would propagate slower as the child worm copies are more likely to be wasted on repeating their parents' scanning trails.

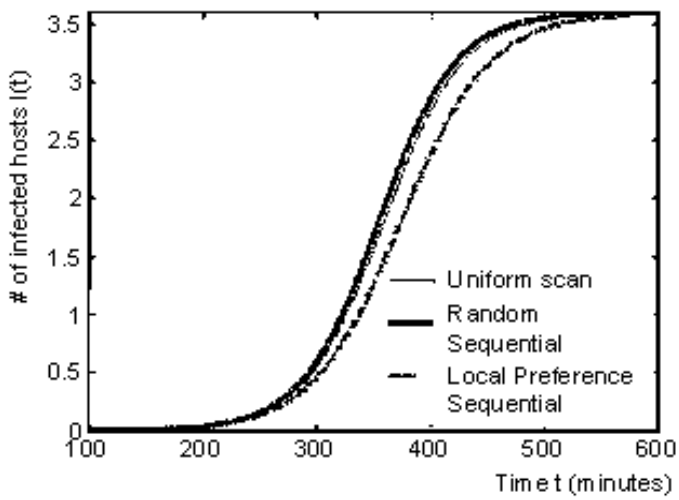

Figure 3. Comparison of a random sequential scan worm, a sequential scan worm with 0.4 local preference, and a uniform scan worm (100 simulation runs; vulnerable hosts uniformly distributed in entire IPv4 space).

A recent example of a sequential scan worm is Win32.Doomjuice.a (01/2004). It used a random sequential scan targeted to 160 Class A networks ($\Omega = \Omega 160 * 2^{24}$) [4].

4. CONCLUSIONS

In terms of the scanning strategies, the summary results of the analysis presented in this paper are:

- a subnet scan increases a worm's propagation speed when vulnerable hosts are not uniformly distributed. The optimal local preference probability increases when the subnet scan is on larger subnetworks;
- when vulnerable hosts are uniformly distributed, the divide-et-impera scan, the sequential scan, and the uniform scan are equivalent in terms of the total number of infected hosts at any time;
- for a sequential scan worm, using local preference in selecting the starting point slows down the worm's propagation speed.

Also, there are some results that should be considered in designing of a worm defense system:

- it is crucial to prevent attackers from identifying the IP addresses of a large number of vulnerable hosts ($I(0)$), or obtaining the address information that helps them to reduce significantly the worm's scanning space (Ω);
- it is necessary to detect the worm as early as possible to be able to avoid a breakout and contain the propagation;
- a worm monitoring system should cover many well distributed IP blocks in order to accurately monitor the propagation of a non-uniform scan worm, especially a sequential scan worm such as Blaster.

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ROMAI J., **1**, 2(2005),
143–148

FULLY FUZZIFIED LINEAR-FRACTIONAL PROGRAMMING PROBLEM - A TRAPEZOIDAL NUMBERS APPROACH

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Abstract In this paper we solve the fully fuzzified linear-fractional program, where all parameters and variables are trapezoidal fuzzy numbers. We change the problem of maximizing a trapezoidal fuzzy number into a deterministic multi-objective linear-fractional programming problem with quadratic constraints. We use the extended principle of Zadeh to agregate fuzzy numbers and the Kerre's method to evaluate a fuzzy constraint.

1. INTRODUCTION

In the paper we formulate a method for solving the fully fuzzified linear-fractional programming problem (FFLFP).

The model of a FFLFP was introduced by Buckley and Feuring [1] by considering all coefficients and variables of a linear program as being fuzzy quantities. Buckley and Feuring [1] transform the fully-fuzzified programming problem in a multi-objective deterministic problem (MODP) which, in the general case treated, is non-linear. Then, the problem is transformed in a multi-objective fuzzy problem by means of which the autors explore the entire set of the Pareto-optimal solutions of the MODP. The solving of the multi-objective fuzzy problem is being made by using a genetic algorithm leading to feasible solutions for the initial problem. In [6] the case of FFLFP with triangular fuzzy numbers is solved building an equivalent multi-objective deterministic problem.

In this paper we are taking the idea of transforming the fully-fuzzified problem in a multi-objective deterministic problem, by considering the coefficients and the variables of the problem as trapezoidal fuzzy numbers aggregated with fuzzy operators obtained through applying the Zadeh's extension principle [5]. For the fuzzy inequalities evaluated by the Kerre's method [2], we are going to find the general form of the equivalent deterministic problem. Generally, a mathematical programming problem with many objective functions and with disjunctive non-linear constraints is obtained. Using the method proposed by Patkar and Stancu-Minasian in [3] the system of disjunctive constraints is re-

placed by an „and" classical system. The non-linearity of some constraints, kept the same after the transformation, will be treated with classic methods.

2. THE MODEL OF THE FFLFP PROBLEM

Consider the fully-fuzzified linear-fractional programming problem

$$\max\left(\overline{Z}=\frac{\sum\limits_{i=1}^{n}\overline{C_iX_i}+\overline{C_0}}{\sum\limits_{i=1}^{n}\overline{D_iX_i}+\overline{D_0}}\right) \tag{1}$$

$$\begin{cases}\overline{M_i}=\sum\limits_{j=1}^{n}\overline{A_{ij}X_j}-\overline{B_i}\leq 0, i=1,...,m\\ \overline{X_i}\geq 0, i=1,...,n\end{cases} \tag{2}$$

where

(i) $(C_i)_{i=1,...,m}$, C_0 şi $(D_i)_{i=1,...,m}$, D_0 represent the coefficients of the linear-fractional objective function,

(ii) $(A_{ij})_{i=1,...,m}^{j=1,...,n}$ şi $(B_i)_{i=1,...,m}$ represent the coefficients and the right-hand side of the linear constraints respectively,

(iii) $(X_i)_{i=1,...,m}$ represent the variables of the problem.

The notation $\overline{Y}$ means that Y represents a fuzzy quantity. The aggregating operators for fuzzy quantities are defined using the Zadeh's extension principle [5] starting with the classic operators of addition and multiplication of real numbers. In addition, the inequalities in constraints 2 are considered as evaluated according to the Kerre's method [2].

As one will see in the next section, if $\overline{C_i}, \overline{C_0}, \overline{D_i}, \overline{D_0}, \overline{B_i}, \overline{X_i}, \overline{A_{ij}}$ are trapezoidal fuzzy numbers for each $i=1,...,n$ and $j=1,...,m$, then $\overline{Z}, \overline{M_i}$ are also trapezoidal fuzzy numbers for each $i=1,...,n$.

In Section 4 we are proposing a solving method for problem 1-2 when all initial fuzzy quantities are described by trapezoidal fuzzy numbers. Before describing this method we define the aggregating way of the trapezoidal fuzzy numbers and we describe Kerre's method of evaluating an inequality of trapezoidal fuzzy numbers (Section 3).

3. THE TRAPEZOIDAL FUZZY NUMBERS

Definition 3.1. *A trapezoidal fuzzy number* $\overline{Y}$ *is a structure* $\left(y^1,y^2,y^3,y^4\right)\in R^4$. *The membership function of the trapezoidal fuzzy number* $\overline{Y}$ *is defined in*

terms of the real numbers y^1, y^2, y^3, y^4 *as follows*

$$\overline{Y}(x) = \begin{cases} 0, & x \in (\infty, y^1) \\ \dfrac{x - y^1}{y^2 - y^1}, & x \in [y^1, y^2] \\ 1, & x \in [y^2, y^3] \\ \dfrac{x - y^4}{y^3 - y^4}, & x \in (y^3, y^4] \\ 0, & x \in (y^4, \infty) \end{cases}$$

Definition 3.2. *(The principle of extension, [5]) Let us consider function*

$$f : X_1 \times X_2 \times ... \times X_n \to Y$$

defined on a Cartesian product of non-fuzzy sets by values in a non-fuzzy set. Considering the fuzzy sets $\overline{A_i} \overline{\subset} X_i, i = 1, 2, ..., n$, *the result of applying function* f *to the fuzzy sets* $\overline{A_i}, i = 1, 2, ..., n$ *is a fuzzy set* B *in* Y *described by the membership function* $\mu_B : Y \to [0, 1]$,

$$\mu_B(y) = \max\{\min\{\mu_{A_i}(x_i) / i = 1, ..., n\} / f(x_1, x_2, ..., x_n) = y\}$$

where $\overline{\subset}$ *means the fuzzy inclusion relation.*

The principle of extension was formulated by Zadeh [5] in order to extend the known models implying fuzzy elements to the case of fuzzy entities.

Applying this principle of extension, the following definitions of the operations with trapezoidal fuzzy numbers follow.

Given two trapezoidal fuzzy numbers $\overline{A} = \left(a^1, a^2, a^3, a^4\right), \overline{B} = \left(b^1, b^2, b^3, b^4\right)$, $a^1, a^2, a^3, a^4, b^1, b^2, b^3, b^4 > 0$, we have:

i) the addition of two trapezoidal fuzzy numbers

$$\overline{A} + \overline{B} = \left(a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4\right),$$

ii) the symmetry of a trapezoidal fuzzy number $-\overline{A} = \left(-a^4, -a^3, -a^2, -a^1\right)$,

iii) the multiplication of two trapezoidal fuzzy numbers

$$\overline{A} \cdot \overline{B} = \left(a^1 b^1, a^2 b^2, a^3 b^3, a^4 b^4\right),$$

iv) the division of two trapezoidal fuzzy numbers

$$\frac{\overline{A}}{\overline{B}} = \left(\frac{a^1}{b^4}, \frac{a^2}{b^3}, \frac{a^3}{b^2}, \frac{a^4}{b^1}\right),$$

v) the inverse of a trapezoidal fuzzy number $\frac{\overline{1}}{\overline{B}} = \left(\frac{1}{b^4}, \frac{1}{b^3}, \frac{1}{b^2}, \frac{1}{b^1}\right)$, where $\overline{1} = (1, 1, 1, 1)$.

The inequality between two fuzzy numbers $\overline{M}, \overline{N}$ having their membership functions $\overline{M}(x)$, and $\overline{N}(x)$ respectively, is defined by Kerre and presented by Buckley and Feuring [1] as

$$\overline{M} \leq \overline{N} \Leftrightarrow d\left(\overline{N}, \overline{\max}\left(\overline{M}, \overline{N}\right)\right) \leq d\left(\overline{M}, \overline{\max}\left(\overline{M}, \overline{N}\right)\right),$$

where d represents the Hamming distance between the expresions of the membership functions and

$$\left(\overline{\max}\left(\overline{M}, \overline{N}\right)\right)(z) = \max\left\{\min\left(\overline{M}(x), \overline{N}(y)\right) \ / \ z = \max(x, y)\right\}$$

according to the Zadeh's principle of extension.

In the following, we apply this manner of defining an inequality between fuzzy numbers to the trapezoidal fuzzy numbers $\overline{M}$ and $\overline{0} = (0, 0, 0, 0)$.

In [6] we proved that

$$\left(\overline{\max}\left(\overline{M}, \overline{0}\right)\right)(z) = \begin{cases} 0, & z < 0 \\ \max\left\{\left(\overline{M}(x)\right) \mid x \leq 0\right\}, & z = 0 \\ \overline{M}(z), & z > 0 \end{cases}.$$

Let us consider now trapezoidal fuzzy numbers. The graphical representation of the non-zero restriction of the membership function of $\overline{M} = (m_1, m_2, m_3, m_4)$ forms together with the Ox axis a trapeze located above Ox. Depending on the four different possible positions of Oy axis relatively to the interior of the trapeze above we deduce four conjunctive systems of distinct condition leading to satisfying the inequality $(m_1, m_2, m_3, m_4) \leq (0, 0, 0, 0)$.

Thus, the inequality between trapezoidal fuzzy numbers $(m_1, m_2, m_3, m_4) \leq (0, 0, 0, 0)$ is equivalent to the following system of disjunctive deterministic constraints

$$\begin{gathered} m_4 \leq 0, \text{ or } (m_3 \leq 0 \leq m_4) \cap \left((m_1 + m_2)(m_3 - m_4) \leq m_3^2 + m_4^2\right), \\ \text{or } (m_2 \leq 0 \leq m_3) \cap (m_3 m_4 \leq m_1 m_2), \\ \text{or } (m_1 \leq 0 \leq m_2) \cap \left((m_2 - m_1)(m_3 + m_4) \leq m_1^2 + m_2^2\right). \end{gathered} \tag{3}$$

4. THE SOLVING METHOD

In this Section we describe an alternative of building an deterministic problem equivalent to problem 1-2.

After aggregating the fuzzy quantities, we interpret the maximization of the objective function described by a trapezoidal fuzzy number as the maximization of the three components of the fuzzy number, as Buckley and Feuring in [1], and we obtain the following multi-objective deterministic linear-fractional program with non-linear disjunctive restrictions

$$\max\left(\frac{\sum_{i=1}^{n} c_i^1 x_i^1 + c_0^1}{\sum_{i=1}^{n} d_i^4 x_i^4 + d_0^4}, \frac{\sum_{i=1}^{n} c_i^2 x_i^2 + c_0^2}{\sum_{i=1}^{n} d_i^3 x_i^3 + d_0^3}, \frac{\sum_{i=1}^{n} c_i^3 x_i^3 + c_0^3}{\sum_{i=1}^{n} d_i^2 x_i^2 + d_0^2}, \frac{\sum_{i=1}^{n} c_i^4 x_i^4 + c_0^4}{\sum_{i=1}^{n} d_i^1 x_i^1 + d_0^1}\right) \tag{4}$$

$$\begin{cases} \left(\sum\limits_{j=1}^{n} a_{ij}^1 x_j^1 - b_i^4, \sum\limits_{j=1}^{n} a_{ij}^2 x_j^2 - b_i^3, \sum\limits_{j=1}^{n} a_{ij}^3 x_j^3 - b_i^2, \sum\limits_{j=1}^{n} a_{ij}^4 x_j^4 - b_i^1\right) \leq \overline{0}, i = \overline{1,\ m} \\ 0 \leq x_i^1 \leq x_i^2 \leq x_i^3 \leq x_i^4, i = 1, ..., n \end{cases} \tag{5}$$

The disjunctivity and the non-linearity of the system of constraints from 5 come from the evaluation of the inequalities of fuzzy numbers as it was described in Section 4.

Applying the Kerre's method to the inequalities 5, problem 1-2 is reduced to the following deterministic MOLFP with disjunctive constraints

$$\max \left(\frac{\sum\limits_{i=1}^{n} c_i^1 x_i^1 + c_0^1}{\sum\limits_{i=1}^{n} d_i^4 x_i^4 + d_0^4}, \frac{\sum\limits_{i=1}^{n} c_i^2 x_i^2 + c_0^2}{\sum\limits_{i=1}^{n} d_i^3 x_i^3 + d_0^3}, \frac{\sum\limits_{i=1}^{n} c_i^3 x_i^3 + c_0^3}{\sum\limits_{i=1}^{n} d_i^2 x_i^2 + d_0^2}, \frac{\sum\limits_{i=1}^{n} c_i^4 x_i^4 + c_0^4}{\sum\limits_{i=1}^{n} d_i^1 x_i^1 + d_0^1} \right), \tag{6}$$

$$R_{i1}^1\left(x_j^4\right) = \textstyle\sum_{j=1}^{n}\left(a_{ij}^4 x_j^4\right) - B_1 \leq 0,$$

or

$$R_{i1}^2\left(x_j^3\right) = \sum_{j=1}^{n}\left(a_{ij}^3 x_j^3\right) - b_4 \leq 0,\ R_{i2}^2\left(x_j^4\right) = -\sum_{j=1}^{n}\left(a_{ij}^4 x_j^4\right) + b_1 \leq 0,$$

$$R_{i3}^2\left(x_j^1, x_j^2, x_j^3, x_j^4\right) = \left(\sum_{j=1}^{n}\left(a_{ij}^1 x_j^1 + a_{ij}^2 x_j^2\right) - b_3 - b_4\right) \cdot \left(\sum_{j=1}^{n}\left(a_{ij}^3 x_j^3 - a_{ij}^4 x_j^4\right) - b_2 + b_1\right) -$$

$$- \left(\sum_{j=1}^{n}\left(a_{ij}^3 x_j^3\right) - b_2\right)^2 - \left(\sum_{j=1}^{n}\left(a_{ij}^4 x_j^4\right) - b_1\right)^2 \leq 0,$$

or

$$R_{i1}^3\left(x_j^2\right) = \sum_{j=1}^{n}\left(a_{ij}^2 x_j^2\right) - b_3 \leq 0,\ R_{i2}^3\left(x_j^3\right) = -\sum_{j=1}^{n}\left(a_{ij}^3 x_j^3\right) + b_2 \leq 0, \tag{7}$$

$$R_{i3}^3\left(x_j^1, x_j^2, x_j^3, x_j^4\right) = \left(\sum_{j=1}^{n}\left(a_{ij}^3 x_j^3\right) - b_2\right)\left(\sum_{j=1}^{n}\left(a_{ij}^4 x_j^4\right) - b_1\right) -$$

$$- \left(\sum_{j=1}^{n}\left(a_{ij}^2 x_j^2\right) - b_3\right)\left(\sum_{j=1}^{n}\left(a_{ij}^1 x_j^1\right) - b_4\right) \leq 0,$$

or

$$R_{i1}^4\left(x_j^1\right) = \sum_{j=1}^{n}\left(a_{ij}^1 x_j^1\right) - b_4 \leq 0,\ R_{i2}^4\left(x_j^2\right) = -\sum_{j=1}^{n}\left(a_{ij}^2 x_j^2\right) + b_3 \leq 0,$$

$$R_{i3}^4\left(x_j^1, x_j^2, x_j^3, x_j^4\right) = \left(\sum_{j=1}^{n}\left(a_{ij}^3 x_j^3 + a_{ij}^4 x_j^4\right) - b_2 - b_1\right) \cdot \left(\sum_{j=1}^{n}\left(a_{ij}^2 x_j^2 - a_{ij}^1 x_j^1\right) - b_3 + b_4\right) -$$

$$- \left(\sum_{j=1}^{n}\left(a_{ij}^1 x_j^1\right) - b_4\right)^2 - \left(\sum_{j=1}^{n}\left(a_{ij}^2 x_j^2\right) - b_3\right)^2 \leq 0,$$

for each $i = 1, ..., m$ and $0 \le x_i^1 \le x_i^2 \le x_i^3 \le x_i^4, i = 1, ..., n$.

According to the method described by Patkar and Stancu-Minasian in [4], we shall consider the variable $\left(\delta_i^j\right)_{i=1,...,m}^{j=1,2,3,4}$ in order to eliminate the disjunctivity and to obtain the system of constraints

$$R_{i1}^1\left(x_j^4\right) \le \left(1-\delta_i^1\right) M_i^1,$$
$$R_{i1}^2\left(x_j^3\right) \le \left(1-\delta_i^2\right) M_{i1}^2,\ R_{i2}^2\left(x_j^4\right) \le \left(1-\delta_i^2\right) M_{i2}^2,\ R_{i3}^2\left(x_j^1, x_j^2, x_j^3, x_j^4\right) \le \left(1-\delta_i^2\right) M_{i3}^2,$$

$$R_{i1}^3\left(x_j^2\right) \le \left(1-\delta_i^3\right) M_{i1}^3,\ R_{i2}^3\left(x_j^3\right) \le \left(1-\delta_i^3\right) M_{i2}^3,\ R_{i3}^3\left(x_j^1, x_j^2, x_j^3, x_j^4\right) \le \left(1-\delta_i^3\right) M_{i3}^3, \tag{8}$$

$$R_{i1}^4\left(x_j^1\right) \le \left(1-\delta_i^4\right) M_{i1}^4,\ R_{i2}^4\left(x_j^2\right) \le \left(1-\delta_i^4\right) M_{i2}^4,\ R_{i3}^4\left(x_j^1, x_j^2, x_j^3, x_j^4\right) \le \left(1-\delta_i^4\right) M_{i3}^4,$$
$$\delta_i^1 + \delta_i^2 + \delta_i^3 + \delta_i^4 \ge 1,\ \delta_i^1, \delta_i^2, \delta_i^3, \delta_i^4 \in \{0, 1\},$$

for each $i = 1, ..., m$ and $0 \le x_i^1 \le x_i^2 \le x_i^3 \le x_i^4, i = 1, ..., n$.

Solving the problem 6-8 will allow us to obtain the solution $\left(x_i^1, x_i^2, x_i^3, x_i^4\right)_{\overline{i=1,\ n}}$, $\left(\delta_i^j\right)_{i=1,...,m}^{j=1,2,3,4}$, namely the trapezoidal fuzzy numbers $(\overline{x_i})_{\overline{i=1,\ n}}$ representing the solution of problem 1-2.

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ROMAI J., **1**, 2(2005),
149–155

INTERPOLATION POLYNOMIALS. APPLICATION IN FINDING SOME COMBINATORIAL FORMULAS

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Abstract A way of solving some counting problems using interpolation polynomials, starting with some values determined by the backtracking method is found. The obtained formulas are implicitly present in the programmes that solve the problems.

1. THE INTERPOLATION WITH FINITE DIFFERENCES

We intend to solve the following

Problem 1.1. *Given m points in the plane, $A_i(x_i, y_i)$, i=1,2,...,m, find the polynomial function associated with a polynomial of degree m-1, the graph of which contains the given points.*

Theorem 1.1. *Let $A_i(x_i, y_i)$, i=1,2,...,m be m points in the plan, such that $x_i \neq x_j$ for every $i \neq j$. Then there exists a unique polynomial P of maximum degree m-1 so that the graph of the polynomial function associated with the polynomial contains the given points.*

The proof of the theorem can be found in [4].

The polynomial P, whose existence and uniqueness follow from the theorem, is called the interpolation polynomial through the points $A_i(x_i,y_i)$, i=1,2,...,m.

From now on $g(X; x_1,\ldots,x_m)$ will denote the interpolation polynomial through the points $A_i(x_i, y_i)$, i=1,2,...,m.

Also, $C(x_1, x_2, \ldots, x_m)$ denotes the coefficient of X^{m-1} from $g(X; x_1, x_2, \ldots, x_m)$. The number $C(x_1, x_2, \ldots, x_m)$ is called the *finite difference* associated with the points $A_1, A_2, \ldots, A_m$. We divide the polynomial $g(X; x_1, x_2, \ldots, x_m)$ by $(X-x_1)(X-x_2)\ldots(X-x_{m-1})$ and using the equality $g(X; x_1, x_2, \ldots, x_m)= C(x_1, x_2, \ldots, x_m).\ (X-x_1)(X-x_2)\ldots(X-x_{m-1})+ R(X)$, where R(X) is a polynomial of maximum degree m-2, we obtain

$R(X)= g(X; x_1, x_2, \ldots, x_m)- C(x_1, x_2, \ldots, x_m).\ (X-x_1)(X-x_2)\ldots(X-x_{m-1})$. So $R(x_i)=y_i$, i=1,2,...,m, thus R(X) is an interpolation polynomial through the points $A_1, A_2, \ldots, A_{m-1}$. In this way we obtained the following formula

$g(X; x_1, x_2, \ldots, x_m) = g(X; x_1, x_2, \ldots, x_{m-1}) + C(x_1, x_2, \ldots, x_m).\ (X-x_1)(X-x_2)\ldots(X-x_{m-1})$,
which will be applied recursively to
$g(X; x_1, x_2, \ldots, x_{m-1}), g(X; x_1, x_2, \ldots, x_{m-2}), \ldots, g(X; x_1)$.

The last interpolation polynomial $g(X;x_1)$ is actually the constant polynomial $g(X;x_1)=g_1$. Thus: $g(X; x_1, x_2, \ldots, x_m)=C(x_1)+C(x_1, x_2)\cdot(X-x_1)+\ldots+C(x_1, x_2, \ldots, x_m).\ (X-x_1)(X-x_2)\ldots(X-x_{m-1})$, which is called the *Newton's formula of interpolation with finite differences.*

Let us compute the finite differences $C(x_1)$, $C(x_1, x_2),\ldots, C(x_1, x_2, \ldots, x_m)$. To this aim let us consider the polynomial

$$f(X) = \frac{X - x_m}{x_1 - x_m} \cdot g(X; x_1, ..., x_{m-1}) + \frac{x_1 - X}{x_1 - x_m} \cdot g(X; x_2, x_3, ..., x_m)$$

It has the following properties

$$1)\ f(x_1) = \frac{x_1 - x_m}{x_1 - x_m} \cdot g(x_1; x_1, ..., x_{m-1}) + \frac{x_1 - x_1}{x_1 - x_m} \cdot g(x_1; x_2, x_3, ..., x_m) = y_1$$

$$2)\ f(x_m) = y_m$$

$f(x_i) = \frac{x_i - x_m}{x_1 - x_m} \cdot g(x_i; x_1, ..., x_{m-1}) + \frac{x_1 - x_i}{x_1 - x_m} \cdot g(x_i; x_2, x_3, ..., x_m) = y_i$,for $i=2,\ldots,m-1$.

Remark that f is an interpolation polynomial through the points (x_1,y_1), $(x_2, y_2), \ldots, (x_m, y_m)$. Due to the fact that the interpolation polynomial is unique, it follows that

$$g(X; x_1, ..., x_m) = \frac{X - x_m}{x_1 - x_m} \cdot g(X; x_1, ..., x_{m-1}) + \frac{x_1 - X}{x_1 - x_m} \cdot g(X; x_2, x_3, ..., x_m).$$

By identifying the coefficients of the highest degree, we obtain

$$C(x_1, ..., x_m) = \frac{C(x_1, ..., x_{m-1}) - C(x_2, ..., x_m)}{x_1 - x_m}.$$

This formula shows that the finite differences can be computed using the following diagram

$C(x_1)=y_1$ $C(x_2)=y_2$ $C(x_3)=y_3$ $\ldots$ $C(x_{m-1})=y_{m-1}$ $C(x_m)=y_m$

$C(x_1,x_2)$ $C(x_2,x_3)$ $\ldots$ $C(x_{m-1},x_m)$

$$\begin{array}{ccc}
\cancel{1\ \ 44\ 2\ \ 4\ 4B} & & \cancel{1\ \ 44\ 2\ \ 4\ 4B} \\
C(x_1,x_2,x_3) & \ldots & C(x_{m-2},x_{m-1},x_m) \\
 & \ldots & \\
 & \cancel{1\ \ 44\ 2\ \ 4\ 4B} & \\
 & C(x_1,\ldots,x_m) &
\end{array}$$

Remark 1.1. *Taking a closer look to the diagram, we notice that every column with finite differences is determined using the column prior to the current one. This fact will be used in the program which determines the interpolation polynomial with finite differences.*

2. SOLVING THE COUNTING PROBLEMS

In this section we solve the following type of counting problems

Given n a positive integer, different from zero, and, eventually, other input data, determine the number of vectors x=($x_1, x_2, \ldots, x_n$), which satisfy some conditions.

Remark 2.1. *If the number wanted is a polynomial formula which depends on n, then the problem can be solved using the following algorithm:*

We determine k values of the number wanted y_1, y_2, ..., y_k, using the backtracking method (for n=1, n=2, ..., n=k).

sw←0; j←1
repeat
We determine the interpolation polynomial P for the points (1, y_1), (2, y_2),..., (j, y_j)
sw←1
for *i←j+1, k* **do**
if *P(i)¡¿y_i* **then**
sw←0
break
endif
endfor
j←j+1
until *(sw=1)***or***(j¿k)*
if *j¡=k* **then**
read *n*
write *P(n)*

else
write *"Is necesary more number y with backtracking method"*
endif

The complexity of the algorithm

-in the first part of the algorithm (y_1, y_2, ..., y_k), the complexity depends on the input data, having an exponential growth because of the backtracking method;

-the second part of the algorithm has a complexity of $O(k^3)$, because the algorithm that finds the interpolation polynomial with finite differences for k points is executed in $O(k^2)$.

The correctness of the algorithm

If we consider that the formula of counting the solutions is of the following type: $f(n)=a_p n^p + a_{p-1} n^{p-1} + \ldots + a_1 n + a_0$, $p \mid k$ and P(n) is an interpolation polynomial which follows after exiting the while cycle, then we obtain

$f(1)=y_1$
$f(2)=y_2$
...
$f(k)=y_k$
$P(1)=y_1$
$P(2)=y_2$
...
$P(k)=y_k$

Form these equalities it follows that the polynomial f-P has as roots the numbers 1, 2, ..., k. From the fact that the polynomials f and P have degrees smaller than k, it follows that the polynomial f-P is null. Thus, f=P. In this case, any value computed by the interpolation polynomial gives the exact solution (if the calculations do not exceed the predefined types; otherwise, operations with big numbers must be implemented).

3. APPLICATION

Given n, a positive integer from the interval [1,1000], how many positive integers of n digits have the product of their digits equal to 8 ?

Exemple. For n=2 we obtain 4 numbers.

Solution. Let x_1, x_2, ..., x_n be the digits number (x_1 is the first digit). From $x_1 x_2 \ldots x_n=8$ and x_1, x_2, ..., x_2 are digits we have three events:

$x_i=x_j=x_k= 2$ and $x_h=1$, $h \notin \{i,j,k\}$;
$x_i= 2$, $x_j=4$, and $x_h=1$, $h \notin \{i,j\}$;
$x_i= 8$, $x_h=1$, $h \neq i$.

We obtain the wanted number : $\binom{n}{3} + A_n^2 + n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ (*).

Having a value for n, we can compute the wanted number using the formula (*).

Another way of solving the problem, without explicitly finding the formula, proceeds as follows. With the backtracking method, we determine the wanted number for the first values of n (for exmple for n=1,. . . ,5) and then, using the interpolation polynomial with finite differences, we solve the problem. The following C++ program determines the value of the interpolation polynomial with finite differences for the given n, and compares the found value with the one from the formula (*).

```
#include <iostream.h>
#include <conio.h>
int sw,j,i,m,cif[4]={1,2,4,8},v[10];
float aux,x0,c[1000],p;
long x[10],y[10],numara,n;
long produs(int k){
long i,p;
p=1;
for(i=1;i<=k;i++)
p*=v[i];
return p;
}
void back(int k){
int i;
long p;
if (k==n+1)
{
if (produs(n)==8)
numara++;
}
else
for(i=0;i<4;i++)
{
v[k]=cif[i];
p=produs(k);
if ((p!=0)&&(p<=8))
back(k+1);
}
}
void puncte(int m){
int i;
for(i=1;i<=m;i++)
{
n=i;
numara=0;
x[i]=i;
back(1);
y[i]=numara;
}
}
float polinom_cu_d_f(float x0, int m){
```

```
int i,k,p,pas;
float c1[100],c2[100],f,temp;
for(i=1;i<=m;i++)
c1[i]=y[i];
p=m;pas=1;
c[1]=y[1];
for(k=2;k<=m;k++)
{
for(i=1;i<p;i++)
c2[i]=(c1[i]-c1[i+1])/(x[i]-x[i+pas]);
c[k]=c2[1];
for(i=1;i<=p;i++)
c1[i]=c2[i];
pas++;
p–;
}
f=0;
temp=1;
for(i=1;i<=m;i++)
{
f+=c[i]*temp;
temp*=(x0-x[i]);
}
return f;
}
void main(){
puncte(5);
for(i=1;i<=5;i++)
cout<<x[i]<<" "<<y[i]<<"\n";
j=1;
do{
sw=1;
for(i=j+1;j<=5;j++){
p=polinom_cu_d_f(i,j);
if (p!=y[i])
{
sw=0;
break;
}
}
j++;
} while((sw==0)&&(j<5));
if (sw)
{
cout<<"n=";
cin>>n;
p=polinom_cu_d_f(n,j-1);
cout<<"With interpolation polynomials with f. d. ";
cout<<p<<"\n";
aux=n*(n+1)*(n+2)/6;
```

```
cout<<"With formula "<<aux<<"\n";
}
getch();
}
```

4. TASKS

1. What is the maximum number of sides, in which the plane can be divided using:

n lines ?

n circles ?

where n is a positive integer smaller than 1000.

2. How many rectangles (different in shape or position), consisting of an integer number of squares, can be drawn on a nxn chess board? (n<300).

3. The same task as 2, if we replace rectangles with squares.

4. Given a nxn chess board (n<10000, n even), find the maximum number of kings that can be placed on the board, without attacking themselves.

5. Given a nxn chess board (n<10000, n multiple of 3), find the maximum number of knights that can be placed on the board, without attacking themselves.

6. Given two positive integers different from zero, n and k, find how many binary sequences (a_1,...,a_n) have exactly $k$monotone sequences.

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ROMAI J., **1**, 2(2005),
157–162

A FREQUENCY ASSIGNMENT PROBLEM

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Abstract A frequency assignment problem is introduced and a binary programming model for solving it is formulated. The model is implemented using AIMMS modelling language. As there are situations where AIMMS fails to solve the integer linear programming problem, a new dynamic model for finding an optimal frequency assignment is developed and also implemented in C++ programming language.

1. PROBLEM FORMULATION

As a result of the large number of mobile communication systems, there is an increasing need to allocate and re-allocate frequencies for point to point communications. Frequency allocations remain operational for seconds/minites in cellular communications, days/weeks in military communication systems or months/years in television systems. During these operational periods, usually the volume of traffic changes significantly, which causes point-to-point capacity and interference problems. Hence, in practice frequency assignment is a recurring process. In this paper a specific frequency allocation problem is modelled as described next.

We consider a satellite communication system in which a station A sends a signal through a satellite to a station B. A *link* in such a communication system is any pair A, B of communicating ground stations. The *frequency domain* is a specific range of frequencies available for allocation. Usually the frequency span is continuous, but for modelling reasons it can be divided up into fixed-width sufficiently small portions, referred to as *channels*. Any specific link requires a pre-specified number of adjacent channels, which is referred to as *frequency interval* or *link width*. The concepts of *channel* and *frequency interval* are represented in fig. 1. *Link interference* represents a combined measure of the transmitter and receiver interference as caused by other existing communication systems.

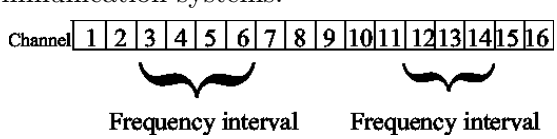

Fig. 1. Channels and intervals in a frequency span.

A *frequency allocation* is the assignment of a frequency interval to at most one link in the communication system. An *optimal frequency allocation* is one in which some measure of total interference is minimized.

For a given link, the overall level of interference is depending on the interference over its entire frequency interval. In our model formulation, we assume that interference data are available on a per channel basis, for each link between two stations. Furthermore, for each interval-link combination, it is assumed that the overall interference of the link is equal to the value of the mean channel interference (one can also consider the value of maximum) that is found in the interval.

To illustrate this, we consider a small example data set consisting of three links with seven adjacent channels available for transmission. The first link requires one channel for transmission, while the other two links must be allocated a frequency interval containing three channels. Table 1 presents the interference level for each link on a per channel basis. Using this data, the overall interference of each interval-link is found by averaging the interferences in the corresponding frequency interval. The values are also presented in Table 1.

	Channel	link	interference	Interval	link	interference
	Link 1	Link 2	Link 3	Link 1	Link 2	Link 3
Channel 1	4	4	3	4	5	3
Channel 2	3	5	4	3	6	4
Channel 3	6	6	2	6	7	3
Channel 4	8	7	6	8	8	3
Channel 5	3	8	1	3	9	2
Channel 6	2	9	2	2	-	-
Channel 7	1	10	3	1	-	-

Table 1: Channel and interval interference data.

2. THE INTEGER PROGRAMMING MODEL AND ITS EXTENSION

In order to formulate our model, the following notations are introduced:
Sets:
Number of links L with index $l \in 1, 2, ..., L$;
Number of channels F with index $f \in 1, 2, ..., F$;
Parameters:

The width of each link $w_l, \forall l \in 1, 2, ..., L$;
The interference function $Loss_l(f), \forall l \in 1, 2, ..., L, \forall f \in 1, 2, ..., F$;
Variable:
The decision variable x_{lf} takes value 1 if the link l starts at channel f and 0 otherwise.

Having the above notations, the frequency assignment problem can be formulated with the aid of the following integer programming model:

$$\text{Minimize:} \quad \sum_l \sum_f x_{lf} \cdot Loss_l(f) \tag{1}$$

$$\text{Subject to:} \quad \sum_l c_{lf} \leq 1 \qquad \forall f \in 1, 2, ..., F \tag{2}$$

$$\sum_f x_{lf} = 1 \qquad \forall l \in 1, 2, ..., L \tag{3}$$

$$c_{lf} = \sum_{i=f-w_l+1}^{f} x_{li} \qquad \forall l \in 1, 2, ..., L, \forall f \in 1, 2, ..., F \tag{4}$$

$$x_{lf} \in \{0, 1\} \qquad \forall l \in 1, 2, ..., L, \forall f \in 1, 2, ..., F \tag{5}$$

$$c_{lf} \in \{0, 1\} \qquad \forall l \in 1, 2, ..., L, \forall f \in 1, 2, ..., F \tag{6}$$

The objective of the model is to minimize the total loss (1), as defined by the loss-function. The first constraint (2) assures that every frequency can be used maximal one time and, thus, there is no overlapping of links. Each link has to start exactly one time and this is stated in the second constraint (3). The third constraint (4) takes care of the fact that every link l has to be placed in a continuous interval of width w_l. In this way all the constraints of frequency allocation are fulfilled.

3. IMPLEMENTATION IN AIMMS

The model is implemented in AIMMS as a mixed integer problem (MIP). AIMMS is an optimization modelling language developed by Paragon Decision Technology B.V., the Netherlands. Reasonable instances of the frequency assignment problem for this model are instances with 20 links, a frequency span of 100 to 200 and a link-width of 5 to 10 channels. Though, in practice, about 1.5 times this amount of channels are needed. Therefore, we are interested in the maximum instance size that can still be solved by this MIP, using AIMMS. In Table 2 the results of some run instances are shown.

From these results we conclude that in general, apart from large instances (frequency-span > 500 and > 20 links), the program has problems in solving instances of the MIP where the links cover almost the whole frequency span.

For instances that cannot be solved by MIP, we use the linear program (LP) to find a feasible solution. Though in many cases even the LP, where the integrality conditions (5) are relaxed (i.e. $0 \leq x_{lf} \leq 1$), cannot be solved. This is possibly due to the fact that the constraint for width-cover still has to be fulfilled.

In case that a solution for the LP problem is found, this solution might be noninteger and, thus, not an optimal solution to the initial problem. The solution is a lower bound and we can use this lower bound to find a feasible integer solution.

From the noninteger solution, we can generate sequences of links. As the decision variables in the LP problem are between 0 and 1 and sum up to 1, they can be interpreted as probabilities.

Example. Suppose we found the following noninteger solution for the LP problem:

	channel →																	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
link ↓ 1		0.5									0.5							
2					1													
3								0.25								0.75		
4													1					

Table 3: A noninteger solution of the LP problem

From Table 3 we see that, with probability 0.5, link 1 is sequenced first and, with probability 0.5, it is sequenced after link 2. Link 3 is sequenced with probability 0.25 before link 4 and, with probability 0.75, after link 4. We find four possible sequences out of this LP solution:

Sequence 1: Link 1, Link 2, Link 3, Link 4;

Sequence 2: Link 1, Link 2, Link 4, Link 3;
Sequence 3: Link 2, Link 3, Link 1, Link 4;
Sequence 4: Link 2, Link 1, Link 4, Link 3.
Given such a sequence of links, we develop a dynamic procedure such that, using the frequency interval width for each link, no link overlapping occurs and the total interference to be minimum. This model is presented in the next section.

4. A DYNAMIC MODEL FOR FINDING AN OPTIMAL FREQUENCY ASSIGNMENT

In this section we give a method for big size instances of frequency assignment problem that can not be solved by AIMMS. The method is implemented and tested in C++ programming language.

For such instances, AIMMS collapses to solve the MIP problem. However, it will be able to solve the LP problem (in most of the cases), so the solution that AIMMS will provide is fractional. From this solution, it can be derived a so called "reasonable" sequence of links; it means that we know that the links should be assigned on the frequency spectrum in a certain order.

Let $\mathcal{F} = \{1, 2, \ldots, F\}$ be the set of frequencies , $\mathcal{L} = \{1, 2, \ldots, N\}$ the set of links, $loss(l, f)$ the loss when link l starts at frequency f and w_l the width of a link.

As input to our method, we consider the sequence of links $\sigma = (L_N, L_{N-1}, \ldots, L_1)$ after a reindexation (for example if we know that the links should be assigned in the order (link9, link5, ..., link7), then $L_N = 9, L_{N-1} = 5, \ldots, L_1 = 7$).

Our problem is to find the optimal frequency assignment for σ, given that $f_{L_N} < f_{L_{N-1}} < \ldots < f_{L_1}$, with f_{L_i} the starting frequency of link L_i , $i = 1, \ldots, N$, so that to have no overlapping and the total loss be minimum.

We develop an inductive procedure that finds an optimal frequency assignment for the sub-sequence $(L_i, \ldots, L_1)$, given an optimal frequency assignment for the sub-sequence $(L_{i-1}, \ldots, L_1)$ for $i = 2, \ldots, N$.

We denote by $C(L_i, f)$ the minimum cost of a frequency assignment for $(L_i, \ldots, L_1)$, knowing that the first link L_i starts at frequency f.

It follows that $C(L_1, f)$ is the minimum cost of assigning the sequence (L_1) to start from frequency f, and is given by the formula

$$C(L_1, f) = \begin{cases} loss(L_1, f), & 1 \leq f \leq F - w_{L_1} + 1 \\ \infty \quad , & \text{otherwise} \end{cases}$$

In case that $i = 2, \ldots, N$, because link L_i starts from frequency f, it means that we just have to assign sub-sequence $(L_{i-1}, \ldots, L_1)$ in an optimal way. But this sub-sequence can start from any of the frequency $g \in \{f + w_{L_i}, \ldots, F\}$.

It follows that the minimum cost is defined by the recursive formula

$$C(L_i, f) = loss(L_i, f) + \min_{g \in \{f + w_{L_i}, \ldots, F\}} C(L_{i-1}, g)$$

for $i = 2, \ldots, N$ and $1 \leq f \leq F$.

For the initial sequence $\sigma = (L_N, L_{N-1}, \ldots, L_1)$ the minimum cost is given by

$$min\{C(L_N, 1), C(L_N, 2), \ldots, C(L_N, F)\}.$$

It means that we calculate the cost when the first link L_N starts from all possible frequencies $1, 2, \ldots, F$, and then we take the minimum.

The above presented method gives the optimal frequency assignment for an initial sequence of links σ.

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ROMAI J., **1**, 2(2005),
163–168

A MATHEMATICAL MODEL IN GYMNASTICS

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Abstract A mathematical model for a gymnast is presented. Acted by forces and moments both external and internal, the sportsman tries to perform some elements of gymnastics. There are three kinds of generalized forces: gravitational forces, forces induced by the man to realize the exercise and biologic forces performed by the human body to avoid some non-usual positions.

1. INTRODUCTION

Consider a gymnast modelled by 3 bars: OA, AB and BC (their significations are: hands, body and feet) (fig. 1).

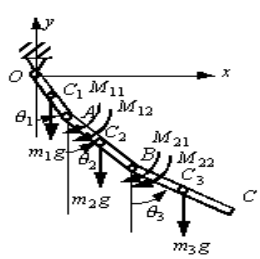

Fig. 1. The model for a gymnast.

The bar AB has the mass m_1, the weight center C_1 is situated at the distance $OA = r_1$, the length $OA = l_1$ and the inertial moment relative to the point C_1 is J_1 . Analogously, for the bars AB and BC the following parameters are known: m_2, m_3, J_2, J_3, l_2, l_3, r_2 and r_3.

The moments M_{11} and M_{21} are due to the gymnast's wish to perform some elements. They read

$$M_{i1}(t) = \begin{cases} m_{i1}^{1} \text{ for } t\epsilon I_{i1}^{1}, \\ \ldots\ldots\ldots\ldots \\ m_{i1}^{j} \text{ for } t\epsilon I_{i1}^{j}, \\ \ldots\ldots\ldots\ldots \\ m_{i1}^{ki} \text{ for } t\epsilon I_{i1}^{k_1}, \\ 0 \text{ otherwise}, \end{cases} \tag{1}$$

where I_{i1}^{j} are real disjoint intervals, $I_{i1}^{j} \bigcap I_{i1}^{l} = \emptyset$, $j \neq l$, the number of these intervals being k_i. In (1) t stands for time, and m_{i1} are constant moments, $i = 1, 2$.

In fig. 2 we have drawn an example for the moments $M_{i1}(t)$.

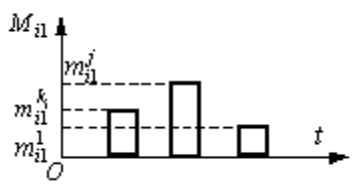

Fig. 2. A variation for $M_{i1} = M_{i1}(t)$.

The moments M_{i2} have as cause the human body's reaction to protect itself against non-normal position. Assume that these moments have the form

$$M_{i2} = C_{i1}(\theta_{i+1} - \theta_i) f_{1i}(\theta_{i+1}, \theta_i) + C_{i2}\left(\dot{\theta}_{i+1} - \dot{\theta}_i\right) f_{2i}\left(\dot{\theta}_{i+1}, \dot{\theta}_i\right), \tag{2}$$

with C_{i1}, C_{i2} constants, $i = 1, 2$, and the functions f_{1i} and f_{2i} read

$$f_{1i} = \begin{cases} 1 \text{ if } \alpha_i < \theta_{i+1} - \theta_i < \alpha_i' \\ 0 \text{ otherwise} \end{cases}, \tag{3}$$

$$f_{2i} = \begin{cases} 1 \text{ if } \beta_i < \dot{\theta}_{i+1} - \dot{\theta}_i < \beta_i' \\ 0 \text{ otherwise} \end{cases}, \tag{4}$$

where α_i, α_i', β_i, β_i', $i = 1, 2$ are physiological constants.

A simplified variant of these conditions is

$$\begin{cases} \text{- if } \theta_{i+1} > \theta_i \text{ and } \dot{\theta}_{i+1} > \dot{\theta}_i \text{ then } \theta_{i+1} = \theta_i \text{ and } \dot{\theta}_{i+1} = \dot{\theta}_i, \\ \text{- if } \theta_{i+1} > \theta_i \text{ then } \theta_{i+1} = \theta_i, \\ \text{- if } \theta_{i+1} < \theta_i - \pi \text{ and } \dot{\theta}_{i+1} < \dot{\theta}_i \text{ then } \theta_{i+1} = \theta_i - \pi \text{ and } \dot{\theta}_{i+1} = \dot{\theta}_i, \\ \text{- if } \theta_{i+1} < \theta_i - \pi \text{ then } \theta_{i+1} = \theta_i - \pi. \end{cases} \tag{5}$$

2. EQUATIONS OF MOTION

These equations are deduced with the aid of the Lagrange equation. In Euler-Lagrange formalism, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_c}{\partial \dot{\theta}_i}\right) - \frac{\partial E_c}{\partial \theta_i} = F_i, \tag{6}$$

where F_i are the generalized forces given by the following expressions

$$F_1 = -\left(m_1 r_1 + m_2 l_1 + m_3 l_1\right) g \sin\theta_1;\ F_2 = -\left(m_2 r_2 + m_3 l_2\right) g \sin\theta_2 - M_{11} - M_{12};$$

$$F_3 = -m_3 g r_3 \sin\theta_3 - M_{21} - M_{22},$$

and the kinetic energy E_c reads

$$E_c = \frac{1}{2}\left(J_1\dot{\theta}_1^2 + m_1 v_{C_1}^2 + J_2\dot{\theta}_2^2 + m_2 v_{C_2}^2 + J_3\dot{\theta}_3^2 + m_3 v_{C_3}^2\right),$$

where

$$v_{C_1}^2 = r_1^2\dot{\theta}_1^2;\ v_{C_2}^2 = l_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + 2 l_1 r_2 \dot{\theta}_1\dot{\theta}_2 \cos\left(\theta_2 - \theta_1\right);$$

$$v_{C_3}^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + r_3^2\dot{\theta}_3^2 + 2 l_1 l 2 \dot{\theta}_1\dot{\theta}_2 \cos\left(\theta_2 - \theta_1\right) +$$
$$+ 2 l_1 r_3 \dot{\theta}_1\dot{\theta}_3 \cos\left(\theta_3 - \theta_1\right) + 2 l_2 r_3 \dot{\theta}_2\dot{\theta}_3 \cos\left(\theta_3 - \theta_2\right).$$

Introducing the notation

$$a_{11} = J_1 + m_1 r_1^2 + \left(m_2 + m_3\right) l_1^2;\ a_{12} = a_{21} = \left(m_2 r_2 + m_3 l_2\right) l_1 \cos\left(\theta_2 - \theta_1\right);$$

$$a_{13} = a_{31} = m_3 l_1 r_3 \cos\left(\theta_3 - \theta_1\right);\ a_{22} = J_2 + m_2 r_2^2 + m_3 l_2^2;$$

$$a_{23} = a_{32} = m_3 l_2 r_3 \cos\left(\theta_3 - \theta_2\right);\ a_{33} = J_3 + m_3 r_3^2;$$

$$b_{11} = b_{22} = b_{33} = 0;\ b_{12} = -b_{21} = -\left(m_2 r_2 + m_3 l_2\right) l_1 \sin\left(\theta_2 - \theta_1\right);$$

$$b_{13} = -b_{31} = -m_3 l_1 r_3 \sin\left(\theta_3 - \theta_1\right);\ b_{23} = -b_{32} = -m_3 l_2 r_3 \sin\left(\theta_3 - \theta_2\right);$$

$$k_1 = -\left(m_1 r_1 + m_2 l_1 + m_3 l_1\right) g \sin\theta_1;\ k_2 = -\left(m_2 r_2 + m_3 l_2\right) g \sin\theta_2 - M_{11} - M_{12};$$

$$k_3 = -m_3 g r_3 \sin\theta_3 - M_{21} - M_{22};$$

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};\quad [\mathbf{B}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix};\quad [\mathbf{K}] = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix};$$

$$\left[\ddot{\theta}\right] = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix};\quad \left[\dot{\theta}\right] = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

the equations of motion read

$$[\mathbf{A}]\left[\ddot{\theta}\right]+[\mathbf{B}]\left[\dot{\theta}\right]=[\mathbf{K}]\,. \tag{7}$$

This equation can be seen as a linear system of three equations with three unknowns: $\ddot{\theta}_1$, $\ddot{\theta}_2$ and $\ddot{\theta}_3$. Its solution is

$$\left[\ddot{\theta}\right]=-\,[\mathbf{A}]^{-1}\,[\mathbf{B}]\left[\dot{\theta}\right]+[\mathbf{A}]^{-1}\,[\mathbf{K}]\,. \tag{8}$$

3. NUMERICAL SIMULATION

Equivalently, relation (8) can be written as

$$\begin{cases} \ddot{\theta}_1=H_1\left(\theta_1,\theta_2,\theta_3,\dot{\theta}_1,\dot{\theta}_2,\dot{\theta}_3\right), \\ \ddot{\theta}_2=H_2\left(\theta_1,\theta_2,\theta_3,\dot{\theta}_1,\dot{\theta}_2,\dot{\theta}_3\right), \\ \ddot{\theta}_3=H_1\left(\theta_1,\theta_2,\theta_3,\dot{\theta}_1,\dot{\theta}_2,\dot{\theta}_3\right), \end{cases} \tag{9}$$

where the notation is evident.

Denoting

$$\begin{cases} \xi_1=\theta_1 \quad \xi_2=\theta_2 \quad \xi_3=\theta_3, \\ \xi_4=\dot{\theta}_1 \quad \xi_5=\dot{\theta}_2 \quad \xi_6=\dot{\theta}_3, \end{cases} \tag{10}$$

from (9) one obtains

$$\begin{cases} \dfrac{\mathrm{d}\xi_1}{\mathrm{d}t}=\xi_4, \quad \dfrac{\mathrm{d}\xi_4}{\mathrm{d}t}=H_1\left(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\right), \\ \dfrac{\mathrm{d}\xi_2}{\mathrm{d}t}=\xi_5, \quad \dfrac{\mathrm{d}\xi_5}{\mathrm{d}t}=H_2\left(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\right), \\ \dfrac{\mathrm{d}\xi_3}{\mathrm{d}t}=\xi_6, \quad \dfrac{\mathrm{d}\xi_6}{\mathrm{d}t}=H_3\left(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\right). \end{cases} \tag{11}$$

This system can be integrated by the fourth order Runge-Kutta method.

Simulation was performed for the following parameters:

- $m_1=8$[kg], $m_2=50$[kg], $m_3=16$[kg];
- $l_1=0.5$[m], $l_2=0.5$[m], $l_3=0.7$[m];
- $r_1=0.25$[m], $r_2=0.25$[m], $r_3=0.35$[m];
- $J_1=0.1666$[kgm^2], $J_2=1.0416$[kgm^2], $J_3=0.6533$[kgm^2].

The initial conditions are:

- $\theta_1^0=\xi_1^0=3.124139$[rad], $\theta_2^0=\xi_2^0=3.124139$[rad], $\theta_3^0=\xi_3^0=3.124139$[rad];
- $\dot{\theta}_1^0=\xi_4^0=0[\frac{\mathrm{rad}}{\mathrm{s}}]$, $\dot{\theta}_2^0=\xi_5^0=0[\frac{\mathrm{rad}}{\mathrm{s}}]$, $\dot{\theta}_3^0=\xi_6^0=0[\frac{\mathrm{rad}}{\mathrm{s}}]$.

The constraints are considered to be offered by the relations (6). In these conditions the constraints impose only the avoiding of non-natural positions of the human body. In fact, the human body cannot reach these limit positions

instantaneously and it cannot correct them in such situations. Biologically speaking, in the space of all possible positions there exist zones where the human body is warned that it is closed to a limit position.

The active moments defined by relation (1) are

$$M_{11} = \begin{cases} 0.08\,[\mathrm{Nm}] \ \text{ if } 0.7\,[\mathrm{s}] \leq t \leq 0.78\,[\mathrm{s}]\,, \\ -0.27\,[\mathrm{Nm}] \ \text{ if } 0.89\,[\mathrm{s}] \leq t \leq 1.16\,[\mathrm{s}]\,, \\ 0 \text{ otherwise }, \end{cases} \tag{12}$$

$$M_{21} = \begin{cases} 0.08\,[\mathrm{Nm}] \ \text{ if } 0.14\,[\mathrm{s}] \leq t \leq 0.22\,[\mathrm{s}]\,, \\ -0.27\,[\mathrm{Nm}] \ \text{ if } 0.92\,[\mathrm{s}] \leq t \leq 1.19\,[\mathrm{s}]\,, \\ 0 \text{ otherwise }. \end{cases} \tag{13}$$

For the active moments these expressions are approximations of the real situation. The experimental data are too few and they depend on a lot of factors. For a single gymnast, every experiment can lead to another set of data. Taking into account all these, we must work only with averages of the experimental data.

In figs. 3, 4, 5 we represented the variation of the parameters of motion versus time.

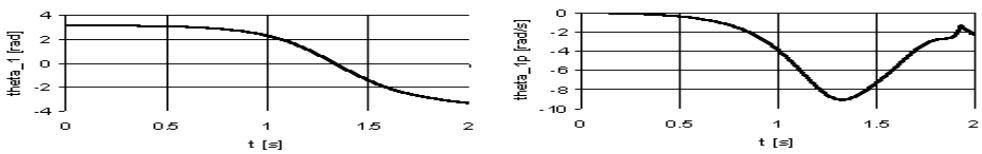

Fig. 3. Variations of $\theta_1 = \theta_1\,(t)$ and $\dot{\theta}_1 = \dot{\theta}_1\,(t)$ for $0 \leq t \leq 2\,[\mathrm{s}]$.

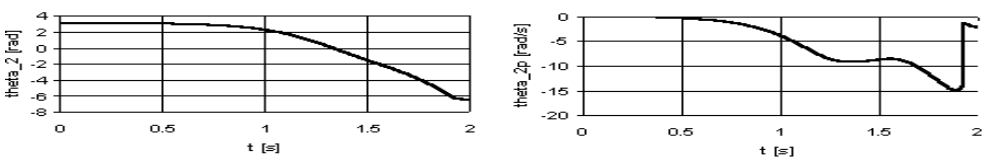

Fig. 4. Variations of $\theta_2 = \theta_2\,(t)$ and $\dot{\theta}_2 = \dot{\theta}_2\,(t)$ for $0 \leq t \leq 2\,[\mathrm{s}]$.

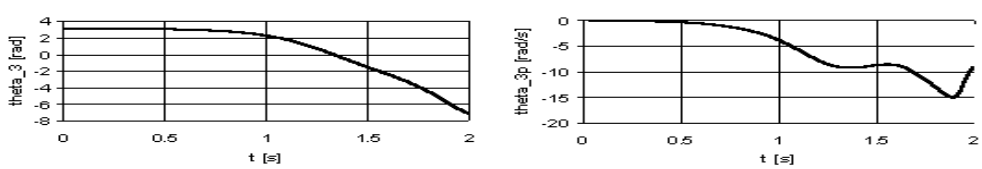

Fig. 5. Variations of $\theta_3 = \theta_3\,(t)$ and $\dot{\theta}_3 = \dot{\theta}_3\,(t)$ for $0 \leq t \leq 2\,[\mathrm{s}]$.

4. CONCLUSIONS

We presented a simplified mathematical model for a gymnast. The corresponding equations are too complex even in this situation. A more detailed model, taking into account more than three elements for the gymnast, can be made. Validation of the model follows the comparison of the results of the simulation with the practical results.

Last years the research in this field has just begun and there are not many available data. For this reason, a more complex model is not necessary at this stage.

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THE EQUATIONS OF THE IDEAL LATCHES

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Abstract The latches are simple circuits with feedback from the digital electrical engineering. We have included in our work the C element of Muller, the RS latch, the clocked RS latch, the D latch and also circuits containing two interconnected latches: the edge triggered RS flip-flop, the D flip-flop, the JK flip-flop, the T flip-flop. Our purpose is to model by equations the previous circuits, considered to be ideal, i.e. non-inertial. The technique of analysis is the pseudoboolean differential calculus.

Keywords: latch, flip-flop, pseudo-boolean equations.

1. LATCHES, THE GENERAL EQUATION

$\mathbf{B} = \{0, 1\}$ is the Boole algebra with two elements. The (normal) signals are, by definition, the functions $x : \mathbf{R} \longrightarrow \mathbf{B}$ of the form

$$x(t) = x(\tau_0 - 0) \cdot \varphi_{(-\infty,\tau_0)}(t) \oplus x(\tau_0) \cdot \varphi_{[\tau_0,\tau_1)}(t) \oplus x(\tau_1) \cdot \varphi_{[\tau_1,\tau_2)}(t) \oplus ...,$$

where $\mathbf{R}$ is the time set, $\varphi_{()} : \mathbf{R} \to \mathbf{B}$ is the characteristic function and $0 \leq \tau_0 < \tau_1 < \tau_2 < ...$ is an unbounded sequence. The equations of the (ideal) latches consist in the next system

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot u(t),\ x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot v(t),\ u(t) \cdot v(t) = 0, \quad (1)$$

where u, v, x are signals and x is the unknown. The last equation of the system is called the *admissibility condition* (of the inputs). In order to solve the system (1.1) we associate with the functions u, v the next sets U_{2k}, V_{2k+1} and respectively numbers t_k:

$$\begin{array}{rcll} U_0 & = & \{t | \overline{u(t-0)} \cdot u(t) = 1\}, & t_0 = \min U_0 \\ V_1 & = & \{t | \overline{v(t-0)} \cdot v(t) = 1, t > t_0\}, & t_1 = \min V_1 \\ U_2 & = & \{t | \overline{u(t-0)} \cdot u(t) = 1, t > t_1\}, & t_2 = \min U_2 \\ V_3 & = & \{t | \overline{v(t-0)} \cdot v(t) = 1, t > t_2\}, & t_3 = \min V_3 ... \end{array}$$

and the next inclusions, respectively inequalities are true:

$$U_0 \supset U_2 \supset U_4 \supset ...\ V_1 \supset V_3 \supset V_5 \supset ...,\ 0 \leq t_0 < t_1 < t_2 < ...$$

For each of U_{2k} (V_{2k+1}) we have the possibilities:

- it is empty. Then t_{2k} (t_{2k+1}) is undefined and all U_{2k}, V_{2k+1}, t_k of higher rank are undefined;

- it is non-empty, finite or infinite. t_{2k} (t_{2k+1}) is defined.

If U_{2k} (V_{2k+1}) are defined for all $k \in \mathbf{N}$, then the sequence (t_k) is unbounded.

A similar discussion is related with the sets V'_{2k}, U'_{2k+1} and respectively numbers t'_k :

$$\begin{array}{rcll} V'_0 & = & \{t|\overline{v(t-0)} \cdot v(t) = 1\}, & t'_0 = \min V'_0 \\ U'_1 & = & \{t|\overline{u(t-0)} \cdot u(t) = 1, t > t'_0\}, & t'_1 = \min U'_1 \\ V'_2 & = & \{t|\overline{v(t-0)} \cdot v(t) = 1, t > t'_1\}, & t'_2 = \min V'_2 \\ U'_3 & = & \{t|\overline{u(t-0)} \cdot u(t) = 1, t > t'_2\}, & t'_3 = \min U'_3 ... \end{array}$$

For solving the system (1.1) we note that the unbounded sequence $0 \leq t''_0 < t''_1 < t''_2 < ...$ exists with the property that u, v, x are constant in each of the intervals $(-\infty, t''_0), [t''_0, t''_1), [t''_1, t''_2), ...$ where the first two equations of (1.1) take one of the forms

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)},\ x(t-0) \cdot \overline{x(t)} = 0, \tag{2}$$

$$\overline{x(t-0)} \cdot x(t) = 0,\ x(t-0) \cdot \overline{x(t)} = x(t-0), \tag{3}$$

$$\overline{x(t-0)} \cdot x(t) = 0,\ x(t-0) \cdot \overline{x(t)} = 0, \tag{4}$$

as $u(t), v(t)$ are equal to $1, 0; 0, 1; 0, 0$ in those intervals. The solutions are written in Table 1.

	eq (1.2) $u(t) = 1, v(t) = 0$	*eq* (1.3) $u(t) = 0, v(t) = 1$	*eq* (1.4) $u(t) = v(t) = 0$
$t \in (-\infty, t''_0)$	$x(t) = 1$	$x(t) = 0$	$x(t) = 0$ $x(t) = 1$
$t \in [t''_k, t''_{k+1})$	$x(t) = 1$	$x(t) = 0$	$x(t) = x(t''_k - 0)$

Table 1.

Theorem 1.1. *Equation (1.1) is equivalent with the equation*

$$x(t) \cdot u(t) \cdot \overline{v(t)} \cup \overline{x(t)} \cdot \overline{u(t)} \cdot v(t) \cup \tag{5}$$

$$\cup (\overline{x(t-0)} \cdot \overline{x(t)} \cup x(t-0) \cdot x(t)) \cdot \overline{u(t)} \cdot \overline{v(t)} = 1.$$

Proof The proof is elementary and it is omitted. ■

Equation (1.5) contains three exclusive possibilities: $x(t) \cdot u(t) \cdot \overline{v(t)} = 1$, $\overline{x(t)} \cdot \overline{u(t)} \cdot v(t) = 1$, respectively $(\overline{x(t-0)} \cdot \overline{x(t)} \cup x(t-0) \cdot x(t)) \cdot \overline{u(t)} \cdot \overline{v(t)} = 1$ equivalent with (1.2), (1.3), (1.4).

Let us solve the system (1.1).

Case a) $u(0-0)=0, v(0-0)=0$. $x(0-0)=0$ and $x(0-0)=1$ are both possible. In order to make a distinction between the two solutions of (1.1) corresponding to the initial value 0, respectively to the initial value 1 we shall denote them by x and x', respectively.

a.i) $x(0-0)=0$. a.i.1) $U_0=\emptyset$, the solution of (1.1) is $x(t)=0$, a.i.2) $U_0\neq\emptyset$ and $\exists\varepsilon>0, x(t)=\varphi_{[t_0,\infty)}(t)$ for $t<t_0+\varepsilon$. This fact results by solving (1.4) for $t<t_0$ and then (1.2) followed perhaps by a finite sequence of (1.4), (1.2), (1.4),... in some interval $[t_0,t_0+\varepsilon)$. Furthermore a.i.2.1) $V_1=\emptyset$, the solution of (1.1) is $x(t)=\varphi_{[t_0,\infty)}(t)$. a.i.2.2) $V_1\neq\emptyset$ and $\exists\varepsilon>0, x(t)=\varphi_{[t_0,t_1)}(t)$ for $t<t_1+\varepsilon$. In some interval $[t_1,t_1+\varepsilon)$, we solved (1.3) followed perhaps by a finite sequence of (1.4), (1.3), (1.4),... a.i.2.2.1) $U_2=\emptyset$, the solution of (1.1) is $x(t)=\varphi_{[t_0,t_1)}(t)$, a.i.2.2.2) $U_2\neq\emptyset$ and $\exists\varepsilon>0, x(t)=\varphi_{[t_0,t_1)}(t)\oplus\varphi_{[t_2,\infty)}(t)$ for $t<t_2+\varepsilon$. a.i.2.2.2.1) $V_3=\emptyset$, the solution of (1.1) is $x(t)=\varphi_{[t_0,t_1)}(t)\oplus\varphi_{[t_2,\infty)}(t)$, a.i.2.2.2.2) $V_3\neq\emptyset$,... a.ii) $x'(0-0)=1$, a.ii.1) $V_0'=\emptyset$, the solution of (1.1) is $x'(t)=1$, a.ii.2) $V_0'\neq\emptyset$, $\exists\varepsilon>0, x'(t)=\varphi_{(-\infty,t_0')}(t)$ for all $t<t_0'+\varepsilon$, a.ii.2.1) $U_1'=\emptyset$, the solution of (1.1) is $x'(t)=\varphi_{(-\infty,t_0')}(t)$, a.ii.2.2) $U_1'\neq\emptyset$, $\exists\varepsilon>0, x'(t)=\varphi_{(-\infty,t_0')}(t)\oplus\varphi_{[t_1',\infty)}(t)$ for all $t<t_1'+\varepsilon$ ...

In figs. 1 and 2 we have drawn the solutions x, x' corresponding to case a) in the situation when $t_0<t_0'$, respectively when $t_0>t_0'$ (the equality $t_0=t_0'$ is impossible, because it implies $u(t_0)=v(t_0')=1$, contradiction with (1.1)). We note the fact that $x_{|[t_0,\infty)}=x'_{|[t_0,\infty)}$, respectively $x_{|[t_0',\infty)}=x'_{|[t_0',\infty)}$. Thus after the first common value of the (distinct) solutions x, x' they coincide

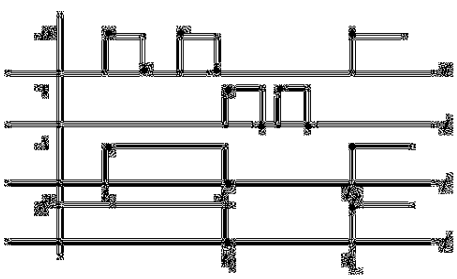

Fig. 1. Case a) $t_0<t_0'$.

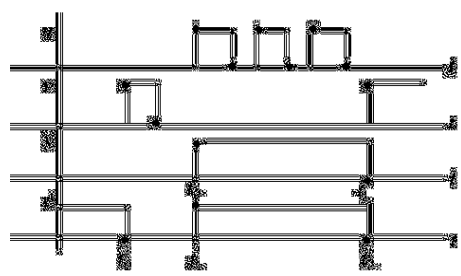

Fig. 2. Case a) $t_0>t_0'$.

Case b) $u(0-0)=1, v(0-0)=0$, the only possibility is $x(0-0)=1$, b.1) $V_0'=\emptyset$, the solution of (1.1) is $x(t)=1$, b.2) $V_0'\neq\emptyset$, $\exists\varepsilon>0, x(t)=\varphi_{(-\infty,t_0')}(t)$ for all $t<t_0'+\varepsilon$...

Case c) $u(0-0)=0, v(0-0)=1$, the only possibility is $x(0-0)=0$, c.1) $U_0=\emptyset$, the solution of (1.1) is $x(t)=0$, c.2) $U_0\neq\emptyset$, $\exists\varepsilon>0, x(t)=\varphi_{[t_0,\infty)}(t)$ for $t<t_0+\varepsilon$...

We have proved the next

Theorem 1.2. *If $u(t)=v(t)=0$, the system (1.1) has two solutions $x(t)=0$ and $x(t)=1$. If $u(0-0)=v(0-0)=0$ but $\exists t>0, u(t)\cup v(t)=1$, then (1.1)*

has two distinct solutions corresponding to $x(0-0)=0$ *and* $x(0-0)=1$, *that become equal at the first time instant* $t>0$ *when* $u(t)\cup v(t)=1$. *If* $u(0-0)\cup v(0-0)=1$, *then the solution is unique.*

2. C ELEMENT

We call the equations of the C element of Muller any of the next equivalent statements

$$\begin{cases} \overline{x(t-0)}\cdot x(t)=\overline{x(t-0)}\cdot u(t)\cdot v(t) \\ x(t-0)\cdot\overline{x(t)}=x(t-0)\cdot\overline{u(t)}\cdot\overline{v(t)} \end{cases} \tag{6}$$

and respectively

$$x(t)\cdot u(t)\cdot v(t)\cup\overline{x(t)}\cdot\overline{u(t)}\cdot\overline{v(t)}\cup \tag{7}$$

$$\cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot(\overline{u(t)}\cdot v(t)\cup u(t)\cdot\overline{v(t)})=1, \tag{8}$$

where u,v,x are signals, the first two called inputs and the last – state. Equations (2.1), (2.2) are the equations of a latch (1.1), (1.5) where $u(t)$ is replaced by $u(t)\cdot v(t)$ and $v(t)$ is replaced by $\overline{u(t)}\cdot\overline{v(t)}$. It is to note the satisfaction of the admissibility condition of the inputs. The analysis of (2.2) is obvious: $x(t)$ is 1 if $u(t)=v(t)=1$, $x(t)$ is 0 if $u(t)=v(t)=0$ and $x(t)=x(t-0)$, $x(t)$ keeps its previous value otherwise. The general form of equations (2.1), (2.2) for m inputs $u_1,...,u_m$ is
$\overline{x(t-0)}\cdot x(t)=\overline{x(t-0)}\cdot u_1(t)\cdot...\cdot u_m(t), x(t-0)\cdot\overline{x(t)}=x(t-0)\cdot\overline{u_1(t)}\cdot...\cdot\overline{u_m(t)}x(t)\cdot u_1(t)\cdot...\cdot u_m(t)\cup\overline{x(t)}\cdot\overline{u_1(t)}\cdot...\cdot\overline{u_m(t)}\cup(\overline{x(t-0)}\cdot\overline{x(t)}\cup x(t-0)\cdot x(t))\cdot\overline{u_1(t)\cdot...\cdot u_m(t)}\cdot(u_1(t)\cup...\cup u_m(t))=1$

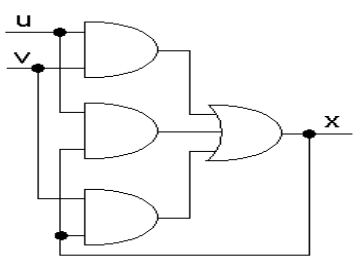

Fig. 3. The C element of Muller.

Fig. 4. The symbol of the C element of Muller.

3. RS LATCH

The equations of the RS latch are given by
$\overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot S(t), Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot R(t), R(t)\cdot S(t)=0$
and equivalently by

$$Q(t)\cdot\overline{R(t)}\cdot S(t)\cup\overline{Q(t)}\cdot R(t)\cdot\overline{S(t)}\cup \tag{9}$$

$$\cup(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot\overline{R(t)}\cdot\overline{S(t)}=1.$$

In (3.1), (3.2) R, S, Q are signals. R, S are called inputs: the reset input and the set input and Q is the state, the unknown relative to which the equations are solved. These equations coincide with (1.1) and (1.5) but the notations are different and traditional. To conclude with, we make the following statements related to equation (3.2). At the RS latch, $Q(t) = 1$ if $R(t) = 0, S(t) = 1$; $Q(t) = 0$ if $R(t) = 1, S(t) = 0$; and $Q(t) = Q(t-0), Q$ keeps its previous value if $R(t) = 0, S(t) = 0$.

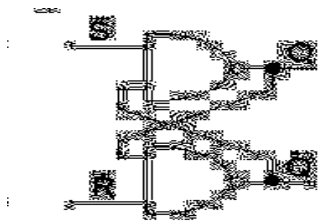

Fig. 5. The RS latch circuit.

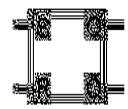

Fig. 6. The symbol of the RS latch.

4. CLOCKED RS LATCH

The equivalent statements
$\overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot S(t) \cdot C(t), Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot R(t) \cdot C(t), R(t) \cdot S(t) \cdot C(t) = 0$ and

$$\begin{gathered} C(t) \cdot (Q(t) \cdot \overline{R(t)} \cdot S(t) \cup \overline{Q(t)} \cdot R(t) \cdot \overline{S(t)} \cup \\ \cup (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) \cdot \overline{R(t)} \cdot \overline{S(t)}) \cup \\ \cup \overline{C(t)} \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) = 1 \end{gathered} \tag{10}$$

are called the *equations of the clocked* RS latch. R, S, C, Q are signals:. the reset, the set and the clock input, the state respectively. The equations (4.1), (4.2) follow from (1.1) and (1.5), where $u(t) = S(t) \cdot C(t)$, $v(t) = R(t) \cdot C(t)$. The clocked RS latch behaves like an RS latch when $C(t) = 1$ and keeps the state constant $Q(t) = Q(t-0)$ when $C(t) = 0$.

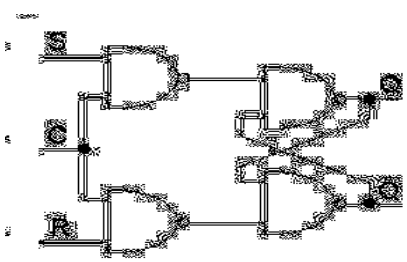

Fig. 7. The clocked RS latch circuit.

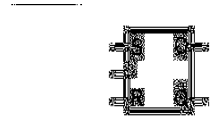

Fig. 8. The symbol of the clocked RS latch.

5. D LATCH

We call the equations of the D latch any of the next equivalent statements

$$\begin{cases} \overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot D(t) \cdot C(t) \\ Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot \overline{D(t)} \cdot C(t) \end{cases} \tag{11}$$

and respectively

$$C(t) \cdot (\overline{Q(t)} \cdot \overline{D(t)} \cup Q(t) \cdot D(t)) \cup \overline{C(t)} \cdot (\overline{Q(t-0)} \cdot \overline{Q(t)} \cup Q(t-0) \cdot Q(t)) = 1 \tag{12}$$

D, C, Q are signals: the data input D, the clock input C and the state Q. On one hand, from (5.1) it is seen the satisfaction of the admissibility condition of the inputs. On the other hand, (5.1), (5.2) follow from the equations of the clocked RS latch (4.1), (4.2) where $R = \overline{S \cdot C}$ and we have used the traditional notation D for the data input, instead of S.

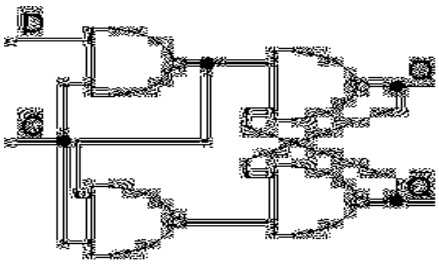

Fig. 9. The D latch circuit.

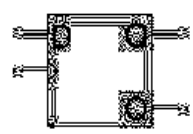

Fig. 10. The symbol of the D latch.

If $C(t) = 1$, the D latch it $Q(t) = D(t)$; if $C(t) = 0$, Q is constant.

6. EDGE TRIGGERED RS FLIP-FLOP

Any of the equivalent statements

$$\begin{cases} \overline{P(t-0)} \cdot P(t) = \overline{P(t-0)} \cdot S(t) \cdot C(t) \\ P(t-0) \cdot \overline{P(t)} = P(t-0) \cdot R(t) \cdot C(t) \\ R(t) \cdot S(t) \cdot C(t) = 1 \\ \overline{Q(t-0)} \cdot Q(t) = \overline{Q(t-0)} \cdot P(t) \cdot \overline{C(t)} \\ Q(t-0) \cdot \overline{Q(t)} = Q(t-0) \cdot \overline{P(t)} \cdot \overline{C(t)} \end{cases} \tag{13}$$

and respectively

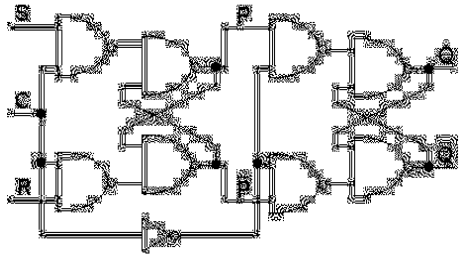

Fig. 11. The edge trigged RS flip-flop circuit.

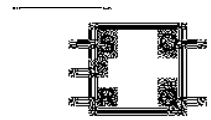

Fig. 12. The symbol of the edge triggered RS flip-flop.

$$C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(P(t)\cdot\overline{R(t)}\cdot S(t)\cup$$
$$\cup\overline{P(t)}\cdot R(t)\cdot\overline{S(t)}\cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot\overline{R(t)}\cdot\overline{S(t)})\cup \quad (14)$$
$$\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1$$

is called the *equation of the edge triggered RS flip-flop.* R, S, C, P, Q are signals: the reset input R, the set input S, the clock input C, the next state P and the state Q. In (6.1), (6.2) the signals R, S, C, P and $P, \overline{C}, Q$ satisfy the equations of a clocked RS latch and of a D latch and (6.2) represents the term by term product of (4.2) by (5.2) written with these variables. The two latches are called *master* and *slave*. The name of edge triggered RS flip-flop refers to the fact that $Q(t)$ is constant at all time instances except $C(t-0)\cdot\overline{C(t)}=1$, when $Q(t)=P(t-0)=\begin{cases}1, if & R(t-0)=0, S(t-0)=1\\ 0, if & R(t-0)=1, S(t-0)=0\end{cases}$, this is the so called *falling edge* of the clock input.

7. D FLIP-FLOP

We call the equations of the D flip-flop any of the next equivalent conditions

$$\begin{cases}\overline{P(t-0)}\cdot P(t)=\overline{P(t-0)}\cdot D(t)\cdot C(t)\\ P(t-0)\cdot\overline{P(t)}=P(t-0)\cdot\overline{D(t)}\cdot C(t)\\ \overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)}\\ Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)}\end{cases} \quad (15)$$

and respectively

$$C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(\overline{P(t)}\cdot\overline{D(t)}\cup P(t)\cdot D(t))\cup \quad (16)$$
$$\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1$$

D, C, P, Q are signals, called: the data input D, the clock input C, the next state P and the state Q. We note that the equations of the D flip-flop represent the special case of edge triggered RS flip-flop when $R=\overline{S\cdot C}$ and S was

denoted by D. The D flip-flop has the state Q constant except for the time instants when $C(t-0)\cdot\overline{C(t)}=1$; then $Q(t)=D(t-0)$.

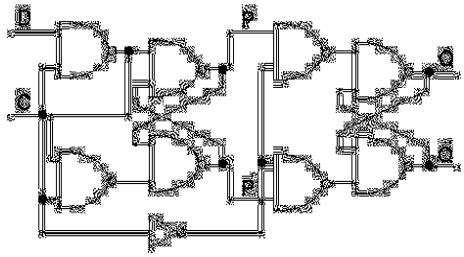

Fig. 13. The D flip-flop circuit.

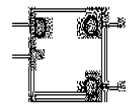

Fig. 14. The symbol of the D flip-flop.

8. JK FLIP-FLOP

The equivalent statements

$$\begin{cases} \overline{P(t-0)}\cdot P(t)=\overline{P(t-0)}\cdot J(t)\cdot\overline{Q(t)}\cdot C(t) \\ P(t-0)\cdot\overline{P(t)}=P(t-0)\cdot K(t)\cdot Q(t)\cdot C(t) \\ \overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)} \\ Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)} \end{cases} \tag{17}$$

and

$$C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(P(t)\cdot J(t)\cdot\overline{Q(t)}\cup\overline{P(t)}\cdot K(t)\cdot Q(t)\cup$$

$$\cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot(\overline{J(t)}\cdot\overline{K(t)}\cup\overline{J(t)}\cdot\overline{Q(t)}\cup\overline{K(t)}\cdot Q(t)))\cup \tag{18}$$

$$\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1$$

are called the *equations of the JK flip-flop.* J,K,C,P,Q are signals: the J input, the K input, the clock input C, the next state P and the state Q. The first two equations of (8.1) (modeling the master latch) coincide with the first two equations of the edge triggered RS flip-flop where $S(t)=J(t)\cdot\overline{Q(t)}$, $R(t)=K(t)\cdot Q(t)$ and the last two equations of (8.1) (modeling the slave latch) coincide with the last two equations of the edge triggered RS flip-flop. We notice that the conditions of admissibility of the inputs of the master and of the slave latch are fulfilled. To be compared (8.2) and (6.2). The JK flip-flop is similar to the edge triggered flip-flop, for example Q changes value only when $C(t-0)\cdot\overline{C(t)}=1$. Let $C(t)=1$; because $Q(t)=Q(t-0)$, i.e. Q is constant, in the union

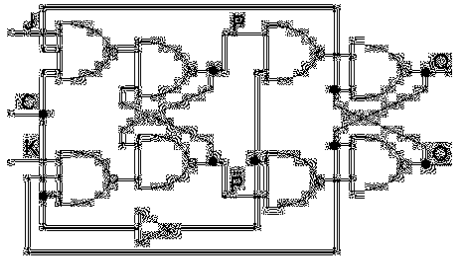

Fig. 15. The JK flip-flop circuit.

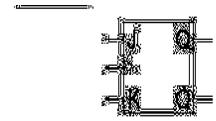

Fig. 16. The symbol of the JK flip-flop.

$$P(t)\cdot J(t)\cdot\overline{Q(t)}\cup\overline{P(t)}\cdot K(t)\cdot Q(t)\cup$$
$$\cup(\overline{P(t-0)}\cdot\overline{P(t)}\cup P(t-0)\cdot P(t))\cdot(\overline{J(t)}\cdot\overline{K(t)}\cup\overline{J(t)}\cdot\overline{Q(t)}\cup\overline{K(t)}\cdot Q(t))$$

only one of $P(t)\cdot J(t)\cdot\overline{Q(t)}$, $\overline{P(t)}\cdot K(t)\cdot Q(t)$ can be 1, thus P changes value at most once and this was not the case at the edge triggered RS flip-flop. Let us make $D(t)=J(t)\cdot\overline{Q(t)}\cup\overline{K(t)}\cdot Q(t)$ in the equations of D flip-flop. We get

$$C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cup Q(t-0)\cdot Q(t))\cdot(P(t)\cdot J(t)\cdot\overline{Q(t)}\cup\overline{P(t)}\cdot K(t)\cdot Q(t)\cup$$
$$\cup\overline{P(t)}\cdot\overline{J(t)}\cdot\overline{Q(t)}\cup P(t)\cdot\overline{K(t)}\cdot Q(t))\cup \tag{19}$$
$$\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1$$

Equations (8.2) and (8.3) have similarities and sometimes the equation of the JK flip-flop is considered to be (8.3).

9. T FLIP-FLOP

The next equivalent statements $\overline{P(t-0)}\cdot P(t)=\overline{P(t-0)}\cdot\overline{Q(t)}\cdot C(t), P(t-0)\cdot\overline{P(t)}=P(t-0)\cdot Q(t)\cdot C(t)$
$\overline{Q(t-0)}\cdot Q(t)=\overline{Q(t-0)}\cdot P(t)\cdot\overline{C(t)}, Q(t-0)\cdot\overline{Q(t)}=Q(t-0)\cdot\overline{P(t)}\cdot\overline{C(t)}$ respectively

$$C(t)\cdot(\overline{Q(t-0)}\cdot\overline{Q(t)}\cdot P(t)\cup Q(t-0)\cdot Q(t)\cdot\overline{P(t)})\cup \tag{20}$$
$$\cup\overline{C(t)}\cdot(\overline{Q(t)}\cdot\overline{P(t-0)}\cdot\overline{P(t)}\cup Q(t)\cdot P(t-0)\cdot P(t))=1$$

are called the *equations of the T flip-flop.* C, P, Q are signals: the clock input, the next state and the state. The conditions of admissibility of the inputs are fulfilled for the first two and for the last two equations from (9.1) (the master and the slave latch). At each falling edge $C(t-0)\cdot\overline{C(t)}=1$ of the clock input, the state Q of the T flip-flop toggles to its complementary value, otherwise it is constant. The equations of the T flip-flop represent the next special cases: in the equations of the edge triggered RS flip-flop, $S(t)=\overline{Q(t)}, R(t)=Q(t)$;

in the equations of the D flip-flop $D(t) = \overline{Q(t)}$; in the equations of the JK flip-flop (any of (9.2), (9.3)) $J(t) = 1, K(t) = 1$.

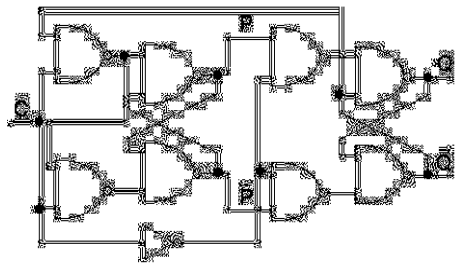

Fig. 17. The T flip-flop circuit.

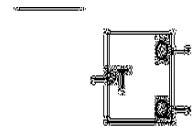

Fig. 18. The symbol of the T flip-flop.

10. CONCLUSIONS

Digital electrical engineering is a non-formalized theory, where the latches are fundamental circuits. In our work we have given the general form of the equations that model the ideal latches, together with the theorem that characterizes the existence and the uniqueness of the solution. Furthermore, we have shown particular forms taken by this system of equations in the case of the most well-known latches and flip-flops. The bibliography dedicated to the latches is rich and descriptive (non-formalized). We quoted references which were a source of inspiration creating some order in our thoughts.

A possibility of continuing the present ideas is that of considering models of inertial latches, for example we can replace (1.1) by $\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d,t)} u(\xi), x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d,t)} v(\xi), \bigcap_{\xi \in [t-d,t)} u(\xi) \cdot \bigcap_{\xi \in [t-d,t)} v(\xi) = 0$, where $d > 0$. We remark that this model replaces u (v) by $\bigcap_{\xi \in [t-d,t)} u(\xi)$ (by $\bigcap_{\xi \in [t-d,t)} v(\xi)$) we meant that the 1 value of u (of v) continues to produce the switch of x from 0 to 1 (from 1 to 0), but this happens only if it is persistent, i.e. if it lasts at least d time units.

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SADDLE-NODE BIFURCATION IN AN EPIDEMIC MODEL

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Abstract A mathematical epidemic model consisting of a Cauchy problem for a system of two first order ordinary differential equations (ode's) is studied. For some particular parameters saddle-node singularities are found. The normal forms of the governing equations are derived using the method of Arrowsmith and Place [1].

Keywords: dynamical system, normal form, saddle-node, epidemic model.

1. MATHEMATICAL MODEL

From the epidemiological point of view, the individuals of a population can be in one of the following states: susceptibles (which were no infected till the considered moment, but can be contaminated in the future), contagious (source of contamination for the neighbor susceptible individuals), immunized (which make the virosis and gained the immunity against the respective virus). In the following sections we investigate the bifurcations of the mathematical epidemic model, described by the Cauchy problem

$$\begin{cases} x_t|_{t=0} = x_0, \\ y_t|_{t=0} = y_0, \end{cases} \tag{1}$$

for the ode's [2]

$$\begin{cases} \dot{x} = -\mu xy + \rho, \\ \dot{y} = \mu xy - \nu y, \end{cases} \tag{2}$$

where $x,\ y$ are the state functions, x are *susceptible individuals*, y are *infected individuals*; $\mu, \rho,\ \nu$ are nonnegative parameters.

2. THE EQUILIBRIUM SET

The equilibrium states are the stationary (in fact, constant) solutions of (1), (2), i.e. they satisfy the system of algebraic equations

$$\begin{cases} -\mu xy + \rho = 0, \\ \mu xy - \nu y = 0, \end{cases} \tag{3}$$

The number of the solutions of (3), i.e. the cardinal of the equilibrium set of (1), (2), depends on the three parameters $\mu,\ \rho$ and ν. More precisely, the following cases occur:

- $\mu = 0,\ \rho,\ \nu \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\rho = 0,\ \mu,\ \nu \neq 0 \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ 0),\ x_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = \mu x_0 - \nu$. We recall that $p_\pm$ are the eigenvalues of the matrix defining the system obtained by linearizing (2) about the equilibrium point;
- $\nu = 0,\ \mu,\ \rho \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\mu = \rho = 0,\ \nu \neq 0\ \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ 0),\ x_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = -\nu$;
- $\mu = \nu = 0,\ \rho \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\rho = \nu = 0,\ \mu \neq 0\ \Rightarrow$ (2) has two equilibrium points $\mathbf{e}_1 = (x_0,\ 0)$ and $\mathbf{e}_2 = (0,\ y_0),\ x_0,\ y_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = \mu x_0$ and $p_+ = 0,\ p_- = y_0$ respectively;
- $\mu = \rho = \nu = 0\ \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ y_0),\ x_0,\ y_0 \in \mathbb{R}$, possessing the eigenvalues $p_\pm = 0$;
- $\mu,\ \rho,\ \nu \neq 0\ \Rightarrow$ (2) has an unique equilibrium $\mathbf{e} = (\nu\mu^{-1},\ \rho\nu^{-1}),\ \mu,\ \rho,\ \nu \in \mathbb{R}$, possessing the eigenvalues $p_\pm = \dfrac{\mu\rho}{2\nu}\left(-1 \pm \sqrt{1 - \dfrac{4\nu^2}{\mu\rho}}\right)$.

Correspondingly, the cardinal of the equilibrium set is $0, \infty, 0, \infty, 0, 2, \infty$ and 1 respectively.

Moreover, for only one or two vanishing parameters the existing equilibria are saddle-nodes, for all vanishing parameters the equilibria are of the double zero type while for no vanishing parameters the unique equilibrium can be either a saddle, or a node, or a focus. Consequently the single nonhyperbolic equilibria are saddle-nodes, and this occurs for: 1) $\rho = 0,\ \mu,\ \nu \neq 0,\ \mu x_0 - \nu \neq 0$; 2) $\mu = \rho = 0,\ \nu \neq 0$; 3) $\rho = \nu = 0,\ \mu \neq 0,\ x_0,\ y_0 \neq 0$; 4)

$\rho = 0, \mu, \nu \neq 0,\ \mu x_0 - \nu = 0$; 5) $\rho = \nu = 0,\ \mu \neq 0,\ x_0 = 0$ and/or $y_0 = 0$ and a double zero equilibrium for 6) $\mu = \rho = \nu = 0$. Moreover, since in the case 2) the system (2), becomes linear and its phase portrait corresponds to the phase trajectories defined by the equations $x = x_0,\ y = y_0 e^{-\nu t}$, it follows that the only interesting cases are 1) and 3). In the following we study them, while the cases 4), 5) and 6) will be treated elsewhere.

3. NORMAL FORM FOR SADDLE-NODE SINGULARITY

As we saw, in the cases $\rho = 0$ and $\rho = \nu = 0$ the singularities have one zero eigenvalue, i.e. they are the saddle-nodes. In the following we deduce the particular normal forms of the governing equations at these singularities.

Theorem 3.1. *For $\rho = 0$ and for given nonvanishing values of μ and ν, namely $\mu = \mu_0,\ \nu = \nu_0,\ \mu_0, \nu_0 \neq 0$ such that $\mu_0 x_0 - \nu_0 \neq 0$, the Cauchy problem (1), (2) has the following normal form at the singularity $\mathbf{e} = (x_0, 0)$*

$$\begin{cases} \dot{s_1} = 0, \\ \dot{s_2} = (\mu_0 x_0 - \nu_0) s_2 + s_1 s_2. \end{cases} \tag{4}$$

In addition, the equilibrium point $\mathbf{e}$ is a degenerated saddle-node.

Proof. Consider two fixed values $\mu = \mu_0,\ \nu = \nu_0$ and let us carry the stationary point $(x_0,\ 0)$ at the origin of coordinates by the change of coordinates $q_1 = x - x_0,\ q_2 = y - 0$. Then (2) becomes

$$\begin{cases} \dot{q_1} = -\mu_0 q_1 q_2 - \mu_0 x_0 q_2, \\ \dot{q_2} = \mu_0 q_1 q_2 + \mu_0 x_0 q_2 - \nu_0 q_2. \end{cases} \tag{5}$$

Applying the transformation $\mathbf{r} \to P^{-1}\mathbf{q}$, with $P = (v_+,\ v_-)$, where v_+ and v_- are eigenvectors corresponding to the eigenvalues p_+ and p_- respectively, ode's (5) become

$$\begin{cases} \dot{r_1} = -\mu_0 \nu_0 r_2 (r_1 + \mu_0 x_0 r_2), \\ \dot{r_2} = \mu_0 r_2 (r_1 + \mu_0 x_0 r_2) + (\mu_0 x_0 - \nu_0) r_2. \end{cases} \tag{6}$$

In order to eliminate the nonresonant terms of order two from (6) we use the transformation (deduced by the normal form method [1])

$$\begin{cases} r_1 = s_1 - \dfrac{\mu_0 \nu_0}{\mu_0 x_0 - \nu_0}\, s_1 s_2 - \dfrac{\mu_0^2 x_0 \nu_0}{2(\mu_0 x_0 - \nu_0)}\, s_2^2, \\ r_2 = s_2 + \dfrac{\mu_0^2 x_0}{\mu_0 x_0 - \nu_0}\, s_2^2, \end{cases} \tag{7}$$

and we obtain the normal form (4). Relations (7) are valid for $\mu_0 x_0 - \nu_0 \neq 0$, i.e. case 2). Comparing (4) with the general normal form at a saddle-node singularity, it follows that in our case the singularity is degenerated. □

Theorem 3.2. *For $\rho = \nu = 0,\ \mu \neq 0$ the Cauchy problem (1), (2) has the following normal form at the singularities $\mathbf{e}_1 = (x_0, 0),\ \mathbf{e}_2 = (0, y_0)$, with $x_0,\ y_0 \neq 0$:*

- *for $\mathbf{e}_1$ it is*
$$\begin{cases} \dot{m_1} = 0, \\ \dot{m_2} = \mu_0 x_0 m_2 + \mu_0 m_1 m_2; \end{cases} \tag{8}$$
- *for $\mathbf{e}_2$ it is*
$$\begin{cases} \dot{n_1} = -(\mu_0 y_0 n_1 + \mu_0 n_1 n_2), \\ \dot{n_2} = 0. \end{cases} \tag{9}$$

Both equilibria are degenerated saddle-nodes.

Proof. **Equilibrium $\mathbf{e_1}$.** The change of coordinates $k_1 = x - x_0,\ k_2 = y - 0,\ \mu = \mu_0$ leads to the ode's

$$\begin{cases} \dot{k_1} = -\mu_0 k_1 k_2 - \mu_0 x_0 k_2, \\ \dot{k_2} = \mu_0 k_1 k_2 + \mu_0 x_0 k_2. \end{cases} \tag{10}$$

Applying the transformation $\mathbf{l} \to P^{-1}\mathbf{k}$, with $P = (v_+,\ v_-)$, where v_+ and v_- are eigenvectors corresponding to the eigenvalues p_+ and p_- respectively, ode's (10) become

$$\begin{cases} \dot{l_1} = 0, \\ \dot{l_2} = \mu_0(k_1 k_2 - k_2^2) + \mu_0 x_0 k_2. \end{cases} \tag{11}$$

Elimination of the nonresonant terms of order two from (11) leads immediately to the normal form (8).

Equilibrium $\mathbf{e_2}$. Similarly, we obtain the normal form (9). □

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ROMAI J., **1**, 2(2005),
183–190

EXPERIMENTAL RESULTS ON SIMULATION MODELS OF HYDRO ENERGETIC EQUIPMENTS DYNAMIC BEHAVIOUR

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Abstract The paper deals with the applied side of the models of simulation of the dynamic behavior of the hydro energetic equipments by generating some autoregressive models (ARX and ARMAX) which, applied to some bearing systems, show that the model approximates with a reasonable error the experimental results, allowing, at the same time, the comparative analysis of the results.

1. INTRODUCTION

Simulations were made based on the values measured experimentally of the sliding bearing and in the case of the bearing with hydraulic sustenation. In both cases we generated ARX and ARMAX models testing the functioning for some sets of parameters. As a simulation program we used the module „System Identification Toolbox" of the programming environment MATLAB of MathWorks. Excitation signals were sinusoids generated by a computer program, taking into account the amplitude and frequency of a real excitation.

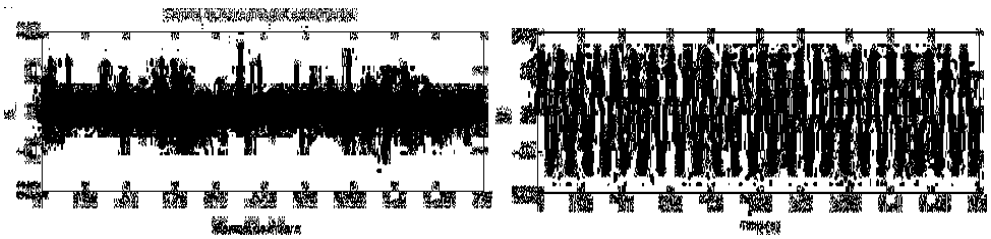

Fig. 1. The input and output signals of the model.

2. RESULTS OF SIMULATION IN THE CASE OF USING SLIDING BEARING

The input and output signals from this case are presented in fig. 1.
Usage of ARX model. Results of simulation and the experimentally measured signal are presented in fig. 2. One may notice that the model ap-

proximates relatively correct the experimental results, except for the random peaks.

Fig. 2. Results of simulation using the ARX model. The blue zone represents the simulation; the black zone represents the experimentally measured signal.

The equation for the model is given by $A(q)y(t) = B(q)u(t) + e(t)$ where: $A(q)$ is the coefficient of the input value; $B(q)$ is the coefficient of the output value; $e(t)$ is the error.

The identification procedure leads to the model given by equations

$$\begin{aligned} A(q) &= 1 - 0,6599q^{-1} + 0,1429q^{-2} + 0,1537q^{-3} - 0,1683q^{-4} \\ &\quad -0,1946q^{-5} - 0,038q^{-6} - 0,009411q^{-7} + 0,1326q^{-8} + 0,04448q^{-9} \\ B(q) &= -4,985 * 10^6 q^{-2} + 1,495 * 10^7 q^{-3} - 1,495 * 10^7 q^{-4} + 4,983 * 10^6 q^{-5} \end{aligned}$$

where the coefficients $A(q)$ and $B(q)$ are automatically determined numerically by the aforementioned computer program. In figs. 3, 4 we present the results of testing the model with an excitation of type step and with an excitation with sinusoid signal of different frequency. The diagram of the response at step signal shows that the system is amortized, coming back to the equilibrium position after an interval of two milliseconds from the excitation.

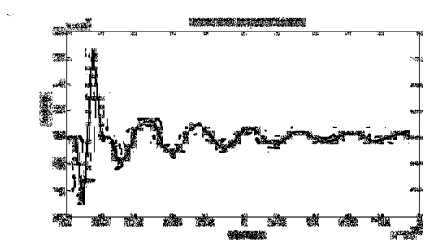

Fig. 3. The response of the model to a step type excitation.

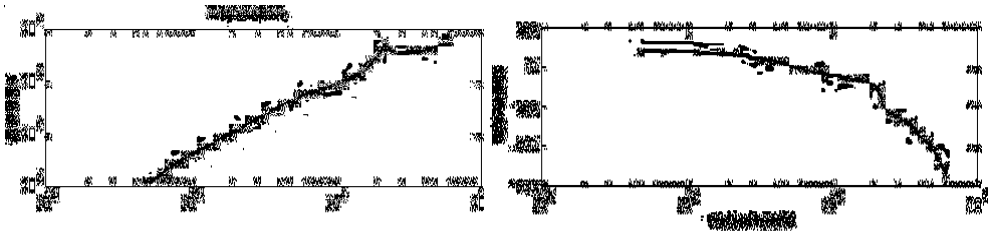

Fig. 4. The response of the model to an excitation with sinusoid signal.

Usage of ARMAX model. The result of the simulation measured experimentally are presented, comparatively in fig. 5. One may notice that the simulated results do not differ very much from the ones obtained with the ARX model. Even if we increase the order (accuracy) of the model, the results do not improve significantly. As a consequence, we recommend using the ARX model for this type of bearing.

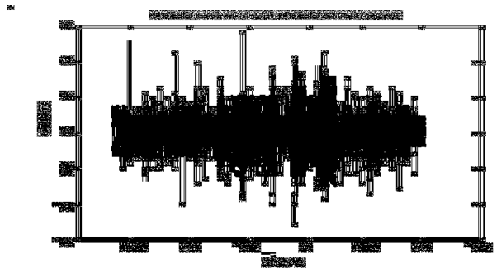

Fig. 5. The result of simulation using the ARMAX model. The yellow zone represents the simulations. The black zone represents the experimentally measured signal.

The equation of the model is given by $A(q)y(t) = B(q)u(t) + C(q)e(t)$.The identification procedure leads to the model given by the relations

$$A(q) = 1 - 1,632q^{-1} + 1,603q^{-2} - 0,664q^{-3},$$
$$B(q) = -0,06334q^{-1} + 0,06356q^{-2}, C(q) = 1 - 0,9315q^{-1} + 0,7581q^{-2},$$

In figs. 6 and 7 we present the results of the testing of the model with step excitation and with sinusoidal signal excitation of different frequencies. The response diagram at step signal shows that the system is less amortized that the one for the amortized ARX model, the tendency to enter in a resonance regime being quite big. We do not observe that with the experimental system.

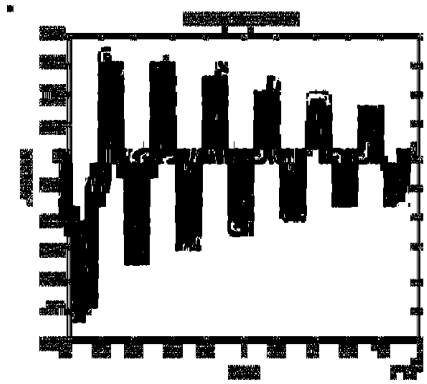

Fig. 6. The response of the model to a step like excitation.

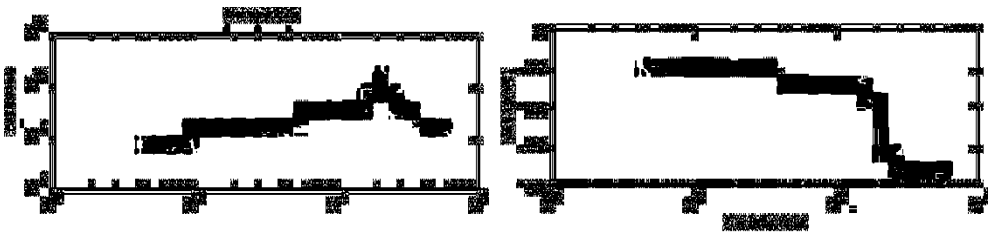

Fig. 7. The response of the model to a sinusoid signal excitation.

3. RESULTS OF SIMULATION IN THE CASE OF USING BEARING WITH HYDRAULIC SUSTENATION

The input and output signals for this case are presented in fig. 8. We notice a functioning smoother than in the case of using sliding bearing. The exaggeratedly large peeks, that denote the presence of shocks in the case of sliding bearing, are not present in this case. Also, we notice an important diminution of the periodic frequency variances that induce in the preceding case the phenomenon of „beating".

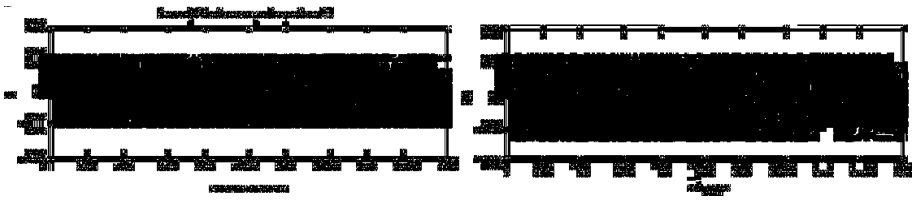

Fig. 8. Input and output signal of the model.

Usage of ARX model. The result of the simulation and the measured signal are presented in fig. 9. One may notice that in this case, the model

approximates correctly the experimental results, the differences between the output value of the model and of the physical system is approximately 10% (amplitude).

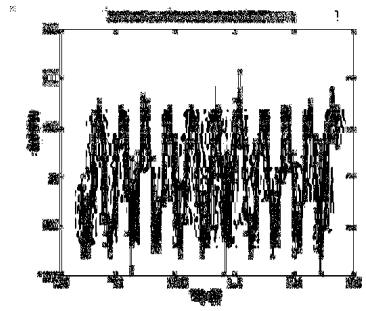

Fig. 9. Results of the simulation using the ARX model.

The equation of the model is given by $A(q)y(t) = B(q)u(t) + e(t)$The identification procedure leads to the model

$$\begin{aligned} A(q) &= 1 - 0,8875q^{-1} + 0,3177q^{-2} + 0,06909q^{-3} - 0,4121q^{-4}, \\ B(q) &= 8,649 * 104q^{-1} - 2,594 * 105q^{-2} + 2,594 * 105q^{-3} - 8,645 * 104q^{-4}. \end{aligned}$$

In figs. 10 and 11 we present the results of the testing of the model with excitation of step type with sinusoid signal of different frequencies. The response diagram at step signal shows that the system is thoroughly amortized, coming back to the balance position after an interval of 1, 2 milliseconds from the excitation.

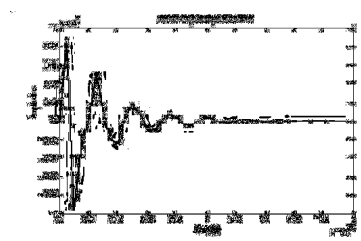

Fig. 10. The response of the model to excitation of type step.

The more powerful amortization from this case is due to the hydro-static sustenation, the oil under pressure taking a significant part of the generated vibration energy in the bearing. Since a great amount of that energy is transformed in heat, it is recommended to cool the oil in the hydraulic circuit.

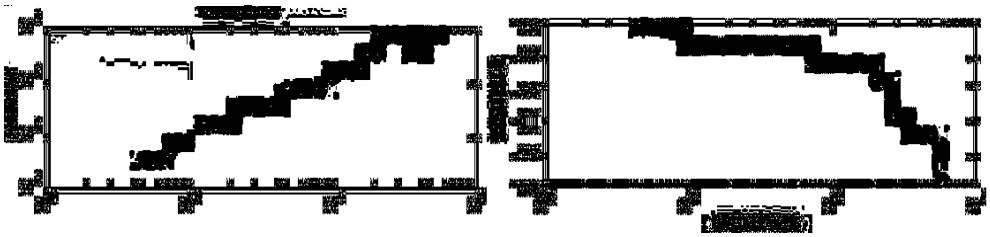

Fig. 11. The response of the model to an excitation with sinusoidal signal.

Usage of ARMAX model. The results of the simulation and the experimentally measured signal are presented in fig. 12. We notice that the ARMAX2221 model used in this case approximates better the physical system, the differences between the output signal of the model and the output signal of the real system being of only 5%.

Fig. 12. The result of the simulation using the ARMAX model.

The equation of the model is given by the relation $A(q)y(t) = B(q)u(t) + C(q)e(t)$.The identification procedure leads to the model

$$A(q) = 1 - 1,254q^{-1} + 0,2719q^{-2},$$
$$B(q) = 0,829q^{-1} + 0,823q^{-2}, C(q) = 1 - 0,3559q^{-1} + 0,3224q^{-2}.$$

In figs. 13, fig. 14 we present the results of testing the model with excitation of type step and with sinusoidal signal of different frequencies. The response diagram at step signal shows that the ARMAX model used has the most important characteristic of amortization, being close to the values of viscous amortization parameters.

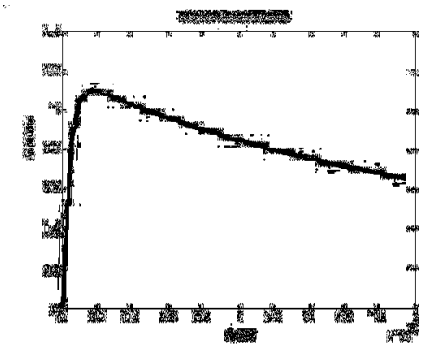

Fig. 13. The response of the model to a step like excitation.

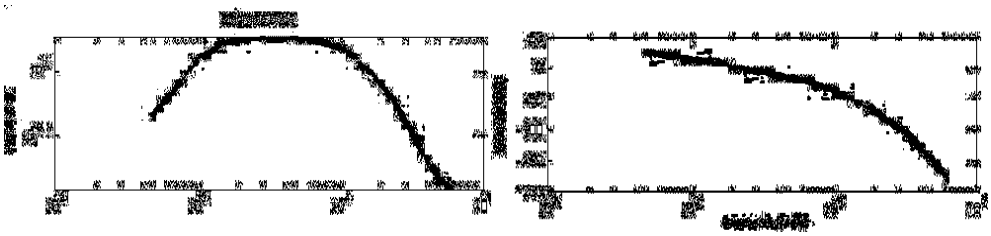

Fig. 14. The response of the model to a sinusoid signal excitation.

4. CONCLUSIONS

The modelling of the function of the experimental system in the case of using sliding bearing and, respectively, of bearing with hydrostatic sustenation, allows, in the case that we obtain a correct model, the prediction of the functioning of the system and in different conditions that the ones tested experimentally. With the aid of the models we can establish the response of the system in limit situation that can lead to major a malfunction. For simulation we must use an adequate software, for example the simulation program „System Identification Toolbox" from the modelling program MATLAB of MathWorks and sinusoidal excitation signals generated by a computer program; in correlation with the amplitude and the frequency of the real excitation. In order to identify the characteristic dynamic processes AHE, we recommend using the Auto Regressive Models (ARX, ARMAX), tested for an adequate number of sets of values for the functioning parameters. The result of the simulation and the experimentally measured signal in the case of using the ARX model, shows that the model approximates with an acceptable error the experimental results, with the exception of random peaks. In the case of the ARMAX model, the results do not differ very much from the results obtained with the ARX model, even if we increase de order (accuracy) of the

model. As a consequence, we recommend using the ARX model, for sliding bearing with film of autoportant lubricant.

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ROMAI J., **1**, 2(2005),
191–200

ARITHMETIZING CONTIUOUS DISTRIBUTIONS: NUMERICAL ASPECTS OF THE ROUNDING METHOD

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Abstract This paper deals with the arithmetization of non-negative univariate distributions, i.e. with their transformation into discrete distributions defined on non-negative integers. The rounding method is used for the arithmetization and some stochastic order relations between the exact and the resulting arithmetized distributions are investigated.

1. INTRODUCTION

An **arithmetic distribution** is a discrete distribution defined on non-negative integers, while an equispaced arithmetic distribution is defined only on multiples of an unit of measurement $h > 0$, called the span [3]. It is sometimes necessary to transform a non-arithmetic distribution into an arithmetic one and such a transformation is called the discretization or arithmetization. For example, in actuarial mathematics, the total claim distribution of a portfolio is frequently evaluated using recursive methods which need an arithmetic form for the claim severity distribution. Since this claim severity distribution is usually absolutely continuous (e.g. Exponential, Lognormal, Pareto etc.), one must first arithmetize it.

There exist several methods for the arithmetization of continuous distributions [2], [3], [6], [7], [8]. In this paper we investigate some practical aspects of the widely used rounding method. The main purpose is to determine an order relation between the exact and the arithmetized distributions. Thus, in Section 2 we recall some well-known stochastic orders (i.e. the stop-loss and the increasing concave orders). Since in insurances the total claim distribution is often obtained by compounding, Section 2 ends with the study of the closure of these order relations under compounding.

Section 3 starts with the presentation of the rounding method. Then we establish order relations between two particular continuous distributions, Exponential and Pareto (which are frequently used to model cost distributions), and their arithmetized forms. As an application of these order relations we consider the premium estimation in insurances, which is often based on com-

pounding. From Section 2 it follows that the order relations established in Section 3 will propagate to the corresponding compound distributions. Therefore, we can deduce an order between the premiums corresponding to the initial distribution and to its arithmetized form. We can conclude if the premium will be underestimated by arithmetization, which of course is not good for the insurance company. Other applications of the order relations can be found in evaluating the ruin probability etc.

2. STOCHASTIC ORDERS

Stochastic orders are binary relations defined on classes of probability distributions. They aim to mathematically translate intuitive ideas like being larger or being more variable for random quantities. They thus extend the classical mean-variance approach to compare riskiness.

2.1. INTEGRAL STOCHASTIC ORDERS

Most stochastic order relations used in the classical expected utility theory and in actuarial sciences are of integral form. Let us recall that an integral stochastic ordering $\preceq_{\mathcal{F}}$ is a stochastic order relation generated by a class $\mathcal{F}$ of measurable functions. To be specific, given two random variables (or two vectors) X and Y, X **is said to precede** Y in the $\preceq_{\mathcal{F}}$ sense if

$$E\phi(X) \leq E\phi(Y), \text{ for all functions } \phi \in \mathcal{F}, \tag{1}$$

provided the expectations involved in (1) exist. Such stochastic orders have been studied e.g. in [1], [4] and [5] in a general setting. Note that $X \preceq_{\mathcal{F}} Y$ means that the distribution functions F_X and F_Y corresponding to X and Y are ordered, and not the particular versions of these random variables. The notation $X \preceq_{\mathcal{F}} Y$ and $F_X \preceq_{\mathcal{F}} F_Y$ will be used interchangeably in the remainder.

If $\mathcal{F}$ is the class of the non-decreasing functions on the real line $\mathbb{R}$, then $\preceq_{\mathcal{F}}$ is the **stochastic dominance** $\preceq_{st}$; the latter expresses the common preference of all decision-makers thinking that more money is better. If one further assumes that $\mathcal{F}$ contains all the non-decreasing and concave functions on $\mathbb{R}$, then $\preceq_{\mathcal{F}}$ is the **increasing concave order** $\preceq_{icv}$ (named **second degree stochastic dominance** in economics); it expresses the common preference of all the profit-seeking decision-makers who are risk averse (i.e. those who always prefer a constant fortune to a random fortune with the same mean, in view of Jensen's inequality).

Contrary to economists, actuaries are mainly concerned with the comparison of risks (and not of random fortunes). Risks are non-negative random variables representing the random amounts of money the insurance company will have to pay out in order to indemnify the policyholder and/or the third

party. This explains the duality existing between the stochastic order relations used in economics (of concave-type) and those used in actuarial sciences (of convex-type).

The dual version of $\preceq_{icv}$ is the **stop-loss order** $\preceq_{sl}$, generated through (1) with $\mathcal{F}$ the class of the non-decreasing and convex functions.

It can be shown that

Theorem 2.1. *The following relations hold:*
(i) if $EX = EY$, then $X \preceq_{sl} Y \Leftrightarrow Y \preceq_{icv} X$, where $\preceq_{sl}$ has to be interpreted as $_1$less variable than $_J$.
(ii) $X_1 \preceq_{icv} X_2 \Leftrightarrow \int_{-\infty}^{x} F_1(u)\,du \geq \int_{-\infty}^{x} F_2(u)\,du$ for all x.
(iii) $X_1 \preceq_{sl} X_2 \Leftrightarrow E(X_1 - d)_+ \leq E(X_2 - d)_+$ for all d, where $(X)_+ = \max(X, 0)$.

2.2. CLOSURE PROPERTIES

Theorem 2.2. *Both $\preceq_{sl}$ and $\preceq_{icv}$ are closed under convolution of non-negative random variables and under compounding.*

Proof . These properties were proved in [6] for the $\preceq_{sl}$ order.

Let us now show that the $\preceq_{icv}$ order is closed under convolution. Suppose that $X_i,\ Y_i, i = 1, 2$ are non-negative independent random variables and $X_i \preceq_{icv} Y_i, i = 1, 2$. We will prove that $X_1 + X_2 \preceq_{icv} Y_1 + Y_2$.

Consider $x > 0$ and the function

$$I_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases} \quad A \subseteq \mathbb{R}\,.$$

Then

$$\int_0^x F_{X_1+X_2}(u)\,du = \int_0^x \int_0^u F_{X_1}(u-y)\,dF_{X_2}(y)\,du =$$

$$= \int_0^x \int_0^x I_{[0,u]}(y)\,F_{X_1}(u-y)\,dF_{X_2}(y)\,du \overset{Fubini}{=}$$

$$= \int_0^x \int_0^x I_{[0,u]}(y)\,F_{X_1}(u-y)\,du\,dF_{X_2}(y) =$$

$$= \int_0^x \int_y^x F_{X_1}(u-y)\,du\,dF_{X_2}(y) \overset{t=u-y}{=} \int_0^x \int_0^{x-y} F_{X_1}(t)\,dt\,dF_{X_2}(y) \geq$$

$$\geq \int_0^x \int_0^{x-y} F_{Y_1}(t)\,dt\,dF_{X_2}(y) \overset{t=u-y}{=} \int_0^x \int_y^x F_{Y_1}(u-y)\,du\,dF_{X_2}(y) =$$

$$= \int_0^x F_{Y_1+X_2}(u)\,du = \int_0^x \int_0^u F_{X_2}(u-y)\,dF_{Y_1}(y)\,du.$$

Using similar arguments, one can show that

$$\int_0^x F_{X_1+X_2}(u)\,du \geq \int_0^x F_{Y_1+Y_2}(u)\,du,$$

and so $X_1 + X_2 \preceq_{icv} Y_1 + Y_2$. This can be extended to n random variables.

The closure under compounding means that if $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are two sequences of non-negative independent and identically distributed random variables, and N is a positive integer-valued random variable, independent of the X_n's and Y_n's, then if $X_i \preceq_{icv} Y_i$ for all i, we have $S_X = \sum_{i=1}^{N} X_i \preceq_{icv} S_Y = \sum_{i=1}^{N} Y_i$. This follows from the relations

$$F_{S_X}(x) = \sum_{n=0}^{\infty} P(N=n)\,F_X^{*n}(x),$$

where $F_X^{*n} = F_{X_1+\ldots+X_n}$, and, for $x > 0$,

$$\begin{aligned}\int_0^x F_{S_X}(u)\,du &= \sum_{n=0}^{\infty} P(N=n) \int_0^x F_X^{*n}(u)\,du \geq \sum_{n=0}^{\infty} P(N=n) \int_0^x F_Y^{*n}(u)\,du = \\ &= \int_0^x F_{S_Y}(u)\,du.\end{aligned}$$

■

3. ORDERING THE EXACT AND THE ARITHMETIZED DISTRIBUTIONS

In this section we establish a stochastic order relation between the exact distribution and the arithmetized distribution obtained by the rounding method. Two particular cases of distributions are considered, namely Exponential and Pareto; they are frequently used in actuarial calculations as claim distributions for small and high costs, respectively.

3.1. ROUNDING METHOD

We first recall the rounding method (mass dispersal). We denote by X the random variable (r.v.) to be arithmetized, by F its distribution function (d.f.) and by f_j the probability placed at jh, $j = 0, 1, ...,$ $h > 0$ being the span. Then we set

$$f_0 = P\left(X < \frac{h}{2}\right) = F\left(\frac{h}{2} - 0\right) \tag{2}$$

$$f_j = P\left(jh - \frac{h}{2} \leq X < jh + \frac{h}{2}\right) = F\left(jh + \frac{h}{2} - 0\right) - F\left(jh - \frac{h}{2} - 0\right), \tag{3}$$

for $j \geq 1$. The notation $F(x - 0)$ indicates that the discrete probability at x should not be included. For continuous distributions this make no difference.

3.2. STOCHASTIC ORDER RELATIONS

Let X be a non-negative absolutely continuous r.v. and $\hat{X}$ its arithmetized corresponding r.v. Denote by F and $\hat{F}$ their d.f.'s respectively.

We now investigate the *Stop-Loss* and *ICV* orders between F and $\hat{F}$. Remark that once an order relation is established, it will propagate to the corresponding compound distributions also, as proved in Theorem 2. In insurances, this is important for the premium calculation, i.e. we can see if the premium for the arithmetized compound distribution will be underestimated (or not).

3.2.1. *Stop-Loss ordering*

We have $E(\hat{X} - d)_+ = \sum\limits_{\substack{k \in \mathbb{N} \\ kh - d > 0}} (kh - d) P(\hat{X} - d = kh - d) =$

$$= \sum_{\substack{k \in \mathbb{N} \\ k > [d/h]}} (kh - d) P(\hat{X} = kh) = \sum_{\substack{k \in \mathbb{N} \\ k > [d/h]}} (kh - d) \left[F\left(kh + \frac{h}{2}\right) - F\left(kh - \frac{h}{2}\right)\right]. \tag{4}$$

Consider the two particular cases.

1. $X \sim Exponential\,(\theta)\,,\ \theta > 0$

In this case, the density of X is $f(x) = \theta e^{-\theta x}, x > 0$. Its distribution function is $F(x) = 1 - e^{-\theta x}$, $x > 0$. Then

$$E(X - d)_+ = \int_d^\infty (x - d)\theta e^{-\theta x} dx = \frac{1}{\theta} e^{-\theta d},$$

and, for the arithmetized distribution, from (4) we have

$$\begin{aligned} E(\hat{X}-d)_+ &= \sum_{k\geq i} khe^{-\theta hk}\left(e^{\theta h/2}-e^{-\theta h/2}\right)-de^{-\theta h(i-1/2)} = \\ &= e^{-\theta hi}\left[\left(\frac{hi}{1-e^{-\theta h}}+\frac{he^{-\theta h}}{(1-e^{-\theta h})^2}\right)\left(e^{\frac{\theta h}{2}}-e^{-\frac{\theta h}{2}}\right)-de^{\frac{\theta h}{2}}\right], \end{aligned}$$

where $i=[d/h]+1$.

Numerical evaluations show that in the Exponential case we have $\hat{X}\preceq_{sl} X$ (Table 1), but the values $E(X-d)_+$ and $E(\hat{X}-d)_+$ are very close (the differences appear after the second decimal). The last column in Table 1 contains the *absolute error*$=\left|E(\hat{X}-d)_+-E(X-d)_+\right|$.

Table 1. Comparison of $E(\hat{X}-d)_+$ and $E(X-d)_+$ when $X\sim Exponential\,(\theta)$ and $h=0.5$

θ	d	$E(\hat{X}-d)_+$	$E(X-d)_+$	*abs. error*
0.1	1	9.04743	9.04837	0.0009424
	2	8.18645	8.18730	0.0008527
	3	7.40741	7.40818	0.0007716
	4	6.70250	6.70320	0.0006981
0.5	1	1.20990	1.21306	0.0031532
	2	0.73384	0.73575	0.0019125
	3	0.44510	0.44626	0.0011600
	4	0.26996	0.27067	0.0007035
1	1	0.36407	0.36787	0.0038043
	2	0.13393	0.13533	0.0013995
	3	0.04927	0.04978	0.0005148
	4	0.01812	0.01831	0.0001894
	5	0.00666	0.00673	0.0000696
	6	0.00245	0.00247	0.0000256
	7	0.00090	0.00091	0.0000094
	8	0.00033	0.00033	0.0000034
	9	0.00012	0.00012	0.0000012
	10	0.00004	0.00004	0.0000004

2. $X\sim Pareto\,(\alpha,\theta)\,,\ \alpha,\theta>0$

The density of X is $f(x) = \dfrac{\alpha\theta^\alpha}{x^{\alpha+1}}, x > \theta$. Its d.f. is $F(x) = 1 - \left(\dfrac{\theta}{x}\right)^\alpha$, $x > \theta$. Then $E(X-d)_+ = \dfrac{\theta^\alpha}{(\alpha-1)d^{\alpha-1}}, \alpha > 1$, and, from (4), we have

$$E(\hat{X}-d)_+ = \frac{\theta^\alpha}{h^{\alpha-1}}\left[\sum_{k\geq i} k\left(\frac{1}{\left(k-\frac{1}{2}\right)^\alpha} - \frac{1}{\left(k+\frac{1}{2}\right)^\alpha}\right) - \right.$$

$$\left. -\frac{d}{\alpha}\sum_{k\geq i}\left(\frac{1}{\left(k-\frac{1}{2}\right)^\alpha} - \frac{1}{\left(k+\frac{1}{2}\right)^\alpha}\right)\right] =$$

$$= \frac{\theta^\alpha}{h^{\alpha-1}}\left[\frac{hi-d}{h}\cdot\frac{2^\alpha}{(2i-1)^\alpha} + \sum_{j=i}^{\infty}\frac{2^\alpha}{(2j+1)^\alpha}\right], \text{where } i = \left[\frac{d}{h}\right] + 1.$$

Numerical evaluations show that in the Pareto case we have $\hat{X} \preceq_{sl} X$ (Table 2), and the difference between $E(X-d)_+$ and $E(\hat{X}-d)_+$ is also very small for a step $h = 0.01$.

Table 2. Comparison of $E(\hat{X}-d)_+$ and $E(X-d)_+$ when $X \sim Pareto\,(\alpha,\theta)$.

h	α	θ	d	$E(\hat{X}-d)_+$	$E(X-d)_+$	*abs. error*
0.1	1.5	0.1	1	0.0623	0.0632	0.0009
			2	0.0436	0.0447	0.0011
0.1	2	0.1	1	0.0100	0.0100	0
			4	0.0025	0.0025	0
			1	0.9992	1.0000	0.0008
0.1	2	1	2	0.4999	0.5000	0.0001
			4	0.2493	0.2500	0.0007
			1	0.9999	1.0000	0.0001
0.01	2	1	2	0.5000	0.5000	0
			4	0.2500	0.2500	0

3.2.2. *ICV*

In this case, for $x > 0$ and $a_{\max} \in \mathbb{N}$ such that $a_{\max}h \leq x < (a_{\max}+1)h$, we have

$$\int_{-\infty}^{x} \widehat{F}(u)du = \sum_{\substack{a\in\mathbb{N}\\ ah\leq x}} \int_{(a-1)h}^{ah} F\left(ah - \frac{h}{2}\right) dx + \int_{a_{\max}h}^{x} F\left(a_{\max}h - \frac{h}{2}\right) dx =$$

$$= \sum_{\substack{a\in\mathbb{N}\\ ah\leq x}} h\, F\left(ah - \frac{h}{2}\right) + (x - a_{\max}h)\, F\left(a_{\max}h + \frac{h}{2}\right). \tag{5}$$

Let us have a look at some particular cases.

1. $X \sim Exponential(\theta)$, $\theta > 0$

Here, for $x > 0$, $\int_{-\infty}^{x} F(u)du = \int_{0}^{x}(1 - e^{-\theta u})du = x + \frac{1}{\theta}\left[e^{-\theta x} - 1\right]$, and, from (5) we have

$$\begin{aligned}\int_{-\infty}^{x} \widehat{F}(u)du &= \sum_{a=1}^{[x/h]} h\left(1 - e^{-\theta(ah-h/2)}\right) + \left(x - h\left[\frac{x}{h}\right]\right)\left[1 - e^{-\theta h\left(\left[\frac{x}{h}\right]+\frac{1}{2}\right)}\right] = \\ &= x - \left(x - h\left[\frac{x}{h}\right]\right) e^{-\theta h\left(\left[\frac{x}{h}\right]+\frac{1}{2}\right)} - h\, e^{-\theta\frac{h}{2}} \cdot \frac{e^{-\theta h\left[\frac{x}{h}\right]} - 1}{e^{-\theta h} - 1}.\end{aligned}$$

Numerical evaluations show that in this case we have $\hat{X} \preceq_{icv} X$, and the values $\int_{-\infty}^{x} F(u)du$ and $\int_{-\infty}^{x} \widehat{F}(u)du$ are very close (the differences occur from the fourth decimal, on see Table 3.a).

Table 3. Comparison of $\int_{-\infty}^{x} \widehat{F}(u)du$ and $\int_{-\infty}^{x} F(u)du$ for

a) $X \sim Exponential\,(\theta)$, $\theta = 0.1$ and $h = 0.1$

x	$\int_{-\infty}^{x} \widehat{F}(u)du$	$\int_{-\infty}^{x} F(u)du$
1	0.048378	0.048374
2	0.187315	0.187307
3	0.408193	0.408182
4	0.703214	0.703200
5	1.065322	1.065306
6	1.488135	1.488116
7	1.965874	1.965853
8	2.493312	2.493289
9	3.065721	3.065696
10	3.678820	3.678794

b) $X \sim Pareto\,(\alpha, \theta)$, $\alpha = 1$, $\theta = 0.1$ and $h = 0.1$

x	$\int_{-\infty}^{x} \widehat{F}(u)du$	$\int_{-\infty}^{x} F(u)du$
1	0.57334	0.669741
2	1.50406	1.600426
3	2.46352	2.559880
4	3.43476	3.531112
5	4.41244	4.508797
6	5.39422	5.490565
7	6.37880	6.475150
8	7.36544	7.461797
9	8.35366	8.450019
10	9.34314	9.439482

2. $X \sim Pareto(\alpha, \theta),\ \alpha, \theta > 0$
For $x > \theta$ we have

$$\begin{aligned}\int_{-\infty}^{x} F(u)du &= \int_{\theta}^{x} \left[1 - \left(\frac{\theta}{u}\right)^{\alpha}\right] du = \\ &= \begin{cases} (x-\theta) + \dfrac{\theta}{\alpha - 1}\left[\left(\dfrac{\theta}{x}\right)^{\alpha-1} - 1\right], & \text{if } \alpha \neq 1 \\ (x-\theta) - \theta \ln \dfrac{x}{\theta}, & \text{if } \alpha = 1 \end{cases}.\end{aligned}$$

From (5), denoting $i = [x/h]$, we also have

$$\int_{-\infty}^{x} \widehat{F}(u)du = \sum_{a=1}^{i} h\left[1 - \left(\frac{\theta}{ah - h/2}\right)^{\alpha}\right] + (x - ih)\left[1 - \left(\frac{\theta}{ih + h/2}\right)^{\alpha}\right] =$$

$$= x - (x - ih)\left(\frac{2\theta}{h}\right)^{\alpha} \frac{1}{(2i+1)^{\alpha}} - \frac{(2\theta)^{\alpha}}{h^{\alpha-1}} \sum_{a=1}^{i} \frac{1}{(2a-1)^{\alpha}}.$$

Numerical evaluations show that in this case we have $X \preceq_{icv} \hat{X}$ (Table 3.b).

3.3. CONCLUSION

$X \preceq_{sl} Y$ or $X \preceq_{icv} Y$ mean that any risk-averse decision maker would prefer to pay X instead of Y, which, in insurances, implies that replacing the total claim X by Y leads to a prudent decision.

But $X \preceq_{sl} Y$ implies $EX \leq EY$ (see (*iii*) from Theorem 1). In a simplified actuarial model, the pure premium corresponding to a risk X is defined by its expected value, EX. Then in both Exponential and Pareto cases, from $\hat{X} \preceq_{sl} X$ we have $E\hat{X} \leq EX$, which will underestimate the pure premium for the arithmetized distribution. Since the two distributions are very closed, the difference between the two means might be insignificant, but one must be very careful when using the rounding method.

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