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ROMAI J., **5**, 1(2009), 1–6

A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN AND RUSCHEWEYH OPERATOR

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Abstract By means of a Sălăgean differential operator and Ruscheweyh derivative we define a new class $\mathcal{BL}(p,m,\mu,\alpha,\lambda)$ involving functions $f \in \mathcal{A}(p,n)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

Keywords: analytic function, starlike function, convex function, Sălăgean operator, Ruscheweyh derivative.

2000 MSC: 30C45.

1. INTRODUCTION AND DEFINITIONS

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$A(p,n) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\}, \tag{1}$$

with $A(1,n) = A_n$ and

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\},$$

where $p, n \in \mathbb{N}$, $a \in \mathbb{C}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}_n^*(p,\alpha)$ we denote a subclass of $A(p,n)$ consisting of p-valently starlike functions of order α, $0 \le \alpha < p$ that satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be p-valently convex of order α in U, if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha, \quad z\in U \tag{3}$$

for some α, $(0\leq\alpha<p)$. We denote by $\mathcal{K}_n(p,\alpha)$, the class of functions in $\mathcal{S}$ which are p-valently convex of order α in U and denote by $\mathcal{R}_n(p,\alpha)$ the class of functions in $A(p,n)$ which satisfy

$$\operatorname{Re} f'(z)>\alpha, \quad z\in U. \tag{4}$$

It is well-known that $\mathcal{K}_n(p,\alpha)\subset\mathcal{S}_n^*(p,\alpha)\subset\mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$, if there is a function w analytic in U, with $w(0)=0$, $|w(z)|<1$, for all $z\in U$ such that $f(z)=g(w(z))$ for all $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$.

Let S^m be the Sălăgean differential operator [7], $S^m: A(p,n)\to A(p,n)$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, defined as

$$\begin{aligned} S^0f(z) &= f(z), \\ S^1f(z) &= Sf(z)=zf'(z), \\ S^mf(z) &= S(S^{m-1}f(z))=z(S^{m-1}f(z))', z\in U. \end{aligned}$$

In [6] Ruscheweyh has defined the operator $R^m: A(p,n)\to A(p,n)$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned} R^0f(z) &= f(z), \\ R^1f(z) &= zf'(z), \\ (m+1)R^{m+1}f(z) &= z\left[R^mf(z)\right]'+mR^mf(z), \quad z\in U. \end{aligned}$$

Let D_λ^m be a generalized Sălăgean and Ruscheweyh operator introduced by A. Alb Lupaş in [1], $D_\lambda^m: A(p,n)\to A(p,n)$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, defined as

$$D_\lambda^mf(z)=(1-\lambda)R^mf(z)+\lambda S^mf(z), \quad z\in U,\ \lambda\geq 0.$$

We note that if $f\in A(p,n)$, then

$$D_\lambda^mf(z)=z^p+\sum_{j=n+p}^{\infty}\left(\lambda j^m+(1-\lambda)C_{m+j-1}^m\right)a_jz^j, \quad z\in U,\lambda\geq 0.$$

For $\lambda = 1$, we get the Sălăgean operator [7] and for $\lambda = 0$ we get the operator [6].

To prove our main theorem we shall need the following lemma.

Lemma 1.1. [5] *Let u be analytic in U, with $u(0) = 1$, and suppose that*

$$\operatorname{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $\operatorname{Re} u(z) > \alpha$ *for* $z \in U$ *and* $1/2 \leq \alpha < 1$.

2. MAIN RESULTS

Definition 2.1. *We say that a function $f \in A(p,n)$ is in the class $\mathcal{BL}(p,m,\mu,\alpha,\lambda)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\lambda \geq 0$, $\alpha \in [0,1)$ if*

$$\left| \frac{D_\lambda^{m+1} f(z)}{z^p} \left(\frac{z^p}{D_\lambda^m f(z)} \right)^\mu - p \right| < p - \alpha, \qquad z \in U. \tag{6}$$

Remark 2.1. *The family $\mathcal{BL}(p,m,\mu,\alpha,\lambda)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BL}(1,0,1,\alpha,1) \equiv \mathcal{S}_n^*(1,\alpha)$, $\mathcal{BL}(1,1,1,\alpha,1) \equiv \mathcal{K}_n(1,\alpha)$ and $\mathcal{BL}(1,0,0,\alpha,1) \equiv \mathcal{R}_n(1,\alpha)$. Another interesting subclasses are the special case $\mathcal{BL}(1,0,2,\alpha,1) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus* [4]*, the class $\mathcal{BL}(1,0,\mu,\alpha,1) \equiv \mathcal{B}(\mu,\alpha)$ introduced by Frasin and Jahangiri* [5]*, the class $\mathcal{BL}(1,m,\mu,\alpha,1) = \mathcal{BS}(m,\mu,\alpha)$ introduced and studied by A. Cătaş and A. Alb Lupaş* [2] *and the class $\mathcal{BL}(1,m,\mu,\alpha,0) = \mathcal{BR}(m,\mu,\alpha)$ introduced and studied by A. Cătaş and A. Alb Lupaş* [1].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BL}(p,m,\mu,\alpha,\lambda)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2.1. *If for the function $f \in A(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\lambda \geq 0$, $1/2 \leq \alpha < 1$ we have*

$$\frac{(m+2)(1-\lambda) R^{m+2} f(z) - (m+1)(1-\lambda) R^{m+1} f(z) + \lambda S^{m+2} f(z)}{(1-\lambda) R^{m+1} f(z) + \lambda S^{m+1} f(z)} - \tag{7}$$

$$\mu \frac{(m+1)(1-\lambda) R^{m+1} f(z) - m(1-\lambda) R^m f(z) + \lambda S^{m+1} f(z)}{(1-\lambda) R^m f(z) + \lambda S^m f(z)} + p\mu - p + 1 \prec 1 + \beta z,$$

$z \in U$, where

$$\beta = \frac{3\alpha - 1}{2\alpha},$$

then $f \in \mathcal{BL}(p, m, \mu, \alpha, \lambda)$.

Proof. If we consider

$$u(z) = \frac{D_\lambda^{m+1} f(z)}{z^p} \left(\frac{z^p}{D_\lambda^m f(z)} \right)^\mu \tag{8}$$

then $u(z)$ is analytic in U with $u(0) = 1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{(m+2)(1-\lambda) R^{m+2} f(z) - (m+1)(1-\lambda) R^{m+1} f(z) + \lambda S^{m+2} f(z)}{(1-\lambda) R^{m+1} f(z) + \lambda S^{m+1} f(z)} - \tag{9}$$
$$\mu \frac{(m+1)(1-\lambda) R^{m+1} f(z) - m(1-\lambda) R^m f(z) + \lambda S^{m+1} f(z)}{(1-\lambda) R^m f(z) + \lambda S^m f(z)} + p\mu - p.$$

Using (7) we get

$$\operatorname{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 16, we deduce that

$$\operatorname{Re}\left\{ \frac{D_\lambda^{m+1} f(z)}{z^p} \left(\frac{z^p}{D_\lambda^m f(z)} \right)^\mu \right\} > \alpha.$$

Therefore, $f \in \mathcal{BL}(p, m, \mu, \alpha, \lambda)$, by Definition 2.1. ∎

As consequences of the above theorem we have the following corollaries.

Corolar 2.1. *If $f \in A_n$ and*

$$\operatorname{Re}\left\{ \frac{9zf''(z) + \frac{7}{2}z^2 f'''(z)}{f'(z) + \frac{1}{2}zf''(z)} - \frac{2zf''(z)}{f'(z)} \right\} > -\frac{5}{2}, \quad z \in U \tag{10}$$

then

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3}{7}, \quad z \in U. \tag{11}$$

That is, f is convex of order $\frac{3}{7}$.

Corolar 2.2. *If $f \in A_n$ and*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U \tag{12}$$

then

$$\operatorname{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{13}$$

In other words, if the function f *is convex of order* $\frac{1}{2}$ *then* $f \in \mathcal{BL}(1,0,0,\frac{1}{2},1) \equiv \mathcal{R}_n\left(1,\frac{1}{2}\right)$.

Corolar 2.3. *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U \tag{14}$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{15}$$

That is, f *is a starlike function of order* $\frac{1}{2}$.

Corolar 2.4. *If* $f \in A_n$ *and*

$$\operatorname{Re}\left\{\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U \tag{16}$$

then $f \in \mathcal{BL}(1,1,1,1/2,1)$ *hence*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{17}$$

That is, f *is convex of order* $\frac{1}{2}$.

References

[1] A. Alb Lupaş, *Some diferential subordinations using Săjăgean and Ruscheweyh operators*, Proceedings of International Conference on Fundamental Sciences, ICFS 2007, Oradea, 58-61.

[2] A. Alb Lupaş, A. Cătaş, *A note on a subclass of analytic functions defined by differential Ruscheweyh derivative*, Journal of Mathematical Inequalities, (to appear).

[3] A. Alb Lupaş, A. Cătaş, *On a subclass of analytic functions defined by a generalized Sălăgean and Ruscheweyh operator*, Journal of Approximation Theory and Applications, **4**, 1-2(2008), 1-5.

[4] A. Alb Lupaş, A. Cătaş, *A note on a subclass of analytic functions defined by a generalized Sălăgean and Ruscheweyh operator*, General Mathematics, (to appear).

[5] A. Cătaş, A.Alb Lupaş, *A note on a subclass of analytic functions defined by differential Sălăgean operator*, submitted.

[6] B. A. Frasin, M. Darus, *On certain analytic univalent functions*, Int. J. Math. and Math. Sci., **25**(5)(2001), 305-310.

[7] B. A. Frasin, J. M. Jahangiri, *A new and comprehensive class of analytic functions*, Analele Universităţii din Oradea, **XV**, 2008.

[8] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[9] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer, Berlin, **1013**(1983), 362-372.

ROMAI J., **5**, 1(2009), 7–20

SEMIREFLEXIVE SUBCATEGORIES

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Abstract In the topological locally convex Hausdorff vector spaces category, the semireflexive subcategories - a categorial notion which generates some well-known cases of semireflexivity, are examined.

Keywords: Lie triple system, homogeneous system, LT-algebra.

2000 MSC: 17Dxx, 18G60.

1. INTRODUCTION

The results of the article are formulated and proved for the category $\mathcal{C}_2\mathcal{V}$ of topological locally convex Hausdorff vector spaces. We denote by $\mathbb{R}$ the lattice of the non-null reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$. Supposing that $\mathbb{R}_m$ is the sublattice of the lattice $\mathbb{R}$ of those $\mathcal{R}$ elements that possess the property: $\mathcal{R}$-replique of the category $\mathcal{C}_2\mathcal{V}$ objects can be realized in two steps - first the topology is weakened, second it is completed somehow. The definded semireflexive spaces have such a property defined in different ways.

In the lattice $\mathbb{R}$ two more complete sublattices are indicated.

$\mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}$, $\mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}$ where $\mathcal{S}$ (respectively Γ_0) is the subcategory of the weak topology spaces (respectively-complete).

In Section 2 some properties of the three lattices $\mathbb{R}_b$, $\mathbb{R}_p$ and $\mathbb{R}_m$ are examined. In Section 3 the next issues are discussed (3.5 - 3.8).

1. Which elements of the lattice $\mathbb{R}_m$ can be realized as a semireflexive product of one element of the lattice $\mathbb{R}_b$ and of one element of the lattice $\mathbb{R}_p$?

2. Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$, where $\mathcal{R} \in \mathbb{R}_b$, and $\mathcal{A} \in \mathbb{R}_p$. The subcategory $\mathcal{R}$ is compulsory c - reflective, does $\mathcal{S} \subset \mathcal{R}$ and that mean the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact?

3. Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$. Are the factors $\mathcal{R}$ and $\mathcal{A}$ determined in a unique way?

4. Let $\mathcal{R}_1$, $\mathcal{R}_2 \in \mathbb{R}_b$, $\Gamma_1, \Gamma_2 \in \mathbb{R}_p$ and $\mathcal{R}_1 \times_{sr} \Gamma_1 = \mathcal{R}_2 \times_{sr} \Gamma_2$. What are the relations of inclusion between subcategories $\mathcal{R}_1$ and $\mathcal{R}_2$ or Γ_1 and Γ_2?

1.1. TERMINOLOGY AND NOTATIONS IN LOCALLY CONVEX SPACES THEORY

The c-reflective subcategories were studied in [9] and [7]. Left and right products were defined and studied in [6]. Other authors results concerning semireflexive subcategories can be found in [2].

In the category $\mathcal{C}_2\mathcal{V}$ we consider the following bicategory structures:

$(\mathcal{E}pi, \mathcal{M}_f)$=(the class of epimorphisms, the class of strict monomorphisms);

$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of precise monomorphisms)=(the class of surjective mappings, the class of topological embedings);

$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of precise epimorphisms, the class of universal monomorphisms) [3], [7];

$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of strict epimorphisms, the class of monomorphisms).

We will consider the following subcategories:

Π, the subcategory of complete spaces with weak topology [8];

$\mathcal{S}$, the subcategory of spaces with weak topology [8];

$s\mathcal{N}$, the subcategory of strict nuclear spaces [5];

$\mathcal{N}$, the subcategory of nuclear spaces [10];

$\mathcal{S}c$, the subcategory of Schwartz spaces [8];

Γ_0, the subcategory of complete spaces [11];

$q\Gamma_0$, the subcategory of quasicomplete spaces [12];

$s\mathcal{R}$, the subcategory of semireflexive spaces [8];

$i\mathcal{R}$, the subcategory of inductive semireflexive spaces [4];

$\mathcal{M}$, the subcategory of spaces with Mackey topology [11];

The last subcategory is coreflective and the others are reflective.

Definition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is $\mathcal{B}$-hereditary, if $fg \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$.*

Dual notion: the class $\mathcal{B}$-cohereditary.

2. THE FACTORIZATION OF THE REFLECTOR FUNCTORS

The results of this section can be found in [1] in Russian (see also [14]).

2.1. The lattice $\mathbb{R}$ of the non-null subcategories of the category $\mathcal{C}_2\mathcal{V}$ is divided into three complete sublattices:

a) The sublattice $\mathbb{R}_b$ of $\mathcal{E}_u$-reflective subcategories. A subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective if the $\mathcal{R}$-replique of any object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Moreover,

$$\mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}.$$

b) The sublattice $\mathbb{R}_p$ of $\mathcal{M}_p$-reflective subcategories, the class of those reflective subcategories $\mathcal{R}$ for $\mathcal{R}$-replique for any object of the category $\mathcal{C}_2\mathcal{V}$ is a topological embeding:

$$\mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}.$$

c) $\mathbb{R}_m = (\mathbb{R} \setminus (\mathbb{R}_b \cup \mathbb{R}_p)) \cup \{\mathcal{C}_2\mathcal{V}\}$.

We mention that $\mathbb{R}_m$ is a complete sublattice with the first element Π and the last element $\mathcal{C}_2\mathcal{V}$.

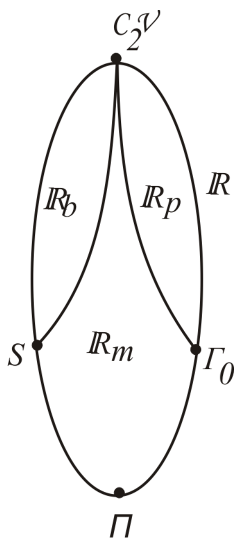

Figure 2.1

2.2. Let $\mathcal{L}$ be an element of lattice $\mathbb{R}_m$. For any object X of category $\mathcal{C}_2\mathcal{V}$ let

$$\begin{tikzcd} X \arrow[r, "b^X"] \arrow[rr, bend left, "l^X"] & bX \arrow[r, "p^X"] & lX \end{tikzcd}$$

Figure 2.2

$l^X : X \longrightarrow lX$ be its $\mathcal{L}$-replique, and $l^X = p^X b^X$ its $(\mathcal{E}_u, \mathcal{M}_p)$-factorization. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{L})$ the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ consisting of all bX form objects and those isomorphic to these. We also can say that $\mathcal{B}$ is the subcategory of all $\mathcal{M}_p$-subobjects of the objects $\mathcal{L}$. It is clear that $\mathcal{B}$ is a $\mathcal{E}_u$-reflective subcategory, and b^X is $\mathcal{B}$-replique of the objects X. Therefore $\mathcal{B} \in \mathbb{R}_b$.

2.3. Let $\Gamma'' = \Gamma''(\mathcal{L})$ be the full subcategory of all objects Y of the category $\mathcal{C}_2\mathcal{V}$, having the property:

Any morphism $f : bX \longrightarrow Y$ is extended through p^X:

$$f = gp^X$$

for some morphism g. The subcategory Γ'' is closed under $\mathcal{M}_f$-subobjects and products. Further, $\Gamma_0 \subset \Gamma''$, therefore $\Gamma'' \in \mathbb{R}_p$. It is obvious that p^X is Γ''-replique of the object bX.

We denote by $G(\mathcal{L})$ the class of all the $\mathcal{M}_p$-reflective subcategories for which p^X is the replique of the object bX. The class $G(\mathcal{L})$ has a minimal element

$$\Gamma' = \Gamma'(\mathcal{L}) = \cap\{\Gamma \mid \Gamma \in G(\mathcal{L})\}.$$

Thus $G(\mathcal{L})$ is a complete lattice with first element $\Gamma'(\mathcal{L})$ and the last element $\Gamma''(\mathcal{L})$.

We can write

$$G(\mathcal{L}) = \{\Gamma \in \mathbb{R}_p \mid \Gamma'(\mathcal{L}) \subset \Gamma \subset \Gamma''(\mathcal{L})\}.$$

2.4. For any element $\Gamma \in G(\mathcal{L})$ the morphism p^X is Γ-replique of the object bX. Therefore if $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{B}$ and $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ are the reflective functors, then

$$l = gb.$$

Theorem. *Let* $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ *and* $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ *be two reflective functors with* $\mathcal{R} \in \mathbb{R}_b$ *and* $\Gamma \in \mathbb{R}_p$. *The following affirmations are equivalent:*

1. $l = gr$.
2. $\mathcal{R} = \mathcal{B}$ *and* $\Gamma \in G(\mathcal{L})$.

2.5. Example. Let us examine the case $\mathcal{L} = \Pi$. Then

$$\mathcal{B}(\Pi) = S$$

and

$$\Gamma'(\Pi) = \Gamma_0.$$

Theorem. *The subcategory* $\Gamma''(\Pi)$ *contains all the normal spaces.*

Proof. Let X be a weak topology space: $X \in \mid \mathcal{S} \mid$, and $g_0^X : X \longrightarrow g_0X$ its Γ_0-replique. Then g_0^X is also the Π-replique of object X. In this case $g_0X \sim K^\tau$, where K is the field of numbers over which the vector spaces from the category $\mathcal{C}_2\mathcal{V} : K = R$ or $K = \mathbb{C}$ are examined. Let $f : X \longrightarrow Y \hookrightarrow \widehat{Y}$, where Y is a normal space, and $\widehat{Y}$ is his completion.

$$\begin{tikzcd}
X \arrow[r, "g_0^X"] \arrow[d, "f"'] & g_0X=K^{\tau} \arrow[dl, dashed, "g_1"'] \arrow[d, dashed, "g"] \\
Y \arrow[r, "i"] & \hat{Y}
\end{tikzcd}$$

Figure 1.3

Then

$$if = gg_0^X$$

for some morphism g, where i is a canonical embedding. Since $g_0X \sim K^\tau$, we conclude that $g(g_0X)$ is a finite dimensional subspace in $\widehat{Y}$. Then subspace $f(X)$ of the Y space as a finite dimensional space is complete and

$$f = g_1g_0^X$$

for some morphism g_1. The theorem is proved.

2.6. Let X and Y be two normal incomplete subspaces, the algebraical dimension of which is:

$$\aleph_0 \leq dimX < dimY$$

Let Γ_1 (respectively Γ_2) be the smallest reflective subcategory which contains the subcategory Γ_0 and X space (respectively Y space). Then the subcategory Γ_1 is not contained in the subcategory Γ_2.

Theorem. *Lattice $G(\Pi)$ contains a proper class of elements.*

2.7. Remark. *On another side we have*

$$\mathcal{B}(\mathcal{C}_2\mathcal{V}) = \mathcal{C}_2\mathcal{V}, \quad G(\mathcal{C}_2\mathcal{V}) = \{\mathcal{C}_2\mathcal{V}\}.$$

3. SEMIREFLEXIVE SUBCATEGORIES

3.1. Definition. *Let $\mathcal{R}$ and $\mathcal{A}$ be two subcategories of the category $\mathcal{C}_2\mathcal{V}$, where $\mathcal{R}$ is a reflective subcategory. Object X of the category $\mathcal{C}_2\mathcal{V}$ is called $(\mathcal{R}, \mathcal{A})$-semireflexive, if his $\mathcal{R}$-replique belongs to the subcategory $\mathcal{A}$.*

We denote by

$$\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$$

the subcategory of all $(\mathcal{R}, \mathcal{A})$-semireflexive objects. The subcategory $\mathcal{L}$ is called the semireflexive product of the subcategories $\mathcal{R}$ and $\mathcal{A}$.

3.2 In the lattice $\mathbb{R}$ there are elements $\mathcal{R}$ such that their reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact. These kind of elements that belong to sublattice $\mathbb{R}_b$ are called the c-reflective subcategories.

The subcategories $\mathcal{S}$, $s\mathcal{N}$ and $\mathcal{S}c$ are c-reflective. The subcategory $\mathcal{N}$ belongs to class $\mathbb{R}_b$ but is not c-reflective.

We mention that also in lattices $\mathbb{R}_p$ and $\mathbb{R}_m$ there are elements of which reflector functor is left exact. For example, the functors

$$g_0 : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_0,$$

$$\pi : \mathcal{C}_2\mathcal{V} \longrightarrow \Pi$$

have this property.

3.3. Theorem. *Let $\mathcal{R}$ and $\mathcal{A}$ be two reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$ and the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact. Then the subcategory*

$$\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$$

is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.

Proof. It is easily to verify that $\mathcal{L}$ is closed under $\mathcal{M}_f$-subobjects and products (see [2]). So it is reflective.

3.4. From the definition we can deduce:

1. Let $\mathcal{R}$, $\mathcal{A} \in \mathbb{R}_b$. Then $\mathcal{S} \subset \mathcal{R} \times_{sr} \mathcal{A}$.
2. Let $\mathcal{R}$, $\mathcal{A} \in \mathbb{R}_p$. Then $\Gamma_0 \subset \mathcal{R} \times_{sr} \mathcal{A}$.
3. Let $\mathcal{R} \in \mathbb{R}_b$, $\mathcal{A} \in \mathbb{R}_p$ and $\mathcal{R} \times_{sr} \mathcal{A}$ be a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. As a rule $\mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$.

3.5. Well known examples of semireflexive subcategories are represented by a semireflexive product of an element of the lattice $\mathbb{R}_b$ and of one element of the lattice $\mathbb{R}_p$. Thus we formulate the following problem.

Problem. *Which elements of the lattice $\mathbb{R}_m$ can be realized as a semireflexive product of one element of the lattice $\mathbb{R}_b$ and of one element of the lattice $\mathbb{R}_p$?*

3.6. Another problem concerning this topic is the following one.

Problem. *Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$, where $\mathcal{R} \in \mathbb{R}_b$, and $\mathcal{A} \in \mathbb{R}_p$. Is the subcategory $\mathcal{R}$ necessarily c-reflective?*

3.7. Problem. *Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$. Are the factors $\mathcal{R}$ and $\mathcal{A}$ determined in a unique way?*

3.8. Problem. *Let $\mathcal{R}_1$, $\mathcal{R}_2 \in \mathbb{R}_b$, Γ_1, $\Gamma_2 \in \mathbb{R}_p$ and $\mathcal{R}_1 \times_{sr} \Gamma_1 = \mathcal{R}_2 \times_{sr} \Gamma_2$. What relations of inclusion are between the subcategories $\mathcal{R}_1$ and $\mathcal{R}_2$ or Γ_1 and Γ_2?*

3.9. Let (E, t) be a locally convex Hausdorff space, $m(t)$- Mackey topology [11] compatible with t topology. Thus $(E, m(t))$ is $\mathcal{M}$-coreplique of the object (E, t). For the elements of the lattice $\mathbb{R}$ we will analyze the following condition

$(\mathcal{SR})$. Let $(E, t) \in\mid \mathcal{L} \mid$, $\mathcal{L} \in \mathbb{R}$. Then for any locally convex topology u on the vector spaces E

$$t \leq u \leq m(t),$$

the space (E, u) belongs to the subcategory $\mathcal{L}$.

3.10 Categorial, the condition $(\mathcal{SR})$ can be written this way

$(\mathcal{SR})$. Let $X \in\mid \mathcal{L} \mid$, and $b : Y \longrightarrow X \in \mathcal{E}_u \cap \mathcal{M}_u$. Then $Y \in\mid \mathcal{L} \mid$.

3.11. a) In the lattice $\mathbb{R}_b$ the elements $\mathcal{S}$, $s\mathcal{N}$, $\mathcal{N}$, $\mathcal{S}c$ do not satisfy the condition $(\mathcal{SR})$. There are elements that satisfy this condition.

b) In the lattice $\mathbb{R}_m$ there are both elements that have the $(\mathcal{SR})$ property and elements that do not have this property.

The subcategory Π has the $(\mathcal{SR})$ property.

Indeed, let $(E,t) \in\mid \Pi \mid$. Then the topology t is a Mackey topology: $t = m(t)$ [12].

3.12. Theorem. *Any element of the lattice* $\mathbb{R}_p$ *has the property* $(\mathcal{SR})$.

Proof. Let $\Gamma \in \mathbb{R}_p$, $X \in\mid \Gamma \mid$ and $b : Y \longrightarrow X \in \mathcal{E}_u \cap \mathcal{M}_u$. Further, let $g^Y : Y \longrightarrow gY$ be the Γ-replique of the object Y. Then

$$b = fg^Y$$

for some morphism f.

$$\begin{tikzcd} Y \arrow[rr, "g^Y"] \arrow[drr, "b"'] & & gY \arrow[d, dashed, "f"] \\ & & X \end{tikzcd}$$

Figure 2.1

Since $b \in \mathcal{M}_u$, $g^Y \in \mathcal{E}pi$ and the class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary it follows that $f \in \mathcal{M}_u$. Also, from the above equality it follows that $f \in \mathcal{E}_u$. Thus in this equality the mappings b and f are bijections. So g^Y also is a bijection, in particulary $g^Y \in \mathcal{E}_u$. Therefore $g^Y \in \mathcal{M}_p \cap \mathcal{E}_u = \mathcal{I}so$. The theorem is proved.

3.13. Theorem. *Given an element* $\mathcal{L} \in \mathbb{R}_m$, *the following affirmations are equivalent*

1. *the subcategory* $\mathcal{L}$ *satisfies condition* $(\mathcal{SR})$;
2. $\mathcal{L} = \mathcal{B} \times_{sr} \Gamma$, *where* $\mathcal{B} = \mathcal{B}(\mathcal{L})$ *and* $\Gamma \in G(\mathcal{L})$;
3. *there is an element* $\mathcal{R} \in \mathbb{R}_b$ *and an element* $\Gamma \in \mathbb{R}_p$ *such that* $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.

Proof. We prove the following implications $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

$1 \Longrightarrow 2$. We verify the embedding $\mathcal{L} \subset \mathcal{B} \times_{sr} \Gamma$. Let $X \in\mid \mathcal{L} \mid$. Then in $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of the morphism $l^X = p^X b^X$ both b^X and p^X morphisms are isomorphisms.

$$\begin{tikzcd}
X \arrow[r, "b^X=1"] \arrow[rr, bend left, "l^X=1"] & bX \arrow[r, "p^X=1"] & lX
\end{tikzcd}$$

Figure 2.2

Thus $bX \in\mid \Gamma \mid$. So $X \in\mid \mathcal{B} \times_{sr} \Gamma \mid$.

Converse. We verify the embedding $\mathcal{B} \times_{sr} \Gamma \subset \mathcal{L}$. Let $(E,t) \in\mid \mathcal{B} \times_{sr} \Gamma \mid$. Then $b(E,t) = (E, b(t)) \in\mid \Gamma \mid$, where the topologies t and $b(t)$ are compatible with the same duality.

$$\begin{tikzcd}
(E,t) \arrow[r, "b^E"] \arrow[rr, bend left, "l^E"] & (E,b(t)) \arrow[r, "p^E"] & l(E,t)
\end{tikzcd}$$

Figure 2.3

Thus $p^E \in \mathcal{I}so$, and $(E, b(t)) \in\mid \mathcal{L} \mid$. By condition $(\mathcal{SR})$, we have $(E,t) \in\mid \mathcal{L} \mid$.

$2 \Longrightarrow 3$. Obviously.

$3 \Longrightarrow 1$. Let $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$, $(E,t) \in\mid \mathcal{L} \mid$, and (E,u) be a locally convex space where $t \le u \le m(t)$

$$\begin{tikzcd}
 & & (E,r(u))=r(E,u) \arrow[dr] & \\
(E,m(t)) \arrow[r] & (E,u) \arrow[ur] \arrow[r] & (E,t) \arrow[r] & (E,r(t))
\end{tikzcd}$$

Figure 2.4

Let $(E, r(u))$ be the $\mathcal{R}$-replique of the object (E,u). Then

$$r(t) \le r(u) \le m(t)$$

and Theorem 3.12 implies that the space $(E, r(u))$ belongs to the subcategory Γ. So $(E,u) \in\mid \mathcal{L} \mid$. Theorem is proved.

3.14. Theorem. *Assume* $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3$, *where* $\mathcal{R}_i \in \mathcal{R}_b$, $i = 1,2,3$, *and* $\Gamma \in \mathbb{R}_p$. *Then*

1. $\mathcal{R}_1 \times_{sr} \Gamma \subset \mathcal{R}_2 \times_{sr} \Gamma$;
2. *if* $\mathcal{R}_1 \times_{sr} \Gamma = \mathcal{R}_3 \times_{sr} \Gamma$, *then* $\mathcal{R}_1 \times_{sr} \Gamma = \mathcal{R}_2 \times_{sr} \Gamma$ *also.*

Proof. Let X be a object of the category $\mathcal{C}_2\mathcal{V}$. Since $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3$, we deduce that between the respective repliques of the object X the following relations

$$\begin{tikzcd}
X \arrow[r, "r_3^X"] \arrow[rrr, bend left, "r_1^X"] \arrow[rr, bend right, "r_2^X"'] & r_3X \arrow[r, dashed, "f"] & r_2X \arrow[r, dashed, "g"] & r_1X
\end{tikzcd}$$

Figure 2.5

$$r_2^X = fr_3^X, \tag{1}$$

$$r_1^X = gr_2^X = gfr_3^X, \tag{2}$$

for some morphisms f and g, hold.

1. Let $X \in\mid \mathcal{R}_1 \times_{sr} \Gamma \mid$. Then $r_1X \in\mid \Gamma \mid$, and from equality (2), and Theorem 3.12 we deduce that $r_2X \in\mid \Gamma \mid$. Thus $X \in\mid \mathcal{R}_2 \times_{sr} \Gamma \mid$.

2. Assume $X \in\mid \mathcal{R}_2 \times_{sr} \Gamma \mid$. Then $r_2X \in\mid \Gamma \mid$, and from Theorem 3.12 and equality (1) it follows that $r_3X \in\mid \Gamma \mid$, so $X \in\mid \mathcal{R}_3 \times_{sr} \Gamma \mid$.

3.15. Theorem. *For any reflective subcategory* $\mathcal{R}$ *with the property* $\mathcal{S} \subset \mathcal{R} \subset \mathcal{N}$, *we have*

$$\mathcal{R} \times_{sr} q\Gamma_0 = s\mathcal{R},$$

in particular,

$$\mathcal{S} \times_{sr} q\Gamma_0 = s\mathcal{N} \times_{sr} q\Gamma_0 = \mathcal{N} \times_{sr} q\Gamma_0 = s\mathcal{R}.$$

Proof. Following the previous theorem, it is enough to prove that $\mathcal{N} \times_{sr} q\Gamma_0 = s\mathcal{R}$, since, from the definition of the subcategory $s\mathcal{R}$ we have

$$\mathcal{S} \times_{sr} q\Gamma_0 = s\mathcal{R}.$$

Let $X \in\mid \mathcal{N} \times_{sr} q\Gamma_0 \mid$. Then $\mathcal{N}$-replique nX of the object X belongs to the subcategory $q\Gamma_0$.

Thus nX is a quasicomplete nuclear space. So it is semireflexive ([12] III 7.2. corollary 2, and also [12] IV 5.8 example 4).

3.16. Theorem. *Assume that* $\mathcal{R} \in \mathbb{R}_b$, $\Gamma, \Gamma_1 \in \mathbb{R}_p$, $\Gamma \subset \Gamma_1$, *and* $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$, $g_1 : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_1$ *are the reflector functors.*

1. *If* $\mathcal{R}$ *is a c-reflective subcategory, then* $g(\mathcal{R}) \subset \mathcal{R}$.
2. *If* $g(\mathcal{R}) \subset \mathcal{R}$, *then* $g_1(\mathcal{R}) \subset \mathcal{R}$.

Proof. 1. Let $X \in \mid \mathcal{R} \mid$, $g^X : X \longrightarrow gX$ be the Γ-replique of the object X, and $r^{gX} : gX \longrightarrow rgX$ the $\mathcal{R}$-replique of object gX. Then

$$r^{gX} g^X = r(g^X) \in \mathcal{M}_p,$$

since $r(\mathcal{M}_p) \subset \mathcal{M}_p$ for a c-reflective subcategory ([2], theorem 2.8). In the above equality $r(g^X) \in \mathcal{M}_p$, and the class $\mathcal{M}_p$ is the $\mathcal{E}pi$-cohereditary. So $r^{gX} \in \mathcal{M}_p \cap \mathcal{E}_u = \mathcal{I}so$.

$$\begin{tikzcd} X \arrow[r, "g^X"] \arrow[d, equal] & gX \arrow[d, "r^{gX}"] \\ X{=}rX \arrow[r, "r(g^X)"] & rgX \end{tikzcd}$$

Figure 2.6

2. Let $X \in \mid \mathcal{R} \mid$, and $g^X : X \longrightarrow gX$ and $g_1^X : X \longrightarrow g_1X$ be the respective repliques of the object X. Since $\Gamma \subset \Gamma_1$ it follows that

$$g^X = f g_1^X$$

for some morphism f.

$$\begin{tikzcd} X \arrow[r, "g_1^X"] \arrow[dr, "g^X"'] & g_1X \arrow[d, dashed, "f"] \\ & gX \end{tikzcd}$$

Figure 2.7

Just like above we deduce that $f \in \mathcal{M}_p$. The hypothesis implies that $gX \in \mid \mathcal{R} \mid$ and $\mathcal{R}$ is a $\mathcal{E}_u$-reflective subcategory. So it is closed under $\mathcal{M}_p$-subobjects. It follows $g_1X \in \mid \mathcal{R} \mid$.

3.17. Theorem. *Assume*

$$\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$$

where $\mathcal{R} \in \mathbb{R}_b$ *and* $\Gamma \in \mathbb{R}_p$. *If* $g(\mathcal{R}) \subset \mathcal{R}$, *then*

$$\mathcal{R} \subset \mathcal{B} = \mathcal{B}(\mathcal{L}).$$

Proof. Let X be an arbitrary object of the category $\mathcal{C}_2\mathcal{V}$, $r^X : X \longrightarrow rX$ and $g^{rX} : rX \longrightarrow grX$ - $\mathcal{R}$ and Γ-replique of the respective objects.

$$\begin{tikzcd}
X \arrow[rr, "b^X"] \arrow[ddr, "l^X"] \arrow[dd, "r^X"'] & & bX \arrow[d, "p^X"] \arrow[ddll, dashed, "t"] \\
 & & lX \arrow[d, dashed, "f"] \\
rX \arrow[rr, "g^{rX}"] & & grX
\end{tikzcd}$$

Figure 2.8

Since $grX \in \mid \mathcal{R} \mid$ we deduce that $grX \in \mid \mathcal{L} \mid$.

Thus

$$fl^X = g^{rX}r^X \tag{1}$$

for some morphism f. Supposing that the $\mathcal{E}_u, \mathcal{M}_p$-factorization of the morphism l^X,

$$l^X = p^Xb^X \tag{3}$$

holds, we deduce

$$g^{rX}r^X = fp^Xb^X \tag{4}$$

where $b^X \in \mathcal{E}_u$, and $g^{rX} \in \mathcal{M}_p$, i.e. $b^X \perp g^{rX}$. Thus

$$r^X = tb^X, \tag{5}$$

for some morphism t,

$$g^{rX}t = fp^{X}. \tag{6}$$

The equality (4) indicates that $\mathcal{R} \subset \mathcal{B}$.

3.18. Conclusions. Returning to problems 3.5-3.8 we can make the following assertions.

1. The $\mathcal{L}$ elements of the lattice $\mathbb{R}_m$ can be presented as a semireflexive product

$$\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$$

with $\mathcal{R} \in \mathbb{R}_b$ and $\Gamma \in \mathbb{R}_p$ having the property $(\mathcal{S}\mathcal{R})$ (Theorem 3.13).

2. $s\mathcal{R} = \mathcal{N} \times_{sr} q\Gamma_0$ and $\mathcal{N}$ is not a c-reflective subcategory.

3. Let $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$. Then neither the first nor the second factor is determined in a unique way (Theorem 3.13 and 3.15).

4. A partially answer is given to question 3.8 by Theorem 3.17.

2.19. Examples. The right product of two subcategories and following examples are examined in more detail in the article [2].

1. Since $(\mathcal{M}, \mathcal{S})$ is a pair of conjugated subcategories in the category $\mathcal{C}_2\mathcal{V}$ and $\Pi = \mathcal{S} \cap \Gamma_0$ we have ([2])

$$\mathcal{S} \times_{sr} \Gamma_0 = \mathcal{M} \times_d \Pi.$$

2. Let $q\Gamma_0$ be a subcategory of the quasicomplete spaces, and $s\mathcal{R}$ the subcategory of the semireflexive spaces [12]. Then

$$\mathcal{S} \times_{sr} (q\Gamma_0) = \mathcal{M} \times_d (\mathcal{S} \cap q\Gamma_0) = s\mathcal{R}.$$

3. The subcategory $\mathcal{S}c$ of Schwartz spaces is c-reflective. Let $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ for which $(\mathcal{K}, \mathcal{S}c)$ is a pair of conjugated subcategories. Then

$$\mathcal{S}c \times_{sr} \Gamma_0 = i\mathcal{R} = \mathcal{K} \times_d (\mathcal{S}c \cap \Gamma_0);$$

$i\mathcal{R}$ is a subcategory of the inductive semireflexive spaces ([4], theorem 1.5).

References

[1] Botnaru D., *The factorization and commutativity and reflexive functors*, Functionals Analysis, Ulianovsk, **21**(1983), 59-71. (in Russian)

[2] Botnaru D., Cerbu O., *Semireflexive product of two subcategories* (in print).

[3] Botnaru D., Gysin V. B., *Stable monomorphisms in the category of separating locally convex spaces*, Bulletin. Acad. Sc. Moldova., **1**(1973), 3-7. (in Russian)

[4] Berezansky I.A., *Inductive reflexive locally convex spaces*, DAN SSSR, v.182, Nr.1, 1968, p.20-20 (in Russian).

[5] Brudovsky B. S., *The additional nuclear topology, transformation of the type s- reflexive spaces and the strict nuclear spaces*, Dokl. Acad. Nauk, SSSR, **178**, 2(1968), 271-273.

[6] Botnaru D., Ţurcanu A., *Les produis de gauche et de droite de deux souscategories*, Acta et Com., Chişinău, **VIII**(2003), 57-73.

[7] Botnaru D., Ţurcanu A., *On Giraux subcategories in locally convex spaces*, ROMAI Journal, **1**, 1(2005), 7-30.

[8] Grothendieck A., *Topological vector spaces*, Gordon and Breach, New York, 1965.

[9] Geyler V.A., Gysin V.B., *Generalized duality for locally convex spaces*, Functionals Analysis, Ulianovsk, **11**(1978), 41-50 (in Russian).

[10] Pietsch R., *Nukleare lokal konvexe raume.* Academie-Verlag, Berlin, 1965.

[11] Robertson A.P., Robertson W.J., *Topological vector spaces*, Cambridge University Press, England, 1964.

[12] Schaeffer H.H., *Topological vector spaces*, The Macmillan Company, New York, 1966.

[13] Raïcov D.A., *Some properties of the bounded linear operators*, The Sciences Bulletin, Pedagogical State College, Moscow "V.I Lenin", **188**(1962), p.171-191.

[14] Ţurcanu A., *The factorization of reflexive functors*, "Al.I. Cuza" University, Iaşi, 2007.

ROMAI J., **5**, 1(2009), 21–27

THE FRATTINI THEORY FOR P - LIE ALGEBRAS

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Abstract The aim of this paper is to present $F_p(L)$-the Frattini p-ideal of L and $\Phi_p(L)$-the Frattini p-subalgebra of L. The basic properties are presented in the second section. The third section is concerned with the main result $F(L) \subset F_p(L) \subset C_L(F(L))$ and in the last section we present some results about p-c-supplemented subalgebras of p-Lie algebras.

Keywords: p-Lie algebra, Frattini p-ideal, p-subalgebra.

2000 MSC: 17B05, 17B20, 17B66.

1. INTRODUCTION

In this paper we denote by $a \to a^p$, $p > 0$ the application which corresponds to a p- Lie algebra. Throughout the article L is a finite-dimensional p-Lie algebra over a field K of characteristic $p > 0$. We denote by $\Phi(L)$ the Frattini subalgebra of L, that is the intersection of the maximal subalgebras of L and by $F(L)$ the Frattini ideal of L, that is the largest ideal of L which is contained in $\Phi(L)$. Analogously we denote by $F_p(L)$ the Frattini p-ideal of L i.e. the largest p-ideal of L that is contained in $\Phi_p(L)$, where $\Phi_p(L)$ is the Frattini p-subalgebra of L, that is the intersection of the maximal p-subalgebras of L. If A is a p-subalgebra of L, the p-core of A is the largest ideal of L contained in A and we denote that with ${}_pA_L$. We say that A is p-core-free in L if ${}_pA_L = 0$. A p-subalgebra A of L is p-c-suplemented in L if there is a p-subalgebra B of L such that $L = A + B$ and $A \cap B$ is a p-subalgebra of ${}_pA_L$. We say that L is p-c-supplemented if every p-subalgebra of L is p-c-supplemented in L.

2. BASIC PROPERTIES

We present some notions and results that we use in the sequel:

$[x, y]$ *is the product of* x y *in* L;
$L^{(1)}$ *is the derived algebra of* L;
$(A)_p = (\{x^{p^n} \mid x \in A, n \in N\})$ *where* $x^{p^n} = (x^{p^{n-1}})^p$;
$A^p = (\{x^p \mid x \in A\})$ *where* A *is a subalgebra of* L;
$L_1 := \bigcap_{i=1}^{\infty} L^{p^i}$;
$L_0 := \{x \in L \mid x^{p^n} = 0,\ n \in N\}$;
$C_L(M) := \{x \in L \mid [x, M] = 0\}, \quad M \subset L$;
$N_L(A) = \{x \in L \mid [x, A] \subseteq A\},$ *where* $A \subset L$.

The first properties of the p-ideals and p-subalgebras of an p-algebra are stated by M. Lincoln and D.A. Towers [9] and we present them in the following.

Lemma 2.1. [9] *Let* A *and* B *be p-subalgebras of* L*, such that* A *is an ideal of* L *. Then* $A+B$ *is a p-subalgebra of* L*.*

Lemma 2.2. [9] *Let* A *be a subalgebra of* L*. Then* $(A)_p^{(1)} \subseteq A^{(1)}$*.*

Lemma 2.3. [9] *If* I *is an ideal of* L*, then* $(I)_p \subseteq C_L(I)$*. In particular,* $(I)_p$ *is an ideal of* L*.*

Lemma 2.4. [9] *If* $A \subseteq L$ *then* $N_L(A)$ *is p-closed.*

We give below some results relative to $F_p(L)$ and $\Phi_p(L)$ inspired by results relative to $F(L)$ and $\Phi(L)$ obtained by Stitzinger [8] for nilpotent Lie algebras.

Lemma 2.5. *For any p-Lie algebra* L*, and* A *a p-subalgebra of* L*, the following statements are true*
(i) if $A + \Phi_p(L) = L$ *then* $A = L$*;*
(ii) if I *is an ideal of* L *such that* $I \subset \Phi_p(A)$*, then* $I \subset \Phi_p(L)$*.*

Proof. (i) We assume that $A \neq L$. Then there exists a maximal p-subalgebra M of L such that $A \subset M$. Now $\Phi_p(L) \subset M$ and so $L = M$, fact contradicting the maximality of M. The result follows.
(ii) Because A is a p-subalgebra of L and I is an ideal of L, Lemma 2.1 implies that $I + A$ is a p-subalgebra of L with $I \subset \Phi_p(A)$. If $I \not\subseteq \Phi p(L)$, then there is

a maximal p-subalgebra M of L such that $I \subset M$, hence $M \subset \Phi_p(A)$, which is in contradiction with the maximality of I, so $I \subset \Phi_p(L)$.

Lemma 2.6. *If I is a p-ideal of L, then*
(i) $(\Phi_p(L) + I)/I \subset \Phi_p(L/I)$*;*
(ii) $(F_p(L) + I)/I \subset F_p(L/I)$*;*
(iii) if $I \subset \Phi_p(L)$*, then (i) and (ii) are true with equality; moreover, if* $F_p(L/I) = 0$*, then* $F_p(L) \subset I$*;*
(iv) if A is a minimal p-subalgebra of L such that $L = I+A$ *then* $I \cap A \subset F_p(A)$*;*
(v) if $I \cap F_p(L) = 0$*, then there is a p-subalgebra A of L such that*
$L = I\dot{+}A$ *(where* $\dot{+}$ *denotes a vector space direct sum).*

Proof. The assertions from (i), (ii) and (iii) are similar with those from Proposition 4.3 [9].
(iv) If $I \cap A \not\subseteq F_p(A)$ then there is a maximal p-subalgebra M of A such that $I \cap A + M = A$. Hence $L = I + M$, contradicting the minimality of A. It follows that $I \cap A \subset F_p(A)$.
(v) Let A be minimal with the property $L = I+A$. Then by (iv), $I \cap A \subset F_p(A)$. But $I \cap A$ is a p-ideal of L hence, by Lemma 2.5(ii), $I \cap A \subset F_p(L) \cap I = 0$. We conclude that $L = I\dot{+}A$.

Definition 2.1. *(i) If L is a p-Lie algebra we denote by Sp(L) the sum of the minimal abelian p-ideals of L and we call it the abelian p-socle of L.*
(ii) We say that L p-splits over an p-ideal I of p-Lie algebra L, if there is a p-subalgebra A of L such that $L = I\dot{+}A$*, where* $\dot{+}$ *represents the direct sum of vector spaces.*

Lemma 2.7. *The abelian p-socle Sp(L) is a p-ideal of L.*

Proof. Let $x \in Sp(L)$, $l \in L$. Then $x^{p^n} = 0$, or $\left[l, x^{p^n}\right] = [l, x]\,(adx)^{p^{n-1}} = 0$ and so (x^{p^n}) is a minimal abelian p-ideal of L. Hence $x^{p^n} \in Sp(L)$.

3. F_P-FREE LIE ALGEBRA

In this section we will present the relationship between F(L) and $F_p(L)$.

Definition 3.1. *A p-Lie algebra is called F-free (respectively, F_p-free) if $F(L) = 0$ (respectively, $F_p(L) = 0$).*

Theorem 3.1. *If L is F_p-free, then L p-splits over its abelian p-socle.*

Proof. If L is F_p-free, then $F_p(L) = 0$. Let $I = Sp(L)$ be the abelian-p-socle of L which is an abelian p-ideal of L. Then $I \cap F_p(L) = 0$. In accord with Lemma 2.7(iv), there is a p-subalgebra A of L such that $L = I \dot{+} A$, so L p-splits over I.

Theorem 3.2. *For any p-Lie algebra L, the relation $F(L) \subset F_p(L)$ holds.*

Proof. It is clear that $L/F_p(L)$ is $F_p(L)$-free and hence the previous theorem implies that $L/F_p(L)$ splits over $S(L/F_p(L))$. So $L/F_p(L)$ is F-free and it follows that $F(L) \subset F_p(L)$.

In the following we introduce two further subalgebras of L.

Definition 3.2 *If L is a p-Lie algebra we note by $T(L)$-the intersection of all maximal subalgebras of L that are not ideals of L and correspondingly by $T_p(L)$ the intersection of all maximal p-subalgebras of L which are not p-ideals of L. We also define $\tau(L)$ (respectively, $\tau_p(L)$), to be the largest ideal (respectively, p-ideal) of L that is contained in T(L) (respectively, $T_p(L)$).*

In these conditions the following statements hold.

Lemma 3.1. *If I is a p-ideal of L, then:*

(i) $(T_p(L) + I) \subset T_p(L/I)$;

(ii) $(\tau_p(L) + I)/I \subset \tau_p(L/I)$;

(iii) if $I \subset T_p(L)$, then statements (i) and (ii) occur with equality; moreover, if $\tau_p(L/I) = 0$ then $\tau_p(L) \subset I$.

Lemma 3.2. *Let A be a maximal p-subalgebra of L. Then*
(i) if A is an ideal of L we have $L^{(1)} \subset A$;
(ii) if A is not an ideal of L, it is a p-subalgebra of L.

Proof. (i) Let $x \notin A$. Then $L = A + (x)_p$ and according to Lemma 2.2 it follows that $L^{(1)} \subset A$.
(ii) We assume that A is not p-close, then $(A)_p = L$, from where, according to Lemma 2.2 we deduce that $L^{(1)} = (A)_p^{(1)} \subset A^{(1)} \subset A$, hence A is an ideal of L, contradicting the hypothesis, so any maximal subalgebra of L which is not an ideal of L is a p-subalgebra of L.

Theorem 3.3 *For any p-Lie algebra L, the following statements hold:*
(i) $\tau_p(L) = C_L(F_p(L))$;
(ii) $\tau(L) = \tau_p(L)$;
(iii) if N is the nilradical of L, then $F_p(L) = \Phi_p(L) \cap N$;
(iv) if L is perfect (that is $L = L^{(1)}$), then $\Phi_p(L) = \Phi(L)$.

Proof. (i) According to Lemma 3.2(i) we have $[\tau_p(L), L] \subset L^{(1)} \cap \tau_p(L) \subset F_p(L)$, hence $\tau_p(L) \subset C_L(F_p(L))$. Now we assume that $C_L(F_p(L)) \not\subseteq \tau_p(L)$. Then there is a maximal p-subalgebra A of L which is not an ideal of L such that $C_L(F_p(L)) \not\subseteq A$. Now it is easy to prove that $C_L(F_p(L))$ is p-closed, so $L = C_L(F_p(L)) + A$. Then $L^{(1)} \subset F_p(L) + A \subset A$ and A is an ideal of L- a contradiction. Therefore $C_L(F_p(L)) \subset \tau_p(L)$.
(ii) According to Lemma 3.2(ii) it immediately follows that $\tau_p(L) \subset \tau(L)$. Consider now a $x \in \tau(L)$. Then, according with Theorem 2.8 [10] and with Theorem 3.2, $[x, L] \subset F(L) \subset F_p(L)$. Therefore, according to Theorem 3.3(i) $x \in C_L(F_p(L)) = \tau_p(L)$ and hence we obtain the conclusion.
(iii) Let $A = \Phi_p(L) \cap N$. Then according to Theorem 3.2, $N^{(1)} \subset F(L) \subset F_p(L)$ and so $N^{(1)} \subset A$. Let us assume that A is not an ideal of L. Then, since $AL \subset NL \subset N$, we have that $AL \not\subseteq \Phi_p(L)$. Hence, there is a maximal p-subalgebra M of L such that $AL \not\subseteq M$. It follows that $N \not\subseteq M$ and hence $L = N + M$. So, $AL = A(N + M) \subset N^{(1)} + M \subset M$, a contradiction. From these we can say that A is a p-ideal of L which is contained in $\Phi_p(L)$ and thus $A \subset F_p(L)$. The reverse assertion is immediate.
(iv) It is clear that any maximal p-subalgebra of L is a maximal subalgebra

of L, hence $\Phi_p(L) \subset \Phi(L)$. According to Lemma 3.2(ii) we have that $\Phi(L) \subset \Phi_p(L)$, and $\Phi_p(L) = \Phi(L)$ follows. In accord with Theorem 3.2 and 3.3 we obtain $F(L) \subset F_p(L) \subset C_L(F(L)) = C_L(F_p(L))$.

4. P-C-SUPPLEMENTED SUBALGEBRAS OF P-LIE ALGEBRAS

In this section we present some results about p-c-Supplemented subalgebras of p-Lie algebras that take into account the results similar to those already presented by Ballester-Bolinches, Wang and Xiuyun in [1].

Lemma 4.1 *Let L be a p-Lie algebra and A a p-subalgebra of L. The following statements hold*
(i) if B is a p-subalgebra of A and it is p-c-supplemented in L, then B is p-c-supplemented in A;
(ii) if I is a p-ideal of L and a p-subalgebra of A then A is p-c-supplemented in L if and only if A/I is p-c-supplemented in L/I.

Proof. (i) Assume that A is a p-subalgebra of L and B is p-c-supplemented in L. Then there is a p-subalgebra C of L such that $L = B + C$ and $B \cap C$is a p-subalgebra of ${}_pB_L$. It follows that $A = (B + C) \cap A$ and $B \cap C \cap A$ is a p-subalgebra of ${}_pB_L \cap A$ which is a p-subalgebra of ${}_pB_A$, and so B is p-c-supplemented in A.
(ii) First we suppose that A/I is p-c-supplemented in L/I. Then there is a p-subalgebra B/I of L/I such that $L/I = A/I + B/I$ and $(A/I) \cap (B/I)$ is a p-subalgebra of ${}_p(A/I)_{L/I} =_p A_L/I$. It follows that $L = A + B$ and $A \cap B$ is a p-subalgebra of ${}_pA_L$, whence A is p-c-supplemented in L.
Now, conversely, we assume that I is an p-ideal of L, more than that, I is a p-subalgebra of A and p-c-supplemented in L. In these circumstances there is a p-subalgebra B of L such that $L = A + B$ and $A \cap B$ is a p-c-subalgebra of ${}_pA_L$. Hence $L/I = A/I + (B+I)/I$ and $(A/I) \cap (B+I)/I = (A \cap (B+I))/I = (I + A \cap B)/I$ but $(I + A \cap B)/I$ is a p-subalgebra of ${}_pA_L/I =_p (A/I)_{L/I}$, and so A/I is p-c-supplemented in L/I.

Lemma 4.2 *Let L be a p-Lie algebra and A, B be p-subalgebras of L such that A is a p-subalgebra of $F_p(B)$. If A is p-c-supplemented in L then A is an ideal of L and A is p-subalgebra of $F_p(L)$.*

Proof. First of all we assume that $L = A + C$ and $A \cap C$ is a p-subalgebra of ${}_pA_L$. Then $B = B \cap L = B \cap (B + C) = A + B \cap C = B \cap C$ since A is a p-subalgebra of $F_p(B)$. From these we obtain that A is a p-subalgebra of B that is a p-subalgebra of C, and $A = A \cap C$ that is a p-c subalgebra of A_L and A is an ideal of L. It then follows from Lemma 4.1 [9] that A is a p-subalgebra of $F_p(L)$.

The results obtained in this last section can be extended and give way to many other results.

References

[1] A. Ballester-Bolinches, Y. Wang, G. Xiuyun, *C-supplemented subgrups of finite groups*, Glasgow Math. J., **42**(2000), 383-389.

[2] G. P. Hochschild, *Basic of algebraic groups and Lie algebras*, Graduate Texts in Math., **75**, Springer, New York, 1981.

[3] J. E. Jacobson, *Lie algebras*, Dover Publ., New York, 1979.

[4] H. C. Myung, *Lie-admissible algebra*, Soviet Encyclopedia Math., Kluver, Dordrecht, **5**, 1990.

[5] C. Reutenauer, *Free Lie algebras*, Clarendor Press, Oxford, 1995.

[6] K. W. Roggenkamp, *Cohomology of Lie algebras, groups and algebras*, Seminar Series in Mathematics, Ovidius University Constanta, 1994.

[7] M. Lincoln, D. A. Towers, *Frattini theory for restricted Lie algebras*, Arch. Math., **45**(1985), 451-457.

[8] E. Stitzinger, *On the Frattini subalgebras of a Lie algebra*, J. Lond. Math. Soc., 1970, 177-182.

[9] D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc. **27**(1973), 440-462.

[10] D. A. Towers, *Two ideals of an algebra closely related to its Frattini ideal*, Arch. Math., **35**(1980), 112-120.

ROMAI J., **5**, 1(2009), 29–44

A CHARACTERIZATION OF N-TUPLES OF COMMUTING GRAMIAN SUBNORMAL OPERATORS

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Abstract The paper presents a Halmos-Bram criterion for gramian subnormality of n-tuples of commuting linear gramian bounded operators which admit adjoint.

Keywords: Commuting gramian subnormal n-tuples of operators, Loynes spaces, positive definiteness.

2000 MSC: 47A20, 44A66.

1. INTRODUCTION

Loynes introduced the notions of VE-space and VH-space (or LVH-space) respectively, in [6] as generalizations of pre-Hilbert, Hilbert space, respectively. LVH-spaces have also been named pseudo-Hilbert spaces in [10] and later Loynes spaces in [11]. In [10] and [11] these spaces are used in the abstract study of the stochastic processes.

The Halmos-Bram criterion for subnormality given in [1] was proved before for a subnormal operator that admits adjoint on a Loynes space. The same characterization is given in this paper for a n-tuple of gramian commuting operators (Theorem 3.1).

2. PRELIMINARIES

For our purposes, we need to recall some definitions and known results.

Definition 2.1. **[6]** *A locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:*

(A.1) *Z is complete;*

(A.2) *in Z there is a closed convex cone, noted as Z_+, by which an order relation is currently introduced on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);*

(A.3) *in Z there is an involution $Z \ni z \to z^* \in Z$ (i.e. $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$;*

(A.4) *the topology of Z is compatible with the order (i.e. there exists a basis of solid (convex) neighbourhoods of the origin);*

(A.5) *the monotone decreasing nets from Z_+ are convergent.*

A set $C \subset Z$ is called *solid* if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.

Definition 2.2. **[6]** *For an admissible space Z in the Loynes sense, a $\mathbb{C}$-linear topological space $\mathcal{H}$ is called pre-Loynes Z-space if the following properties are satisfied:*

(L.1) *$\mathcal{H}$ is endowed with a Z-valued inner product (gramian) with the properties:*

(G.1) $[h,h] \geq 0$; $[h,h] = 0$ *implies* $h = 0$;

(G.2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G.3) $[\lambda h, k] = \lambda [h, k]$;

(G.4) $[h,k]^* = [k,h]$;

for each $h, k, h_1, h_2 \in \mathcal{H}$, $\lambda \in \mathbb{C}$;

(L.2) *the topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h,h] \in Z$ is continuous.*

If, in addition $\mathcal{H}$ is complete with this topology, then $\mathcal{H}$ is called Loynes Z-space.

Definition 2.3. **[6]** *A subspace $\mathcal{M}$ in Loynes Z-space $\mathcal{H}$ is called accessible, if for any $h \in \mathcal{H}$ there exist an unique $h_1 \in \mathcal{M}$ and $h_2 \perp \mathcal{M}$ (i.e. $[h_2, h'] = 0$, $\forall h' \in \mathcal{M}$) such that $h = h_1 + h_2$. The operator $\mathcal{P}_{\mathcal{M}}$ defined by $\mathcal{P}_{\mathcal{M}} h := h_1$ is called the gramian projection associated to the accessible space $\mathcal{M}$.*

Let $\mathcal{H}$ be a Loynes Z-space. We denote by $\mathcal{P}_Z$ the set of monotonic semi-norms that generates the topology of Z and by $q_p : \mathcal{H} \to [0, \infty)$, the

semi-norm defined by $q_p(h) = (p[h,h])^{1/2}$, where $p \in \mathcal{P}_Z$.
Let $\mathcal{H}$ and $\mathcal{K}$ two Loynes Z-spaces. We denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the space of all linear operators from $\mathcal{H}$ to $\mathcal{K}$. We say that an operator $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ admits adjoint and we denote that by $T \in \mathcal{L}^*(\mathcal{H},\mathcal{K})$ if there exists $S \in \mathcal{L}(\mathcal{K},\mathcal{H})$ such that

$$[Th,k]_{\mathcal{K}} = [h,Sk]_{\mathcal{H}}\ ;\ h \in \mathcal{H}\ ,\ k \in \mathcal{K}.$$

If S exists, it is unique determined and we denote by $T^* = S$ the gramian adjoint of T.([4])

We also introduce the following notations:

$\mathcal{L}^*(\mathcal{H})$ - the set of all linear operators on $\mathcal{H}$ that admit adjoint,

$\mathcal{C}^*(\mathcal{H})$ - the set of operators $T \in \mathcal{L}^*(\mathcal{H})$ with the property that for any $p \in \mathcal{P}_Z$ there exist $M_p > 0$ and $p_0 \in \mathcal{P}_Z$ such that $p([Th,Th]) \le M_p \cdot p_0([h,h])$ for any $h \in \mathcal{H}$,

$\mathcal{CQ}^*(\mathcal{H})$ - the set of operators $T \in \mathcal{C}^*(\mathcal{H})$ with the property that for any $p \in \mathcal{P}_Z$ there exists $M_p > 0$ such that $p([Th,Th]) \le M_p \cdot p([h,h])$ for any $h \in \mathcal{H}$,

$\mathcal{CU}^*(\mathcal{H})$ - the set of operators $T \in \mathcal{C}^*(\mathcal{H})$ with the property that there exists $M > 0$ for any $p \in \mathcal{P}_Z$ such that $p([Th,Th]) \le M \cdot p([h,h])$ for any $h \in \mathcal{H}$,

$\mathcal{B}^*(\mathcal{H})$ - the set of operators $T \in \mathcal{L}^*(\mathcal{H})$ with the property that there exists $M > 0$ such that $[Th,Th] \le M \cdot [h,h]$ for any $h \in \mathcal{H}$.

It is known that in the Loynes spaces the following inclusions hold

$$\mathcal{B}(\mathcal{H}) \subseteq \mathcal{CU}(\mathcal{H}) \subseteq \mathcal{CQ}(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})\ ,$$

$$\mathcal{B}^*(\mathcal{H}) \subseteq \mathcal{CU}^*(\mathcal{H}) \subseteq \mathcal{CQ}^*(\mathcal{H}) \subseteq\subseteq \mathcal{C}^*(\mathcal{H}) \subseteq \mathcal{L}^*(\mathcal{H}),$$

and $\mathcal{B}^*(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, $\mathcal{CU}^*(\mathcal{H}) \subset \mathcal{CU}(\mathcal{H})$, $\mathcal{CQ}^*(\mathcal{H}) \subset \mathcal{CQ}(\mathcal{H})$, $\mathcal{C}^*(\mathcal{H}) \subset \mathcal{C}(\mathcal{H})$, $\mathcal{L}^*(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$. This classes appear first in the A. Michael's books.

We remark that the classical Fuglede-Putnam theorem holds in the case of Loynes spaces.

Theorem 2.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Loynes Z-spaces and N_1, N_2 two gramian normal operators on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. If there exists $T \in \mathcal{B}^*(\mathcal{H}_1, \mathcal{H}_2)$ such that $TN_1 = N_2T$ then $TN_1^* = N_2^*T$.*

For further considerations we state the following theorem.

Theorem 2.2. **[6]** *Let Γ be a $*$-semigroup, and T_ε $(\varepsilon \in \Gamma)$ a family of continuous linear operators in the VH-space H satisfying the following conditions:*

(a) $T_\varepsilon = I$, $T_{\varepsilon^*} = (T_\varepsilon)^*$;

(b) T_ε *is positive-definite as a function of ε, in the sense that if g_ε $(\varepsilon \in \Gamma)$ is a function from Γ to $\mathcal{H}$ which vanishes except for a finite number of indices then*

$$\sum_{\varepsilon,\eta} [T_{\varepsilon^*\eta} g_\eta, g_\varepsilon] \geq 0;$$

(c) *for a given α in Γ and a given neighbourhood N_0 of the origin in Z, there exists a neighbourhood N_0^α of the origin in Z such that, if g_ε is such a function,*

$$\sum_{\varepsilon,\eta} [T_{\varepsilon^*\eta} g_\eta, g_\varepsilon] \in N_0^\alpha$$

implies that

$$\sum_{\varepsilon,\eta} [T_{\varepsilon^*\alpha^*\alpha\eta} g_\eta, g_\varepsilon] \in N_0.$$

Then there exist a VH-space $\widehat{H}$, in which H can be isomorphically embedded as an accessible subspace, and a representation D_ε of Γ in $\widehat{H}$, such that if P is the projection onto H then

$$T_\varepsilon = prD_\varepsilon,$$

where by prD_ε we mean the restriction to H of the operator PD_ε. There is, moreover, such an $\widehat{H}$ which is minimal in the sense that it is generated by elements of the form $D_\varepsilon f$,where $f \in H$ and $\varepsilon \in \Gamma$, and this minimum

$\widehat{H}$ is uniquely determined up to isomorphism. The following properties are valid in this minimum $\widehat{H}$:

(1) *if N_0 in condition (c) is closed, then $[h,h] \in N_0^*$ implies that*

$$[D_\alpha h, D_\alpha h] \in N_0,$$

for h in $\widehat{H}$;

(2) *if $T_{\varepsilon\alpha\eta} = T_{\varepsilon\beta\eta} + T_{\varepsilon\gamma\eta}$ for some fixed $\alpha, \beta, \gamma,$ and all ε, η in Γ, then*

$$D_\alpha = D_\beta + D_\gamma;$$

(3) *if $\{\alpha(\lambda)\}$ is a net such that $\cap_\alpha N_0^{\alpha(\lambda)}$ is a neighbourhood of the origin in Z, then $T_{\varepsilon\alpha(\lambda)\eta} \to T_{\varepsilon\alpha\eta}$ weakly for all ε, η implies that $D_{\alpha(\lambda)} \to D_\alpha$ weakly.*

3. CHARACTERIZATIONS OF N-TUPLES OF COMMUTING GRAMIAN SUBNORMAL OPERATORS

Definition 3.1. *If $n \in \mathbb{N}^*$ and $\{T_i\}_{i=1}^n \subset \mathcal{L}^*(\mathcal{H})$, $\mathcal{H}$ being a Loynes Z-space, we say that the n-tuple $(T_1, T_2, ..., T_n)$ of commuting operators is gramian subnormal if there exist a Loynes Z-space $\mathcal{K}$, $\mathcal{K} \supset \mathcal{H}$ and the gramian normal commuting operators $N_i \in \mathcal{B}^*(\mathcal{K})$, $i \in \{1, .., n\}$ such that $\mathcal{H}$ is accessible in $\mathcal{K}$, $\mathcal{H}$ is invariant under each N_i, $i \in \{1, ..., n\}$ and $N_i \mid_{\mathcal{H}} = T_i$ for any $i \in \{1, ..., n\}$.*

Examples1. As in the Hilbert space, the gramian isometries on $\mathcal{B}^*(\mathcal{H})$ are gramian subnormal operators on Loynes spaces. For that we use Proposition 3.1.9 concerning partial isometries from [4].

2. As in the Hilbert space, the natural gramian shift $S_{\mathcal{L}}$ defined in $l^2[Z_+, \mathcal{L}]$. (see, [4]) by

$$S_{\mathcal{L}}(l_0, l_1, ..., l_m, ...) = (0, l_0, ..., l_{m-1}, ...)$$

is gramian subnormal operator (also see [4]).

In the sequel, the following lemma will be useful.

Lemma 3.1. *For any $h_1, h_2 \in \mathcal{H}$ and $\alpha > 0$ the following inequality holds*

$$[h_1, h_2] + [h_2, h_1] \leq \alpha[h_1, h_1] + \frac{1}{\alpha}[h_2, h_2].$$

Proof. From Definition 2.2, condition (G.1), we have

$$[\alpha h_1 - h_2, \alpha h_1 - h_2] \geq 0.$$

It is clear that

$$[\alpha h_1 - h_2, \alpha h_1 - h_2] = \alpha^2[h_1, h_1] + [h_2, h_2] - \alpha[h_1, h_2] - \alpha[h_2, h_1].$$

Thus

$$\alpha[h_1, h_2] + \alpha[h_2, h_1] \leq \alpha^2[h_1, h_1] + [h_2, h_2]$$

for any $\alpha > 0$, whence the conclusion.

As in the case of Hilbert space we have

Lemma 3.2 *Let T, S be two operators from $\mathcal{B}^*(\mathcal{H})$ with $S \leq T$ and $T, S \geq 0$. Then $\|S\| \leq \|T\|$.*

Proof. By using the definition, it follows that $0 \leq [Sh, h] \leq [Th, h]$, $\forall\, h \in \mathcal{H}$. By Lemma 1, [7], T and S are gramian self-adjoint operators, therefore by [2], from $[Th, Th] \leq \|T\|^2[h, h]$, we obtain $[Th, h] \leq \|T\|[h, h]$, $\forall\, h \in \mathcal{H}$. Thus $0 \leq [Sh, h] \leq \|T\|[h, h]$, $h \in \mathcal{H}$. Applying again the same theorem see [2], we have, $[Sh, Sh] \leq \|T\|^2[h, h]$, $\forall\, h \in \mathcal{H}$. This means that $\|S\|^2 \leq \|T\|^2$, i.e. $\|S\| \leq \|T\|$. ∎

The following lemma generalizes Lemma 1 for the Loynes spaces [5].

Lemma 3.3 *Let $\mathcal{H}$ be a Loynes Z-space and A_l be n commutative operators from $\mathcal{B}^*(\mathcal{H})$. If for every non-negative integer M and element $h_{i_1,\ldots,i_n} \in \mathcal{H}$ $(0 \leq i_l \leq M,\ l = 1, 2, \ldots, n)$*

(1)
$$\sum_{i_l, j_l = 0;\, l = 1, 2, \ldots, n}^{M} [A_1^{i_1} \ldots A_n^{i_n} h_{j_1,\ldots,j_n}, A_1^{j_1} \ldots A_n^{j_n} h_{i_1,\ldots,i_n}] \geq 0,$$

then for any $\nu_l \in \mathbb{N}, l = 1, 2, \ldots, n$ we have

(2)
$$\sum_{i_l, j_l = 0;\, l = 1, 2, \ldots, n}^{M} [A_1^{i_1+\nu_1} \ldots A_n^{i_n+\nu_n} h_{j_i,\ldots,j_n}, A_1^{j_1+\nu_1} \ldots A_n^{j_n+\nu_n} h_{i_1,\ldots,i_n}] \leq$$

$$\leq \| A_1 \|^{2\nu_1} \cdot \| A_2 \|^{2\nu_2} ... \| A_n \|^{2\nu_n} \sum_{i_l,j_l=0;\, l=1,2,...,n}^{M} [A_1^{i_1}...A_n^{i_n} h_{j_1,...,j_n}, A_1^{j_1}...A_n^{j_n} h_{i_1,...,i_n}].$$

Proof. Let $\mathcal{H}_{i_1,...,i_n}$ be spaces isomorphic to $\mathcal{H}$ ($i_l = 0,1,2,...; l = 1,2,3,...,n$) and

$$\mathcal{K} = \sum_{i_l \geq 0;\, l=1,2,...,n} \boxplus \mathcal{H}_{i_1,...,i_n},$$

i.e. the set of sequences $(h_j)_{j \in \mathbb{Z}_+^n}$ with $h_j \in \mathcal{H}_{j_1,...,j_n}$ ($j = (j_1,...,j_n)$) for which

$$\sum_{j_l \geq 0;\, l=1,2,...,n} [h_{j_1,...,j_n}, h_{j_1,...,j_n}]$$

is convergent in Z. We shall denote the sum of the series by $[h,h]_{\mathcal{K}}$, where $h = (h_j)_{j \in \mathbb{Z}_+^n}$ is an element from $\mathcal{K}$. We define by polarity

$$[h,h^{'}]_{\mathcal{K}} = \frac{1}{4} \cdot ([h+h^{'},h+h^{'}]_{\mathcal{K}} - [h-h^{'},h-h^{'}]_{\mathcal{K}} + i \cdot [h+i \cdot h^{'}, h+i \cdot h^{'}]_{\mathcal{K}} - i \cdot [h-i \cdot h^{'}, h-i \cdot h^{'}]_{\mathcal{K}})$$

for $h, h^{'} \in \mathcal{K}$.

The set

$$\sum_{i_l \geq 0;\, l=1,2,...,n} \boxplus \mathcal{H}_{i_1,...,i_n}$$

will be formally denoted by $l^2[\mathbb{Z}_+^n, \mathcal{H}]$ and it is a Loynes Z-space with gramian $[.,.]_{\mathcal{K}}$ [4].

For $\varepsilon > 0$, we define $B_l = (\|A_l\| + \varepsilon)^{-1} \cdot A_l$ ($l = 1,2,...,n$). It is obvious that $\|B_l\| < 1$ ($l = 1,...,n$). Now, we define the operator $S : \mathcal{K} \to \mathcal{K}$ by $Sh = k$, $h = (h_j)_{j \in \mathbb{Z}_+^n} \in \mathcal{K}$, $k = (k_j)_{j \in \mathbb{Z}_+^n} \in \mathcal{K}$, where

$$k_i = k_{i_1,...,i_n} = \sum_{j_l \geq 0;\, l-1,2,...,n}^{\infty} B_n^{*j_n} ... B_1^{*j_1} B_1^{i_1} B_2^{i_2} ... B_n^{i_n} h_{j_1,...,j_n}.$$

Using Lemma 3.1 we show that k is correctly defined. We have,

$$[\sum_{j_l=M+1;\, l=1,...,n}^{M+p} B_n^{*j_n} ... B_1^{*j_1} B_1^{i_1} ... B_n^{i_n} h_{j_1,...,j_n}, \sum_{k_l=M+1;\, l=1,...,n}^{M+p} B_n^{*k_n} ... B_1^{*k_1} B_1^{i_1} ... B_n^{i_n} h_{k_1,...,k_n}] =$$

$$= \sum_{j_l,k_l=M+1;\, l=1,...,n}^{M+p} [B_n^{*j_n} ... B_1^{*j_1} B_1^{i_1} ... B_n^{i_n} h_{j_1,...,j_n}, B_n^{*k_n} ... B_1^{*k_1} B_1^{i_1} ... B_n^{i_n} h_{k_1,...,k_n}] =$$

$$= \sum_{j_l=M+1;\, l=1,...,n}^{M+p} [B_n^{*j_n} ... B_1^{*j_1} B_1^{i_1} ... B_n^{i_n} h_{j_1,...,j_n}, B_n^{*j_n} ... B_1^{*j_1} B_1^{i_1} ... B_n^{i_n} h_{j_1,...,j_n}] +$$

$$+\sum_{j_l\neq k_l=M+1;\, l=1,...,n}^{M+p}[B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_{j_1,...,j_n},B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_{k_1,...,k_n}]\leq$$

$$\leq\sum_{j_l=M+1,\, l=1,...,n}^{M+p}\|B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}\|\cdot[h_{j_1,...,j_n},h_{j_1,...,j_n}]+$$

$$+\frac{1}{2}\cdot(\sum_{j_l,k_l=M+1;\, j_l\neq k_l,l=1,...,n}^{M+p}[B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_{j_1,...,j_n},B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_{k_1,...,k_n}]+$$

$$+\sum_{j_l,k_l=M+1;\, j_l\neq k_l;\, l=1,...,n}^{M+p}[B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_{k_1,...,k_n},B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_{j_1,...,j_n}])\leq$$

$$\leq\sum_{j_l=M+1;\, l=1,...,n}^{M+p}[h_{j_1,...,j_n},h_{j_1,...,j_n}]+$$

$$\frac{1}{2}\cdot\sum_{j_l,k_l=M+1;\, k_l\neq j_l;\, l=1,...,n}^{M+p}(\alpha_{j_l,k_l}[B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j,\ B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j]+$$

$$+\frac{1}{\alpha_{j_l,k_l}}\cdot[B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k,B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k])=$$

$$=\sum_{j_l=M+1,\, l=1,...,n}^{M+p}[h_{j_1,...,j_n},h_{j_1,...,j_n}]+\frac{1}{2}\cdot\sum_{j_l,k_l=M+1,\, l=1,...,n}^{M+p}(\frac{(j_1+1)^2...(j_n+1)^2}{(k_1+1)^2...(k_n+1)^2}\cdot$$

$$\cdot[B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j,B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j]+\frac{(k_1+1)^2...(k_n+1)^2}{(j_1+1)^2...(j_n+1)^2}\cdot$$

$$\cdot[B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k,B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k])\leq\sum_{j_l=M+1,\, l=1,...,n}^{M+p}[h_{j_1,...,j_n},h_{j_1,...,j_n}]+$$

$$+\frac{1}{2}\cdot(\sum_{k_l=M+1,\, l=1,...,n}^{M+p}\frac{1}{(k_1+1)^2...(k_n+1)^2}\cdot$$

$$\cdot\sum_{j_l=M+1,\, l=1,...,n}^{M+p}(j_1+1)^2...(j_n+1)^2\cdot[B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j,B_n^{*j_n}...B_1^{*j_1}B_1^{i_1}...B_n^{i_n}h_j]+$$

$$+\sum_{j_l=M+1,\, l=1,...,n}^{M+p}\frac{1}{(j_1+1)^2...(j_n+1)^2}\sum_{k_l=M+1,\, l=1,...,n}^{M+p}(k_1+1)^2...(k_n+1)^2\cdot$$

$$\cdot[B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k,B_n^{*k_n}...B_1^{*k_1}B_1^{i_1}...B_n^{i_n}h_k])\to 0,\text{ as}\, p\longrightarrow\infty.$$

Therefore $k\in\mathcal{K}$ is correctly defined. In the same way one proves that S is correctly defined and bounded.

For $h = (h_{i_1,..,i_n})_{i\in\mathbb{Z}_+^n}$ whose components are equal to zero except for a finite number of $h_{i_1,...,i_n}$, we have by hypothesis

$$[Sh, h] = \sum_{j_l,k_l=0}^{M} [B_1^{j_1}...B_n^{j_n} h_{k_1,...,k_n}, B_1^{k_1}...B_n^{k_n} h_{j_1,...,j_n}] =$$

$$= \sum_{j_l,k_l=0,\, l=1,...,n}^{M} [\frac{A_1^{j_1}...A_n^{j_n} h_{k_1,...,k_n}}{(\|A_1\|+\varepsilon)^{j_1}...(\|A_n\|+\varepsilon)^{j_n}}, \frac{A_1^{k_1}...A_n^{k_n} h_{j_1,...,j_n}}{(\|A_1\|+\varepsilon)^{k_1}...(\|A_n\|+\varepsilon)^{k_n}}] =$$

$$= \sum_{j_l,k_l=0;\, l=1,...,n}^{M} [A_1^{j_1}...A_n^{j_n} z_{k_1,...,k_n}, A_1^{k_1}...A_n^{k_n} z_{j_1,...,j_n}] \geq 0,$$

where $z_{k_1,...,k_n} = \frac{h_{k_1,...,k_n}}{(\|A_1\|+\varepsilon)^{k_1}...(\|A_n\|+\varepsilon)^{k_n}}$. The set of these h is obvious dense in $\mathcal{K}$. This means that $S \geq 0$ on $\mathcal{K}$.

In the same way we define an operator $T : \mathcal{K} \longrightarrow \mathcal{K}$ by $Th = h'$, $h = (h_i)_{i\in\mathbb{Z}_+^n}$ and $h' = (h'_i)_{i\in\mathbb{Z}_+^n}$ where

$$h'_i = h'_{i_1,...,i_n} = \sum_{j_l=0;\, l=1,...,n}^{\infty} B_n^{*j_n+\nu_n}...B_1^{*j_1+\nu_1} B_1^{i_1+\nu_1} B_n^{i_n+\nu_n} h_{j_1,...,j_n}.$$

Analogously we show that T is a positive and bounded operator on $\mathcal{K}$.

We define $B' : \mathcal{K} \longrightarrow \mathcal{K}$ by $B'k = k'$, $k = (k_j)_{j\in\mathbb{Z}_+^n}$ and $k' = (k'_i)_{i\in\mathbb{Z}_+^n}$, where $k'_i = B_1^{\nu_1}...B_n^{\nu_n} k_i$. It is clear that $B' \in \mathcal{B}^*(\mathcal{K})$ and $\|B'\| < 1$. Moreover, $B'' : \mathcal{K} \longrightarrow \mathcal{K}$, $B''k = k''$ where $k'' = (k''_i)_{i\in\mathbb{Z}_+^n}$ and $k''_i = B_n^{*\nu_n}...B_1^{*\nu_1} k_i$ has the same property as B'. Also we have $B''SB' = T$. Therefore $\|T\| \leq \|B''\| \cdot \|S\| \cdot \|B'\| < \|S\|$.

$[Tk, Tk] = [B''SB'k, B''SB'k] \leq [SB'k, SB'k]$.

$$[SB'k, SB'k] = \sum_{j_l=0,l=1,...,n}^{\infty} [k_l, k_l] =$$

$$= \sum_{j_l=0,l=1,...,n}^{\infty} [\sum_{i_l=0}^{\infty} B_n^{*i_n}...B_1^{*i_1} B_1^{j_1+\nu_1}...B_n^{j_n+\nu_n} h_{i_1,...,i_n}, \sum_{k_l=0}^{\infty} B_n^{*k_n}...B_1^{*k_1} B_1^{j_1+\nu_1}...B_n^{j_n+\nu_n} h_{k_1,...,k_n}] =$$

$$= \sum_{j_l\geq 0}^{\infty} \sum_{k_l,i_l=0}^{\infty} [B_n^{*i_n}...B_1^{*i_1} B_1^{j_1+\nu_1}...B_n^{j_n+\nu_n} h_{i_1,...,i_n}, B_n^{*k_n}...B_1^{*k_1} B_1^{j_1+\nu_1}...B_n^{j_n+\nu_n} h_{k_1,...,k_n}] =$$

$$= \sum_{r_l=\nu_l,l=1,...,n}^{\infty} \sum_{k_l,i_l=0}^{\infty} [B_n^{*i_n}...B_1^{*i_1} B_1^{r_1}...B_n^{r_n} h_{i_1,...,i_n}, B_n^{*k_n}...B_1^{*k_1} B_1^{r_1}...B_n^{r_n} h_{k_1,...,k_n}], r_l \in \mathbb{N}, r_l = j_l+\nu_l$$

Hence (where $j_l \in \mathbb{N}, l = 1, ..., n$)

$$[Tk, Tk] \leq \sum_{j_l \geq 0}^{\infty} \sum_{k_l, i_l = 0}^{\infty} [B_n^{*i_n} ... B_1^{*i_1} B_1^{j_1} ... B_n^{j_n} h_{i_1,...,i_n}, B_n^{*k_n} ... B_1^{*k_1} B_1^{j_1} ... B_n^{j_n} h_{k_1,...,k_n}] =$$

$$= [Sk, Sk].$$

Therefore $[Tk, Tk] \leq [Sk, Sk]$ or $[T^2 k, k] \leq [S^2 k, k]$ where $T \geq 0, S \geq 0$. By Proposition 45 of [9], taking $f(x) = \sqrt{x}$ on $[0, \infty)$ we have $\sqrt{T^2} \leq \sqrt{S^2}$ i.e. $T \leq S$.

Thus $T \leq S$ i.e. $[Th, h] \leq [Sh, h]$ for any $h \in \mathcal{K}$.

The last inequality becomes

$$\sum_{j_l, k_l = 0;\, l = 1,...,n}^{M} [B_1^{j_1 + \nu_1} ... B_n^{j_n + \nu_n} h_{k_1,...k_n}, B_1^{k_1 + \nu_1} ... B_n^{k_n + \nu_n} h_{j_1,...j_n}] =$$

$$= \sum_{j_l, k_l = 0;\, l = 1,...,n}^{M} \frac{[A_1^{j_1 + \nu_1} ... A_n^{j_n + \nu_n} z_{k_1,...,k_n}, A_1^{k_1 + \nu_1} ... A_n^{k_n + \nu_n} z_{j_1,...,j_n}]}{(\|A_1\| + \varepsilon)^{2\nu_1} ... (\|A_n\| + \varepsilon)^{2\nu_n}} \leq$$

$$\leq \sum_{j_l, k_l = 0;\, l = 1,...,n}^{M} [B_n^{*k_n} ... B_1^{*k_1} B_1^{j_1} ... B_n^{j_n} h_{k_1,...k_n}, h_{j_1,...j_n}].$$

Because ε was chosen arbitrarily positive, we obtain the inequality (2). ∎

Theorem 3.1 *An n-tuple $(T_1, T_2, ... T_n)$ of commuting operators where $T_l \in \mathcal{B}^*(\mathcal{H}), l = 1, 2, ..., n$ is gramian b-subnormal if and only if:*

$$(3) \qquad \sum_{i_l, j_l \geq 0;\, l = 1,...,n}^{M} [T_1^{i_1} T_2^{i_2} ... T_n^{i_n} h_{j_1, j_2, ..., j_n}, T_1^{j_1} T_2^{j_2} ... T_n^{j_n} h_{i_1, i_2, ..., i_n}] \geq 0$$

Proof. " $\Rightarrow$ " If $\{T_i\}_{i=1}^n \subset \mathcal{L}^*(\mathcal{H})$ is gramian b-subnormal we consider N_i, $i \in \{1, ..., n\}$, a gramian normal extension of n-tuple $(T_1, T_2, ..., T_n)$ to a Loynes Z-space $\mathcal{K} \supseteq \mathcal{H}$, as in the Definition 3.1.

Denoting by P the gramian self-adjoint projection of $\mathcal{K}$ on $\mathcal{H}$, we have $T_i^m h = P N_i^{*m} h$, for any $h \in \mathcal{H}$, $m \geq 1$ and $i \in \{1, ..., n\}$.

Taking into account the equalities $N_i^{*j} N_i^l = N_i^l N_i^{*j}$ and $N_i^{*j} N_k^l = N_k^l N_i^{*j}$, $(i, k \in \{1, ..., n\}, l, j \in \{1, ..., M\})$, obtained by applying Theorem 2.1 to the gramian normal commuting operators N_i and N_k, and using the fact that $\mathcal{H}$ is invariant under $N_i\,(i \in \{1,...,n\})$ for any $h_{i_1,i_2,...,i_n} \in \mathcal{H}$, we obtain

$$\sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [T_1^{i_1}T_2^{i_2}...T_n^{i_n}h_{j_1,j_2,...,j_n}, T_1^{j_1}T_2^{j_2}...T_n^{j_n}h_{i_1,i_2,...,i_n}] =$$

$$= \sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [N_1^{i_1}N_2^{i_2}...N_n^{i_n}h_{j_1,j_2,...,j_n}, N_1^{j_1}N_2^{j_2}...N_n^{j_n}h_{i_1,i_2,...,i_n}] =$$

$$= \sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [N_n^{*j_1}...N_2^{*j_2}N_1^{*j_1}N_1^{i_1}N_2^{i_2}...N_n^{i_n}h_{j_1,j_2,...,j_n}, h_{i_1,i_2,...,i_n}] =$$

$$= \sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [N_1^{i_1}N_2^{i_2}...N_n^{i_n}N_1^{*j_1}N_2^{*j_2}...N_n^{*j_n}h_{j_1,j_2,...,j_n}, h_{i_1,i_2,...,i_n}] =$$

$$= \sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [N_1^{*j_1}N_2^{*j_2}...N_n^{*j_n}h_{j_1,j_2,...,j_n}, N_n^{*i_n}...N_2^{*i_2}N_1^{*i_1}h_{i_1,i_2,...,i_n}] =$$

$$= \sum_{i_l,j_l\geq 0;\, l=1,...,n}^{M} [N_1^{*j_1}N_2^{*j_2}...N_n^{*j_n}h_{j_1,j_2,...,j_n}, N_1^{*i_1}N_2^{*i_2}...N_n^{*i_n}h_{i_1@,i_2,...,i_n}] =$$

$$= [\sum_{j_l\geq 0;\, l=1,...,n}^{M} N_1^{*j_1}N_2^{*j_2}...N_n^{*j_n}h_{j_1,...,j_n}, \sum_{i_l\geq 0;\, l=1,...,n}^{M} N_1^{*i_1}N_2^{*i_2}...N_n^{*i_n}h_{i_1,...,i_n}] =$$

$$= [g,g] \geq 0,$$

where

$$g = \sum_{i_l\geq 0;\, l=1,...,n}^{M} N_1^{*i_1}N_2^{*i_2}...N_n^{*i_n}h_{i_1,...,i_n}.$$

In the previous series of equalities (the third equality from the end) we used the fact that the operators N_i commuted as pairs.

Conversely, let $(T_1,...T_n)$ be an n-tuple which satisfies the condition (3). Let

$$S = \mathbb{N}\times\mathbb{N}\times...\times\mathbb{N} = (\mathbb{N}\times...\times\mathbb{N})\times(\mathbb{N}\times...\times\mathbb{N})$$

be a $*$ semigroup with unit $e = (0, ..., 0)$, the involution $(i_1, ..., i_n, j_1, ..., j_n)^* = (j_1, ..., j_n, i_1, ..., i_n)$ and the operation

$(i_1, ..., i_n, j_1, ..., j_n) + (i_1^l, ..., i_n^l, j_1^l, ..., j_n^l) = (i_1 + i_1^l, ..., i_n + i_n^l, j_1 + j_1^l, ..., j_n + j_n^l).$

The above-defined application satisfies all the conditions from the definition of an involution:

(i 1)

$$((i_1, ..., i_n, j_1, ..., j_n)^*)^* = ((i_1, ..., i_n, j_1, ..., j_n));$$

(i 2)

$$((i_1, ..., i_n, j_1, ..., j_n) + (i_1^l, ..., i_n^l, j_1^l, ..., j_n^l))^* =$$
$$= (i_1 + i_1^l, ..., i_n + i_n^l, j_1 + j_1^l, ..., j_n + j_n^l)^* =$$
$$= (j_1 + j_1^l, ..., j_n + j_n^l, i_1 + i_1^l, ..., i_n + i_n^l) =$$
$$= (j_1, ..., j_n, i_1, ..., i_n) + (j_1^l, ..., j_n^l, i_1^l, ..., i_n^l) =$$
$$= (i_1, ..., i_n, j_1, ..., j_n)^* + (i_1^l, ..., i_n^l, j_1^l, ..., j_n^l)^*;$$
$$= (\lambda \cdot j_1, ..., \lambda \cdot j_n, \lambda \cdot i_1, ..., \lambda \cdot i_n) = \lambda \cdot (i_1, ..., i_n, j_1, ..., j_n)^*.$$

Now, for $T_i \in \mathcal{L}^*(\mathcal{H}), i \in 1, ..., n$ we define the application $S \ni (i_1, ..., i_n, j_1, ..., j_n) \to T(i_1, ..., i_n, j_1, ..., j_n) \in \mathcal{L}^*(\mathcal{H})$ by $T(i_1, ..., i_n, j_1, ..., j_n) = T_1^{*j_1} T_2^{*j_2} ... T_n^{*j_n} T_1^{i_1} ... T_n^{i_n}$.

It is obvious that $T(0, ..., 0) = I$. By using the fact that T_i and T_j commute $(i, j \in 1, ..., n)$, we prove that $T((i_1, ..., i_n, j_1, ..., j_n)^*) = T((i_1, ..., i_n, j_1, ..., j_n))^*$. Indeed

$$T((i_1, ..., i_n, j_1, ..., j_n)^*) = T(j_1, ..., j_n, i_1, ..., i_n) =$$
$$= T_1^{*i_1} T_2^{*i_2} ... T_n^{i_n} T_1^{j_1} ... T_n^{j_n}$$

and

$$T((i_1, ..., i_n, j_1, ..., j_n))^* = (T_1^{*j_1} T_2^{*j_2} ... T_n^{*j_n} T_1^{i_1} ... T_n^{i_n})^* =$$
$$T_n^{*i_n} ... T_1^{*i_1} T_n^{j_n} ... T_1^{j_1} = T_1^{*i_1} ... T_n^{*i_n} T_1^{j_1} ... T_n^{j_n}.$$

We denote $\pi = (i_1, ..., i_n, j_1, ..., j_n)$ and $\pi^l = (i_1^l, ..., i_n^l, j_1^l, ..., j_n^l)$.

Let $\{g_\pi\} \subset \mathcal{H}$ be a finite set. We have,

$$\sum_{\pi,\, \pi^l}{}' \, [T_{\pi^*+\pi^l} \cdot g_{\pi^l}, g_\pi] =$$

$$\sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' [T_{(j_1+i_1^l,\dots,j_n+i_n^l,i_1+j_1^l,\dots,i_n+j_n^l)} \cdot g_{i_1^l,\dots,i_n^l,j_1^l,\dots,j_n^l}, g_{i_1,\dots,i_n,j_1,\dots,j_n}] =$$

$$= \sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' [T_1^{*i_1+j_1^l} \dots T_n^{*i_n+j_n^l} \dots T_n^{j_n+i_n^l} g_{i_1^l,\dots,j_n^l}, g_{i_1,\dots,j_n}] =$$

$$= \sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' [T_1^{j_1+i_1^l} \dots T_n^{j_n+i_n^l} g_{i_1^l,\dots,j_n^l}, T_n^{i_n+j_n^l} \dots T_1^{i_1+j_1^l} g_{i_1,\dots,j_n}] =$$

$$= \sum_{(i_1,\dots,j_n),\, (i_1^l,\dots,j_n^l)}{}' [T_1^{j_1+i_1^l} \dots T_n^{j_n+i_n^l} g_{i_1^l,\dots,j_n^l}, T_1^{i_1+j_1^l} \dots T_n^{i_n+j_n^l} g_{i_1,\dots,j_n}] =$$

$$= \sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' [T_1^{j_1} \dots T_n^{j_n} h_{j_1^l,\dots,j_n^l}, T_1^{j_1^l} \dots T_n^{j_n^l} h_{j_1,\dots,j_n}] \geq 0$$

by relation (3), where we noted

$$h_{j_1,\dots,j_n} = \sum_{(i_l,\dots,i_n)}{}' \, T_1^{i_1} \dots T_n^{i_n} g_{i_1,\dots,j_n}$$

and

$$h_{j_1^l,\dots,j_n^l} = \sum_{(i_l^l,\dots,i_n^l)}{}' \, T_1^{i_1^l} \dots T_n^{i_n^l} g_{i_1^l,\dots,j_n^l}.$$

By using the condition (b), we obtain (here $u = (u_1, \dots, u_n, v_1, \dots, v_n)$)

$$\sum_{\pi,\, \pi^l}{}' \, [T_{\pi^*+u^*+u+\pi^l} g_{\pi^l}, g_\pi] =$$

$$= \sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' \, [T(j_1+v_1+u_1+i_1^l, \dots, i_n+u_n+v_n+j_n^l) g_{i_1^l,\dots,j_n^l}, g_{i_1,\dots,j_n}] =$$

$$= \sum_{(i_1,\dots,j_n);\, (i_1^l,\dots,j_n^l)}{}' \, [T_1^{*i_1+u_1+v_1+j_1^l} \dots T_n^{*i_n+u_n+v_n+j_n^l} T_1^{j_1+v_1+u_1+i_1^l} \dots T_n^{j_n+v_n+u_n+i_n^l} g_{i_1^l,\dots,j_n^l}, g_{i_1,\dots,j_n}]$$

$$= \sum_{(i_1,...,j_n);\,(i_1^l,...,j_n^l)} {}' [T_1^{j_1+v_1+u_1+i_1^l} ... T_n^{j_n+v_n+u_n+i_n^l} g_{i_1^l,...,j_n^l}, T_1^{i_1+u_1+v_1+j_1^l} ... T_n^{i_n+u_n+v_n+j_n^l} g_{i_1,...,j_n}]$$

$$= \sum_{(j_1,...,j_n);\,(j_1^l,...,j_n^l)} {}' [T_1^{j_1+v_1+u_1} ... T_n^{j_n+v_n+u_n} h_{j_1^l,...,j_n^l}, T_1^{u_1+v_1+j_1^l} ... T_n^{u_n+v_n+j_n^l} h_{j_1,...,j_n}].$$

In what follows, we apply the condition (b) and it follows

$$\sum_{\pi,\pi^l} {}' [T_{\pi^*+u^*+u+\pi^l} g_{\pi^l}, g_\pi] \leq$$

$$\leq c \cdot \sum_{(j_1,...,j_n);\,(j_1^l,...,j_n^l)} {}'[T_1^{j_1} ... T_n^{j_n} h_{j_1^l,...,j_n^l}, T_1^{j_1^l} ... T_n^{j_n^l} h_{j_1,...,j_n}] =$$

$$= c \cdot \sum_{\pi,\pi^l} {}' [T_{\pi^*+\pi^l} g_{\pi^l}, g_\pi]$$

Therefore the conditions of Theorem 2.2 are satisfied for T_π so that there exists a representation in $\mathcal{K}$, D_π which will be as in the fundamental theorem (see [6]) and $\mathcal{H}$ is accessible in $\mathcal{K}$.

Because $\pi = (i_1, ..., i_n, j_1, ..., j_n) =$

$= i_1 \cdot (1, 0, ..., 0) + ... + i_n \cdot (0, ..., 0, 1, 0, ..., 0) + j_1 \cdot (0, ..., 0, 1, 0, ..., 0) + ...+,$

$+ j_n \cdot (0, ..., 0, 1) = i_1 \cdot \eta_1 + ... + i_n \cdot \eta_n + ... + j_1 \cdot \eta_1^* + ... + j_n \cdot \eta_n^*$
where $\eta_i = (0, ..., 0, 1, 0, ..., 0)$ with 1 on i place and $\eta_i^* = (0, ..., 0, 1, 0, ..., 0)$ with 1 on $n+i$ place.

Therefore

$$\pi = \sum_{l=1}^{n} (\eta_l \cdot i_l + \eta_l^* \cdot j_l)$$

and denoting $D_{\eta_l} = N_l$ we have

$$D_\pi = \prod_{l=1}^{n} D_{\eta_l i_l} D_{\eta_l^* j_l} = \prod_{l=1}^{n} N_l^{i_l} N_l^{*j_l} =$$

$$= N_1^{*j_1} ... N_n^{*j_n} N_1^{i_1} ... N_n^{i_n}.$$

It is obvious that N_l, $l = 1, ..., n$ is a gramian normal operator and from $T(i_1, ..., i_n, j_1, ..., j_n) = T_1^{*j_1} ... T_n^{*j_n} T_1^{i_1} ... T_n^{i_n}$ and $T_\pi h = P D_\pi h$, $h \in \mathcal{H}$ it follows that

$$(*) T_1^{*j_1}...T_n^{*j_n}T_1^{i_1}...T_n^{i_n}h = PN_1^{*j_1}...N_n^{*j_n}N_1^{i_1}...N_n^{j_n}h$$

$h \in \mathcal{H}$.

Using this last equality we show that $T \subseteq N$.

Taking in $(*)$ $j_1 = ... = j_n = i_2 = ... = i_n = 0, ..., j_1 = ... = j_n = i_1 = ... = i_{n-1} = 0$ and $j_1 = i_1 = 1$ and $j_2 = i_2 = ... = i_n = 0$ respectively,..., $j_n = i_n = 1$ and $j_1 = ... = j_{n-1} = i_1 = ... = i_{n-1} = 0$ respectively, we obtain $T_i^j h = PN_i^j h$ and $T_i^* T_i h = PN_i^* N_i h, (h \in \mathcal{H}, i \in \{1, ..., n\})$.

From $[T_i h, T_i h] = [PN_i h, PN_i h]$ we have $[N_i h - PN_i h, N_i h - PN_i h] = 0$. It is easy to see that $N_i h = T_i h, (h \in \mathcal{H}, i \in \{1, ..., n\})$.

We obtain (b) for N_i instead of T_i and taking only a term we obtain that N_i is from $\mathcal{B}^*(\mathcal{K})$.

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References

[1] J. Bram, *Subnormal operators*, Duke Math. J., **22**(1955), 75-94.

[2] L. Ciurdariu, A. Crăciunescu, *On spectral representation of Gramian normal operators on Pseudo-Hilbert spaces*, Anal. Univ. de Vest Timişoara, **XLV**, 1(2007), 131-149.

[3] D. Gaspar, *Functional analysis*, Facla, Timişoara, 1984. (in Romanian)

[4] P. Gaspar, *Armonic analysis on spaces of random variables*, PhD Thesis, Univ. Vest Timişoara, 2004.(in Romanian)

[5] T. Ito, *On the commutative family of subnormal operators*, J. Faculty Hokkaido Univ. Ser. I. Math., **14**(1958), 1-15.

[6] R.M.Loynes, *On generalized positive definite functions*, Proc. London Math. Soc., **3(15)**(1965), 373-384.

[7] R.M.Loynes, *Linear operators in VH-spaces*, Trans. American Math. Soc. **116**(1965), 167-180.

[8] F.Riesz, B.Sz.-Nagy, *Lecons d'analyse functionnelle*, Budapest, 1952.

[9] Ş. Strătilă, L. Zsido, *Operator algebras*, I, II, T.U.T., Timişoara, 1995.

[10] A. Weron, S. A. Chobanyan, *Stochastic processes on pseudo-Hilbert spaces*, Bull. Acad. Polon., Ser. Math. Astr. Phys., **XX1**, 9(1973), 847-854.(in Russian)

[11] A. Weron, *Prediction theory in Banach spaces*, Proc. of Winter School on Probability, Karpacz, Springer, London, 1975, 207-228.

ROMAI J., **5**, 1(2009), 45–61

DARBOUX INTEGRABILITY IN THE CUBIC DIFFERENTIAL SYSTEMS WITH THREE INVARIANT STRAIGHT LINES

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Abstract For cubic differential systems with three invariant straight lines and a weak focus, conditions for $O(0,0)$ to be a center were found. The presence of a center at $O(0,0)$ is proved by constructing an integrating factor formed from these lines.

Keywords: cubic differential systems, center problem, invariant algebraic curves, Darboux integrability.

2000 MSC: 34C05, 58F14.

1. INTRODUCTION

Consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 = P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) = -Q(x,y), \end{aligned} \tag{1}$$

in which the variables x, y and coefficients a, b, $\ldots$, s are assumed to be real. The origin $O(0,0)$ is a singular point of a center or focus type for (1), i.e. a weak focus. The purpose of this paper is to find verifiable conditions for their distinctions.

It is known [5] that the origin is a center for system (1) if and only if in some neighborhood of $O(0,0)$ it possesses a holomorphic first integral $F(x,y) = C$ or a holomorphic integrating factor of the form

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y),$$

where μ_k are homogeneous polynomials of degree k.

Also, it is known [9] that for (1) a formal power series $\Psi(x, y)$ can be found, such that

$$D(\Psi) = g_3(x^2 + y^2)^2 + g_5(x^2 + y^2)^3 + \cdots,$$

where g_{2j+1} are polynomials in the coefficients of system (1) called focal values. The origin $O(0, 0)$ is a center for (1) if and only if $g_{2j+1} = 0,\ j = 1, 2, \ldots, \infty$.

The problem of the center was solved for quadratic systems and for cubic systems with only homogeneous cubic nonlinearities. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see, for example, [9], [2], [3], [13], [14]–[15]).

In this paper we solve the problem of the centre for cubic differential system (1) assuming that (1) has three invariant straight lines and is Darboux integrable. The paper is organized as follows. The results concerning the relation between Darboux integrability and invariant algebraic curves are presented in sections 2 and 3. In section 4 we find four sufficient series of conditions for the existence of three invariant straight lines. In section 5 we obtain sufficient conditions for the existence of a centre by using the Darboux method of integrability.

2. INVARIANT STRAIGHT LINES AND CENTERS IN CUBIC SYSTEMS

We shall study the problem of the center assuming that (1) has invariant straight lines.

Definition 2.1. *An algebraic invariant curve [1] (or an algebraic particular integral) of (1) is a set of points (considered over $\mathbb{C}^2$) satisfying an equation $f(x, y) = 0$, where f is a polynomial in x and y such that*

$$\frac{df}{dt} = \dot{f} = \frac{\partial f}{\partial x}P - \frac{\partial f}{\partial y}Q = fK,$$

for some polynomial $K = K(x, y)$ called the cofactor of the invariant algebraic curve $f(x, y) = 0$.

By Definition 1 a straight line

$$L \equiv C + Ax + By = 0, \quad A^2 + B^2 \neq 0, \tag{2}$$

is an invariant straight lines for (1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$A \cdot P(x,y) - B \cdot Q(x,y) \equiv (C + Ax + By) \cdot K(x,y). \tag{3}$$

According to [2] the cubic system (1) cannot have more than four nonhomogeneous invariant straight lines, i.e. invariant straight lines of the form

$$1 + Ax + By = 0, \quad A^2 + B^2 \neq 0. \tag{4}$$

The cofactor of (4) is

$$\begin{aligned} K(x,y) \quad = \quad & -Bx + Ay + (aA - gB + AB)x^2 + \\ & (cA - dB + B^2 - A^2)xy + (fA - bB - AB)y^2. \end{aligned} \tag{5}$$

If the cubic system (1) has complex invariant straight lines then obviously they occur in complex conjugated pairs

$$L \equiv C + Ax + By = 0 \quad \text{and} \quad \overline{L} \equiv \overline{C} + \overline{A}x + \overline{B}y = 0.$$

As homogeneous invariant straight lines $Ax + By = 0$ system (1) can have only the lines $x \pm iy = 0,\ i^2 = -1$.

From (3) identifying the coefficients of monomials in x and y, it follows that (4) is an invariant straight line of (1) if and only if A and B are the solutions of the system

$$\begin{aligned} &F_1(A,B) = A^2B + aA^2 - gAB - kA + sB = 0, \\ &F_2(A,B) = AB^2 - fAB + bB^2 + rA - lB = 0, \\ &F_3(A,B) = A^3 - 2AB^2 - cA^2 + (d-a)AB + gB^2 + mA - qB = 0, \\ &F_4(A,B) = B^3 - 2A^2B + fA^2 + (c-b)AB - dB^2 - pA + nB = 0. \end{aligned} \tag{6}$$

In [2] and [3] for cubic differential system (1) coefficient conditions for the existence of four invariant straight lines was found. It was proved that the system (1) with four invariant straight lines (real, complex, real and complex) has a singular point of a center type at the origin, if and only if the first two focal values vanish.

3. DARBOUX INTEGRABILITY AND INTEGRATING FACTORS

The problem of integrating a differential equation by using invariant algebraic curves was considered for the first time by Darboux in 1878 in [4]. One of the main applications of the Darboux method is proving the existence of a center.

Definition 3.1. *A real system (1) is integrable on an open set U of $\mathbb{R}^2$ if there exists a nonconstant analytic function $F : U \to \mathbb{R}$ which is constant along all solution curves $(x(t), y(t))$ in U, i.e. $F(x(t), y(t)) =$ constant, for all values of t where the solution is defined. Such an F is called a first integral of (1) or a constant of motion on U.*

Definition 3.2. *An integrating factor for a system (1) on some open set U of $\mathbb{R}^2$ is a $\mathbb{C}^1$ function $\mu = \mu(x, y)$ defined on U, not identically zero on U, such that*

$$\frac{\partial(\mu P)}{\partial x} - \frac{\partial(\mu Q)}{\partial y} = 0.$$

This condition also can be written as follows:

$$\frac{\partial \mu}{\partial x}P - \frac{\partial \mu}{\partial y}Q + \mu\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) = 0. \tag{7}$$

Theorem 3.1. *Suppose that $f(x, y) \in \mathbb{C}[x, y]$ and let $f(x, y) = f_1^{n_1} \cdots f_q^{n_q}$ be the factorization of f in irreducible factors over $\mathbb{C}[x, y]$. Then the curve $f(x, y) = 0$ is invariant if and only if the curves $f_j(x, y) = 0$ are invariant for $j = 1, \ldots q$.*

If the cubic system (1) has sufficiently many invariant algebraic curves $f_j(x, y) = 0,\ j = 1, \ldots, q$, then in most cases a first integral (an integrating factor) can be constructed in the Darboux form [4]

$$f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q}, \tag{8}$$

where $\alpha_i \in \mathbb{C}$ and $f_i(x, y) \in \mathbb{C}[x, y]$. In this case we say that the system (1) is *Darboux integrable.*

If (8) is a first integral or an integrating factor, then necessarily the curves $f_i(x, y) = 0$ are invariant algebraic curves of (1).

Theorem 3.2. *The expression (8) is an integrating factor for (1) if and only if*

$$\sum_{i=1}^{N} \alpha_i K_i(x,y) \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}, \tag{9}$$

where $\alpha_i,\ i = \overline{1,N}$ *are numbers do not simultaneously equal to zero.*

Let the cubic system (1) have three invariant straight lines $L_j = 0,\ j = 1,2,3$ of the form (2). According to Theorem 2 system (1) has an integrating factor of the form

$$\mu(x,y) = L_1^{\alpha_1} L_2^{\alpha} L_3^{\alpha_3} \tag{10}$$

if and only if

$$\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3 \equiv \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}, \tag{11}$$

where $\alpha_j,\ j = 1,2,3$ are numbers do not simultaneously equal to zero.

Integrating factors of the form (8) were successfully used solving the problem of the center for quadratic systems [6] and for cubic system with only homogeneous cubic nonlinearities [7].

By [2] and [3] if the cubic system (1) has four invariant straight lines and the first two focal values vanish, then it is Darboux integrable.

4. CONDITIONS FOR THE EXISTENCE OF THREE INVARIANT STRAIGHT LINES

It is easy to see that for the relative positions of three invariant straight lines three different cases can occur:

1) two of the invariant straight lines are parallel;

2) all the invariant straight lines pass through the same point (forming a bundle);

3) the invariant straight lines are in a generic position (forming a triangle).

4.1. CONDITIONS FOR THE EXISTENCE OF THREE INVARIANT STRAIGHT LINES TWO OF WHICH ARE PARALLEL

Let the invariant straight lines L_1, L_2 be real and parallel for (1). Then by a rotation of axes we can make them parallel to the axis of ordinates (Oy). Note that by a rotation of axes of coordinates the differential system (1) does not change the form.

If L_1 and L_2 are complex, then the straight lines $\overline{L_1}, \overline{L_2}$ conjugate with L_1 and L_2 are also invariant for (1) (the coefficients in (1) are real). As for (1) the problem of the center with at least four invariant straight lines is solved, it remains to consider only the case when $L_2 \equiv \overline{L_1}$. From $L_1 || \overline{L_1}$, it follows that L_1 looks as $1 + A(x + By) = 0$, where A is a complex number and B is real. In this case, via a rotation of axes about the origin, it is also possible to make the straight lines L_1 and L_2 to be parallel to the axis Oy. Notice that the case: the cubic system (1) has a weak focus and three parallel invariant straight lines is not realized.

In order for the cubic system (1) have two invariant straight lines L_1, L_2 parallel to the axis Oy, it is necessary that the right-hand sides of the first equation from (1) look as $P(x, y) = y(1 + cx + mx^2)$, i.e.

$$a = f = k = p = r = 0, \quad m(c^2 - 4m) \neq 0. \tag{12}$$

If conditions (12) are fulfilled, then (1) has the following invariant straight lines

$$L_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0.$$

We pass now to the problem of finding conditions for the existence of the third invariant straight line, which for above reasons is assumed to be real. Let L_3 be given by the equation $1 + Ax + By = 0$, where A, B are real numbers and $B \neq 0$. In this case A, B should satisfy the algebraic system of equations (6).

Rescaling on coordinate axes and time t, we can make $B = 1$ and the system (1) under conditions (12) does not change the form.

Assume that $B = 1$, then from (6) and (12) we find $A = l - b$ in conditions

$$
\begin{aligned}
&a = f = k = p = r = 0, \quad n = b^2 + bc - 3bl - cl + d + 2l^2 - 1,\\
&q = -b^3 - b^2c + 3b^2l + 2bcl - bd - 3bl^2 - bm + 2b - cl^2 +\\
&\ + dl + g + l^3 + lm - 2l, \quad s = -b^2 - bg + 2bl + gl - l^2.
\end{aligned} \tag{13}
$$

So, in order for the system (1) have three invariant straight lines L_1, L_2, L_3, $L_1||L_2||Oy$, $L_3 \equiv 1 + Ax + y = 0$, it is necessary and sufficient that conditions (13) be fulfilled. The invariant straight lines are

$$
L_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2}x = 0, \quad L_3 \equiv 1 + (l - b)x + y = 0 \tag{14}
$$

and have, respectively, the cofactors

$$
\begin{aligned}
&K_{1,2} = y(2mx + c \pm \sqrt{c^2 - 4m})/2, \quad K_3 = -x + (l - b)y +\\
&+(l - b - g)x^2 + (1 - b^2 - bc + 2bl + cl - d - l^2)xy - ly^2.
\end{aligned} \tag{15}
$$

4.2. CONDITIONS FOR THE EXISTENCE OF A BUNDLE OF THREE INVARIANT STRAIGHT LINES

Assume that the cubic system (1) has three invariant straight lines which pass through the same point (x_0, y_0). By a rotation and rescaling of coordinate axes we can make $x_0 = 0,\ y_0 = 1$. In this case, the equation of each invariant straight line forming a bundle has the form

$$
1 + Ax - y = 0. \tag{16}
$$

Obviously the point $(0, 1)$ of the intersection of these invariant straight lines is a singular point for (1), i.e. $P(0, 1) = Q(0, 1) = 0$. From these equalities we find $r = -f - 1,\ l = -b$. Substituting $B = -1,\ r = -f - 1$ and $l = -b$ in (6) we obtain

$$
\begin{aligned}
&F_1(A, B) = (a - 1)A^2 + (g - k)A - s = 0, \quad F_2(A, B) \equiv 0,\\
&F_3(A, B) = A^3 - cA^2 + (a - d + m - 2)A + g + q = 0,\\
&F_4(A, B) = (f + 2)A^2 + (b - c - p)A - d - n - 1 = 0.
\end{aligned}
$$

From these equalities we can see that the system (1) has three distinct invariant straight lines of the form (16) if and only if the following conditions

$$a = 1,\ f = -2,\ k = g,\ l = -b,\ n = -d - 1,\ p = b - c,\ r = 1,\ s = 0, \quad (17)$$

$$\begin{aligned} &4(g + q)c^3 + (d - m + 1)^2c^2 + 18(d - m + 1)(g + q)c + 4d^3- \\ &12(m - 1)d^2 + 12(m - 1)^2d - 27(g + q)^2 - 4(m - 1)^3 \neq 0. \end{aligned} \quad (18)$$

hold.

In the conditions (17), (18) the straight line (16) is invariant for (1) if and only if A satisfies the equation

$$A^3 - cA^2 + (m - d - 1)A + g + q = 0. \quad (19)$$

The left-hand side of inequality (18) coincides with the discriminant of the equation (19), and (18) implies the roots $A_1,\ A_2,\ A_3$ of the equation (19) to be distinct: $A_i \neq A_j,\ \ \forall i \neq j$.

4.3. CONDITIONS FOR THE EXISTENCE OF THREE INVARIANT STRAIGHT LINES FORMING A TRIANGLE

Assume that the differential system (1) has exactly three invariant straight lines

$$L_j \equiv A_jx + B_jy + C_j = 0, \quad j = 1, 2, 3;\ A_j, B_j, C_j \in \mathbb{C} \quad (20)$$

such that no pair of the lines is parallel and no more than two lines pass through the same point (in generic position), i.e.

$$\Delta_{jl} = \begin{vmatrix} A_j & B_j \\ A_l & B_l \end{vmatrix} \neq 0,\ \ j \neq l,\ j, l = 1, 2, 3;\ \ \Delta_{123} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0. \quad (21)$$

The invariant straight line L_3 can be considered real.

Conditions (21) allow us to write system (1) in the form [8]

$$\begin{aligned} \frac{dx}{dt} &= \Big(\sum_{j=1}^{3} \frac{T_jL_{jy}}{L_j} + p_1\Big) \prod_{j=1}^{3} L_j \equiv P(x, y), \\ \frac{dy}{dt} &= -\Big(\sum_{j=1}^{3} \frac{T_jL_{jx}}{L_j} + q_1\Big) \prod_{j=1}^{3} L_j \equiv Q(x, y), \end{aligned} \quad (22)$$

where $L_{jx} = \partial L_j/\partial x,\ L_{jy} = \partial L_j/\partial y;\ p_1,\ q_1 \in \mathbb{R}$ and $T_j,\ j = 1, 2, 3$ are linear in x and y. Let $T_j = m_j x + n_j y + s_j,\ j = 1, 2, 3$.

The straight lines L_1, L_2, L_3 have respectively the cofactors

$$\begin{aligned} K_1(x, y) &= \Delta_{12} L_3 T_2 + \Delta_{13} L_2 T_3 + (p_1 A_1 - q_1 B_1) L_2 L_3, \\ K_2(x, y) &= \Delta_{23} L_1 T_3 + \Delta_{21} L_3 T_1 + (p_1 A_2 - q_1 B_2) L_1 L_3, \\ K_3(x, y) &= \Delta_{31} L_2 T_1 + \Delta_{32} L_1 T_2 + (p_1 A_3 - q_1 B_3) L_1 L_2. \end{aligned} \tag{23}$$

By affine transformations of coordinates and time rescaling

$$x \to \alpha_1 x + \beta_1 y + \gamma_1, \quad y \to \alpha_2 x + \beta_2 y + \gamma_2, \quad t \to \alpha t \tag{24}$$

system (22) does not change the form.

Let (x^*, y^*) be a singular point for (22) with pure imaginary eigenvalues. By transformations of the form (24), first we translate (x^*, y^*) at the origin, i.e.

$$P(0, 0) = Q(0, 0) = 0 \tag{25}$$

and then transform the linear part of $P(x, y)$ to be equal with y, and of $Q(x, y)$ to be equal with $-x$, i.e.

$$P'_x(0, 0) = Q'_y(0, 0) = 0, \quad P'_y(0, 0) = -Q'_x(0, 0) = 1. \tag{26}$$

The intersection point of the straight lines L_1 and L_2 is a singular point for (22) and has real coordinates. In particular, this point can be $(0, 0)$.

We shall consider two cases: **1)** $(0, 0) \notin L_1 \cap L_2$ and **2)** $(0, 0) \in L_1 \cap L_2$.

1). Let $(0, 0) \notin L_1 \cap L_2$. By rotating the system of coordinates ($x \to x\cos\varphi - y \sin\varphi$, $y \to x \sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x$, $y \to \alpha y$), we obtain

$$L_1 \cap L_2 = (0, 1). \tag{27}$$

In this case the invariant straight lines (20) can be written as

$$L_j = A_j x - y + 1,\ L_3 = A_3 x + B_3 y + 1, \quad A_j \in \mathbb{C},\ j = 1, 2;\ A_3, B_3 \in \mathbb{R}, \tag{28}$$

and (21):

$$\Delta_{12} = A_2 - A_1 \neq 0,\ \Delta_{j3} = A_j B_3 + A_3 \neq 0,\ j = 1, 2,\ \Delta_{123} = B_3 + 1 \neq 0. \tag{29}$$

The relations (25), (26) and $P(0,1) = Q(0,1) = 0$ induce the following conditions on the coefficients of system (22):

$$
\begin{aligned}
& p_1 = s_1 + s_2 - s_3 B_3, \quad q_1 = -s_1 A_1 - s_2 A_2 - s_3 A_3, \\
& m_1 = (s_1 A_1^2 - s_1 A_1 A_2 + s_3 A_2 A_3 B_3 - m_3 A_2 B_3 + \\
& \quad + s_3 A_3^2 - m_3 A_3 + 1)/(A_1 - A_2), \\
& n_1 = (s_3 A_2 B_3^2 - s_1 A_1 - n_3 A_2 B_3 + s_1 A_2 + A_2 + \\
& \quad + s_3 A_3 B_3 - n_3 A_3)/(A_1 - A_2), \\
& m_2 = (s_2 A_1 A_2 - s_3 A_1 A_3 B_3 + m_3 A_1 B_3 - s_2 A_2^2 - s_3 A_3^2 + \\
& \quad + m_3 A_3 - 1)/(A_1 - A_2), \\
& n_2 = (n_3 A_1 B_3 - s_3 A_1 B_3^2 - s_2 A_1 - A_1 + s_2 A_2 - s_3 A_3 B_3 + \\
& \quad + n_3 A_3)/(A_1 - A_2).
\end{aligned}
\tag{30}
$$

2). Let now $(0,0) \in L_1 \cap L_2$. In this case

$$L_{1,2} \equiv x \pm iy = 0,\ i^2 = -1 \quad \text{and} \quad L_3 \equiv A_3 x + B_3 y + 1 = 0.$$

The identity (3) for $L_{1,2} = x \pm iy$ gives the following coefficient conditions for (1)

$$b + c - g = a + d - f = p - q + l - k = m + n - r - s = 0. \tag{31}$$

Via a rotation of axes about the origin and under the transformation $x \to \gamma x,\ y \to \gamma y,\ \gamma \in R \setminus 0$, the invariant straight line $L_3 = 0$ becomes $1 - x = 0$. Remark that under the above-mentioned transformation the curve $x^2 + y^2 = 0$ remains invariant.

For $L_3 = 1 - x$ the identity (3) yields

$$k = -a, \quad m = -c - 1, \quad p = -f, \quad r = 0. \tag{32}$$

From (31) and (32) we obtain that $L_{1,2} \equiv x^2 + y^2 = 0,\ L_3 \equiv 1 - x = 0$ are invariant straight lines for (1) if and only if the following relations

$$
\begin{aligned}
& f = a + d, \quad g = b + c, \quad k = -a, \quad l = d + q, \quad m = -c - 1, \\
& n = c + s + 1, \quad p = -a - d, \quad r = 0,
\end{aligned}
\tag{33}
$$

are satisfied. These curves have respectively the cofactors

$$
\begin{aligned}
& K_{1,2} = 2(ax - by - ax^2 - (c + s + 1)xy - (d + q)y^2), \\
& K_3 = -y - ax^2 - (c + 1)xy - (a + d)y^2.
\end{aligned}
\tag{34}
$$

5. DARBOUX INTEGRABILITY IN THE CUBIC SYSTEMS WITH THREE INVARIANT STRAIGHT LINES

In this section by using the identity (11) we shall construct an integrating factor of the Darboux type (10) in each of the cases (13), (17), (30) and (33). In this way we prove the presence of a center at the origin.

5.1. SOLUTION OF THE PROBLEM OF THE CENTER FOR CUBIC DIFFERENTIAL SYSTEMS WITH THREE INVARIANT STRAIGHT LINES OF WHICH TWO ARE PARALLEL

Let us consider the invariant straight lines (14) with cofactors (15). From the identity (11), we find that

$$\begin{aligned}
x:&\quad \alpha_3=-d,\\
y:&\quad \sqrt{c^2-4m}\,(\alpha_1-\alpha_2)+(\alpha_1+\alpha_2+2)c+2bd-2ld-4b=0,\\
xy:&\quad (b^2+bc-lc+d-1)(d-2)+(d-4)l^2-2bl(d-3)+\\
&\quad +(\alpha_1+\alpha_2+2)m=0,\\
x^2:&\quad l^3-l^2(3b+c)+l(3b^2+2bc+2d-2)+(l-b)m-\\
&\quad -b^3-b^2c-2bd+2b-dg+g=0,\\
y^2:&\quad l(d-3)=0.
\end{aligned}$$

We shall consider two cases: **1)** $l=0$ and **2)** $d-3=0,\ l\neq 0$.

1) Let $l=0$, then $x^2: b^3+b^2c+2bd+bm-2b+dg-g=0$.

If $b=g=0$, the cubic system (1) along with three invariant straight lines (14) has also one more invariant straight line $1+(d-1)y=0$.

If $b=0,\ d=1,\ g\neq 0$, we have the following conditions

$$a=b=f=k=l=n=p=r=s=0,\quad d=1,\quad q=g \tag{35}$$

for the existence of three invariant straight lines

$$L_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0, \quad L_3 \equiv 1 + y = 0$$

and a Darboux integrating factor of the form (10) with $\alpha_1 = \alpha_2 = \alpha_3 = -1$.

If $b \neq 0,\ m = (-b^3 - b^2c - 2bd + 2b - dg + g)/b$, then along with three invariant straight lines (14), the system (1) has also one more invariant straight line $1 - bx + (d-1)y = 0$.

2) Let now $d = 3,\ l \neq 0$, then

$$x^2 : l^3 - (3b + c)l^2 + (3b^2 + 2bc + 4)l + (l - b)m - b^3 - b^2c - 4b - 2g = 0.$$

If $l = b$, then $g = 0$ and we have the following conditions

$$a = f = g = k = p = q = r = s = 0, \quad d = 3, \quad l = b, \quad n = 2 \tag{36}$$

for the existence of three invariant straight lines

$$L_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0, \quad L_3 \equiv 1 + y = 0$$

and a Darboux integrating factor of the form (10) with

$$\alpha_{1,2} = \frac{(m+1)(4m - c^2) \pm (c + 2bm)\sqrt{c^2 - 4m}}{m(c^2 - 4m)}, \quad \alpha_3 = -3$$

If $l - b \neq 0$, then we obtain the following conditions

$$\begin{aligned}
&a = f = k = p = r = 0, \quad d = 3, \quad q = 3(b + g - l), \\
&m = (-b^3 - b^2c + 3b^2l + 2bcl - 3bl^2 - 4b - cl^2 - 2g + l^3 + 4l)/(b - l), \\
&s = -b^2 - bg + 2bl + gl - l^2, \quad n = b^2 + bc - 3bl - cl + 2l^2 + 2,
\end{aligned} \tag{37}$$

for the existence of three invariant straight lines

$$L_{1,2} \equiv 1 + \frac{c(b - l) \pm \sqrt{(b - l)\,\delta}}{2(b - l)} x = 0, \quad L_3 \equiv 1 + (l - b)x + y = 0$$

and a Darboux integrating factor of the form (10) with

$$\alpha_{1,2} = \left(\delta_1 \pm \delta_2 \sqrt{(b - l)\,\delta}\right)/\delta_3, \quad \alpha_3 = -3,$$

where

$$\begin{aligned}
\delta\ &= 4b^3 + 4b^2c - 12b^2l + bc^2 - 8bcl + 12bl^2 + 16b - c^2l + 4cl^2 + 8g - 4l^3 - 16l,\\
\delta_1\ &= \delta(b^3 + b^2c - 5b^2l - 2bcl + 7bl^2 + 6b + cl^2 + 4g - 3l^3 - 6l),\\
\delta_2\ &= 2b^4 + 3b^3c - 12b^3l + b^2c^2 - 11b^2cl + 24b^2l^2 + 8b^2 - 2bc^2l + 13bcl^2 + 2bc\\
&\quad + 4bg - 20bl^3 - 32bl + c^2l^2 - 5cl^3 - 2cl - 12gl + 6l^4 + 24l^2,\\
\delta_3\ &= -2\delta(b^3 + b^2c - 3b^2l - 2bcl + 3bl^2 + 4b + cl^2 + 2g - l^3 - 4l).
\end{aligned}$$

Theorem 5.1. *The conditions (35), (36) and (37) are sufficient conditions in order for the cubic system (1) to have three invariant straight lines of two which are parallel, and a center at the origin.*

5.2. SOLUTION OF THE PROBLEM OF THE CENTER FOR CUBIC DIFFERENTIAL SYSTEMS WITH A BUNDLE OF THREE INVARIANT STRAIGHT LINES

Let the coefficient conditions (17) are satisfied and the inequality (18) holds. In this case, we have three distinct invariant straight lines $L_j \equiv 1 + A_j x - y = 0,\ j = 1, 2, 3$ with cofactors $L_j = x + A_j y + gx^2 + (cA_j - A_j^2 + d + 1)xy + (b - A_j)y^2$.

In order to find an integrating factor composed from these invariant straight lines we use the identity (11), which yields: $q = g(d + 1),\ bd = 0$ and

$$\begin{aligned}
\alpha_1 &= [(d-2)A_2A_3 + (c-2b)(A_2 + A_3 - c) + d^2 + d + 2m]/(A_1 - A_2)(A_1 - A_3),\\
\alpha_2 &= [(d-2)A_1A_3 + (c-2b)(A_1 + A_3 - c) + d^2 + d + 2m]/(A_1 - A_2)(A_3 - A_2),\\
\alpha_3 &= [(d-2)A_1A_2 + (c-2b)(A_1 + A_2 - c) + d^2 + d + 2m]/(A_1 - A_3)(A_1 - A_3).
\end{aligned}$$

Denote by A_1, A_2 and A_3 the roots of equations (19). Then

$c = A_1 + A_2 + A_3,\ m = A_1A_2 + A_1A_3 + A_2A_3 + d + 1,\ g(d+2) = -A_1A_2A_3.$

Let $b = 0$. Then along with these three invariant straight lines system (1) has also the invariant straight line $1 + (d + 1)y = 0$.

If $d = 0,\ b \neq 0$, then we have the following center conditions

$$a = r = 1,\ f = -2,\ k = q = g,\ l = -b,\ n = -1,\ p = b - c,\ d = s = 0 \quad (38)$$

and an integrating factor of the form (10) with

$$\alpha_1 = (A_1^2 - A_1A_2 - A_1A_3 - 2bA_1 - 2)/(A_1 - A_2)(A_3 - A_1),$$
$$\alpha_2 = (A_2^2 - A_1A_2 - A_2A_3 - 2bA_2 - 2)/(A_1 - A_2)(A_2 - A_3),$$
$$\alpha_3 = (A_3^2 - A_1A_3 - A_2A_3 - 2bA_3 - 2)/(A_1 - A_3)(A_3 - A_2).$$

Theorem 5.2. *The conditions (38) are sufficient conditions in order for system (1) to have a bundle of three invariant straight lines and a center at the origin.*

5.3. SOLUTION OF THE PROBLEM OF THE CENTER FOR CUBIC DIFFERENTIAL SYSTEMS WITH THREE INVARIANT STRAIGHT LINES FORMING A TRIANGLE

1) Consider the cubic system (22) and let the conditions (30) hold. Let $(0,0) \notin L_1 \cup L_2 \cup L_3$. In this case, a singular point $(0,0)$ is a weak focus for (22) and the invariant straight lines L_1, L_2, L_3 are given by formulas (28) and (29).

Denote

$$\begin{aligned} h_1 = {} & (B_3+1)(n_3 - s_3B_3)[A_3(n_3 - s_3B_3) - (m_3 - s_3A_3)\cdot(B_3 + A_1A_2)] \\ & + (n_3 - s_3B_3)(m_3 - s_3A_3)(A_1A_2 - A_1A_3 - A_2A_3 + A_3^2) + (B_3+1) \\ & \cdot (m_3 - s_3A_3)(1 - m_3A_3 + s_3A_3^2) - A_3(n_3 - s_3B_3) \\ & + A_1A_2(m_3 - s_3A_3). \end{aligned} \tag{39}$$

Theorem 5.3. *The cubic system (22) with conditions (30) and $h_1 = 0$ has a center at the origin.*

Proof. The system (22) with the invariant straight lines $L_1,\ L_2,\ L_3$ and cofactors $K_1,\ K_2,\ K_3$ has a Darboux integrating factor of the form (10) if and only if the identity (11) holds. Let $\Delta \equiv A_1\Delta_{23} + A_2A_3 - A_3^2 + B_3^2 + B_3 \neq 0$.

From (11) by taking into account (23), (30) and $h_1 = 0$, we obtain

$$\begin{aligned}
\alpha_1 &= (n_3 - s_3B_3)(A_1\Delta_{23} + A_2A_3 - A_3^2) + A_3(B_3+1)(m_3 - s_3A_3) + \\
&\quad + B_3 - A_1A_2 - 3 - \alpha_2 + \alpha_3B_3; \\
\alpha_2 &= -[(n_3 - s_3B_3)(A_1^2\Delta_{23} + A_1A_2A_3 - A_1A_3^2 + A_3B_3 + A_3) + (m_3 - s_3A_3)\cdot \\
&\quad (B_3+1)(A_1A_3 - B_3) - A_1^2A_2 + A_1B_3 - 2A_1 + A_2 + A_3 + \Delta_{13}\alpha_3]/\Delta_{12}; \\
\alpha_3 &= [-(B_3+1)^2(n_3 - s_3B_3)^2(A_3^2 + A_1^2A_2^2) + 2A_1A_2(B_3+1)(n_3 - s_3B_3)^2\cdot \\
&\quad (A_1 - A_3)(A_2 - A_3) - (n_3 - s_3B_3)^2(A_1 - A_3)^2(A_2 - A_3)^2 + (B_3+1)\cdot \\
&\quad (n_3 - s_3B_3)(A_1^2A_2^2 + A_1A_2 - A_1A_3 - A_2A_3 + 2A_3^2) - (A_1 - A_3)(A_2 - A_3)\cdot \\
&\quad (1 + A_1A_2)(n_3 - s_3B_3) + (B_3+1)^2(A_3^2 + B_3^2)(m_3 - s_3A_3)^2 - (B_3+1)\cdot \\
&\quad (m_3 - s_3A_3)(A_1A_2A_3 - A_1B_3 - A_2B_3 + 2A_3B_3 + A_3) - \Delta]/\Delta.
\end{aligned}$$

2) Assume now that the coefficient conditions (33) hold. In this case $(0,0) \in L_1 \cup L_2 \cup L_3$. Let us find the coefficient conditions under which the system (1) has an integrating factor of the form

$$\mu = (x^2 + y^2)^{\alpha_1}(1 - x)^{\alpha_2}, \tag{40}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. Let us write down the identity (11) with K_1, K_2 from (34). Equating in this identity to zero the coefficient of y, we find that $\alpha_2 = -2b\alpha_1 - 2b + c$. Next, we equate to zero the coefficients of x, x^2, xy and y^2 in (11), we obtain, respectively

$$\begin{aligned}
&2a\alpha_1 + 2a - d = 0, \quad 2a(b-1)\alpha_1 + 2ab - ac - 3a - q = 0, \\
&2(bc + b - c - s - 1)\alpha_1 + 2bc + 2b - c^2 - 5c - 2s - 4 = 0, \\
&2(ab + bd - d - q)\alpha_1 + 2ab - ac - a + 2bd - cd - 4d - 3q = 0.
\end{aligned} \tag{41}$$

2.1) The case $a \neq 0$. From the first equation of (41) we find $\alpha_1 = (d - 2a)/(2a)$. The second equation gives $q = bd - ac - a - d$. In this condition the fourth equation of (41) becomes an identity and the third one looks like

$$G \equiv a(c+1)(c+2) - d(c+1)(b-1) + ds = 0.$$

Taking into consideration that $\frac{\partial(\mu P)}{\partial x} - \frac{\partial(\mu Q)}{\partial y} = \frac{xyG}{a}$, we conclude that under coefficient conditions

$$\begin{aligned}
&f = a + d, \quad g = b + c, \quad k = -a, \quad l = bd - ac - a, \\
&m = -c - 1, \quad n = c + s + 1, \quad p = -a - d, \quad r = 0, \\
&q = bd - ac - a - d, \quad a(c+1)(c+2) - d(c+1)(b-1) + ds = 0
\end{aligned} \tag{42}$$

and $a \neq 0$ the system (1) has an integrating factor of the form (40) with

$$\alpha_1 = (d - 2a)/(2a), \quad \alpha_2 = (ac - bd)/a.$$

2.2). The case $a = 0$. From the first equation of (41) we get $d = 0$ and from the second one, $q = 0$. Remark that conditions (33) together with $a = d = q = 0$ are contained in (42). Hence in the case 2.2) by constructing an integrating factor of the form (40) we do not obtain new sufficient center conditions for (1) different from (42).

Theorem 5.4. *The cubic system (1) with conditions (42) has a center at the origin.*

References

[1] Amel'kin, V. V., Lukashevich, N.A., Sadovsky, A.P., *Non-linear oscillations in the systems of second order*, Belarusian Univer. Press, Belarus, 1982 (Russian).

[2] Cherkas, L. A., Romanovskii, V. G., Zoladek, H., *The centre conditions for a certain cubic system*, Differential Equations Dynam. Systems, **3–4**, 5 (1997), 299–302.

[3] Christopher, C. J., *Polynomial systems with invariant algebraic curves,* Preprint 1991, University College of Wales.

[4] Cozma, D., Şubă, A., *Partial integrals and the first focal value in the problem of centre*, NoDEA, **1**, 2 (1995), 21–34.

[5] Cozma, D., Şubă, A., *The solution of the problem of center for cubic differential systems with four invariant straight lines*, Scientific Annals of the "Al.I.Cuza" University (Romania), Mathematics, **XLIV** (1998), s.I.a, 517–530.

[6] Cozma, D., Şubă, A. , *Solution of the problem of the centre for a cubic differential system with three invariant straight lines*, Qualitative Theory of Dynamical Systems, **1**, 2 (2001), 129–143.

[7] Darboux, G., *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré.* Bull. des Sc. Math. 1878, pp. 60–96; 13–144; 155-200.

[8] Koij, R. E., Christopher, C. J., *Algebraic invariant curves and the integrability of polynomial systems*, Appl. Math. Lett., **4**, 6 (1993), 51–53.

[9] Liapunov, A. M., *Probléme général de la stabilité du mouvement*, Ann. of Math. Stud., **17** (1947), Princeton University Press.

[10] Lloyd, N. G., Pearson, J. M., Romanovsky, V. G., *Computing integrability conditions for a cubic differential system*, Computers Math. Applic., **10**, 32 (1996), 99–107.

[11] Romanovskii, V. G., Şubă, A., *Center of some cubic systems*, Annals of Differential Equations, **4**, 17 (2001), 363–376.

[12] Rousseau, C., Schlomiuk, D., *Cubic vector fields, symmetric with respect to a center*, J. Differential Equations, **2**, 123 (1995), 388–436.

[13] Sadovskii, A. P., *Solution of the center and focus problem for a cubic system of non-linear oscillations*, Differential Equations, **2**, 33 (1997), 236–244.

[14] Sadovskii, A. P., *On center conditions for one cubic system of differential equations*, Diff. Equations, **1**, 36 (2000), 98–102 (Russian).

[15] Schlomiuk, D., Guckenheimer, J., Rand, R., *Integrability of planar quadratic vector fields*, Exposition. Math., **8** (1990), 3–25.

ROMAI J., **5**, 1(2009), 63–76

STABILITY CRITERIA FOR QUASIGEOSTROFIC FORCED ZONAL FLOWS
I. ASYMPTOTICALLY VANISHING LINEAR PERTURBATION ENERGY

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Abstract This paper opens a series of studies regarding stability criteria for quasi-geostrophic forced zonal flows in the presence of lateral diffusion and bottom dissipation of the vertical vorticity.

The criteria, implying the asymptotic vanishing of the perturbation kinetic energy, are expressed in terms of the maximum shear of the basic flow and/or its meridional derivative and they are independent on the perturbation wavenumber. Some stability regions are enlarged with respect to some linear asymptotic stability criteria found in the literature. A comparison with the inviscid case is made.

Keywords: hidrodynamic stability, turbulence.

2000 MSC:76E20, .

1. INTRODUCTION

When we deal with oceanic or atmospheric fluids the flow is characterized by many parameters and is very complex presenting large spatial variations in the vertical direction and characteristics of turbulence with a extremely variable spectrum. For this reason, we limit our considerations to only a part of the turbulent scale (the synoptic scale) and a well determined spatial region, taking the rest of the atmosphere into account by means of initial and boundary conditions. Consequently, we obtained the quasigeostrophic approximation [1] [2] [3] [4] used to describe the dynamics of planetary fluids, namely ocean and atmosphere, with the synoptical scale. In the inviscid case, Vamos and Georgescu [5] proved that it represents the model of the fifth order asymptotic approximation of the primitive equations [4], as the Rossby number tends to zero. All other small parameters were expressed with by powers of this number. Thus, the main parameters remaining to govern the flow are the Reynolds number related to the lateral dissipation and the parameter r related to the bottom dissipation. The wind stress *curl* enters through the forcing term F which is assumed to be longitude-independent. In this approximation, we consider "channelled flows", that is, flows ideally placed between rigid walls of a constant latitude which isolate the region. They are interesting from a geophysical point of view, as, in a rotating planet, they maintain themselves without any external forcing; therefore they are the simplest flow configuration in the presence of longitude-independent force. The Antartic Circumpolar current is a meaningful example of this dynamics!

Non stationary forcing, implying non steady basic flows $\Psi_0(y,t)$, were used to study the case of barotropic flows by Kuo [1] [2], Haidvogel and Holland [6], Wolanski [7] and Crisciani and Mosetti [8]. A more complicated forcing, and, correspondingly a two dimensional basic flow $\Psi_0(x,y,t)$ was suitable for baroclinic instability studies in channel geometry (e.g. for Antarctic flows)or in rectangular closed basins (e.g. for Northern Hemisphere flows. Among these studies, for various types of basin geometries, we quote those of Stommel [9], Munk [10], Mc Williams and Chow [11], Le Provost and Verron [12], Crisciani [13], and Crisciani and Mosetti [14][15][16]. The linear as well as the non linear cases were considered.

Many mathematical methods applicable to fluid flow stability are available in the monographs by Lin [17], Ladyzenskaya [18], Joseph [19], Drazin and Reid [20], Georgescu [21], Straughan [22] and Chossat and Ioos [23]. We shall apply many of them to investigate the stability of wind driven flows in various circumstances.

In this paper we present linear stability criteria for the case of the stationary and longitude independent basic flow $\Psi_0(y)$, corresponding to the forcing $F(y)$. Some of these criteria will extend those of Crisciani and Mosetti [24], while others will use some results of these authors.

In Section 2 we present the mathematical model (equation and boundary conditions) governing the perturbation.

In Section 3 the inequality of the kinetic energy K is deduced ; three methods to derive conditions for negativity of the rate of change of K are presented, and several linear stability criteria are deduced. A special attention is focused on choosing the best among an infinite set of criteria and on giving closed form expressions for the curves bounding the stability domain.

In Section 4 we examine our results to determine the best linear stability criteria in order to compare it with the non-linear stability, suggesting that the increase of stability values permits a more reasonable comparison with the non linear case and consequently, the discussion of possible subcritical instabilities [21].

2. THE BAROTROPIC CHANNEL MATHEMATICAL MODEL. BASIC STATE AND PERTURBATIONS

In the ambit of quasigeostrophic circulation of barotropic flows with longitude independent forcing, the stability problem of zonal flows leads to the formulation of the following vorticity balance

$$\frac{\partial \nabla^2 \Psi}{\partial t} + \mathcal{J}(\Psi, \nabla^2 \Psi + \beta y) = F(y,t) - r\nabla^2 \Psi + A\nabla^4 \Psi. \tag{1}$$

where Ψ is the stream function,(y being latitude and x longitude), $\nabla^2 \Psi$ is the vertical vorticity, i.e. the vertical component of the *curl* of the geostrophic current, β is the planetary vorticity gradient due to latitudinal variation of the Coriolis parameter, F is a forcing term, $-r\nabla^2\Psi$ is the bottom dissipation term and $A\nabla^4\Psi$ is related to lateral vorticity diffusion [3] [4]. In the following both the constants A and r will be assumed to be greater than zero. The boundary conditions on Ψ are:

$$\Psi_x = 0 \qquad y = y_1 \quad y = y_2 \tag{2}$$

$$\nabla^2 \Psi = 0 \qquad y = y_1 \quad y = y_2 \tag{3}$$

where (2) is the condition of zero mass flux across the wall latitude, and (3) represents the zero lateral vorticity diffusion. The subscript indicates the differentiation. If we specify the forcing term $F(y)$, the problem (1)-(2)-(3) has the zonal solution $\Psi_0(y)$ [8], apart from a vorticity vanishing term. The basic flow, characterized by the local vorticity $\nabla^2\Psi_0 = q_0$, is the unique solution of the following two-point problem $q_0(y_1) = q_0(y_2) = 0$ for the ordinary differential equation $Aq_{0yy} - rq_0 + F(y) = 0$.

Therefore the linear perturbation $\phi(x, y, t) = \Psi - \Psi_0$, induced by the perturbation of the initial condition , satisfies the following equation [8]

$$(\nabla^2\phi)_t - \Psi_{0y}(\nabla^2\phi)_x + (\Psi_{0yyy} + \beta)\phi_x + r\nabla^2\phi - A\nabla^4\phi = 0, \qquad (x, y, z, t) \in \Omega\times\mathbf{R}_+, \tag{4}$$

the boundary conditions

$$\phi_x = 0, \quad \nabla^2\phi = 0 \qquad \text{at} \quad \mathrm{y} = \mathrm{y}_1 \quad \text{and} \quad \mathrm{y} = \mathrm{y}_2, \tag{5}$$

and some initial conditions $\phi = \phi_0$ for $t = 0$. In (4) Ω is the closed basin $\Omega = \{\, (x, y, z) \in \mathbf{R}^3 \mid 0 \le x \le L, y_1 \le y \le y_2, 0 \le z \le D \,\}$, and $\mathbf{R}_+ = \{\, t \in \mathbf{R} \mid t \geq 0\}$.

Let $K(t) = \dfrac{1}{2}\displaystyle\int_\Omega (\phi_x^2 + \phi_y^2)dxdydz$ which represents the corresponding perturbation kinetic energy. The basic flow Ψ_0 is asymptotically stable (in the mean) if $lim_{t\to\infty}K(t) = 0$. It is stable in the mean if $\dfrac{dK}{dt} < 0$.

In order to deduce criteria for asymptotic stability, we need inequalities of the form

$$\frac{dK}{dt} + aK \le b|g(t)|, \tag{6}$$

where $a > 0$ and $b < 0$ are constants and g is a bounded function. For stability criteria we have $a = 0$, that is,

$$\frac{dK}{dt} \le b|g(t)|. \tag{7}$$

In the next sections we shall obtain criteria for asymptotic stability exploiting three ideas. The first is to retain terms as small as possible on the left-hand side of (6). This will imply larger negative terms in b. The second concerns a better correlation between the Schwarz and imbedding inequalities and the general form of the Young inequality; the third is to use, instead of (6) and (7), inequalities of the form

$$\frac{dK}{dt} + aK \le b_1\big(g(t)\big)^2 + 2b_2|g(t)h(t)| + b_3\big(h(t)\big)^2, \tag{8}$$

where $a > 0, b_1 < 0, b_2$ and b_3 are real constants. In this case the asymptotic stability criterion follows from the condition of negative definiteness of the quadratic form from the right-hand side of (8).

3. LINEAR STABILITY CRITERIA

Let us start with the most general inequality for $\frac{dK}{dt}$. In the hypothesis that

$$\phi(x,y,t) = B(y,t)e^{ikx} \tag{9}$$

where $k \in \mathbf{R}$ is the x wavenumber, we have

$$\begin{aligned} K(t) &= \frac{1}{2}\int_\Omega (\phi_x\phi_x^* + \phi_y\phi_y^*)dxdydz \\ &= \frac{1}{2}DL(\|B_y\|^2 + k^2\|B\|^2) \end{aligned} \tag{10}$$

where $*$ stands for the complex conjugate and $\|.\|^2 = \int_{y_1}^{y_2}(|.|)^2 dy$.

Therefore, the evolution equation for K follows by multiplying (4) by ϕ^*, integrating the result over Ω and taking into account (5). It reads:

$$\begin{aligned} \frac{1}{DL}\frac{dK}{dt} + r(\|B_y\|^2 + k^2\|B\|^2) + A(\|B_{yy}\|^2 + 2k^2\|B_y\|^2 + k^4\|B\|^2) = \\ = -k\int_{y_1}^{y_2} q_o \mathfrak{Im}(B^*B_y)dy \end{aligned} \tag{11}$$

and $\mathfrak{Im}$ indicates the imaginary part.

By using the Schwarz inequality, the most general inequality for $\frac{dK}{dt}$ follows

$$\begin{aligned} \frac{1}{DL}\frac{dK}{dt} + r(\|B_y\|^2 + k^2\|B\|^2) + A(\|B_{yy}\|^2 + 2k^2\|B_y\|^2 + k^4\|B\|^2) \leq \\ \leq \mu_2|k|\|B\|\|B_y\|, \end{aligned} \tag{12}$$

where $\mu_2 = max_{y\in[y_1,y_2]}|q_0(y)|$. Inequality (12) implies that

$$\frac{1}{DL}\Big[\frac{dK}{dt} + 2(r + k^2A)K\Big] \leq \mu_2|k|\|B\|\|B_y\| - k^2A\|B_y\|^2, \tag{13}$$

the inequality (13) is equivalent to

$$\frac{1}{DL}\Big[\frac{dK}{dt} + 2(r+k^2A)\epsilon K\Big] + (r+k^2A)(1-\epsilon)(\|B_y\|^2 + k^2\|B\|^2) \leq \mu_2|k|\|B\|\|B_y\| - k^2A\|B_y\|^2, \tag{14}$$

where $0 < \epsilon \leq 1$ is an arbitrary number. Thus (14) has the form (8) where

$$\begin{aligned} a = (r + k^2 A)\epsilon, \quad & b_1 = -\Big[Ak^2(2-\epsilon) + r(1-\epsilon)\Big], \\ b_2 = \tfrac{\mu_2 |k|}{2}, \quad & b_3 = -k^2(r + k^2 A)(1-\epsilon). \end{aligned} \tag{15}$$

So, (14) is negatively defined if

$$-4k^4 A^2 (1-\epsilon)(2-\epsilon) - 4k^2 Ar(3-2\epsilon)(1-\epsilon) + \mu_2^2 - 4r^2(1-\epsilon)^2 < 0. \tag{16}$$

This occurs if

$$\mu_2 < 2r(1-\epsilon) \tag{17}$$

or if

$$\mu_2 \geq 2r(1-\epsilon) \qquad \text{and} \qquad \mathrm{k}^2 \geq \frac{-(3-2\epsilon)\mathrm{r} + \sqrt{\mathrm{r}^2 + \mu_2^2 (2-\epsilon)(1-\epsilon)^{-1}}}{2\mathrm{A}(2-\epsilon)}. \tag{18}$$

Inequalities (17) represent criteria for asymptotic stability. They do not depend on k or the Reynolds number $\mathcal{R}$ (which is proportional to A). As will be shown in the next Section, they are the best for large $\mathcal{R}$ and μ_3, where

$$\mu_3 = max_{y \in [y_1, y_2]} |q_{0y}|. \tag{19}$$

When $\epsilon \to 0$, the asymptotic stability criteria (17) become better and better and tend to the limit criterion:

$$\mu_2 < 2r. \tag{20}$$

For $\epsilon = 0$, the above reasonings shows that among the criteria (17)

$$\mu_2 \leq 2r \tag{21}$$

is the best criterion for stability only.

Criteria (18) depend on k. In order to obtain k-independent criteria, opposite inequalities for k are needed, such sufficient conditions will be obtained in the following by using the opposite inequalities.

Let us write (14) in the equivalent form

$$\frac{1}{DL}\Big(\frac{dK}{dt} + 2k^2 A\epsilon K\Big) \leq \mu_2 |k| \|B\| \|B_y\| - \|B_y\|^2 \Big[r + k^2 A(2-\epsilon)\Big] - k^2 \Big[r + k^2 A(1-\epsilon)\Big] \|B\|^2, \tag{22}$$

where $0 < \epsilon \leq 1$. If in (22) we neglect all terms in $\|B\|^2$ and use the Poincaré inequality $\|B\| \leq \alpha^{-1}\|B_y\|$, where $\alpha = \pi(y_2 - y_1)^{-1}$, we get an inequality of the form

$$\frac{1}{DL}(\frac{dK}{dt} + a'K) \leq b'\|B_y\|^2, \tag{23}$$

where

$$a' = 2k^2 A\epsilon, \qquad b' = |k|\mu_2\alpha^{-1} - r - k^2 A(2-\epsilon). \tag{24}$$

Therefore b' in (24) is negative for all k^2 either if

$$1) \quad \alpha^{-1}\mu_2 < 2\sqrt{(2-\epsilon)rA} \tag{25}$$

or if

$$2) \quad \alpha^{-1}\mu_2 \geq 2\sqrt{(2-\epsilon)rA} \quad \text{and} \quad |\mathrm{k}| \leq \frac{1}{2\mathrm{A}(2-\epsilon)}\left[\alpha^{-1}\mu_2 - \sqrt{\alpha^{-2}\mu_2^2 - 4(2-\epsilon)\mathrm{rA}}\right]. \tag{26}$$

The inequalities (25) are criteria independent of k; the best criteria correspond to $\epsilon \to 0$. Among the criteria (25), the limit inequality

$$\alpha^{-1}\mu_2 < 2\sqrt{2rA} \tag{27}$$

represents the limit of criteria of asymptotic stability and is the best criterion of stability.

For $\epsilon = 0$ the inequality

$$\alpha^{-1}\mu_2 \leq 2\sqrt{2rA} \tag{28}$$

is, among (25), the best criterion for stability.

For $\epsilon = 1$, we obtain the criteria of Crisciani and Mosetti [24]

$$\alpha^{-1}\mu_2 < 2\sqrt{rA}.$$

Multiplying (4) by $B^*_{yy} - k^2 B^*$ and integrating over Ω, Crisciani and Mosetti [24] obtained the following criteria:

$$|k| \geq \frac{\mu_3}{r}, \tag{29}$$

$$|k| \geq \sqrt[3]{\frac{\mu_3}{A}}. \tag{30}$$

From $(26)_2$ and (29) it follows that

$$\frac{\mu_3}{r} \leq \frac{1}{2A(2-\epsilon)}\left[\alpha^{-1}\mu_2 - \sqrt{\alpha^{-2}\mu_2^2 - 4(2-\epsilon)rA}\right], \tag{31}$$

which, in the plane $(\mu_3, \alpha^{-1}\mu_2)$, represents a region of asymptotic stability. The inequality (31) is equivalent to:

$$\alpha^{-1}\mu_2 - \frac{2A(2-\epsilon)}{r}\mu_3 \geq \sqrt{\alpha^{-2}\mu_2^2 - 4(2-\epsilon)rA}, \tag{31'}$$

which shows that the left hand side of (31)' must be positive. However, in the $(\mu_3, \alpha^{-1}\mu_2)$ plane, the stability region is above the straightline b_ϵ

$$\alpha^{-1}\mu_2 = \frac{2(2-\epsilon)A}{r}\mu_3. \tag{32}$$

Moreover, taking the square of $(31)'$ we find that the stability region is under the curve $\mathcal{C}_\epsilon$:

$$\alpha^{-1}\mu_2 = \frac{2A}{r}\mu_3(2-\epsilon) - \frac{r^2}{\mu_3}. \tag{33}$$

At the intersection of the curve (33) with the straightline (32), there is the point $P_\epsilon = \Big(\frac{1}{\sqrt{(2-\epsilon)}} r\sqrt{\frac{r}{A}}, 2\sqrt{(2-\epsilon)rA}\Big)$ which is the minimum of the curve (33).Therefore, the region of asymptotic stability corresponding to (25) and (31) is the union of the part of the $(\mu_3, \alpha^{-1}\mu_2)$-plane located between the curves (33), (32) and the $\alpha^{-1}\mu_2$axis, with that situated between the μ_3-axis and the straightline r_ϵ:

$$\alpha^{-1}\mu_2 = 2\sqrt{(2-\epsilon)rA} \tag{34}$$

The largest such region is obtained for $\epsilon = 0$ and it is bounded by the curve $\mathcal{C}_0$ and the straightline r_0

$$\alpha^{-1}\mu_2 = \frac{2A}{r}\mu_3 + \frac{r^2}{\mu_3}, \qquad 0 < \mu_3 \leq \frac{1}{\sqrt{2}} r\sqrt{\frac{r}{A}} \tag{35}$$

$$\alpha^{-1}\mu_2 = 2\sqrt{2rA}, \qquad \mu_3 \geq \frac{1}{\sqrt{2}} r\sqrt{\frac{r}{A}}. \tag{36}$$

The criterion defined by (35), (36) is independent of k^2, because, if (31) is satisfied, k^2 must verify at least one of $(26)_2$ or (29).

From $(26)_2$ and (30) it follows that

$$\sqrt[3]{\frac{\mu_3}{A}} \leq \frac{1}{2A(2-\epsilon)}\Big[\alpha^{-1}\mu_2 - \sqrt{\alpha^{-2}\mu_2^2 - 4(2-\epsilon)rA}\Big], \tag{37}$$

which, in the plane $(\mu_3, \alpha^{-1}\mu_2)$, represents a region of asymptotic stability. As above, it follows that the asymptotic stability region corresponding to (37) is located between the curves $\mathcal{C}'_\epsilon$

$$\alpha^{-1}\mu_2 = (2-\epsilon)\sqrt[3]{\mu_3 A^2} + r\sqrt[3]{\frac{A}{\mu_3}}, \tag{38}$$

and $\mathcal{C}''_\epsilon$

$$\alpha^{-1}\mu_2 = 2(2-\epsilon)\sqrt[3]{\mu_3 A^2} \tag{39}$$

that have the intersection point $P_\epsilon = \Big(\frac{1}{(2-\epsilon)\sqrt{(2-\epsilon)}} r\sqrt{\frac{r}{A}},\ 2\sqrt{(2-\epsilon)rA}\Big)$. Let us note that the ordinate of P_ϵ is the same for both cases (31) and (37) and, for the points whose abscissa is less than that of P_ϵ, (38) is above (39),which, in the same interval, is under the straightline (34). Therefore, the region of asymptotic stability is the union of the region delimited by the curve (38) and the $\alpha^{-1}\mu_2$ axis, with the part of the $(\mu_3, \alpha^{-1}\mu_2)$-plane which is situated between the μ_3-axis and the straightline (34). The largest such region is obtained for $\epsilon = 0$ and it is bounded by the curves $\mathcal{C}'_0$ and r_0

$$\alpha^{-1}\mu_2 = 2\sqrt[3]{\mu_3 A^2} + r\sqrt[3]{\frac{\mu_3}{A}}, \qquad \mu_3 \leq \frac{1}{2\sqrt{2}} r\sqrt{\frac{r}{A}}, \tag{40}$$

$$\alpha^{-1}\mu_2 = 2\sqrt{2rA}, \qquad \mu_3 \geq \frac{1}{2\sqrt{2}} r\sqrt{\frac{r}{A}}. \tag{41}$$

Crisciani and Mosetti [24] treated only the case $\epsilon = 1$, obtaining the curves $\mathcal{C}'_1$ and $\mathcal{C}''_1$ As above, the criterion provided by (40), (41), is independent on k^2 because from (37) it follows that k^2 must satisfy at least one of $(26)_2$ and (30).

If we associate, for $\epsilon = 0$, (26) with (18), we obtain:

$$\frac{1}{16A^2}\left[\alpha^{-1}\mu_2 - \sqrt{\alpha^{-2}\mu_2^2 - 8rA}\right]^2 \geq \frac{-3r + \sqrt{r^2 + 2\mu_2^2}}{4A}. \tag{42}$$

Inequality (42) is valid for

$$\mu_2 \geq 2r, \qquad \alpha^{-1}\mu_2 \geq 2\sqrt{2rA}, \tag{43}$$

and leads to the following asymptotic stability criterion

$$\alpha^{-1}\mu_2 < \frac{1}{\alpha}\sqrt{2r^2 + rA\alpha^2 + \alpha^4 A^2 + \sqrt{(2r^2 + rA\alpha^2 + \alpha^4 A^2)^2 + 4Ar^3\alpha^2}} = \alpha^{-1}\mu_2^* \tag{44}$$

which must be considered together with (43) and the restrictions

$$\frac{r}{\alpha^2 A} \geq \frac{7+\sqrt{41}}{4}. \tag{45}$$

In the next section we will clarify, in physical terms, the inequality (45).

4. DISCUSSION AND CONCLUDING REMARKS

We shall summarize our main results in mathematical and physical terms. In this paper we obtained some wavenumber independent stability criteria, namely (21) and (28), which are in terms of the maximum shear of the basic flow. The criterion (28) is better than (21) for $r/A\alpha^2 \leq 2$ and worse for $r/A\alpha^2 \geq 2$.

Let us assume a typical interval for the oceanic values (in S.I. units) of lateral vorticity diffusion

$$10^2 \leq A \leq 10^4, \tag{46}$$

for bottom dissipation r the value 10^{-7} and for L the value 10^6; therefore, we have: $(\alpha = \dfrac{\pi}{2L})$

$$\frac{r}{A\alpha^2} = 0\big(4\frac{10^2}{\pi^2}\big) \tag{47}$$

and thus the criterion (21) is better than (28).

The criterion (28) is obtained from the inequality (19) when $\epsilon = 0$, from the same inequality, when $\epsilon = 1$, we recover the criterion of Crisciani and Mosetti $\mu_2 \leq 2\sqrt{rA}$.

Moreover, from (47) it follows that the asymptotic stability criterion (44) holds for

$$\mu_2 \geq 2r,$$

since, in physical terms, the typical values of $r/A\alpha^2$ satisfy (45) and $(43)_2$. Some other asymptotic stability regions are defined in terms of the maximum shear of the basic flow and of the maximum of its meridional derivative, namely, in the $(\mu_3, \alpha^{-1}\mu_2)$-plane, the region

$$\begin{array}{ll} \alpha^{-1}\mu_2 = \frac{2A}{r}\mu_3 + \frac{r^2}{\mu_3}, & \text{if} \quad 0 < \mu_3 \leq \frac{1}{\sqrt{2}} r\sqrt{\frac{r}{A}}, \\ \alpha^{-1}\mu_2 = 2\sqrt{2rA}, & \text{if} \quad \mu_3 \geq \frac{1}{\sqrt{2}} r\sqrt{\frac{r}{A}}. \end{array}$$

and the region

$$\begin{aligned} \alpha^{-1}\mu_2 &= 2\sqrt[3]{\mu_3 A^2} + r\sqrt[3]{\tfrac{\mu_3}{A}}, \quad \text{if} \quad \mu_3 \leq \tfrac{1}{2\sqrt{2}} r\sqrt{\tfrac{r}{A}}, \\ \alpha^{-1}\mu_2 &= 2\sqrt{2rA}, \qquad\qquad\quad \text{if} \quad \mu_3 \geq \tfrac{1}{2\sqrt{2}} r\sqrt{\tfrac{r}{A}}. \end{aligned}$$

which are independent from k^2. We again find , from the asymptotic stability region defined from (35)-(36) and (40)-(41),the curves $\mathcal{C}_1$ and $\mathcal{C}'_1$ obtained from Crisciani and Mosetti [24], corresponding to the case $\epsilon = 1$. We note that the curve $\mathcal{C}_{\epsilon_1}$ is below the curve $\mathcal{C}_{\epsilon_2}$ if $\epsilon_1 > \epsilon_2$. So, for $0 \leq \epsilon \leq 1$, the family $\mathcal{C}_\epsilon$ is situated between the disjoint curves $\mathcal{C}_0$ and $\mathcal{C}_1$.Similarly, the curve $\mathcal{C}'_{\epsilon_1}$ is below the curve $\mathcal{C}'_{\epsilon_2}$ if $\epsilon_1 > \epsilon_2$ and, for $0 \leq \epsilon \leq 1$, the family $\mathcal{C}'_\epsilon$ is situated between $\mathcal{C}'_0$ and $\mathcal{C}'_1$. In addition, the curve $\mathcal{C}_1$ is above the curve $\mathcal{C}'_0$ and they do not intersect each other. Thus the best asymptotic stability region, bounded by the curves $\mathcal{C}_0$ and r_0,namely the first of the two previous regions, is given by (35), (36) and becomes better and better for large R and μ_3.

If we assume [2] [4], the basic flow in nondimensional form, is :

$$U_0 = \frac{1}{2}(1 + \cos \pi y) \tag{48}$$

where $U_0 = u_0/V$, $u_0 = \frac{\partial \Psi_0}{\partial y}$, V being a typical velocity. Our criterion (21) becomes:

$$\pi \leq 4r' \tag{49}$$

where $r' = \frac{L}{V} r$ is the non dimensional bottom friction coefficient.

The asymptotic stability region given by (35)-(36), in the specific case of the basic flow (48), takes the form

$$\begin{aligned} 1 &\leq \tfrac{\pi^2}{Rr'} + \tfrac{2r'^2}{\pi^2} \quad \text{if} \quad \tfrac{\pi^2}{2} \leq r'\sqrt{\tfrac{Rr'}{2}}, \\ 1 &\leq 2\sqrt{\tfrac{2r'}{R}} \qquad\quad \text{if} \quad \frac{\pi^2}{2 \geq r'\sqrt{\tfrac{Rr'}{2}}} \end{aligned} \tag{50}$$

where $R = \frac{VL}{A}$is the Reynolds number.

We observe that the inequality $(50)_1$ is satisfied for

$$R \leq 8r' \tag{51}$$

and (51) is equivalent to $(50)_2$. Therefore, in the plane (r', R), (51) gives the region for which the asymptotic stability is ensured. A necessary condition for instability can be obtained, of course, by reversing the inequality (51). In this case, from (31), for $\epsilon = 0$, it follows that the wavenumbers k which give instability belong to the interval

$$\frac{1}{4A}\left[\alpha^{-1}\mu_2 - \sqrt{\alpha^{-2}\mu_2^2 - 8rA}\right] \leq k \leq \frac{\mu_3}{r}, \tag{52}$$

which, in nondimensional form, for the basic flow (48) becomes:

$$\frac{R}{4} - \frac{1}{4}\sqrt{R(R-8r')} \leq k \leq \frac{\pi^2}{2r'}. \tag{53}$$

We observe that the left hand side of (53) is greater than r', that is, the interval defined by (53) is a proper subset of the interval (28) from [24]. Moreover , to clarify the effect of the dissipation on the instability of the basic flow (48), we can compare (53) with the interval obtained in [2]:

$$0 \leq k \leq \frac{\pi\sqrt{3}}{2} \tag{54}$$

to which the wavenumbers giving instability for (48) belong, as demonstrated by Kuo with numerical calculations.

Finally, we underline that a better linear stability criterion is of interest because, generally, these criteria are feeble and their values are considerably under the true linear stability limits (neutral curves). In this way they differ greatly from the global (energy) criteria.

Thus, the proof of subcritical situations is not possible for very weak stability criteria but becomes possible if stronger linear stability results are available.

Our methods are generally applicable and lead to considerable improvement of the linear stability domain for cases where functional inequalities at hand are very weak. However all our considerations were limited by the fact that the terms in the second meridional derivative of basic flow was disregarded. Its consideration is of further concern.

Acknowledgments. This work was performed under the auspices of G.N.F.M. -C.N.R. and was supported by the Italian M.U.R.S.T. under the Grant 60% "Algebra lineare ed equazioni alle derivate parziali" in the framework of Professor Roberto Peluso group, and the Grant 90/1994 of the Romanian Academy of Bucharest.

References

[1] Kuo, K.,L., *Dynamic instability of two dimensional non divergent flow in a barotropic atmosphere*, J. Meteor., **6**(1949), 105-122.

[2] Kuo, K. L., *Dynamics of quasigeostrophic flow and instability theory*, Adv. Appl. Mech., **13**(1973), 247-330.

[3] Hendershott, M. C., *Single layer models of the general circulation. General circulation of the ocean*, H.D.I. Abarbanel, W. R. Young Eds., Springer, 1986.

[4] Pedlosky, J., *Geophysical fluid dynamics.* 2d ed. Springer, 1987.

[5] Vamos, C., Georgescu, A. *Models of asymptotic approximation for synoptic-scale flow*, Z. Meteorol., **40**, 1,(1990), 14-20.

[6] Haidvogel, D. B., Holland, W.R., *The stability of currents in eddy-resolving general circulation models*, J. Physics Oceanogr., **8**, 3(1978), 393-413.

[7] Wolansky, G., *Existence, uniqueness and stability of stationary barotropic flow with forcing and dissipation*, Comm. Pure Appl. Math., **41**(1988), 19-46.

[8] Crisciani, F., Mosetti, R., *Linear asymptotic stability of geophysical channeled flows in the presence of arbitrary longitude -shaped perturbations*, Le Matematiche, **46**, 1(1991), 107-115.

[9] Stommel, H., *The westward intensification of wind driven ocean currents.*, Trans. Am. Geophys. Union, **29**, (2)(1948), 202-206.

[10] Munk, W. H., *On the wind-driven ocean circulation*, J. Meteor., **7**(1950), 79-93.

[11] Mc Williams, J. C., Chow, J. H. S., *Equilibrium geostrophic turbulence I: A reference solution in a beta-plane channel*, J.Phys. Oceanogr, **11**(7)(1981), 921-949.

[12] Le Provost, C., Verron, J., *Wind driven ocean circulation: Transition to barotropic instability*, Dyn. Atmos. Oceans, **11** , (2)(1987), 175-201.

[13] Crisciani, F., *Some a priori properties of unstable flows in the homogeneous model of the ocean*, Il Nuovo Cimento C, **11 C**(1988), 105-113.

[14] Crisciani, F., Mosetti, R., *A stability criterion for time- dependent basic zonal flows in the homogeneous model of the ocean*, Il Nuovo Cimento C, **12 C**, (2)(1989), 261-268.

[15] Crisciani, F., Mosetti, R., *On the stability of wind-driven flows in oceanic basins*, J. Physic. Oceanogr., **20**, (11)(1990), 1787-1790.

[16] Crisciani, F., R. Mosetti, *Stability of analytical solutions of Stommel-Veronis ocean circulation models*, J. Phys. Oceanogr., **24**, (1)(1994), 155-158.

[17] Lin, C. C., *The theory of hydrodynamic stability*, Cambridge University Press, 1955.

[18] Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flows*, Gordon and Breach, 1969.

[19] Joseph, D.D., *Stability of Fluid Motions*, I and II, Springer-Verlag, 1976.

[20] Drazin, P. G., Reid, W. H., *Hydrodynamic Stability*, Cambridge University Press, 1981.

[21] Georgescu, A.,:*Hydrodynamic Stability Theory*, Kluwer, 1985.

[22] Straughan, B., *The energy method, stability and non linear convection.* Springer-Verlag, 1992.

[23] Chossat, P., Iooss, G., *The Couette Taylor problem*, Institut Non-Lineaire de Nice, 1994.

[24] Crisciani, F., Mosetti, R., *Stability criteria independent from the perturbation wave-number for forced zonal flows*, J. Phys. Oceanogr., **21**, (7)(1991), 1075-1079.

ROMAI J., **5**, 1(2009), 77–85

A PETRI NETS APPLICATION IN THE MOBILE TELEPHONY

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Abstract Petri nets are used to describe graphically the structure of the distribute systems that need some representations of the concurrent or parallel activities. It represents a modelling language that is applicable to a wide variety of systems thanks to their generality and permissiveness. There are many types of Petri nets; more studied in this paper are CEN(Condition Event Nets), PTN(Place Transition Nets) and CPN (Coloured Petri Nets).

This paper aims to reveal the differences between these three types of Petri nets using an application on a mobile phone. From the several processes that the mobile phone can be used for, it is presented the action of sending the data using infrared or bluetooth system.

Keywords: Petri nets, distribute systems.

2000 MSC: 94C99.

1. INTRODUCTION

A Petri network is a mathematical representation for the discrete distribute systems. As a modelling language, it can describe graphically the structure of a distribute system using a direct graph with labels. This network contains nodes which can determine the places, nodes that can determine the transitions and direct arcs that are connecting the places with the transitions.

A Petri network is a 5-tuple PN=(P, T, F, W, M_0), where

$P = \{p_1, p_2, ..., p_m\}$ is a finite set of places,

$T = \{t_1, t_2, ..., t_{m\}}$ is a finite set of transitions,

$F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs,

$W : F \to \{1, 2, 3, ...\}$ is the weight function,

$M_0 : P \to \{0, 1, 2, 3, ...\}$ is the initial marking,

$P \cap T = \oslash$ and $P \cup T \neq \oslash$.

The Petri networks are using graphical symbols to represent states (usually represented by circles), transitions (usually represented by squares) and arrows from the states to the transitions and from the transitions to the states. $\bigcirc \rightarrow \square \rightarrow \bigcirc$. The states can be called places or conditions, and the transitions are also referred like events.

The condition event networks (CEN) represent the base type for the Petri nets. A CEN is formed through *conditions* (states), *events* (transitions) and *connections* (arrows) from the conditions to the events and from the events to the conditions.

A simple extension of the condition event network is to allow a marking to have more then one token for the conditions. These networks are known as place transition network (PTN). The conditions are now called *places* and the events *transitions*. In a PTN the places are labelled with a positive number that represents the capacity. This means the maximum number of tokens that can be in one place. The arrows can be labelled with a positive number representing the weight.

In the colored Petri nets (CPN) there is a difference between the tokens. The term colored is referring to the fact that the tokens are distinct through the value, that is called color. In a CPN to any place it is associated a colored set that is specifying the set of the colors for that place. A transition may have a sequence of guard expressions that will be evaluated to a Boolean value. The arrows from the places to the transitions and from the transitions to the places are called arcs.

2. SENDING THE DATA USING INFRARED AND BLUETOOTH SYSTEM

Sending the data using infrared or bluetooth system needs to proceed the following steps:

1 select the file that have to be sent;

2 select the type of the sending (infrared or bluetooth);

3 - bluetooth

- looking for the available mobile phones;
- selecting the mobile phone for sending;

- infrared
  - connecting using infrared system to the other mobile phone;

4 sending the file to the mobile phone;

5 receiving the file from the first mobile phone;

6 receiving the message about the successful sending.

The special cases of this process are of interest. Such is the moment when it is possible to send a file through infrared or the moment when it is possible to send through bluetooth, and all the settings that are included.

We suppose that it is not possible to send from a mobile phone one file through infrared and bluetooth system at the same time. Thus, the first rule is that for sending one file through infrared it is necessary that the phone not to send files using bluetooth. Similarly for the bluetooth not sending the file through infrared. In the same time it is not possible to send more than one file using any of the two systems. For sending the second file it means waiting for the finishing process of sending the first file.

The preconditions for the event *sending through infrared* are:

- *system for sending through infrared free;*
- *system for sending though bluetooth free;*
- *ready for sending through infrared.*

The postcondition for this event is *ready to receive through infrared.*

In the same time for occurring bluetooth event *sending through bluetooth* there are preconditions:

- *system for sending through infrared free;*
- *system for sending through bluetooth free;*
- *ready for sending through bluetooth.*

The postcondition for this event is *ready to receive through bluetooth*.

For a PTN we suppose that for sending the data through infrared we need the same resources as for sending the data through the bluetooth system. The number of the available resources is three. One process that doesn't get the necessary number of resources for running will wait until the resources will become free. The place *Keys* will show through a token each resource that is in use. When one process of sending the data is finished, the used resources are being returned in the place Keys and proceed to the other actions which don't need the critical resources. For representing using CPN the number of places

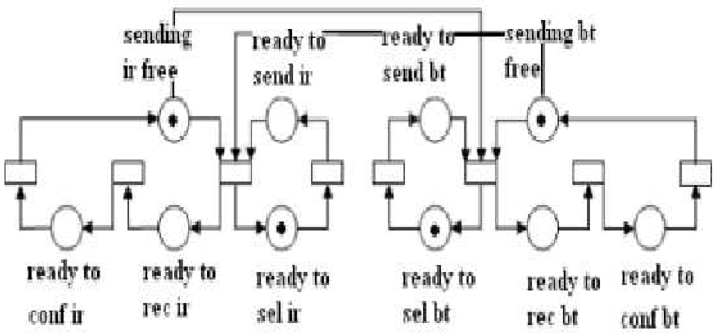

Fig. 1. CEN for Sending the File

will be reduced. It is not necessary to have representation for sending the file through the infrared and a separate one for sending through the bluetooth. Previously there were three preconditions for each case, now there is just one place, and the marking with colors will make the difference between the two cases. The function S(x) is making the connection between the case of sending the data using one system and the possibility of realizing this process.

Further we study the properties of this system. For this we use just the PTN diagrams. The networks above are bounded, because there are no places in which the number of the tokens become infinity. As noted from above the

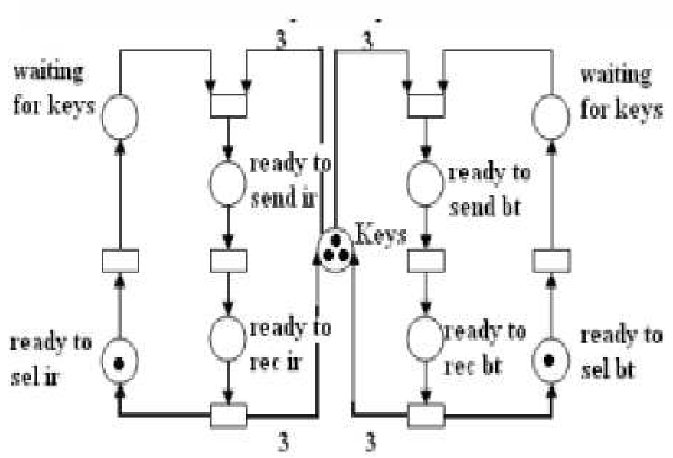

Fig. 2. **PTN for Sending the File**

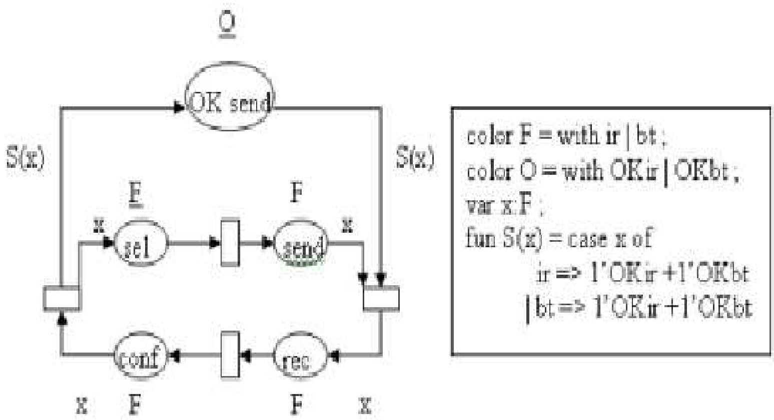

Fig. 3. **CPN for Sending the File**

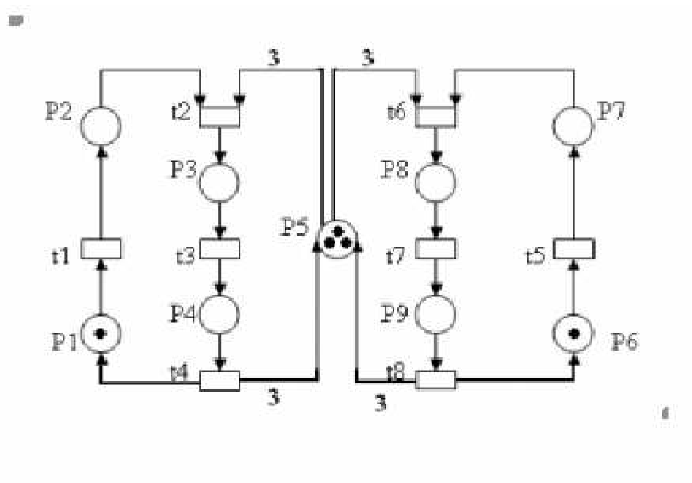

Fig. 4. **Figure 2 with the Places and the Transitions Numbered**

liveness property, we can say there are no dead transition, so the networks are 3-live. In the same time the networks above are reversible, coverable and persistent.

For getting the coverability tree for fig. 2, we numbered the places and the transitions. Thus the initial marking will be M0=(1,0,0,0,3,1,0,0,0). Here two transitions t1 and t5 occur.

If t1 will fire we get the marking M1=(0,1,0,0,3,1,0,0,0), then two transitions t2 and t5 can occur.

If t2 will fire we get the marking M2(0,0,4,0,0,1,0,0,0). For M3 there are two transition that can occur, t3 and t5.

If t3 will fire we get M3(0,0,0,4,0,1,0,0,0) then can occur t4 and t5.

If t4 will fire we get M4(1,0,0,0,3,1,0,0,0)=M0, so the node M4 will become "old".

If t5 will fire in M3 we get the marking M5(0,0,0,4,0,0,1,0,0) then it can occur the transition t4. If t4 will fire we get M6(1,0,0,0,3,0,1,0,0) then t1 and t6 can occur.

If t1 will fire, then we get M7(0,1,0,0,3,0,1,0,0) and t2 and t6 can occur.

If t2 will fire, we get the marking M8(0,0,4,0,0,0,1,0,0), then the transition t3 can occur. After firing the transition t3 we

Fig. 5. **The Coverability Tree for the Figure 2**

get M9(0,0,0,4,0,0,1,0,0). For this marking it can occur t4.
After firing t4 we'll get M10=(1,0,0,0,3,0,1,0,0)=M6,
so the node M10 is "old".
If for M7 will fire t6 we get the marking M11=(0,1,0,0,0,0,0,4,0) for which it can occur t7.
The firing of t7 is getting the marking M12=(0,1,0,0,0,0,0,0,4).
For this marking the transaction t8 can occur. After firing this transition we get the marking M13=(0,1,0,0,3,1,0,0,0)=M1, so the node will be labeled "old".
If for M6 will fire t6 we get the marking M14=(1,0,0,0,0,0,0,4,0) then it can occur t1 and t7.
If t1 will fire we get the marking M15=(0,1,0,0,0,0,0,4,0) = M11 so the node will be "old".
If for M14 will fire t7 we get the marking M16=(1,0,0,0,0,0,0,0,4) then it can occur t1 and t8.
If t1 will fire we get M17=(0,1,0,0,0,0,0,0,4) = M12, so the node M17 will be labeled "old".
If for M16 the transaction t8 will fire we get the marking M18=(1,0,0,0,3,1,0,0,0)=M0, so the node will be "old".
If for M2 will fire t5 we get the marking M19=(0,0,4,0,0,0,1,0,0)=M8, so the node M19 will be "old".
If for M1 will fire t5 we get the marking M20=(0,1,0,0,3,0,1,0,0)=M7, so M20 will become "old".
If for M0 will fire t5 we get the marking M21=(1,0,0,0,3,0,1,0,0)=M6, so the node M21 become "old".

The coverability tree that result is in Figure 5.

References

[1] D. Bj.rner, *Software Engineering*, 3 vols, Springer, 2006.

[2] Tadao Murata, *Petri Nets: Properties, Analysis and Applications*

[3] J,L, Peterson, Englewood Cliffs, *Petri Nets Theory and the Modeling of Systems* NJ: Prentice-Hall, Inc., 1981

[4] $http://www.cse.fau.edu/\ maria/COURSES/CEN4010-SE/C10/10-7.html$

ROMAI J., **5**, 1(2009), 85–87

SOLVING HIGHER-ORDER FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY

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Abstract Higher-order fuzzy differential equations with initial value conditions are considered. We apply the new results to the particular case of second-order fuzzy linear differential equation.

Keywords: fuzzy differential equations, fuzzy initial value problem, generalized differentiability.

2000 MSC:26E50.

1. PRELIMINARIES

Denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis satisfying the following properties:
(i) $\forall u \in \mathbb{R}_F$, u is normal, i.e. there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$;
(ii) $\forall u \in \mathbb{R}_F$, u is fuzzy convex set;
(iii) $\forall u \in \mathbb{R}_F$, u is upper semicontinuous on $\mathbb{R}$;
(iv) $cl\{s \in \mathbb{R} | u(s) > 0\}$ is compact, where cl denotes the closure of a subset.
Then $\mathbb{R}_F$ is called the space of fuzzy numbers.
The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$, $D(u, v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$.

2. GENERALIZED DIFFERENTIABILITY

In [1] a more general definition of derivative for fuzzy-number-valued functions is introduced. Using this differentiability concept we have the following definition.

Definition 2.1. *Let $F : I \to \mathbb{R}_F$ and $m, n = 1, 2$. We say that F is (m)-differentiable on I if F is differentiable as in Definition 5(m)[1] and its derivative is denoted by $D^1_m F$. Also, if $D^1_m F$ is (n)-differentiable as a fuzzy function, $D^2_{m,n} F$ denotes its second derivative on I.*

Theorem 2.1. *Let $D^1_1 F : I \to \mathbb{R}_F$ and $D^1_2 F : I \to \mathbb{R}_F$ be fuzzy functions.*
(i) If $D^1_1 F$ is (1)-differentiable, then $f^{'}_\alpha$ and $g^{'}_\alpha$ are differentiable functions and $[D^2_{1,1} F(t)]^\alpha = [f^{''}_\alpha(t), g^{''}_\alpha(t)]$.
(ii) If $D^1_1 F$ is (2)-differentiable, then $f^{'}_\alpha$ and $g^{'}_\alpha$ are differentiable functions and $[D^2_{1,2} F(t)]^\alpha = [g^{''}_\alpha(t), f^{''}_\alpha(t)]$.
(iii) If $D^1_2 F$ is (1)-differentiable, then $f^{'}_\alpha$ and $g^{'}_\alpha$ are differentiable functions and $[D^2_{2,1} F(t)]^\alpha = [g^{''}_\alpha(t), f^{''}_\alpha(t)]$.
(iv) If $D^1_2 F$ is (2)-differentiable, then $f^{'}_\alpha$ and $g^{'}_\alpha$ are differentiable functions and $[D^2_{2,2} F(t)]^\alpha = [f^{''}_\alpha(t), g^{''}_\alpha(t)]$.

3. SOLVING FUZZY DIFFERENTIAL EQUATIONS

Consider the Cauchy problem for the second-order fuzzy differential equations

$$y^{''}(t) + ay^{'}(t) + by(t) = \sigma(t), \quad y(0) = \gamma_0, \quad y^{'}(0) = \gamma_1, \tag{1}$$

where $a, b \in \mathbb{R}$ and $\sigma(t)$ is a fuzzy function on some interval I. The interval I can be $[0, A]$ for some $A > 0$ or $I = [0, \infty)$.
Our strategy of solving (1) is based on the choice of the derivative in the fuzzy differential equation. In order to solve (1) we have three steps: first we choose the type of derivative and change problem (1) to a system of ODE by using Theorem (2.2) and considering initial values. Second we solve the obtained ODE system. The final step is to find such a domain in which the solution and its derivatives have valid level sets. In view of these we propose the following definition.

Definition 3.1. *Let $y : I \to \mathbb{R}_F$ be a fuzzy function such that $D^1_m y$ and $D^2_{m,n} y$ exist for $m, n = 1, 2$ on I. If $y, D^1_m y$ and $D^2_{m,n} y$ satisfy problem (1) we say that y is a (m,n)-solution of problem (1).*

We have the following alternatives for solving problem (1):

(1,1)-Solution: If we consider $y'(t)$ using the (1)-derivative and then its derivative, $y''(t)$ using the (1)-derivative we have

$$[y''(t)]^\alpha = [\underline{y}^{''}(t;\alpha), \overline{y}^{''}(t;\alpha)] \quad \textit{and} \quad [y'(t)]^\alpha = [\underline{y}^{'}(t;\alpha), \overline{y}^{'}(t;\alpha)].$$

Now we proceed as follows:

(i) solve the differential system

$$\begin{cases} \underline{y}''(t;\alpha) + a\underline{y}'(t;\alpha) + b\underline{y}(t;\alpha) = \underline{\sigma}(t;\alpha), \\ \overline{y}''(t;\alpha) + a\overline{y}'(t;\alpha) + b\overline{y}(t;\alpha) = \overline{\sigma}(t;\alpha), \\ \underline{y}(0) = \underline{\gamma_0}, \overline{y}(0) = \overline{\gamma_0}, \underline{y}'(0) = \underline{\gamma_1}, \overline{y}'(0) = \overline{\gamma_1}, \end{cases}$$

for $\underline{y}$ and $\overline{y}$.

(ii) ensure that $[\underline{y}(t;\alpha), \overline{y}(t;\alpha)]$, $[\underline{y}^{'}(t;\alpha), \overline{y}^{'}(t;\alpha)]$ and $[\underline{y}^{''}(t;\alpha), \overline{y}^{''}(t;\alpha)]$ are valid level sets.

Other cases are similar to (1,1)-solution.

Remark 3.1. *The solution of FDE (1) depends upon the selection of derivatives, in the first or second form.*

Example 3.1. *Consider the second order fuzzy initial value problem*

$$y''(t) = \sigma_0, \quad y(0) = \gamma_0, \quad y'(0) = \gamma_1 \qquad t \geqslant 0,$$

where $\sigma_0 = \gamma_0 = \gamma_1 = [\alpha - 1, 1 - \alpha]$. *It possesses 4 different solutions.*

In order to extend the results to Nth-order fuzzy differential equations, we can follow the proof of Theorem 2.2 to get the same results for derivatives of an arbitrary order N. We have at most 2^N solutions for a Nth-order fuzzy differential equation by choosing the different types of derivatives.

References

[1] B. Bede, S.G. Gal, *Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems. 151 (2005) 581-99.

ROMAI J., **5**, 1(2009), 89–97

FINANCIAL ANALYSIS AND COST OF QUALITY

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Abstract The economic procedures and phenomena are characterized by the fact that their manifestation is very complex, having a wide range of aspects. This feature essentially distinguishes them from the phenomena from other areas. The remarkable complexity of economic phenomena has as causes lots of factors whose specificity depends on the context. This complexity of the social-economic area also has a cause of informational nature. The informational tides from this area have an heterogeneous nature and a relatively low degree of accuracy and relevance, because of the imperfections characteristic to the measuring process. In the decisional act is used a lot of information which has to be registered, analyzed and interpreted. The paper stresses upon the importance of the financial analysis as a tool of appreciating the performances and the financial potential of a company, cooperative statistics and new solutions of tackling with financial analysis.

Keywords: financial analysis, economic models, indicators, financial diagnosis, function, interpreting.

2000 MSC:91B60.

1. INTRODUCTION

Due to a dynamic development of communication means during the past decades, the concept of information is assimilated, in the economic environment, as a signal that can appear at one moment on one of the communication channels and that can be exploited in the benefit of the society or can be ignored, being considered as irrelevant.

When it comes to defining the concept of information, a relevant example is presented by F. Heylighen, on the website entitled “Web Dictionary of Cybernetics and Systems”, where he states that the answer to a question can be considered as a carrier of information as long as it diminishes the interlocutor’s degree of uncertainty.

2. A SYSTEMIC APPROACH TO THE ECONOMIC ENVIRONMENT

In the context of information systems, data are considered as a tangible object, being represented by numeric or alphanumeric symbols and can be memorised by a variety of storing mediums.

In order for financial accounting information to be useful to the users, it is necessary that they meet certain qualitative characteristics. According to the IASB general arrangement, for the elaboration and presentation of financial reports, the four main qualitative characteristics are: intelligibility, relevance, credibility and comparability.

- **Intelligibility** refers to the easy understanding of information by the users. The IASB general arrangement explicitly requires that information about complex issues, that should be included in financial reports, should not be excluded based only on the fact that they can be difficult to comprehend by certain users, because of their importance in decision-making.

- **Relevance** represents the ability of information to allow users to evaluate past, present or future events and, through their content, to facilitate the decisional activity.

- **Credibility** regards the lack of errors amidst information and their ability to present no deformities or subjective points of view, so that they can illustrate a faithful representation of reality.

- **Comparability** assumes that the financial information is compared in time and space. This objective can be accomplished by the permanency of accounting methods that evaluate, classify and present the elements described in financial reports. Any modification of these methods must allow users to identify the differences between the applied accounting methods.

The IASB [3] accounting arrangement also defines the limits of relevant and credible information. Among these we mention:

- the opportunity (referring to obtaining the information in useful time, so that it can be used in the decisional processes).

- the balance between benefit and cost (the benefits that can be quantified as a result of using information need to be bigger that the cost of supplying that information).

Attempting to develop a terminology both simple and rigorous, the initiators of the system theory defined the system as en "ensemble of interacting elements". An abstract concept that has been developed in the economic and informatics field is that which represents systems as structures with input and output. Based on these approaches, the mathematical theory of systems was developed.

Starting with the hypothesis that elements in the system are formed from other elements, and these are organized on hierarchic levels, we can conclude that a system (also called a higher system) consists of a number of several sub-systems.

One principle stated in systems theory claims that, in time, systems evolve and have the ability of integrating in more and more complex systems. In the economic field, integrations have a complex typology.

Therefore, we can speak about:

- a genetic integration in the case of sub-systems that are part of a certain system, because they were created in a certain environment and because of certain dependencies which can not exist outside the system;

- a second type of integration is integration by coercing. In an economic system the integration by coercing assumes forcing the elements to integrate in a certain organizational scheme and can be clearest exemplified by the fiscal regulations that impose certain restrictive norms in the financial accounting field;

- another type of integration among economic systems is the integration by dependence, that refers to certain elements' necessity to remain inside a system, because of the fact that, directly or indirectly, they have relations of dependence with other elements. Therefore, in a society with a production activity, the productive units are dependent on the supplying service that ensures their raw material;

- a form of integration manifesting especially on capital markets, in the economic field, is the integration on choice. This offers to elements (the sub-systems) the possibility of choosing the system they will integrate in.

In the perspective of a hierarchic model proposed by Dumitru Oprea [1] the main components of the economic system are:

✓ *the institutions* (institutions of the state, but also large enterprises);

✓ *the organisations* (sub-systems of institutions, by example, the production units of enterprises);

✓ *the units* (base elements of the economic system that can not be further divided).

During the design of informatics systems, one resorts to specific shaping methods that are supposed to capture the component units and the activities taking place in an enterprize, in an integrated vision, identifying the information flows.

The classic model of presentation of information flows in an enterprize highlights the main role of the accounting department (sub-system).

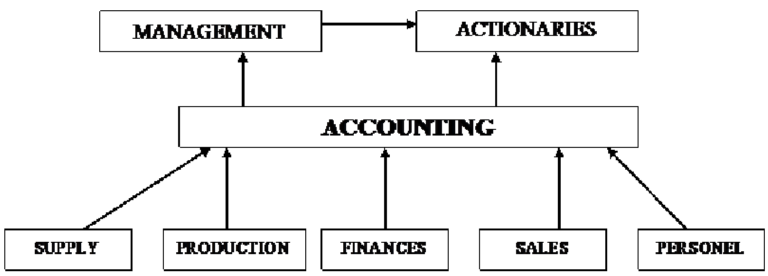

Fig. 1. The information flow in an economic enterprize.

3. CATEGORIES OF ACCOUNTING INFORMATION

According to accounting treaties [2], the accounting informations can be classified in four categories:

✓ operational information;

✓ information supplied by the financial accounting;

✓ information supplied by the administrative accounting;

✓ accounting information regarding the satisfaction of fiscal needs.

Accounting epistemology refers to the way the accounting is organized, in the sense of separating the organization of financial accounting and that of the administrative accounting, thorough the notions of accounting monism and dualism.

In a formal definition, monism or the single-circuit accounting system is the organization form through which the administrative accounting is completely integrated in the accounting system. Formal monism can manifest integrally, where accounting is held entirely by means of accounts or attenuated, where parts of the administrative accounting are treated without using the accounts system, through a number of separate tables.

Accounting dualism is the form of organization that assumes the net dissociation between the financial accounting and the administrative accounting, that is exclusively assigned to the supply of information for the internal environment of the enterprize.

In Romania, the organization of the accounting system is made according to the dualist concept, so that, at the enterprize level, one can distinguish two separate sections of accounting: financial and administrative (or managerial).

4. THE IMPORTANCE OF THE ECONOMICAL-FINANCIAL ACCOUNTING ANALYSIS AND DIAGNOSIS

The economical-financial analysis represents an ensemble of concepts, techniques and tools that ensure the handling of internal and external information, in order to formulate pertinent opinions regarding an economic agent's situation, the level and quality of his performance, the degree of risk imposed by the evolution in a dynamic competition environment.

The process of reporting finances imposes on the enterprize that they complete a full set of financial informing situations. The balance sheet, the loss and gain account, the presentation situations of financial modifications (the treasury flow situation) and the explicative notes are compulsory elements of these reports.

The approach of the economic systems, in the sense of diagnosing them, assumes a specific measure, that integrates the classic vision, based on the knowledge of causality relations and the internal laws of the formation and evolution of phenomena. The practical usefulness of financial analysis meets difficulties in approaching the problem in an accurate and unique manner. The reason for these difficulties resides, most of the time, in the fact that the ensemble of tools and methods required by the financial analysis do not represent a theoretical discipline, but have a specific content.

The results of the economic activities run by an enterprize must be analyzed not only as values, but also with respect to a reference criterion. Analysis methods, as noticeable in the regression analysis method, are methods that use mathematical formulas to enable their definition. Therefore, the interpretation of the results obtained will take into account the mathematical interpretation, in correlation with the economical facts described by the results.

The method of regression analysis. Also know as correlation method, this method can be used when the phenomenon and the factors that are being analyzed are of the stochastic type. Applying this method assumes:

- performing a **qualitative analysis** in order to establish the economic content of the analyzed phenomenon (y) and of the influence factors (x_1, x_2, $\ldots x_n$);
- identifying the causality relations between the phenomenon and the factors, followed by the mathematical formalization (regression equation). A regression equation can be fit into one of these categories:

  linear, $y = a + bx$; hyperbolic, $y = a + \frac{b}{x}$; parabolic, $y = a + bx + cx^2$; exponential, $y = a * b^x$;

- determining, by calculation, the value of the regression equation parameters (you apply the least squares method);

- establishing the intensity of the relation between the analyzed phenomenon and the influence factors, by means of the correlation ratio (r_{xy}) or the correlation rapport, according to the formula

$$r_{xy} = \frac{n\sum yx - \sum x * \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2]} * \sqrt{[n\sum y^2 - (\sum y)^2]}};$$

- quantifying the factors' influence over the analyzed phenomenon through the determined coefficients (dy_x);

There is a series of indicators that can be grouped into:

► indicators calculated on the basis of the accounting balance sheet:

- net patrimony;
- revolving fund;
- necessary capital;
- net treasury;
- cash flow;

► indicators calculated on the basis of the results account:

- turnover;
- commercial margin;
- production of the year;
- the added value;
- gross operating surplus;
- the capacity of auto-financing;
- profit/loss.

The aggregation of data, in a first step, can refrain to cumulating as future "facts" in the data deposit scheme. We must mention that, for the "time" dimension, the aggregation by cumulating will only be done for income and expenses accounts.

5. CONCLUSION

The concept of shaping an enterprise's financial accounting activity assumes a systemic approach that will allow the identification of all the elements that are interacting and are generating influences over the enterprize.

Following a detailed study of the decisional processes characteristic to the financial accounting field and analyzing the activities in the Business Intelligence category that circumscribe to these on different managerial levels, we can classify these activities as:

- activities of report on the information regarding the situation of the indicators that reflect the evolution of the organization;

- the creation of predictions based the historic data extracted from the transactional systems of the enterprize and the external sources data, organized, most of the cases, in the shape of Data Warehouse or Data Mart;

- data analysis activities that, in the field of informatics systems dedicated to assisting the decision based on the Data Warehouse and OLAP technology, are gradually replaced by the multidimensional analysis, which opens new perspectives on exploring the data and has the role of permanently supplying new information to the decision factors;

- identification of correlations between the factors that influence the activities run by the enterprize.

References

[1] Oprea, D., *Analysis and design of economical informational systems*, Polirom, Bucharest, 1999. (in Romanian)

[2] Feleagă, N., Ionaşcu, I., *Treatise on financial accounting*, **II**, Editura Economică, Bucharest, 1998.(in Romanian)

[3] *International Accounting Standards Board*, organism for the elaboration and publication of international accounting standards.

[4] Işfănescu A., Stănescu C., Băicuşi, *Economic-financial analysis*, Editura Economică, Bucharest, 1999. (in Romanian)

[5] Ristea, M., (ed.), *Financial accounting of enterprizes*, Editura Universitară, Bucharest, 2004.(in Romanian)

ROMAI J., **5**, 1(2009), 97–104

EKMAN CURRENTS ON THE ROMANIAN BLACK SEA SHORE

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Abstract Analytical solutions for Ekman currents in shallow marine waters are presented. Navier-Stokes equations, subject to some assumptions, are used as mathematical model. The boundary conditions are specific to some regions on the Romanian Black Sea coast. The magnitude of Ekman currents velocity, its direction, the Ekman net transport and its direction too, for a particular function which gives the sea basin depth are calculated.

Keywords: Eckman currents, marine waters flow models.

2000 MSC:76D05.

1. INTRODUCTION

The ocean currents are real rivers of water (cold or hot) moving in the oceans and seas. They are the result of some forces which act upon the oceans water: the wind, the force due the density gradients, the gravity, the Coriolis force. The ocean currents are very important for the life on our planet. They have a very great influence the earth's energy distribution and hence influence the climate. The ocean currents may be divided into two major circulation systems: surface circulation and deep-water circulation.

The stress of the wind takes action to the sea surface and the energy is transferred from wind to sea, inducing a momentum exchange. Ekman was the first to study the effect of the frictional stress of the wind at the ocean surface. That is why the surface currents due to the wind stress were named Ekman currents. Under some assumptions, the velocity and the direction of the current, the water transport and its direction can be calculated. The motion is relatively steady-state. For eddy friction stress, the kinematic viscosity is used estimating vertical shear. The Ekman currents were calculated in the case when the ocean is infinite and in the case of finite constant depth [1].

The Ekman current is a motion obtained under the effect of two forces: Coriolis force and friction. This motion takes place only in a region named Ekman layer. The width of the Ekman layer is a number named Ekman depth or depth of frictional influence. The Ekamn current, obtained by wind blowing over the sea, has the maximum speed at the surface. Its speed decreases with the depth increase and its direction changes gradually to the right of the wind direction in the northern hemisphere, describing a kind of spiral. The Ekman mass transport is at right angles to the wind direction. When the waters move away from the coast, water below the surface comes up to replace it. It is the so-called upwelling phenomenon. If the waters move towards the coast it must move down and the downwelling phenomenon appears [2].

In the Black Sea basin there is a surface water circulation due to the winds which blow along the sea surface. The Ekman currents have a small speed and variable direction. The most important wind direction during all the year round is north-western one, but the wind blow also from north, west, south, south-west, north-east [3].

On the Romanian Black Sea shore there is a great difference in the Ekman currents development on the account of the small depth of the water for a very large region. That is why the Ekman currents can be studied in the case when the basin depth is not infinite, the wind direction is variable, the wind is not uniform [5].

For each such motion, there are many specific conditions which must be taken into consideration and there are different approximations used in every case.

2. EQUATIONS OF MOTION

In oceanography the system of equations of motion is obtained from the Navier-Stokes ones. The components of motion velocity are: u, v, w. The third equation is used in hydrostatic form. The vertical component of the velocity can be neglected. The equations for horizontal motion are written in the presence of two forces only: Coriolis force and frictional force

$$\frac{du}{dt} = f\ v + \frac{1}{\rho}\frac{\partial p}{\partial x} + A_h\ \left(\frac{\partial^2 u}{\partial x^2}\ + \frac{\partial^2 u}{\partial y^2}\right) + A_z\ \frac{\partial^2 u}{\partial z^2}, \tag{1a}$$

$$\frac{dv}{dt} = -f\ u + \frac{1}{\rho}\frac{\partial p}{\partial y}\ + A_h\ \left(\frac{\partial^2 v}{\partial x^2}\ + \frac{\partial^2 v}{\partial y^2}\right) + A_z\ \frac{\partial^2 v}{\partial z^2}, \tag{1b}$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{1c}$$

Here f is the Coriolis coefficient (assumed constant), p the pressure, $A_h,\ A_z$ eddy viscosities in horizontal and vertical directions. The coordinate system $Ox\ y\ z$, used is a rectangular one, with Ox pointing to the east, Oy to the north and Oz up. It be used only because it was concerned with vertical shear [3].

The effect of the frictional stress at the sea surface due to the wind blowing over it, must be studied under the assumptions:

- no boundaries;
- finite depth H, depending on the horizontal position x, y;
- a steady wind blowing for a long time under a constant angle relatively north direction, with the magnitude of wind stress;
- homogeneous water and barotropic condition;
- f is constant, i. e. the f-plane approximation [1], [2].

For Ekman currents it can be used the system of horizontal equations of motion

$$0 = f\ v\ + A_z\ \frac{\partial^2 u}{\partial z^2}, \tag{2a}$$

$$0 = -f\ u + A_z\ \frac{\partial^2 v}{\partial z^2}, \tag{2b}$$

The boundary conditions are

$$\rho_w A_z \frac{\partial u}{\partial z}\ = \tau_x, \tag{3}$$

$$\rho_w A_z \frac{\partial v}{\partial z}\ = \tau_y; \tag{4}$$

on the water surface, for $z = 0$, where ρ is the water density;

$$u = v = 0, \tag{5}$$

on the bottom of the basin.

3. SOLUTIONS FOR A BASIN WITH CONSTANT DEPTH

If the basin has a constant depth H, the boundary conditions for the bottom are

$$u = v = 0, \text{ for } z = -H.$$

By the Gougenheim and Saint-Guilly method, using complex analysis, the two equations for horizontal motion lead to the equation

$$A_z \frac{\partial^2}{\partial z^2}(u + iv) - i\ f\ (u\ +\ i\ v)\ =\ 0. \tag{6}$$

with which we associate the suitable boundary conditions at the sea surface

$$A_z \left| \frac{\partial}{\partial z}(u + iv) \right|_{z=0} = \frac{\tau}{\rho_w} (\sin \alpha\ +\ i\ \cos \alpha) \tag{7}$$

and on the bottom

$$u\ +\ i\ v\ =\ 0, \text{ for } z = -H. \tag{8}$$

It is assumed that at the surface a wind blows up with a constant stress τ and its direction is given by an angle α with the north direction. The solution of equation (6) is

$$u + i\ v\ = \sqrt{\frac{2\ A_z}{f}} \frac{\tau}{\rho_w\ A_z} \exp\left(i\ \left(\frac{\pi}{4}\ -\ \alpha\right)\right) \tanh\ \left[\sqrt{\frac{f}{2\ A_z}}\ (1\ +\ i)\ (z\ +\ H)\right]. \tag{9}$$

The horizontal components of current speed are:

$$u\ =\ \frac{\tau}{\rho_w} \sqrt{\frac{2}{f\ A_z}} \frac{\cos\left(\frac{\pi}{4}\ -\ \alpha\right) \sinh (2\ a\ H)\ -\ \sin\ \left(\frac{\pi}{4}\ -\ \alpha\right) \sin (2\ a\ H)}{\cos (2\ a\ H)\ +\ \cosh (2\ a\ H)}, \tag{10}$$

$$v\ =\ \frac{\tau}{\rho_w} \sqrt{\frac{2}{f\ A_z}} \frac{\sin\left(\frac{\pi}{4}\ -\ \alpha\right) \sinh (2\ a\ H)\ +\ \cos\ \left(\frac{\pi}{4}\ -\ \alpha\right) \sin (2\ a\ H)}{\cos (2\ a\ H)\ +\ \cosh (2\ a\ H)}, \tag{11}$$

where

$$a\ =\ \sqrt{\frac{f}{2\ A_z}}. \tag{12}$$

4. THE MAGNITUDE OF EKMAN CURRENT VELOCITY

The magnitude of current velocity $V = \sqrt{(u + i\,v)\,\overline{(u + i\,v)}}$ follows

$$V = \frac{\tau}{\rho_w}\sqrt{\frac{2}{f\ A_z}}\sqrt{\frac{\cosh\left(\sqrt{\frac{2\,f}{A_z}}\,(z + H)\right) - \cos\left(\sqrt{\frac{2\,f}{A_z}}\,(z + H)\right)}{\cosh\left(\sqrt{\frac{2\,f}{A_z}}H\right) + \cos\left(\sqrt{\frac{2\,f}{A_z}}H\right)}}.$$

At the surface we have

$$V = \frac{\tau}{\rho_w}\sqrt{\frac{2}{f\ A_z}}\sqrt{\frac{\cosh\left(\sqrt{\frac{2\,f}{A_z}}H\right) - \cos\left(\sqrt{\frac{2\,f}{A_z}}H\right)}{\cosh\left(\sqrt{\frac{2\,f}{A_z}}H\right) + \cos\left(\sqrt{\frac{2\,f}{A_z}}H\right)}}.$$

With

$$D_E = \pi\sqrt{\frac{2\ A_z}{f}},$$

the magnitude of current velocity becomes

$$V = \frac{\tau}{\rho_w}\frac{D_z}{\pi}\sqrt{\frac{\cosh\left(\sqrt{\frac{2\,f}{A_z}}\,(z + H)\right) - \cos\left(\sqrt{\frac{2\,f}{A_z}}\,(z + H)\right)}{\cosh\left(\sqrt{\frac{2\,f}{A_z}}H\right) + \cos\left(\sqrt{\frac{2\,f}{A_z}}H\right)}}. \tag{13}$$

D_z is an important length called Ekman depth or frictional influence. In the case of the basin with great depths, the assumption used is H - infinite. For the Romanian Black Sea shore this depth is greater than the basin depth. That is why the influence of the wind can be felt for all the basin depth.

The wind stress magnitude can be expressed by $\tau = \rho_{air}\ C_D\ (U_{10})^2$, where C_D is the drag coefficient $C_D \simeq 1.4 \times 10^{-3}$. and U_{10} is the wind speed (in m/s). The eddy viscosity A_z ranges from 10^{-5} to $10^{-1} m^2 s^{-1}$. For $A_z = 10^{-1}\ m^2 s^{-1}$, $f = 10^{-4}\ s^{-1}$, $\rho_{air} = 1.3\ kg/m^3$, $\rho_w = 1025\ kg/m^3$, the magnitude of current velocity is

$$V = 0.79415{\times}10^{-2}\,(U_{10})^2\sqrt{\frac{\cosh\ (0.04472\,(z + H)) - \cos\ (0.04472\,(z + H))}{\cosh\ (0.04472\ H) + \cos\ (0.04472\ H)}}. \tag{14}$$

We used $f = 10^{-4}\ s^{-1}$, because $f = 2\,\Omega\ \sin\phi$ where Ω is the magnitude of the angular velocity of the rotation of the earth, $\Omega = 7.29 \times 10^{-5} rad/s$

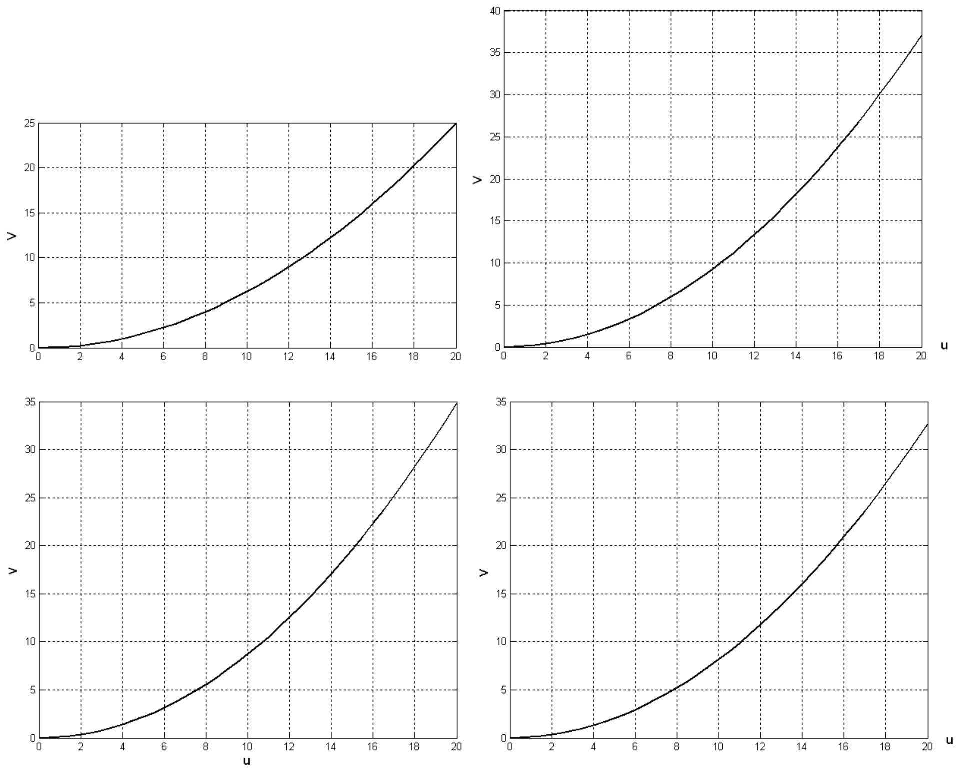

Fig. 1. The variation of surface current magnitude with the wind speed for a) $H = 25$ m; b)$H = 50$ m; c) $H = 75$ m; d)$H = 100$ m (wind speed in m/s, current speed in cm/s).

and ϕ is the latitude. For the Romanian Black Sea shore, $\phi \simeq 45^0$. With all those assumptions, the Ekman surface current has the speed

$$V = 0.79415 \times 10^{-2} \, (U_{10})^2 \sqrt{\frac{\cosh \ (0.04472 \ H) - \cos \ (0.04472 \ H)}{\cosh \ (0.04472 \ H) \ + \cos \ (0.04472 \ H)}} \qquad (15)$$

It can be seen that the surface current speed depends on H and U_{10}.

The variation of surface current magnitude with the wind speed for some values of basin depth can be seen in Fig. 1.

It can be seen that the maximum value for the current speed is reached for $H = 50$ m. The variation of surface current magnitude with the basin depth for a constant wind speed can be seen in Fig. 2.

For H one can use functions like:

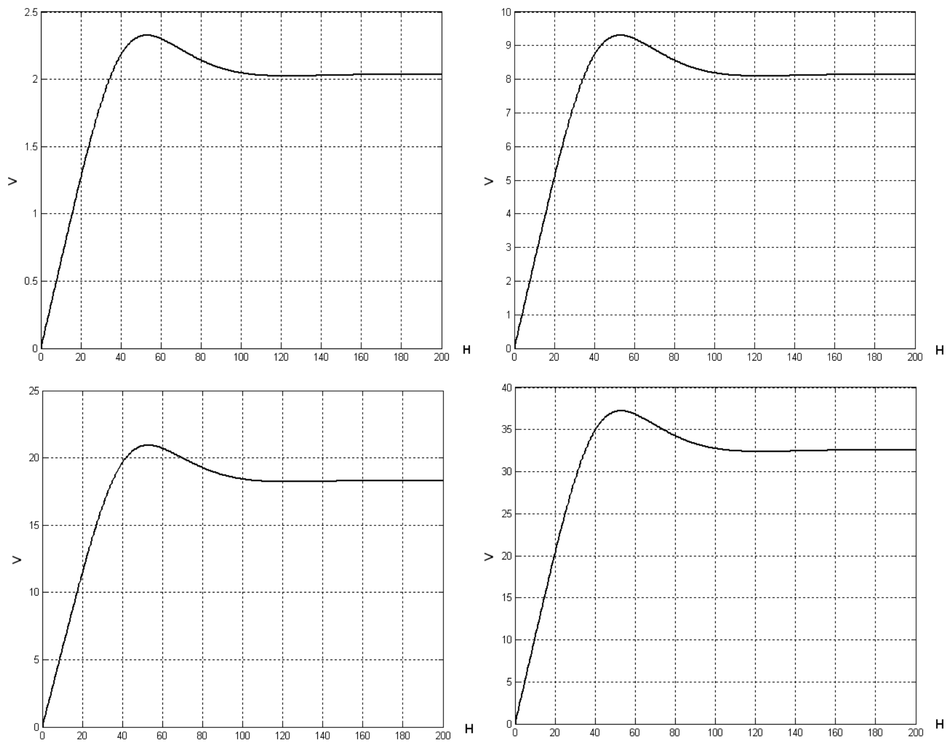

Fig. 2. The variation of surface current magnitude with the basin depth for: a) $U_{10} = 5$ m/s; b) $U_{10} = 10$ m/s; c) U_{10}=15 m/s; d) U_{10}=20 m/s (wind speed in m/s, current speed in cm/s).

- $H = s \cdot x$;
- $H = s \cdot x + t \cdot y$;
- $H = s_1 x$, for $0 < x < l_1$ and $h_1 + s_2 (x - l_1)$, for $l_1 \leq x \leq l_2$.

5. CONCLUSIONS

The Black Sea water circulation on the Romanian sea-shore is a very difficult problem. There are many elements which determine and influence this circulation. In this paper we studied the influence of only one element: the wind blowing up the surface of the basin on the current magnitude. We shall continue our study with the investigation of the influence the same element has on the current direction, the magnitude of the water transport and the direction of this transport. We must complete the study with the influence of density variation, bottom friction, surface level variation.

References

[1] Lacombe, H., *Cours d' oceanographie physique*, Gautier-Villars, Paris, 1965.

[2] Pond, S., Pickard, G., *Introductory Dynamical Oceanography*, Butterworth-Heinemann Publications, London, 2003.

[3] Stewart, R., *Introduction to physical oceanography*, Texas A&M University, 2002.

[4] www.unesco.pub Potenţialul eolian pe litoralul românesc al Mării Negre.

[5] Muntean, A., Introducere in studiul curentilor marini, Editura Universităţii din Piteşti, 2005.

ROMAI J., **5**, 1(2009), 105–112

TRANSVERSALITY CONDITIONS FOR INFINITE HORIZON OPTIMIZATION PROBLEMS: THREE ADDITIONAL ASSUMPTIONS

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Abstract We consider the transversality condition for the optimization problem: $\max\limits_{y(t),u(t)} \int_0^\infty v\left(y\left(t\right), u\left(t\right), t\right) dt$, subject to $\dot{y} = f\left(y\left(t\right), u\left(t\right), t\right)$. In economics, different forms of transversality conditions have been proposed to solve the problem. We show that the most general form of transversality condition can be derived under three additional assumptions following Chiang (1994)'s approach. We also reconsider the famous counterexamples of Halkin (1974) and Shell (1969), in the light of our transversality condition.

Keywords: transversality condition; dynamic optimization; infinite horizon.

JEL Classification Numbers: C61, D90.

1. INTRODUCTION

This note studies the transversality conditions for the model

$$\begin{cases} \max\limits_{y(t),u(t)} \int_0^\infty v\left(y\left(t\right), u\left(t\right), t\right) dt \text{ subject to} \\ y\left(0\right) = y_0,\ \dot{y} = f\left(y\left(t\right), u\left(t\right), t\right), \forall t \geqslant 0,\ u\left(t\right) \in U, \end{cases} \tag{1}$$

where v and f are real-valued continuously differentiable functions. Letting $\lambda\left(t\right)$ be the co-state variable associated with the constraint in (1), the corresponding Hamiltonian is given by:

$$H\left(y\left(t\right), u\left(t\right), \lambda\left(t\right), t\right) \equiv v\left(y\left(t\right), u\left(t\right), t\right) + \lambda\left(t\right) f\left(y\left(t\right), u\left(t\right), t\right).$$

The following equation is generally referred to as the transversality condition to (1):

$$\lim_{T\to\infty}\left[\lambda\left(T\right)y(T)\right]=0, \tag{2}$$

for any optimal path $\{y(t)\}$. By using an elementary perturbation argument, Michel (1982) provides a proof of the necessity of (2). However, the perturbation he considers is rather specific and his results are later generalized in Kamihigashi (2001). In this paper, we aim to provide a straightforward proof of this result, using Chiang (1994)'s approach, which is incorrect without certain assumptions. We show that three additional assumptions that are either implicitly assumed, or entirely overlooked, are required. We also investigate the famous counterexamples, Halkin (1974) and Shell (1969). They have been argued by Caputo (2005) as valid counterexamples, which would disqualify transversality condition in the form of $\lim_{T\to\infty}\lambda\left(T\right)=0$ as a necessary condition. We show that both of them satisfy the three assumptions and hence our transverality condition, $\lim_{T\to\infty}\lambda\left(T\right)\Delta y\left(T\right)=0$. However, since they have implicit fixed terminal states, $\lim_{T\to\infty}\lambda\left(T\right)=0$ cannot be applied to the two examples.

2. TRANSVERSALITY CONDITIONS FOR FINITE HORIZON PROBLEMS (CHIANG, 1992)

Following Chiang (1992), we first consider the finite horizon version of (1):

$$\begin{cases} \max\int_0^T v\left(y\left(t\right),u\left(t\right),t\right)dt \text{ subject to} \\ y\left(0\right)=y_0,\ \dot{y}=f\left(y\left(t\right),u\left(t\right),t\right),\forall t\geqslant 0,\ u\left(t\right)\in U. \end{cases} \tag{1a}$$

We introduce a new functional

$$\begin{aligned} V &\equiv v+\int_0^T\left[\lambda\left(t\right)f\left(y\left(t\right),u\left(t\right),t\right)-\dot{y}\right]dt \\ &=\int_0^T\left\{v\left(y\left(t\right),u\left(t\right),t\right)+\left[\lambda\left(t\right)f\left(y\left(t\right),u\left(t\right),t\right)-\dot{y}\right]\right\}dt. \end{aligned} \tag{3}$$

As in Chiang (1992, pp. 177-181), when we vary V infinitesimally by ε, from the standard variational methods, we have

$$\frac{dV}{d\varepsilon}=\int_0^T\left[\left(\frac{\partial H}{\partial y}+\dot{\lambda}\right)q\left(t\right)+\frac{\partial H}{\partial u}p\left(t\right)\right]dt+\left[H\right]_{t=T}\Delta T-\lambda\left(T\right)\Delta y\left(T\right)=0, \tag{4}$$

that leads to the transversality conditions for the following three cases, with $y\left(T\right)$, $\lambda\left(T\right)$ and $\left[H\right]_{t=T}$ denoting the optimal values

(i) *free terminal state* $(\Delta y\left(T\right)\neq 0)$ *and a fixed* $T(\Delta T=0)$

$$\lambda\left(T\right)=0; \tag{5}$$

(ii) *fixed terminal point* (T, y_T given)

$$y\left(T\right)=y_T; \tag{6}$$

(iii) *fixed-end point* (*fixed terminal state* $(\Delta y\left(T\right)=0)$ *and a free* $T(\Delta T\neq 0)$)

$$\left[H\right]_{t=T}=0. \tag{7}$$

3. INFINITE HORIZON TRANSVERSALITY CONDITIONS AND THE THREE ASSUMPTIONS

We proceed to consider the infinite horizon case. As in most works, we assume that the objective functional converges for all admissible paths.

Assumption (i). $V=\int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt$ is finite.

The objective functional in (1) can then be restated as

$$V=\int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt=\lim_{T\to\infty}\int_0^T v\left(y\left(t\right),u\left(t\right),t\right)dt \tag{8}$$

. We have

$$V=\int_0^\infty\left(v+\lambda\left(f-\dot{y}\right)\right)dt \tag{9}$$

$$=\lim_{T\to\infty}\int_0^T\left(v+\lambda\left(f-\dot{y}\right)\right)dt$$

$$=\lim_{T\to\infty}\left(\int_0^T\left(v+\lambda f+\dot{y}\right)dt-\left[\lambda y\right]_0^T\right)$$

$$=\int_0^\infty\left(v+\lambda f+\dot{y}\right)dt-\left[\lambda y\right]_0^\infty$$

$$= \int_0^\infty \left(H + \dot{\lambda} f\right) dt - (\lambda(\infty) y(\infty)) + (\lambda(0) y(0)).$$

Note that the last equality of (9) will be invalid if either $\int_0^\infty \left(H + \dot{\lambda} y\right) dt$ or $(\lambda(\infty) y(\infty))$ is infinite. Therefore, we consider

Assumption (ii). Both $\int_0^\infty \left(H + \dot{\lambda} y\right) dt$ and $(\lambda(\infty) y(\infty))$ are finite.

Then (9) can be further stated as

$$\begin{aligned} V_\varepsilon &= \int_0^\infty \left[H(t, y^* + \varepsilon q, u^* + \varepsilon p, \lambda) + \dot{\lambda}(y^* + \varepsilon q)\right] dt - \\ &\quad -\lambda(\infty)(y^*(\infty) + \varepsilon q(\infty)) + \lambda(0)(y^*(0) + \varepsilon q(0)). \end{aligned} \tag{10}$$

In general, $\frac{\partial}{\partial y}\left(\int_0^\infty v(x, y) dx\right) = \int_0^\infty \frac{\partial v(x,y)}{\partial y} dx$ only when $\lim_{T\to\infty}\left[\int_0^T \frac{\partial v(x,y)}{\partial y} dx\right]$ converges uniformly for y (Theorem 3.4 (p. 289), Lang, 1983). Hence, we impose

Assumption (iii). $\lim_{T\to\infty} \int_0^T \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right) q + \frac{\partial H}{\partial u} p\right] dt$ converges uniformly for y.

When Assumptions (i)∼(iii) are satisfied, setting

$$\frac{dV_\varepsilon}{d\varepsilon} = \underbrace{\int_0^\infty \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right) q(t) + \frac{\partial H}{\partial u} p(t)\right] dt}_{(I)} + \underbrace{\lim_{T\to\infty} H \Delta T}_{(II)} - \underbrace{\lim_{T\to\infty} \lambda(T) \Delta y(T)}_{(III)} = 0, \tag{11}$$

the first-order condition requires that each of the three component terms in (11) to be set equal to zero, respectively.

Again here we only consider the perturbation around the optimal values. This gives rise to *the general transversality conditions* for the infinite horizon problems:

$$\lim_{T\to\infty} H(T) \Delta T = 0 \text{ and } \lim_{T\to\infty} \lambda(T) \Delta y(T) = 0. \tag{12}$$

Remark $\lim_{T\to\infty} H(T) \Delta T = 0$ vanishes if the time horizon is assumed to be fixed at ∞.

Our general transversality condition is derived by directly following Chiang (1994)'s approach. It thus constitutes a straightforward proof of Kamihigashi (2001)'s fundamental results. Moreover, if the perturbation is to shift the entire optimal path downward by a small fixed proportion $\varepsilon \in [0, 1)$, as is in

Kamihigashi (2001), then $\Delta y(t) = \varepsilon y(t)$. Substituting $\varepsilon y(t)$ into the second equation in (12) leads to

$$\lim_{T\to\infty} \lambda(T)\,\varepsilon y(T) = \varepsilon \lim_{T\to\infty} \lambda(T)\,y(T) = 0. \tag{13}$$

Dividing both sides of (13) by ε, we then have (2).

Moreover, for problems with *a free terminal state*, as both the terminal time and the terminal state are not fixed ($\Delta T \neq 0$ and $\Delta y(T) \neq 0$), the transversality conditions comprise two conditions

$$\lim_{T\to\infty} [H(T)] = 0 \text{ and } \lim_{T\to\infty} \lambda(T) = 0 \tag{12$'$}$$

.

Note that the necessity of the second equation of (12') is derived by Michel (1982), who considers a specific perturbation that shifts the entire optimal path downward by a fixed value.

On the other hand, for problems with *a fixed terminal state* ($\Delta y(T) = 0$), the transversality condition is

$$\lim_{T\to\infty} [H(T)] = 0. \tag{12$''$}$$

Furthermore, from the definition of the Hamiltonian, we have

Remarks. 1. When $T \to \infty$, if the objective function converges to zero and the state equation is nonzero, then $\lim_{T\to\infty} [H(T)] = 0$ is equivalent to $\lim_{T\to\infty} \lambda(T) = 0$. Obviously, as the objective function for most discounted cases does approach zero when $T \to \infty$, either one of the two conditions in (12) can function as the transversality condition for such cases.

2. Sydster et al.'s necessary condition for an infinite horizon optimization problem with discounting (Theorem 9.11.2, 2005), can be readily derived from the Assumptions 2 and 3. Moreover, the "normal" transversality condition, $\lim_{T\to\infty} [\lambda(T) \cdot y(T)] = 0$, being redundant, remains valid.

3. Although economically intuitive, (2) can only be directly derived by variational approach that considers specific perturbations. However, it has to be noticed that there is no need to assume that the present value of the stock at the infinity should be zero.

4. For systems with steady-states, "inefficient overaccumulation of capital stock" does not necessarily imply that the present value of the stock approaches zero when t approaches infinity, $\lim_{t\to\infty}[\lambda(t)\cdot y(t)]=0$, as argued in Weitzman (2003). It only requires that $\lim_{T\to\infty}\lambda(T)\,\Delta y(T)=0$, i.e., the variations in the values of the capital stock should approach zero.

Note that Chiang's (1992) derivation of the transversality conditions for the infinite horizon case is imprecise as the above three assumptions have not been explicitly stipulated, although it can be easily verified that for most discounted problems, Assumptions (i)$\sim$(iii) are satisfied. Next, we consider the famous counterexamples of Halkin (1974) and Shell (1969).

4. APPLICATIONS

Several writers (Caputo (2005), for example) argue that there exist counterexamples that would disqualify transversality condition in the form of $\lim_{T\to\infty}\lambda(T)=0$ as a necessary condition. We shall demonstrate, however, these are not valid counterexamples, because they have implicit fixed terminal states.

4.1. HALKIN'S EXAMPLE (1974)

We first consider Halkin's example:

$$\begin{cases} \max\int_0^\infty(1-y)\,udt \text{ subject to} \\ y(0)=0,\ \dot{y}=(1-y)\,u,\ u(t)\in[0,1]\,. \end{cases} \tag{14}$$

From the equation of motion for y, we see that the definite solution is

$$y(t)=1-e^{-\int_0^t udt}. \tag{15}$$

Hence, problem (14) can be reformulated as

$$\max\int_0^\infty e^{-\int_0^t udt}udt \tag{16}$$

Let $\int_0^t udt=v(t)$, we see that $udt=dv$, and $\int_0^\infty udt=v(\infty)$. Now (16) can be restated as

$$\int_0^{v(\infty)}e^{-v}dv=-e^{-v}\Big|_0^{v(\infty)}=-e^{-v(\infty)}+1. \tag{17}$$

Obviously, (17) is maximized when $v(\infty) = \infty$, with the maximum being 1. In other words, the objective function in (14) is maximized if and only if $\int_0^\infty u(t)\,dt = \infty$.

Caputo (2005) incorrectly argues that Halkin's problem is a valid counterexample. He defines the control as $u(t) = \begin{cases} k \in [0,1] \;\; \forall t \in [0,\tau], \tau < \infty, \\ 0 \; \forall t \in (\tau, \infty), \end{cases}$ and argues that since the value of $\underset{t\to\infty}{}\, y(t)$ depends on k, the problem does not has a fixed terminal state. However, his choice of the control, although feasible, is not optimal as under such a case $\int_0^\infty u dt < \infty$.

Moreover, as $u(t) \in [0,1]$, we see that $e^{-\int_0^t u dt} \in (0,1]$. Consequently, $y(t) \in [0,1)$. It follows that

$$\underset{t\to\infty}{}\, y(t) = 1. \tag{18}$$

On the other hand, as

$$H = (1-y)\,u + \lambda\,(1-y)\,u = (1+\lambda)\,(1-y)\,u, \tag{19}$$

from the condition $\partial H/\partial u = 0$, we have $(1+\lambda)(1-y) = 0$, and the optimal co-state variable $\lambda^*(t) = -1$ inasmuch as y is always less than 1. Therefore, $H = 0$. Note that $\underset{T\to\infty}{}\, \lambda^*(T) \neq 0$.

It is easy to verify that Assumption (i)$\sim$(iii) are satisfied for this problem. Moreover, from (18), we see the problem has an implicit fixed terminal state, (12") applies and $\underset{T\to\infty}{}\, \lambda(T) = 0$ does not apply.

5. THE SHELL PROBLEM (1969)

The Shell problem is a modified version of Ramsey's (1928) model, which maximizes the deviation from a "bliss level". Let $\bar{c}$ be the steady-state consumption level:

$$\begin{cases} \max \int_0^\infty (c(t) - \bar{c})\,dt \text{ subject to} \\ k(0) = k_0,\; \dot{k} = \phi(k(t)) - c(t) - (n+\delta)\,k, 0 \leqslant c(t) \leqslant \phi(k(t)) \; . \end{cases} \tag{20}$$

It is easy to verify that Assumption (i)$\sim$(iii) are satisfied. Hence, (12) applies. However, as pointed in Chiang (1992), the Shell problem contains an implicit fixed terminal state as $c \to \bar{c}$ when $t \to \infty$. Therefore, the transversality condition should be (12").

Conclusions

As shown above, the dispute over the transversality condition for the infinite horizon problems, especially the interpretation concerning the "counterexamples", arises because the transversality conditions vary according to terminal states. Chiang correctly points out that both examples are not valid counterexamples as they all have implicitly fixed terminal states. However, a correct derivation of this result requires three additional assumptions to be explicitly stipulated, although it can be easily verified that Assumptions (i)$\sim$(iii) are satisfied for most discounted problems.

References

[1] Caputo, M. R., 2005, *Foundations of dynamic economic analysis: optimal control theory and applications*, Cambridge University Press, 2005.

[2] Chiang, A. C., 1992, *Elements of dynamic optimization*, McGraw-Hill, Singapore,, 1992.

[3] Halkin, H., *Necessary conditions for optimal control problems with infinite horizons*, Econometrica **42**), 267-272.

[4] Kamihigashi, T., *Necessity of transversality conditions for infinite horizon problems*, Econometrica **69**(2001), 995-1012.

[5] Lang, S., *Undergraduate analysis*, Springer, New York, 1997.(2nd Ed.)

[6] Michel, P., *On the transversality condition in infinite horizon optimal problems*, Econometica, **50**(1982), 975-985.

[7] Ramsey, F. P., *A mathematical theory of saving*, Economic Journal **38**(1928), 543-559.

[8] Shell, K.,*Applications of Pontryagin's maximum principle to economics*, in *Mathematical systems theory and economics*, H. W. Kuhn, G.P. Szegö (eds.), **1**, Springer, Berlin, 1969, 241-292.

[9] Sydster, K., Hammond, P., Seierstad, A., Strom, A., *Further mathematics for economic analysis*, Prentice Hall, 2005.

[10] Weitzman, M., *On the welfare significance of national product in a dynamic economy*, Quarterly Journal of Economics **90**(1976), 156-162.

ROMAI J., **5**, 1(2009), 113–120

A GENERALIZED MATHEMATICAL CAVITATION EROSION MODEL

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Abstract The methods most used to estimate the cavitation erosion resistance pay a special attention to the velocity erosion curve. Depending on the nature and condition of eroded materials, other kind of the volume loss rate curve of erosion cavitations progress is proposed. This model gives a new vision of the volume loss rate curve and generalize some previous mathematical models.

Keywords: erosion, cavitation, loss curve, mathematical model.

2000 MSC:76S05.

1. INTRODUCTION

The cavities are formed into a liquid when the static pressure of the liquid is reduced below the vapor pressure of the liquid in current temperature. If the cavities are carried to higher-pressure region they implode violently and very high pressures can occur. The cavitation phenomenon may cause serious changes in the microstructure and intrinsic stress level of the material. Macroscopically, the change in hardness is often observed; microscopically, the slip bands and deformation twins appear, and the phase transformations may occur in unstable alloys.

Cavitation erosion is a progressive loss of material from a solid due to the impact action of the collapsing bubbles or cavities in the liquid near the material surface. The models describing the mathematical relations between the volume loss and time in the cavitation erosion was intensely studied but the problem of the analytical description of the characteristic curves for cavitation erosion remained open.

We propose a mathematical model, which permits to define the volume loss curve of material and volume loss rate curve of cavitation erosion.

Usually, the volume loss curve [3], [4], [5], [6] is described by the formula

$$V(t) = A \cdot \Im \cdot U(k,t), \tag{1}$$

or formula

$$V(t) = A \cdot U(k, \Im t), \tag{2}$$

with A-the eroded surface area; $\Im$-measure of cavitations intensity; U- erosion progress function resulting out of applied phenomenological model; k- a set of real parameters (usually 3 parameters are quite sufficient) determined by fitting the erosion curve to the experimental data; t-cumulative exposure duration.

For the volume loss curve V and for volume loss rate curve $v = \frac{dV}{dt}$, for unity eroded surface area, usually [4], [5], [6] the pictures from fig. 1, fig. 2 are proposed.

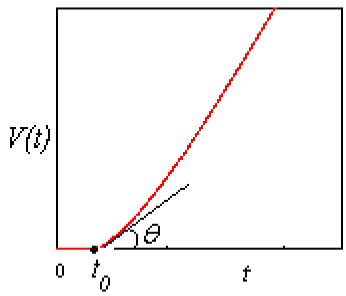

Fig. 1. The volume loss rate curve.

The volume loss rate curve (fig. 2) can be divided into four typical periods:

- incubation period I, is an initial period of damage in which volume loss of material is nearly zero (non-measurable). During the incubation period, a considerable plastic deformation occurs, without any apparent weight loss. In this time interval, the material accumulates energy;

- acceleration period A. In this time interval, the intensification of damage is observed, distinguished by violent increase of volume loss rate of erosion and the volume loss rate reaches the maximal value;

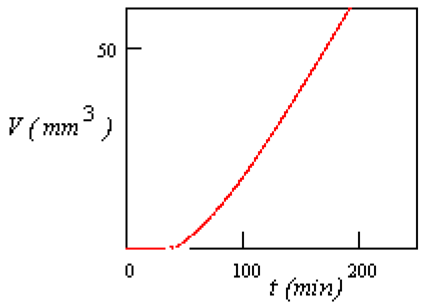

Fig. 2. The volume loss rate curve

- deceleration period D. In this time interval, volume loss rate decreases;

- steady state erosion period S, characterized by almost constant volume loss rate of erosion.

2. A "DAMPED" MODEL OF CAVITATION EROSION

In [1] the volume loss curve is given by formula

$$V(t) = A[v_s t - f(t)], \tag{3}$$

where A is the eroded surface area, v_s is the ultimate value of the volume loss rate and $f(t)$ is the solution of the second-order homogenous linear ordinary differential equations of with constant coefficients

$$\frac{d^2y}{dt^2} + 2\beta\frac{dy}{dt} + \beta^2 = 0 \tag{4}$$

which describes the "damped" oscillations with an "infinite period". Solving the ODE (4) and using (3) it follows that the volume loss is given by formula

$$V(t) = A[v_s t - \lambda t e^{-\beta t}], \tag{5}$$

and the volume loss rate curve is given by formula

$$v(t) = A[v_s - \lambda e^{-\beta t} + \lambda\beta e^{-\beta t}]. \tag{6}$$

The real parameters v_s , λ and β will be determined by fitting the erosion curve to the experimental data (by using the least squares method or another numerical method).

3. A GENERALIZATION OF "DAMPED" MODEL OF CAVITATION EROSION

Remark. The standard curve presented in fig. 2., is acceptable mathematically but it is not very good. During the incubation period, the volume loss V, and the volume loss rate $v = \frac{dV}{dt}$ are null. At the end of the incubation period, the volume loss rate curve $v(t)$ can have a point of discontinuity, because the volume rate curve, may be not smooth at the end point of the incubation period.

Let $(t_0, 0)$ be the end point of the incubation period. If θ is the angle between the time axis and the tangent to the volume loss curve at $(t_0, 0)$, and $\tan\theta \neq 0$ (fig. 3), then we have a discontinuity point for the volume loss rate curve $v(t)$, as in fig. 4.

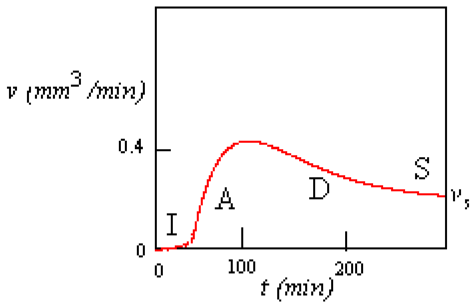

Fig. 3. The volume loss curve near of the end point of the incubation period.

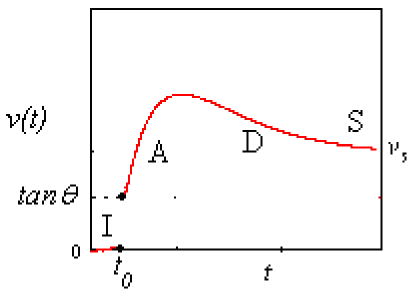

Fig. 4. The volume loss rate curve with a possible discontinuity at the end point of the incubation period.

During the incubation period, the volume loss and the volume loss rate curves are null and the our study is superfluous. In order to simplify the calculations, we choose the time interval $[-\varepsilon, 0]$ as the incubation period and we study the volume loss and the volume loss rate curves for $t \geqslant 0$ ($t_0 = 0$ is the end point of the incubation period)

In most of situations (e.g. in the case of damped vibrations) the volume loss rate curve $v(t)$ can look as in fig. 5 (the incubation period was chosen to be the time interval $[-\varepsilon, 0]$).

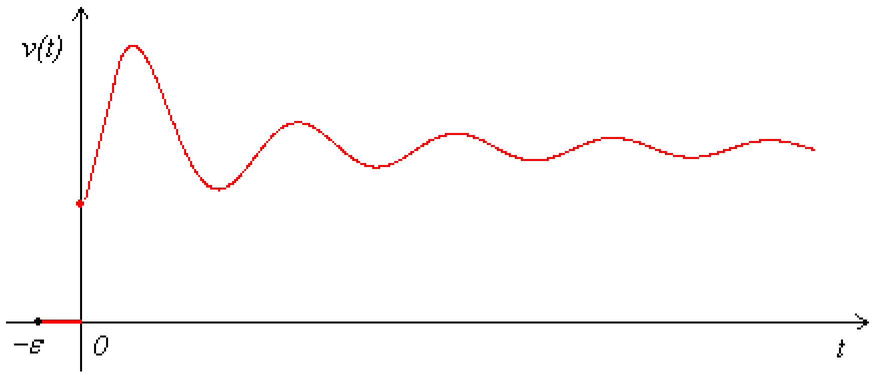

Fig. 5. New model of the volume loss rate curves.

Like in [1], for $v(t)$ we get

$$v(t) = \frac{dV(t)}{dt} = A[v_s t - \frac{df}{dt}(t)],$$

but now, $\frac{df}{dt}$ must satisfy the ODE

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = 0, \tag{7}$$

where α and β are real constants, depending on the eroded material; $\alpha \geq 0$, $\beta > 0$. Equation (7) has a physical interpretation like in the case of damped vibrations.

Since the eroded material, tested in Hydraulic Machinery Laboratory by using a vibratory device with a nickel tube, is subject to a frictional force and to a damping force, Newton's second law reads

$$m\frac{d^2y}{dt^2} = damping force + restoring force = -p\frac{dy}{dt} - qy. \tag{8}$$

Putting in (8) $\alpha = \frac{p}{m}$ and $\beta = \frac{q}{m}$, we obtain the equation (7), which is a second-order linear ordinary differential equation. Its auxiliary equation reads

$$r^2 + \alpha r + \beta = 0. \tag{9}$$

and has the roots $r_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$.

Case I. If $\alpha = 2\sqrt{\beta}$, the roots of equation (9) are $r_1 = r_2 = -\sqrt{\beta}$, and the equation (7) is, like (4), discussed in [1]. Taking into account that $f(0) = 0$, $\lim\limits_{t\to\infty} f(t) = 0$ and $\lim\limits_{t\to\infty} \frac{df(t)}{dt} = 0$ we have the volume loss curve

$$V(t) = A[v_s t - \lambda t e^{-\sqrt{\beta} t}] \tag{10}$$

and the volume loss rate curve is

$$v(t) = A[v_s - \lambda e^{-\sqrt{\beta} t} + \lambda \sqrt{\beta} t e^{-\sqrt{\beta} t}]. \tag{11}$$

Typical graphs of v as a function of t are shown in fig. 4.

Case II. If $\alpha^2 - 4\beta < 0$, then the roots of auxiliary equation (9) are complex: $r_1 = \frac{-\alpha}{2} + \gamma i$, $r_2 = \frac{-\alpha}{2} - \gamma i$, where $\gamma = \sqrt{4\beta - \alpha^2}$.

Since $\frac{df}{dt}$ is the general solution of equation (7) we have:

$$\frac{df}{dt}(t) = e^{\frac{-\alpha}{2} t}(C_1 \cos\gamma t + C_2 \sin\gamma t), \tag{12}$$

and

$$f(t) = \frac{2e^{\frac{-\alpha}{2} t}}{\alpha^2 + 4\gamma^2}(2C_1\gamma \sin\gamma t - C_2\alpha \sin\gamma t - 2C_2\gamma \cos\gamma t - C_1\alpha \cos\gamma t). \tag{13}$$

Obviously, the conditions: $\lim\limits_{t\to\infty} f(t) = 0$ and $\lim\limits_{t\to\infty} \frac{df(t)}{dt} = 0$ are satisfied. Using formula (13), the condition $f(0) = 0$ implies

$$2C_2\gamma + C_1\alpha = 0. \tag{14}$$

Then, using formulae (10) and (14), the volume loss curve is given by

$$V(t) = A\left[v_s t + \frac{2e^{\frac{-\alpha}{2} t}}{\alpha^2 + 4\gamma^2} C_2 \left(\frac{4\gamma^2}{\alpha} \sin\gamma t + \alpha \sin\gamma t\right)\right] \tag{15}$$

and the volume loss rate curve is

$$v(t) = A[v_s - e^{\frac{-\alpha}{2} t} C_2(\sin\gamma t - \frac{2\gamma}{\alpha} \sin\gamma t). \tag{16)]}$$

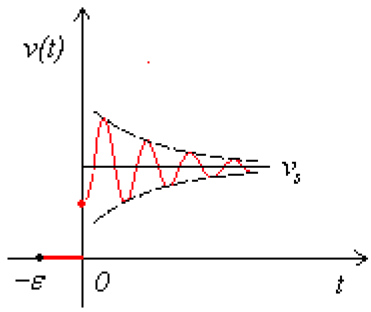

Fig. 6. The volume loss rate curve for case II.

Typical graphs of v as a function of t are shown in fig. 6.

Case III. If $\alpha^2 - 4\beta > 0$, then auxiliary equation (11) has distinct real roots r_1, r_2. Then we have: $\frac{df}{dt}(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ and $f(t) = \frac{C_1}{r_1} e^{r_1 t} + \frac{C_2}{r_2} e^{r_2 t}$. Since α, β are positive and $\sqrt{\alpha^2 - 4\beta} < \alpha$, the roots r_1, r_2 must both be negative and the conditions $\lim_{t\to\infty} f(t) = 0$ and $\lim_{t\to\infty} \frac{df(t)}{dt} = 0$ are satisfied. The condition $f(0) = 0$ implies $\frac{C_2}{r_2} = -\frac{C_1}{r_1}$. Then, the volume loss curve is

$$V(t) = A(v_s t - \frac{C_1}{r_1} e^{r_1 t} + \frac{C_1}{r_1} e^{r_2 t}) \tag{17}$$

and the volume loss rate curve reads

$$V(t) = A(v_s - C_1 e^{r_1 t} + \frac{C_1 r_2}{r_1} e^{r_2 t}).$$

Typical graphs of v as a function of t are shown in fig. 4.

CONCLUSIONS

The advantage of using ODEs for theoretical analytical erosion curves is that the parameters appear in a natural way in the solution of ODEs and it is often not necessary to decide a priori the number of these parameters. Usually, the real parameters which appear in the expressions of $V(t)$ or $v(t)$ can be determined by fitting the erosion curve to the experimental data, using the least squares method or another numerical method.

Depending on nature and condition of eroded material, we have presented a new possible image of the erosion curves according to the experimental data. Practically, the experimental data may suggest the best mathematical model selection for a given material subject to erosion.

References

1. Bordeaşu, I., Pătrăşcoiu, C., Bădărău, V., Sucitu, O., Popoviciu, M. O., *New contribution to cavitation erosion curve modeling*, FME TRANSACTION, Univ. of Belgrade, Faculty of Mechanical Enginering, New Series, **34**, 1(2006), 39-43.

2. Heymann, F. J., *On the time dependence of the rate of erosion due to impingement or cavitation*, The 69th Annual Meeting, ASTM, Atlantic City, 1996, 70-82.

3. Noskievič, J., *Some aspects with regard to cavitation resistance of materials*, HIDROTURBO 76, Brno, **C7**, 1976, 1-16.(Polish)

4. Steller, J., Kaczmarzyk, P., *Phenomenological models of cavitation erosion progress*, Intern. Cavitation Erosion Test Seminar, Sopot, June 1-2 2000, 1-9.

5. Steller, J., *A concept of fractional cavitation erosion resistance*, Intern. Cavitation Erosion Test Seminar, Sopot, June 1-2 2000, F1-F10.

6. Thiruvengadam, A., Preiser, H. S., *On testing materials for cavitation damage resistance*, Journal Ship., **8**, 233-243.

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CLASSIFICATION OF THE CUBIC DIFFERENTIAL SYSTEMS WITH SEVEN REAL INVARIANT STRAIGHT LINES

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Abstract A result regarding the classification of the cubic differential systems with seven real invariant straight lines is presnted.

Keywords:cubic differential systems, invariant lines.

2000 MSC: 34C05.

We consider the cubic differential system

$$\dot{x} = P(x, y),\ \dot{y} = Q(x, y), \tag{1}$$

where P, $Q \in \mathbb{R}[x, y]$, $max\{deg(P), deg(Q)\} = 3$ and $GCD(P, Q) = 1$.

The straight line $Ax+By+C = 0$ is said to be invariant for (1) if there exists a polynomial $K(x, y)$ such that the identity $A \cdot P + B \cdot Q \equiv (Ax + By + C) \cdot K$ holds. Let $K(x, y) \equiv (Ax + By + C)^m \cdot K^*(x, y)$, where $m \in \mathbb{N}$, $K^* \in \mathbb{R}[x, y]$ and $Ax + By + C = 0$ does not divide $K^*(x, y)$. Then we say that the invariant straight line has the degree of invariance $m + 1$.

A set of invariant straight lines can be infinite, finite or empty. In the cases, the number of invariant straight lines is finite, this number is at most eight.

A qualitative investigation of cubic systems with exactly eight and exactly seven invariant straight lines was carried out in [1-3]. In this paper a similar qualitative investigation is done for cubic differential systems with exactly seven real invariant straight lines. It is proved

Theorem. *Any cubic differential system possessing real invariant straight lines with total degree of invariance seven via affine transformation and time rescaling can be written as one of the following seven systems*

$$\begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = y(y+1)(y-a), \\ a > 0,\ a \neq 1; \end{cases} \qquad \begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = y(y+1)((2+a)x-(1+a)y-a), \\ a > 0,\ a \neq 1; \end{cases}$$

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y^2(dx+(1-d)y), \\ d(1-d)(d-3)(2d-3) \neq 0; \end{cases} \qquad \begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = y(y+1)((1-a)x+ay-a), \\ a > 0,\ a \neq 1; \end{cases}$$

$$\begin{cases} \dot{x} = x^2(bx+y), \\ \dot{y} = y^2((2+3b)x-(1+2b)y), \\ b(b+1)(2+3b)(1+2b) \neq 0; \end{cases} \qquad \begin{cases} \dot{x} = x(x+1)(a+(2a-1)x+y), \\ \dot{y} = y(y+1)(a+(3a-1)x+(1-a)y), \\ a(2a-1)(1-a)(3a-1)(3a-2) \neq 0; \end{cases}$$

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y^2(y+1). \end{cases}$$

For the obtained cubic systems the qualitative investigation was performed.

References

[1] P. A. Lyubimova, *About one differential equation with invariant straight lines*, Differential and integral equations, Gorky Universitet **1**(1977), 19-22.(in Russian)

[2] P. A. Lyubimova, *About one differential equation with invariant straight lines*, Differential and integral equations, Gorky Universitet **8**(1984), 66-69.(in Russian)

[3] J. Llibre, N. Vulpe, *Planar cubic polynomial differential systems with the maximum number of invariant straight lines*, Preprint, 1-54, 2006.

[4] A. A. Andronov, E. A. Leontovich, I. I. Gordon, A. L. Maier, *Qualitative investigation of two-dimensional autonomous systems*, Nauka, Moscow, 1966.(in Russian)

ROMAI J., **5**, 1(2009), 123–130

OBJECT CLASSIFICATION METHODS WITH APPLICATION IN ASTRONOMY

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Abstract

This paper reviews some of the methods used in practice for object classification and studies the possibility of combining a well-known object classification technique with other image processing methodologies such as edge detection. An application of this proposed process may be the classification of galaxies, based upon the Hubble classification. The feature taken into consideration is the galaxy shape.

Keywords: object classification, PCA, Hubble galaxy classification.

2000 MSC:85A99.

1. INTRODUCTION

The need of automated object recognition is of great importance nowadays not only because of the practical impact in almost all fields of information manipulation, but also because of the large quantity of data available for processing.

Many approaches have been proposed in this purpose. There are different types of artificial neural network architectures taken into consideration as well as statistical methods to discriminate between classes of objects. Also, there are efficient methods that use fractal dimensions of objects to be classified in order to assign them to a certain class.

The large amount of data received from satellites nowadays makes very difficult their analysis by the human factor manually. That is why there are many attempts to automate this process, to develop ways in order to obtain processed elements out of the information provided by observatories or satellites.

For example, Institut d'Astrophysique & Observatoire de Paris has developed a software [9] to able create catalogues of objects by analyzing images that contain different surfaces of the sky. The main feature of this package is the use of neural networks applied in classification techniques.

Other example of this type of software is YODA (Yet another Object Detection Application) [10]. This type of software represents an important methodology of astronomical image processing. It computes different shape parameters and classifies objects according to more than one approach.

There are a few developing directions regarding the classification of objects in astronomy images. First, there is the need of pre-processing the raw information received from the information capturing device (such as satellites, terrestrial and extra-terrestrial telescopes or NASA spacecrafts). The results of astronomical observations are sometimes one dimensional signals that need to be transformed into digital images. The following step is to process the image obtained, so that the result emphasizes the objects appearance. This is done taking into consideration some of object features in the picture, among which shape plays an important part. The last step is the actual classification.

2. CLASSIFICATION OF GALAXIES

There are several types of features relevant to the classification (of galaxies). They can be divided into: photometry, profile and shape features [2]. The photometric features (which represent the central concentration of light index) and the profile features (the radial distribution of surface brightness, otherwise representing asymmetry) can be used [1] to morphologically classify galaxies.

The shape features are more attractive when dealing with morphology of galaxies because the morphology itself is a visual context for classification.

Here are some important shape features which are successfully used in classification techniques: elongation (the measure of flatness of an object), a form factor which is the ratio between the area and the square perimeter of the galaxy (this is important because elliptical galaxies are more luminous while spiral ones have much broader areas and less luminosity), convexity (which is very large in spiral galaxies and very small for elliptical ones) [2]. As seen in fig. 1, there are a few main types of galaxies according to shape classification:

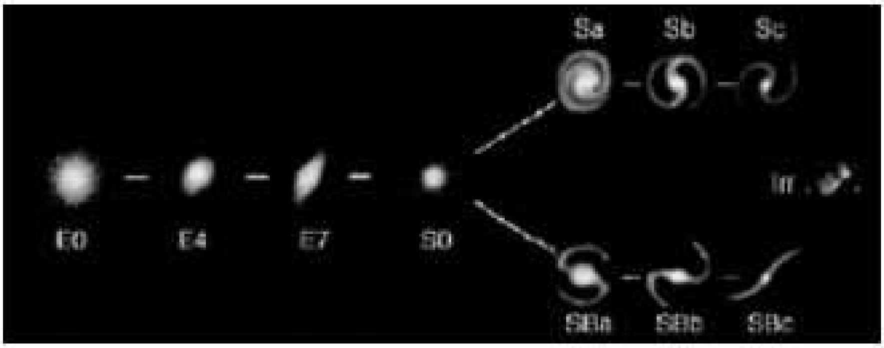

Fig. 1. Hubble's classification scheme.

elliptical galaxies (denoted by **E**), spiral galaxies (denoted by **S**). According to the shape of the spiral, the spiral galaxies can be subdivided into three subclasses: spherical or barred-like; lenticular (denoted by **S0** and characterized by a bright center); and irregular, which can not be assigned to any class mentioned above.

3. ARTIFICIAL NEURAL NETWORKS AND GALAXY CLASSIFICATION

Artificial neural networks have been successfully used in automation of galaxy classification. Both supervised (back propagation method) and unsupervised (self organizing maps such as Kohonen networks) types of neural networks were brought to light by researchers.

The first attempts used the raw image data to train the classifiers, leading to high error rates.

Nielsen and Odewahn (1995) emphasize the idea of using parameters characterizing galaxies instead of straightforward pixels of the source image. Their results showed that the use of profile features-based classifier is the most efficient (as compared to other feature-based methods or raw pixel use) [2]. A description of a back propagation application with two hidden layers is found in [4]. The method assumes the use of some parameters in order to compute the weight vectors of the neural network; first, photometric parameters (such as surface brightness, concentration index and color) are used for classification. Then, parameters are switched to profile-like features (brightness profiles in two band passes [2]). The last experiment was made on raw pixel data obtained from images.

Another approach in this field of research is presented in [5] and it uses an algorithm called DBNN (Difference Boosting Neural Network).

Although this architecture is closely related to the naive Bayesian classifier, it gives some degree of freedom regarding the correlation of the data attributes. The instrument used to this purpose is the association of a threshold window with every attribute. This window influences the decision coefficients of the classifier. The classification system contains a boosting technique, which emphasizes the difference between training data elements.

There are other more recent papers which point to the idea of using also other mathematical tools such as the fractal signature of classified objects [6].

Another approach is the one presented in [3]. It implies the classifier Random Forrest and involves a number of decision tree classifiers. The individual decisions are combined to give the final classification result. The algorithm proposed in [3] is an efficient tool for the galaxy classification problem. The preprocessing part implies geometric transformations of the image, such that shape features of the galaxies are enhanced.

These transformations are:

1 a threshold, in order to extract the the bright part of the image and eliminate some background irrelevant elements;

2 rotation of the image with an angle given by the first principal component of the image, so that the galaxy is brought into a horizontal position (the standard position for all images, in order for the classifier to have higher efficiency);

3 then, the image is resized to standard dimension, in order to be included into the training sample matrix.

The use of the pixel information would be very expensive in terms of resources because of the dimension of the data (e. g. if all images would be standard 128x128, the training matrix would be Nx16384, where N is the number of images taken into consideration). That is why in order to reduce dimensionality, the authors propose the use PCA (Principal Component Analysis). As experimental results show, this is a very effective method which enables significant reduction of computation time and resources use.

The next step is represented by the actual classification, which is done by a classifier that assumes the use of decision trees.

4. PROCESSING DATA

The algorithm reviewed above was implemented as follows:

1 preprocessing techniques described above;

2 result images were resized in standard dimension;

3 application of PCA techniques and use of them in classification at different values;

4 training of the artificial neural network by a number of N digital images. The original algorithm is using RF techniques for classification;

5 testing of the weight matrix by presenting to it new images yet to be classified.

The experimental results presented below were obtained by applying the above algorithm with the following changes:

1 in order to enhance optimally the shape and brightness distribution of the galaxies, preprocessing techniques described above were used, at different values of the threshold;

2 result images were resized in dimension 24x24 (and not 128x128, as the original algorithm suggested).

3 application of PCA techniques and use of them in classification at different values. Improvement of efficiency relative to the increasing number

number of PCs	error
8	0.2821
15	0.2308
20	0.2051
25	0.1282
30	0.0513
35	0

Fig. 2. Classification on training data set.

number of PCs	error
8	80%
15	20%
20	40%
25	20%
30	40%
35	20%

Fig. 3. Classification on 5 new images.

of principal components used in training of the artificial neural network was noticed, as it can be noticed from the results below.

4 training of the artificial neural network by a number of N digital images (N=33) in order to obtain a minimum error rate for classifying the images given to training. The experimental results presented below are obtained by using a standard classifier.

5. EXPERIMENTAL RESULTS

The images used were downloaded from [11]. The implementation was written in Matlab, version 7.6.0.324 and tested on configuration including a 2Ghz processor and 1G of RAM.

6. CONCLUSIONS

In automated galaxy classification good results were obtained by using several types of artificial neural networks. Experimental results show that even on a reduced dimension of space of training samples, the efficiency of the algorithm is satisfying. Future directions in this domain may include the use of different algorithms such as adaptive algorithms for different neural networks, boosting algorithms or combinations of methods generally used in object classification.

Acknowledgements. The images used in this research were taken from www.seds.org. Special thanks are due to Dr. Maria Miroiu for her important collaboration and suggestions.

References

[1] Abraham, B., Valdes, F., Yee, H., Carlberg, R., *Automated morphological classification of faint galaxies*, 1993.

[2] Goderya, S. N., Lolling, S. M., *Morphological classification of galaxies using computer vision and artificial neural networks: a computational scheme astrophysics and space science* **279**(2002), 377 - 387.

[3] Calleja, J., Fuentes, O., *Automated Classification of Galaxy Images. Lecture Notes in Computer Science*, **3215**(2004).

[4] Odewahn, S., *Galaxy Classification Using Artificial Neural Networks*,I nternational Joint Conference on Neural Networks, **6**(1999), 3824 - 3829.

[5] Philip, N. S., Wadadekar, Y., Kembhavi, A.,Joseph K. B., *A difference boosting neural network for automated star-galaxy classification*, Astronomy & Astrophysics, **385**, (2002).

[6] Lekshmi, S., Revathy1, K., Prabhakaran Nayar, S. R., *Galaxy classification using fractal signature*, Astronomy& Astrophysics, (2003).

[7] Miroiu, M., Vaduva, I., *Special Tailored Boosting Methos for Face Recognition*, SEERC, (2008), 101 -112.

[8] Turk, M. A.,; Pentland, A. P., *Face recognition using eigenfaces*, IEEE Computer Society Conference **3-6**(1991) 586 - 591.

[9] http://terapix.iap.fr/rubrique.php?id_rubrique=91

[10] http://www.aanda.org/index.php?option=article&access=standard&Itemid=129 &url=/articles/aa/pdf/2003/01/aah2400.pdf

[11] http://www.seds.org/messier/objects.html#galaxy

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UNIVERSAL REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS: FIXED POINTS, EQUIVALENCIES AND DYNAMICAL BIFURCATIONS

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Abstract The asynchronous systems are the non-deterministic models of the asynchronous circuits from the digital electrical engineering. In the autonomous version, such a system is a set of functions $x : \mathbf{R} \to \{0,1\}^n$ called states ($\mathbf{R}$ is the time set). If an autonomous asynchronous system is defined by making use of a so called generator function $\Phi : \{0,1\}^n \to \{0,1\}^n$, then it is called regular. The regular autonomous asynchronous systems compute in real time the iterates of Φ when these are not made, in general, on all the coordinates $\Phi_1, ..., \Phi_n$ simultaneosly. The property of universality means the greatest in the sense of the inclusion.

The purpose of the paper is that of defining and of characterizing the fixed points, the equivalencies and the dynamical bifurcations of the universal regular autonomous asynchronous systems. We use analogies with the dynamical systems theory.

Keywords: asynchronous system, fixed point, dynamical bifurcation.

2000 MSC: 94C10, 93A30, 47H10, 37G99.

1. INTRODUCTION

Switching theory, more precisely: what we mean by switching theory, has been practiced in the 50's and the 60's by many mathematicians, in dialogue with engineers. The last book from this series was published by Moisil in 1969 **??**. After 1970, the theory of modeling the asynchronous circuits from the digital electrical engineering has developed in a manner suggesting that the main interest of the researchers is to keep away their works from publication. In this context, we have started some years ago the construction of a theory of modeling the asynchronous circuits under the name of asynchronous systems theory. A part of this theory, related with the universal regular autonomous asynchronous systems is presented in this paper. The bibliography that we indicate consists in works on dynamical systems (written as usual on real numbers, we use binary numbers here) that create analogies. They are not relevant to the readers that are familiar with the concepts of orbit, nullclin, dynamical bifurcation etc, except for showing the source of inspiration of the construction. The paper is obviously self-contained.

2. PRELIMINARIES

Definition 2.1. *We denote by* $\mathbf{B} = \{0, 1\}$ *the* ***binary Boole algebra****, endowed with the discrete topology and with the usual laws.*

Definition 2.2. *Let be the Boolean function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n, \Phi = (\Phi_1, ..., \Phi_n)$ *and* $\nu \in \mathbf{B}^n, \nu = (\nu_1, ..., \nu_n)$. *We define* $\Phi^\nu : \mathbf{B}^n \to \mathbf{B}^n$ *by* $\forall \mu \in \mathbf{B}^n$,

$$\Phi^\nu(\mu) = (\overline{\nu_1} \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu), ..., \overline{\nu_n} \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu)).$$

Remark 1. Φ^ν *represents the function resulting from* Φ *when this one is not computed, in general, on all the coordinates* $\Phi_i, i = \overline{1, n}$: *if* $\nu_i = 0$, *then* Φ_i *is not computed,* $\Phi_i^\nu(\mu) = \mu_i$ *and if* $\nu_i = 1$, *then* Φ_i *is computed,* $\Phi_i^\nu(\mu) = \Phi_i(\mu)$.

Definition 2.3. *Let be the sequence* $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \mathbf{B}^n$. *The functions* $\Phi^{\alpha^0 \alpha^1 ... \alpha^k} : \mathbf{B}^n \to \mathbf{B}^n$ *are defined iteratively by* $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}$,

$$\Phi^{\alpha^0 \alpha^1 ... \alpha^k \alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0 \alpha^1 ... \alpha^k}(\mu)).$$

Definition 2.4. *The sequence* $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \mathbf{B}^n$ *is called* ***progressive*** *if*

$$\forall i \in \{1, ..., n\}, \text{ the set } \{k | k \in \mathbf{N}, \alpha_i^k = 1\} \text{ is infinite.}$$

The set of the progressive sequences is denoted by Π_n.

Remark 2.1. *Let be* $\mu \in \mathbf{B}^n$. *When* $\alpha = \alpha^0, \alpha^1, ..., \alpha^k, ...$ *is progressive, each coordinate* $\Phi_i, i = \overline{1, n}$ *is computed infinitely many times in the sequence* $\Phi^{\alpha^0 \alpha^1 ... \alpha^k}(\mu)$, $k \in \mathbf{N}$.

Definition 2.5. *The* ***initial value****, denoted by* $x(-\infty + 0)$ *or* $\lim\limits_{t \to -\infty} x(t) \in \mathbf{B}^n$ *and the* ***final value****, denoted by* $x(\infty - 0)$ *or* $\lim\limits_{t \to \infty} x(t) \in \mathbf{B}^n$ *of the function*

$x : \mathbf{R} \to \mathbf{B}^n$ *are defined by*

$$\exists t' \in \mathbf{R}, \forall t < t', x(t) = x(-\infty + 0),$$

$$\exists t' \in \mathbf{R}, \forall t > t', x(t) = x(\infty - 0).$$

Definition 2.6. *The function $x : \mathbf{R} \to \mathbf{B}^n$ is called (**pseudo**)**periodical with the period** $T_0 > 0$ if*

a) $\lim\limits_{t \to \infty} x(t)$ *does not exist and*
b) $\exists t' \in \mathbf{R}, \forall t \geq t', x(t) = x(t + T_0)$.

Definition 2.7. *The **characteristic function** $\chi_A : \mathbf{R} \to \mathbf{B}$ of the set $A \subset \mathbf{R}$ is defined in the following way:*

$$\chi_A(t) = \begin{cases} 1, \ if \ t \in A \\ 0, otherwise \end{cases}.$$

Notation 2.1. *We denote by Seq the set of the real sequences $t_0 < t_1 < ... < t_k < ...$ which are unbounded from above.*

Remark 2.2. *The sequences $(t_k) \in Seq$ act as time sets. At this level of generality of the exposure, a double uncertainty exists in the real time iterative computations of the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$: we do not know precisely neither the coordinates Φ_i of Φ that are computed, nor when the computation happens. This uncertainty implies the non-determinism of the model and its origin consists in structural fluctuations in the fabrication process, the variations in ambiental temperature and the power supply etc.*

Definition 1. *A **signal** (or $n-$**signal**) is a function $x : \mathbf{R} \to \mathbf{B}^n$ of the form*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \tag{1}$$

$$... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

with $(t_k) \in Seq$. The set of the signals is denoted by $S^{(n)}$.

Remark 2. *The signals $x \in S^{(n)}$ model the electrical signals from the digital electrical engineering. They have by definition initial values and they avoid 'Dirichlet type' properties (called Zeno properties by the engineers) such as*

$$\exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t - \varepsilon, t), \exists t'' \in (t - \varepsilon, t), x(t') \neq x(t''),$$

$$\exists t \in \mathbf{R}, \forall \varepsilon > 0, \exists t' \in (t, t+\varepsilon), \exists t'' \in (t, t+\varepsilon), x(t') \neq x(t'')$$

because these properties cannot characterize the inertial devices.

We can interpret now Definition 2.6 of (pseudo)periodicity in the situation when $x \in S^{(n)}$*. If at b) we would have had* $\forall t \in \mathbf{R}, x(t) = x(t+T_0)$*, then the existence of* $x(-\infty+0)$ *implies that* x *is constant. Similarly, if a) would be false, then* x *would be constant. In other words Definition 2.6 was formulated in a way that makes us work with non-constant functions, a request of non-triviality.*

Notation 2.2. *We denote by* P^* *the set of the non-empty subsets of a set.*

Definition 2.8. *The* ***autonomous asynchronous systems*** *are the non-empty sets* $X \in P^*(S^{(n)})$.

Example 2.1. *We give the following simple example that shows how the autonomous asynchronous systems model the asynchronous circuits. In Figure 1 we have drawn the (logical) gate NOT with the input* $u \in S^{(1)}$ *and the state*

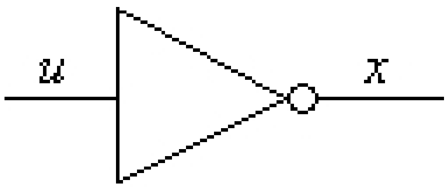

Fig. 1. Circuit with the logical gate NOT.

(the output) $x \in S^{(1)}$*. For* $\lambda \in \mathbf{B}$ *and*

$$u(t) = \lambda,$$

the state x *represents the computation of the negation of* u *and it is of the form*

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_0,t_1)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_1,t_2)}(t) \oplus ... \oplus \overline{\lambda} \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

$$= \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_0,\infty)}(t),$$

where $\mu \in \mathbf{B}$ *is the initial value of* x *and* $(t_k) \in Seq$ *is arbitrary. As we can see,* x *depends on* t_0, μ, λ *only and it is independent on* $t_1, t_2, ...$

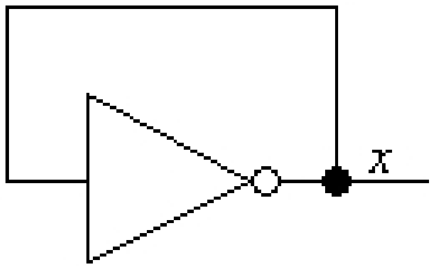

Fig. 2. Circuit with feedback with the logical gate NOT.

In Figure 2, we have

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

$$\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus ...$$

thus this circuit is modeled by the autonomous asynchronous system

$$X = \{\mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus ...$$

$$\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus ... | \mu \in \mathbf{B}, (t_k) \in Seq\} \in P^*(S^{(1)}).$$

Definition 2.9. *The **progressive functions** $\rho : \mathbf{R} \to \mathbf{B}^n$ are by definition the functions*

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \qquad (2)$$

where $(t_k) \in Seq$ and $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \Pi_n$. The set of the progressive functions is denoted by P_n.

Definition 2.10. *For $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and $\rho \in P_n$ like at (2), we define $\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \to \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n, \forall t \in \mathbf{R}$,*

$$\Phi^\rho(\mu, t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \Phi^{\alpha^0 ... \alpha^k}(\mu) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Remark 2.3. *The previous equation reminds the iterations of a discrete time real dynamical system. The time is not exactly discrete in it, but some sort of intermediate situation occurs between the discrete and the real time; on the other hand the iterations of Φ do not happen in general on all the coordinates (synchronicity), but on some coordinates only, such that any coordinate Φ_i is* *computed infinitely many times,* $i = \overline{1,n}$ *(asynchronicity) when* $t \in \mathbf{R}$*. This is the meaning of the progress property, giving the so called 'unbounded delay model' of computation of the Boolean functions.*

3. DISCRETE TIME

Notation 3.1. *We denote by*

$$\mathbf{N}_{\_} = \mathbf{N} \cup \{-1\}$$

the discrete time set.

Definition 3.1. *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\alpha \in \Pi_n, \alpha = \alpha^0, ..., \alpha^k, ...$ *We define the function* $\widehat{\Phi}^\alpha : \mathbf{B}^n \times \mathbf{N}_{\_} \to \mathbf{B}^n$ *by* $\forall (\mu, k) \in \mathbf{B}^n \times \mathbf{N}_{\_}$,

$$\widehat{\Phi}^\alpha(\mu, k) = \begin{cases} \mu, k = -1, \\ \Phi^{\alpha^0 ... \alpha^k}(\mu), k \geq 0 \end{cases}.$$

Notation 3.2. *Let us denote*

$$\widehat{\Pi}_n = \{\alpha | \alpha \in \Pi_n, \forall k \in \mathbf{N}, \alpha^k \neq (0, ..., 0)\}.$$

Definition 3.2. *The equivalence of* $\rho, \rho' \in P_n$ *is defined by:* $\exists (t_k) \in Seq, \exists (t'_k) \in Seq, \exists \alpha \in \widehat{\Pi}_n$ *such that (2) and*

$$\rho'(t) = \alpha^0 \cdot \chi_{\{t'_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t'_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t'_k\}}(t) \oplus ...$$

are true.

Definition 3.3. *The 'canonical surjection'* $s : P_n \to \widehat{\Pi}_n$ *is by definition the function* $\forall \rho \in P_n$,

$$s(\rho) = \alpha$$

where $\alpha \in \widehat{\Pi}_n$ *is the only sequence such that* $(t_k) \in Seq$ *exists, making the equation (2) true.*

Remark 3.1. *The relation between the continuous and the discrete time is the following: for any* $\mu \in \mathbf{B}^n$ *and any* $\rho \in P_n$*, the sequences* $\alpha \in \widehat{\Pi}_n$ *and* $(t_k) \in Seq$ *exist making the equation (2) true and we have*

$$\Phi^\rho(\mu, t) = \widehat{\Phi}^\alpha(\mu, -1) \cdot \chi_{(-\infty, t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu, 0) \cdot \chi_{[t_0, t_1)}(t) \oplus ...$$

$$... \oplus \widehat{\Phi}^{\alpha}(\mu, k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...$$

Equivalent progressive functions $\rho, \rho' \in P_n$ (i.e. $s(\rho) = s(\rho')$) give 'equivalent' functions $\Phi^{\rho}(\mu, t), \Phi^{\rho'}(\mu, t)$ in the sense that the computations $\widehat{\Phi}^{\alpha}(\mu, k), k \in \mathbf{N}_-$ are the same $\forall \mu \in \mathbf{B}^n$, but the time flow is piecewise faster or slower in the two situations.

4. REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS

Definition 4.1. *The **universal regular autonomous asynchronous system** $\Xi_{\Phi} \in P^*(S^{(n)})$ that is generated by the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ is defined by*

$$\Xi_{\Phi} = \{\Phi^{\rho}(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in P_n\}.$$

Definition 4.2. *An autonomous asynchronous system $X \in P^*(S^{(n)})$ is called **regular**, if Φ exists such that $X \subset \Xi_{\Phi}$. In this case Φ is called the **generator function** of X.*

Remark 4.1. ***1.*** *The terminology of 'generator function' is also used in [1], meaning the vector field of a discrete time dynamical system. In [3] the terminology of 'generator' (function) of a dynamical system is mentioned too. Moisil called Φ 'network function' in a non-autonomous, discrete time context; for Moisil, 'network' means 'system' or 'circuit'.*

2. *In the last two definitions, the attribute 'regular' refers to the existence of a generator function Φ and the attribute 'universal' means maximal relative to the inclusion.*

For a regular system, Φ is not unique in general.

Example 4.1. *For any $\mu^0 \in \mathbf{B}^n$ and $\rho^* \in P_n$, the autonomous systems $\{\Phi^{\rho^*}(\mu^0, \cdot)\}$, $\{\Phi^{\rho}(\mu^0, \cdot) | \rho \in P_n\}$, $\{\Phi^{\rho^*}(\mu, \cdot) | \mu \in \mathbf{B}^n\}$ and Ξ_{Φ} are regular.*

For $\Phi = 1_{\mathbf{B}^n}$, the system $\Xi_{1_{\mathbf{B}^n}} = \{\mu | \mu \in \mathbf{B}^n\} = \mathbf{B}^n$ is regular and we have identified the constant function $x \in S^{(n)}, x(t) = \mu$ with the constant $\mu \in \mathbf{B}^n$.

Another example of universal regular autonomous asynchronous system is given by $\Phi = \mu^0$, the constant function, for which $\Xi_{\mu^0} = \{x | x_i = \mu_i \cdot \chi_{(-\infty, t_i)} \oplus \mu_i^0 \cdot \chi_{[t_i, \infty)}, \mu_i \in \mathbf{B}, t_i \in \mathbf{R}, i = \overline{1, n}\}$.

Remark 4.2. *These examples suggest several possibilities of defining the systems $X \subset \Xi_\Phi$ which are not universal. For example by putting appropriate supplementary requests on the functions ρ, one could rediscover the 'bounded delay model' of computation of the Boolean functions. If ρ is fixed, we get the 'fixed delay model' of computation of the Boolean functions.*

5. ORBITS AND STATE PORTRAITS

Definition 5.1. *Let be $\rho \in P_n$. Two things are understood by* ***orbit****, or (****state****, or* ***phase****)* ***trajectory of*** Ξ_Φ ***starting at*** $\mu \in \mathbf{B}^n$*:*

a) the function $\Phi^\rho(\mu, \cdot) : \mathbf{R} \to \mathbf{B}^n$;

b) the set $Or_\rho(\mu) = \{\Phi^\rho(\mu, t) | t \in \mathbf{R}\}$ representing the values of the previous function.

Sometimes the function from a) is called the ***motion*** *(or the* ***dynamic****) of μ through Φ^ρ.*

Definition 5.2. *The equivalent properties*

$$\exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'$$

and

$$\mu' \in Or_\rho(\mu)$$

are called of ***accessibility****; the points $\mu' \in Or_\rho(\mu)$ are said to be* ***accessible****.*

Remark 5.1. *The orbits are the curves in $\mathbf{B}^n$, parametrized by ρ and t. On the other hand $\rho \in P_n$, $t' \in \mathbf{R}$ imply $\rho \cdot \chi_{(t',\infty)} \in P_n$ and we see the truth of the implication*

$$\mu' = \Phi^\rho(\mu, t') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t).$$

Definition 5.3. *The* ***state*** *(or the* ***phase****)* ***portrait*** *of Ξ_Φ is the set of its orbits.*

Example 5.1. *The function* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ *is defined by the following table*

(μ_1, μ_2)	$\Phi(\mu_1, \mu_2)$
$(0,0)$	$(0,0)$
$(0,1)$	$(1,0)$
$(1,0)$	$(1,1)$
$(1,1)$	$(1,1)$

The state portrait of Ξ_Φ *is:*

$$\{(0,1) \cdot \chi_{(-\infty,t_0)} \oplus (0,0) \cdot \chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\} \cup$$

$$\cup \{(0,1) \cdot \chi_{(-\infty,t_0)} \oplus (1,0) \cdot \chi_{[t_0,t_1)} \oplus (1,1) \cdot \chi_{[t_1,\infty)} | t_0, t_1 \in \mathbf{R}, t_0 < t_1\} \cup$$

$$\cup \{(0,1) \cdot \chi_{(-\infty,t_0)} \oplus (1,1) \cdot \chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\} \cup$$

$$\cup \{(1,0) \cdot \chi_{(-\infty,t_0)} \oplus (1,1) \cdot \chi_{[t_0,\infty)} | t_0 \in \mathbf{R}\} \cup \{(0,0)\} \cup \{(1,1)\}.$$

This set is drawn in Figure 3,

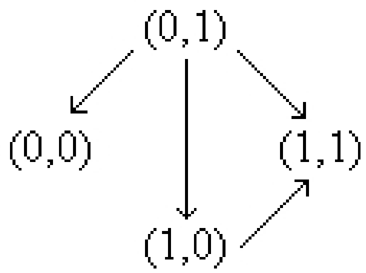

Fig. 3. The state portrait of the system from Example 5.1.

where the arrows show the increase of time. One might want to put arrows from $(0,0)$ *to itself and from* $(1,1)$ *to itself.*

6. NULLCLINS

Definition 6.1. *Let be* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$*. For any* $i \in \{1, ..., n\}$*, the* ***nullclins*** *of* Φ *are the sets*

$$NC_i = \{\mu | \mu \in \mathbf{B}^n, \Phi_i(\mu) = \mu_i\}.$$

If $\mu \in NC_i$*, then the coordinate* i *is said to be* ***not excited****, or* ***not enabled****, or* ***stable*** *and if* $\mu \in \mathbf{B}^n \setminus NC_i$ *then it is called* ***excited****, or* ***enabled****, or* ***unstable****.*

Remark 6.1. *Sometimes, instead of indicating* Φ *by a table like previously, we can replace Figure 3 by Figure 4,*

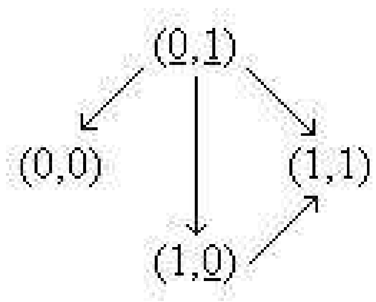

Fig. 4. The state portrait of the system from Example 5.1, version.

where we have underlined the unstable coordinates. For example in Figure 4, $(\underline{0},\underline{1})$ *means that* $\Phi(0,1)=(1,0)$, $(1,\underline{0})$ *means that* $\Phi(1,0)=(1,1)$ *etc.*

In fact Figure 4 results uniquely from Figure 3, one could know by looking at Figure 3 which coordinates should be underlined and which should be not. For example the existence of an arrow from $(0,1)$ *to* $(1,0)$ *shows that in* $(0,1)$ *both coordinates are enabled.*

7. FIXED POINTS. REST POSITION

Definition 7.1. *A point* $\mu\in\mathbf{B}^n$ *that fulfills* $\Phi(\mu)=\mu$ *is called a* ***fixed point*** *(an* ***equilibrium point****, a* ***critical point****, a* ***singular point****), shortly an* ***equilibrium*** *of* Φ. *A point that is not fixed is called* ***ordinary****.*

Theorem 7.1. *The following statements are equivalent for* $\mu\in\mathbf{B}^n$:

$$\Phi(\mu)=\mu, \tag{3}$$

$$\exists\rho\in P_n,\forall t\in\mathbf{R},\Phi^\rho(\mu,t)=\mu, \tag{4}$$

$$\forall\rho\in P_n,\forall t\in\mathbf{R},\Phi^\rho(\mu,t)=\mu, \tag{5}$$

$$\exists\rho\in P_n,Or_\rho(\mu)=\{\mu\}, \tag{6}$$

$$\forall\rho\in P_n,Or_\rho(\mu)=\{\mu\}, \tag{7}$$

$$\mu\in NC_1\cap...\cap NC_n. \tag{8}$$

Proof. (3)$\Longrightarrow$(4) We take $\rho \in P_n$ in the following way

$$\rho(t) = (1, ..., 1) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus (1, ..., 1) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

with $(t_k) \in Seq$. For the sequence

$$\forall k \in \mathbf{N}, \alpha^k = (1, ..., 1)$$

from Π_n we can prove by induction on k that

$$\forall k \in \mathbf{N}, \Phi^{\alpha^0 ... \alpha^k}(\mu) = \mu \tag{9}$$

wherefrom

$$\Phi^{\rho}(\mu, t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \mu \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus \mu \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... = \mu \tag{10}$$

(4)$\Longrightarrow$(3) From (4) we have the existence of $\alpha \in \Pi_n$ and $(t_k) \in Seq$ with the property that (10) is true, thus (9) is true. We denote

$$I_0 = \{i | i \in \{1, ..., n\}, \alpha_i^0 = 1\},$$

$$I_1 = \{i | i \in \{1, ..., n\}, \alpha_i^1 = 1\},$$

$$...$$

$$I_k = \{i | i \in \{1, ..., n\}, \alpha_i^k = 1\},$$

$$...$$

and we have from (9):

$\forall i \in \{1, .., n\}$,

$$\Phi_i^{\alpha^0}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_0 \\ \mu_i, i \in \{1, ..., n\} \setminus I_0 \end{cases} = \mu_i;$$

$$\forall i \in \{1, .., n\}, \Phi_i^{\alpha^0 \alpha^1}(\mu) = \Phi_i^{\alpha^1}(\Phi^{\alpha^0}(\mu)) =$$

$$= \Phi_i^{\alpha^1}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_1 \\ \mu_i, i \in \{1, ..., n\} \setminus I_1 \end{cases} = \mu_i;$$

$$...$$

$$\forall i \in \{1, .., n\}, \Phi_i^{\alpha^0 \alpha^1 ... \alpha^k}(\mu) = \Phi_i^{\alpha^k}(\Phi^{\alpha^0 ... \alpha^{k-1}}(\mu)) =$$

$$= \Phi_i^{\alpha^k}(\mu) = \begin{cases} \Phi_i(\mu), i \in I_k \\ \mu_i, i \in \{1, ..., n\} \setminus I_k \end{cases} = \mu_i;$$

...

with the conclusion that

$$\forall k \in \mathbf{N}, \forall i \in I_0 \cup I_1 \cup ... \cup I_k, \Phi_i(\mu) = \mu_i.$$

As α is progressive, some $k' \in \mathbf{N}$ exists with the property that

$$I_0 \cup I_1 \cup ... \cup I_{k'} = \{1, ..., n\},$$

thus (3) is true.

(3)$\Longrightarrow$(5) Let be

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{11}$$

with $\alpha^0, ..., \alpha^k, ... \in \Pi_n$ and $(t_k) \in Seq$ arbitrary. It is proved by induction on k the validity of (9) and this implies the truth of (10).

(5)$\Longrightarrow$(3) This is true because (5)$\Longrightarrow$(4) and (4)$\Longrightarrow$(3) are true.

(4)$\Longleftrightarrow$(6) and (5)$\Longleftrightarrow$(7) are obvious.

(3)$\Longleftrightarrow$(8) $\Phi(\mu) = \mu \Longleftrightarrow \Phi_1(\mu) = \mu_1$ and...and $\Phi_n(\mu) = \mu_n \Longleftrightarrow \mu \in NC_1$ and...and $\mu \in NC_n \Longleftrightarrow \mu \in NC_1 \cap ... \cap NC_n$. ∎

Definition 2. *If* $\Phi(\mu) = \mu$, *then* $\forall \rho \in P_n$, *the orbit* $\Phi^\rho(\mu, t) = \mu$ *is called* ***rest position***.

8. FIXED POINTS VS. FINAL VALUES OF THE ORBITS

Theorem 8.1. *([8], Theorem 49) The following fixed point property is true*

$$\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim_{t\to\infty} \Phi^\rho(\mu, t) = \mu' \Longrightarrow \Phi(\mu') = \mu'.$$

Proof. Let $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ be arbitrary and fixed. Some $t' \in \mathbf{R}$ exists such that $\forall t \geq t'$,

$$\mu' = \Phi^\rho(\mu, t) \overset{\text{Remark 5.1}}{=} \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t).$$

Because $\forall t < t'$,

$$\Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t) = \Phi^{(0,...,0)}(\mu', t) = \mu',$$

from Theorem 7.1, (4)$\Longrightarrow$(3) we have that $\Phi(\mu') = \mu'$. ∎

Remark 3. *Theorem 8.1 shows that the final values of the states of the system Ξ_Φ are fixed points of Φ.*

Theorem 8.2. *([8], Theorem 50) We have $\forall\mu \in \mathbf{B}^n, \forall\mu' \in \mathbf{B}^n, \forall\rho \in P_n$,*

$$(\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbf{R}, \Phi^\rho(\mu, t') = \mu') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \mu'.$$

Proof. For arbitrary $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ we suppose that $\Phi(\mu') = \mu'$ and $\Phi^\rho(\mu, t') = \mu'$. We have $\forall t \geq t'$,

$$\Phi^\rho(\mu, t) \overset{\text{Remark 5.1}}{=} \Phi^{\rho \cdot \chi_{(t',\infty)}}(\mu', t) \overset{\text{Theorem 7.1, (3)}\Longrightarrow\text{(5)}}{=} \mu'.$$

∎

Remark 4. *As resulting from Theorem 8.2, the accessible fixed points are final values of the states of the system Ξ_Φ.*

The properties of the fixed points that are expressed by Theorems 7.1, 8.1, 8.2 give a better understanding of Example 5.1.

9. TRANSITIVITY

Definition 9.1. *The system Ξ_Φ (or the function Φ) is **transitive**, or **minimal** if one of the following non-equivalent properties holds true:*

$$\forall\mu \in \mathbf{B}^n, \forall\mu' \in \mathbf{B}^n, \exists\rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu', \tag{12}$$

$$\forall\mu \in \mathbf{B}^n, \forall\mu' \in \mathbf{B}^n, \forall\rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu'. \tag{13}$$

Remark 9.1. *The property of transitivity may be considered one of surjectivity or one of accessibility.*

If Φ is transitive, then it has no fixed points. For example $1_{\mathbf{B}^n}$ is not transitive since all $\mu \in \mathbf{B}^n$ are fixed points for this function.

Example 9.1. *The property (12) of transitivity is exemplified in Figure 5 and the property (13) of transitivity is exemplified in Figure 6.*

10. THE EQUIVALENCE OF THE SYSTEMS

Notation 10.1. *Let $h : \mathbf{B}^n \to \mathbf{B}^n$ and $x : \mathbf{R} \to \mathbf{B}^n$ be some functions. We denote by $h(x) : \mathbf{R} \to \mathbf{B}^n$ the function*

$$\forall t \in \mathbf{R}, h(x)(t) = h(x(t)).$$

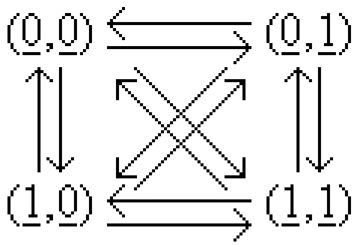

Fig. 5. Transitivity.

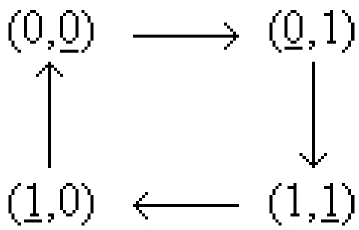

Fig. 6. Transitivity.

Remark 10.1. *If* $h : \mathbf{B}^n \to \mathbf{B}^n$ *and* $x \in S^{(n)}$ *is expressed by*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

then

$$h(x)(t) = h(x(-\infty + 0)) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(x(t_0)) \cdot \chi_{[t_0,t_1)}(t) \oplus ...$$

$$... \oplus h(x(t_k)) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$$

Notation 10.2. *For* $h : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\alpha = \alpha^0, ..., \alpha^k, ... \in \mathbf{B}^n$, *we denote by* $\widehat{h}(\alpha)$ *the sequence* $h(\alpha^0), ..., h(\alpha^k), ... \in \mathbf{B}^n$.

Notation 10.3. *Let be* $k \geq 2$ *arbitrary and we denote for* $\mu^1, ..., \mu^k \in \mathbf{B}^n$,

$$\mu^1 \cup ... \cup \mu^k = (\mu_1^1 \cup ... \cup \mu_1^k, ..., \mu_n^1 \cup ... \cup \mu_n^k).$$

Notation 10.4. *We denote by* Ω_n *the set of the functions* $h : \mathbf{B}^n \to \mathbf{B}^n$ *that fulfill*

i) h *is bijective;*

ii) $h(0, ..., 0) = (0, ..., 0),\ h(1, ..., 1) = (1, ..., 1);$

iii) $\forall k \geq 2, \forall \mu^1 \in \mathbf{B}^n, ..., \forall \mu^k \in \mathbf{B}^n,$

$$\mu^1 \cup ... \cup \mu^k = (1, ..., 1) \Longleftrightarrow h(\mu^1) \cup ... \cup h(\mu^k) = (1, ..., 1).$$

Theorem 10.1. *a)* Ω_n *is group relative to the composition* $'\circ'$ *of the functions;*

b) $\forall h \in \Omega_n, \forall \alpha \in \Pi_n, \widehat{h}(\alpha) \in \Pi_n;$

c) $\forall h \in \Omega_n, \forall \rho \in P_n, h(\rho) \in P_n.$

Proof. a) We can prove the fact that $1_{\mathbf{B}^n} \in \Omega_n, \forall h \in \Omega_n, \forall h' \in \Omega_n, h \circ h' \in \Omega_n$ and $\forall h \in \Omega_n, h^{-1} \in \Omega_n$. For example let be $h \in \Omega_n, k \geq 2$ and $\nu^1, ..., \nu^k \in \mathbf{B}^n$ arbitrary, for which we denote $\mu^1 = h^{-1}(\nu^1), ..., \mu^k = h^{-1}(\nu^k)$. We have:

$$h^{-1}(\nu^1 \cup ... \cup \nu^k) = (1, ..., 1) \Longleftrightarrow \nu^1 \cup ... \cup \nu^k = h(1, ..., 1) = (1, ..., 1)$$

$$\Longleftrightarrow h(\mu^1) \cup ... \cup h(\mu^k) = (1, ..., 1) \Longleftrightarrow \mu^1 \cup ... \cup \mu^k = (1, ..., 1)$$

$$\Longleftrightarrow h^{-1}(\nu^1) \cup ... \cup h^{-1}(\nu^k) = (1, ..., 1),$$

thus h^{-1} fulfills iii) from Notation 10.4.

b) Let $h \in \Omega_n$ and $\alpha = \alpha^0, ..., \alpha^k, ... \in \mathbf{B}^n$ be arbitrary. We denote for $p \geq 1$

$$\{\mu^1, ..., \mu^p\} = \{\mu | \mu \in \mathbf{B}^n, \{k | k \in \mathbf{N}, \alpha^k = \mu\} \text{ is infinite}\}$$

and we remark that

$$\alpha \in \Pi_n \Longleftrightarrow \mu^1, ..., \mu^p, \mu^1, ..., \mu^p, \mu^1, ... \in \Pi_n \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \mu^1 = (1, ..., 1), p = 1 \\ \mu^1 \cup ... \cup \mu^p = (1, ..., 1), p \geq 2 \end{cases},$$

$$\widehat{h}(\alpha) \in \Pi_n \Longleftrightarrow h(\mu^1), ..., h(\mu^p), h(\mu^1), ..., h(\mu^p), h(\mu^1), ... \in \Pi_n \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} h(\mu^1) = (1, ..., 1), p = 1 \\ h(\mu^1) \cup ... \cup h(\mu^p) = (1, ..., 1), p \geq 2 \end{cases}.$$

Case $p = 1$,

$$\alpha \in \Pi_n \Longrightarrow \mu^1 = (1, ..., 1) \Longrightarrow h(\mu^1) = (1, ..., 1) \Longrightarrow \widehat{h}(\alpha) \in \Pi_n.$$

Case $p \geq 2$,

$$\alpha \in \Pi_n \Longrightarrow \mu^1 \cup ... \cup \mu^p = (1, ..., 1) \Longrightarrow h(\mu^1) \cup ... \cup h(\mu^p) = (1, ..., 1) \Longrightarrow$$

$$\Longrightarrow \widehat{h}(\alpha) \in \Pi_n.$$

c) Let us take arbitrarily some $h \in \Omega_n$ and a function $\rho \in P_n$,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$$

where $\alpha \in \Pi_n$ and $(t_k) \in Seq$. We have

$$h(\rho)(t) = h(\rho(t)) =$$

$$= h((0, ..., 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus (0, ..., 0) \cdot \chi_{(t_0,t_1)}(t) \oplus ...$$

$$... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus (0, ..., 0) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus ...)$$

$$= h(0, ..., 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus h(0, ..., 0) \cdot \chi_{(t_0,t_1)}(t) \oplus ...$$

$$... \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus h(0, ..., 0) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus ...$$

$$= h(\alpha^0) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus h(\alpha^k) \cdot \chi_{\{t_k\}}(t) \oplus ...$$

Because $\widehat{h}(\alpha) \in \Pi_n$, taking into account b), we conclude that $h(\rho) \in P_n$. ∎

Theorem 10.2. *Let be the functions $\Phi, \Psi : \mathbf{B}^n \to \mathbf{B}^n$ and the bijections $h : \mathbf{B}^n \to \mathbf{B}^n, h' \in \Omega_n$. The following statements are equivalent:*

a) $\forall \nu \in \mathbf{B}^n$, *the diagram*

$$\begin{array}{ccc} \mathbf{B}^n & \stackrel{\Phi^\nu}{\to} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \stackrel{\Psi^{h'(\nu)}}{\to} & \mathbf{B}^n \end{array}$$

is commutative;

b) $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}_{\_}$,

$$h(\widehat{\Phi}^\alpha(\mu, k)) = \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu), k);$$

c) $\forall \mu \in \mathbf{B}^n, \forall \rho \in P_n, \forall t \in \mathbf{R}$,

$$h(\Phi^\rho(\mu, t)) = \Psi^{h'(\rho)}(h(\mu), t). \tag{14}$$

Proof. a)$\Longrightarrow$b) It is sufficient to prove that $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \forall k \in \mathbf{N}$,

$$h(\Phi^{\alpha^0 ... \alpha^k}(\mu)) = \Psi^{h'(\alpha^0)...h'(\alpha^k)}(h(\mu)) \tag{15}$$

since this is equivalent with b).

We fix arbitrarily some μ and some α and we use the induction on k. For $k=0$ the statement is proved, thus we suppose that it is true for k and we prove it for $k+1$:

$$h(\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu))=h(\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu)))=\Psi^{h'(\alpha^{k+1})}(h(\Phi^{\alpha^0...\alpha^k}(\mu)))=$$

$$=\Psi^{h'(\alpha^{k+1})}(\Psi^{h'(\alpha^0)...h'(\alpha^k)}(h(\mu)))=\Psi^{h'(\alpha^0)...h'(\alpha^k)h'(\alpha^{k+1})}(h(\mu)).$$

b)$\Longrightarrow$c) For arbitrary $\mu\in\mathbf{B}^n$ and $\rho\in P_n$,

$$\rho(t)=\rho(t_0)\cdot\chi_{\{t_0\}}(t)\oplus...\oplus\rho(t_k)\cdot\chi_{\{t_k\}}(t)\oplus...$$

with $(t_k)\in Seq,\rho(t_0),...,\rho(t_k),...\in\Pi_n$, we have that

$$h'(\rho)(t)=h'(\rho(t))=h'(\rho(t_0))\cdot\chi_{\{t_0\}}(t)\oplus...\oplus h'(\rho(t_k))\cdot\chi_{\{t_k\}}(t)\oplus... \quad (16)$$

is an element of P_n (see Theorem 10.1 c)) and

$$h(\Phi^{\rho}(\mu,t))=h(\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus\Phi^{\rho(t_0)}(\mu)\cdot\chi_{[t_0,t_1)}(t)\oplus...$$

$$...\oplus\Phi^{\rho(t_0)...\rho(t_k)}(\mu)\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...)=$$

$$=h(\mu)\cdot\chi_{(-\infty,t_0)}(t)\oplus h(\Phi^{\rho(t_0)}(\mu))\cdot\chi_{[t_0,t_1)}(t)\oplus...$$

$$...\oplus h(\Phi^{\rho(t_0)...\rho(t_k)}(\mu))\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...=$$

$$\overset{(15)}{=}h(\mu)\cdot\chi_{(-\infty,t_0)}(t)\oplus\Psi^{h'(\rho(t_0))}(h(\mu))\cdot\chi_{[t_0,t_1)}(t)\oplus...$$

$$...\oplus\Psi^{h'(\rho(t_0))...h'(\rho(t_k))}(h(\mu))\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...\overset{(16)}{=}\Psi^{h'(\rho)}(h(\mu),t).$$

c)$\Longrightarrow$a) Let $\nu,\mu\in\mathbf{B}^n$ be arbitrary and fixed and we consider $\rho\in P_n$,

$$\rho(t)=\nu\cdot\chi_{\{t_0\}}(t)\oplus\rho(t_1)\cdot\chi_{\{t_1\}}(t)\oplus...\oplus\rho(t_k)\cdot\chi_{\{t_k\}}(t)\oplus...$$

with $(t_k)\in Seq$ fixed too. We have

$$h(\Phi^{\rho}(\mu,t))=h(\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus\Phi^{\nu}(\mu)\cdot\chi_{[t_0,t_1)}(t)\oplus\Phi^{\nu\rho(t_1)}(\mu)\cdot\chi_{[t_1,t_2)}(t)\oplus...)=$$

$$=h(\mu)\cdot\chi_{(-\infty,t_0)}(t)\oplus h(\Phi^{\nu}(\mu))\cdot\chi_{[t_0,t_1)}(t)\oplus h(\Phi^{\nu\rho(t_1)}(\mu))\cdot\chi_{[t_1,t_2)}(t)\oplus...$$

But

$$h'(\rho)(t)=h'(\rho(t))=h'(\nu)\cdot\chi_{\{t_0\}}(t)\oplus h'(\rho(t_1))\cdot\chi_{\{t_1\}}(t)\oplus...,$$

$$\Psi^{h'(\rho)}(h(\mu),t) =$$

$$= h(\mu)\cdot\chi_{(-\infty,t_0)}(t)\oplus\Psi^{h'(\nu)}\cdot\chi_{[t_0,t_1)}(t)\oplus\Psi^{h'(\nu)h'(\rho(t_1))}\cdot\chi_{[t_1,t_2)}(t)\oplus...$$

and from (14), for $t\in[t_0,t_1)$, we obtain

$$h(\Phi^{\nu}(\mu)) = \Psi^{h'(\nu)}(h(\mu)).$$

∎

Definition 10.1. *We consider the functions* $\Phi,\Psi:\mathbf{B}^n\to\mathbf{B}^n$. *If two bijections* $h:\mathbf{B}^n\to\mathbf{B}^n, h'\in\Omega_n$ *exist such that one of the equivalent properties a), b), c) from Theorem 10.2 is satisfied, then* Ξ_Φ,Ξ_Ψ *are called* ***equivalent*** *and* Φ,Ψ *are called* ***conjugated***. *In this case we denote* $\Phi\overset{(h,h')}{\to}\Psi$.

Remark 10.2. *The equivalence of the universal regular autonomous asynchronous systems is indeed an equivalence and it should be understood as a change of coordinates. Thus* Φ *and* Ψ *are indistinguishable.*

Example 10.1. $\Phi,\Psi:\mathbf{B}^2\to\mathbf{B}^2$ *are given by, see Figure 7*

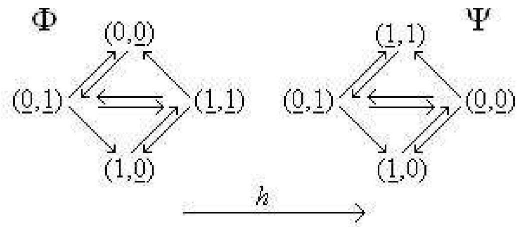

Fig. 7. Equivalent systems.

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,\Phi(\mu_1,\mu_2) = (\mu_1\oplus\mu_2,\overline{\mu_2}),$$

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,\Psi(\mu_1,\mu_2) = (\overline{\mu_1},\overline{\mu_1}\cdot\overline{\mu_2}\cup\mu_1\cdot\mu_2)$$

and the bijection $h:\mathbf{B}^2\to\mathbf{B}^2$ *is*

$$\forall(\mu_1,\mu_2)\in\mathbf{B}^2,h(\mu_1,\mu_2) = (\overline{\mu_2},\overline{\mu_1}).$$

The diagram

$$\begin{array}{ccc}\mathbf{B}^2 & \overset{\Phi^{\nu}}{\to} & \mathbf{B}^2\\ h\downarrow & & \downarrow h\\ \mathbf{B}^2 & \overset{\Psi^{\nu'}}{\to} & \mathbf{B}^2\end{array}$$

commutes for $\nu = \nu' = (0,0)$ *and on the other hand for* $\nu = \nu' = (1,1)$ *we have the assignments*

$$\begin{array}{ccc} (0,0) & \overset{\Phi}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (1,1) & \overset{\Psi}{\rightarrow} & (0,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \overset{\Phi}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (0,1) & \overset{\Psi}{\rightarrow} & (1,0) \end{array}, \quad \begin{array}{ccc} (1,0) & \overset{\Phi}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (1,0) & \overset{\Psi}{\rightarrow} & (0,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \overset{\Phi}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (0,0) & \overset{\Psi}{\rightarrow} & (1,1) \end{array}.$$

We denote $\pi_i : \mathbf{B}^2 \rightarrow \mathbf{B}, \forall(\mu_1, \mu_2) \in \mathbf{B}^2$,

$$\pi_i(\mu_1, \mu_2) = \mu_i, i = \overline{1,2}.$$

For $\nu = (0,1), \nu' = (1,0)$ *we have*

$$\begin{array}{ccc} (0,0) & \overset{(\pi_1,\Phi_2)}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (1,1) & \overset{(\Psi_1,\pi_2)}{\rightarrow} & (0,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \overset{(\pi_1,\Phi_2)}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (0,1) & \overset{(\Psi_1,\pi_2)}{\rightarrow} & (1,1) \end{array},$$

$$\begin{array}{ccc} (1,0) & \overset{(\pi_1,\Phi_2)}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (1,0) & \overset{(\Psi_1,\pi_2)}{\rightarrow} & (0,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \overset{(\pi_1,\Phi_2)}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (0,0) & \overset{(\Psi_1,\pi_2)}{\rightarrow} & (1,0) \end{array},$$

and for $\nu = (1,0), \nu' = (0,1)$ *the assignments are*

$$\begin{array}{ccc} (0,0) & \overset{(\Phi_1,\pi_2)}{\rightarrow} & (0,0) \\ h\downarrow & & \downarrow h \\ (1,1) & \overset{(\pi_1,\Psi_2)}{\rightarrow} & (1,1) \end{array}, \quad \begin{array}{ccc} (0,1) & \overset{(\Phi_1,\pi_2)}{\rightarrow} & (1,1) \\ h\downarrow & & \downarrow h \\ (0,1) & \overset{(\pi_1,\Psi_2)}{\rightarrow} & (0,0) \end{array},$$

$$\begin{array}{ccc} (1,0) & \overset{(\Phi_1,\pi_2)}{\rightarrow} & (1,0) \\ h\downarrow & & \downarrow h \\ (1,0) & \overset{(\pi_1,\Psi_2)}{\rightarrow} & (1,0) \end{array}, \quad \begin{array}{ccc} (1,1) & \overset{(\Phi_1,\pi_2)}{\rightarrow} & (0,1) \\ h\downarrow & & \downarrow h \\ (0,0) & \overset{(\pi_1,\Psi_2)}{\rightarrow} & (0,1) \end{array},$$

respectively. Φ *and* Ψ *are conjugated.*

Example 10.2. *The functions* $h, h' : \mathbf{B}^2 \to \mathbf{B}^2$ *are given in the following table*

(μ_1, μ_2)	$h(\mu_1, \mu_2)$	$h'(\mu_1, \mu_2)$
$(0,0)$	$(0,1)$	$(0,0)$
$(0,1)$	$(1,1)$	$(1,0)$
$(1,0)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,0)$	$(1,1)$

and the state portraits of the two systems are given in Figure 8. Ξ_Φ *and* Ξ_Ψ *are equivalent.*

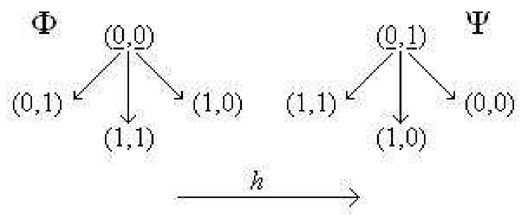

Fig. 8. Equivalent systems.

Theorem 10.3. *If* Φ *and* Ψ *are conjugated, then the following possibilities exist:*

a) $\Phi = \Psi = 1_{\mathbf{B}^n}$;

b) $\Phi \neq 1_{\mathbf{B}^n}$ *and* $\Psi \neq 1_{\mathbf{B}^n}$.

Proof. We presume that $\Phi \overset{(h,h')}{\rightarrow} \Psi$. In the equation

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = \Psi^{h'(\nu)}(h(\mu))$$

we put $\Psi = 1_{\mathbf{B}^n}$ and we have

$$\forall \nu \in \mathbf{B}^n, \forall \mu \in \mathbf{B}^n, h(\Phi^\nu(\mu)) = h(\mu)$$

thus $\forall \nu \in \mathbf{B}^n, \Phi^\nu = 1_{\mathbf{B}^n}$ and finally $\Phi = 1_{\mathbf{B}^n}$. ∎

Theorem 10.4. *We suppose that* Ξ_Φ *and* Ξ_Ψ *are equivalent and let be* h, h' *such that* $\Phi \overset{(h,h')}{\rightarrow} \Psi$.

a) If μ *is a fixed point of* Φ, *then* $h(\mu)$ *is a fixed point of* Ψ.

b) For any $\mu \in \mathbf{B}^n$ *and any* $\rho \in P_n$, *if* $\Phi^\rho(\mu, t)$ *is periodical with the period* T_0, *then* $\Psi^{h'(\rho)}(h(\mu), t)$ *is periodical with the period* T_0.

c) If Ξ_Φ is transitive, then Ξ_Ψ is transitive.

Proof. a) We suppose that $\Phi(\mu) = \mu$. The commutativity of the diagram

$$\begin{array}{ccc} \mathbf{B}^n & \stackrel{\Phi^\nu}{\to} & \mathbf{B}^n \\ h\downarrow & & \downarrow h \\ \mathbf{B}^n & \stackrel{\Psi^{h'(\nu)}}{\to} & \mathbf{B}^n \end{array}$$

for $\nu = (1, ..., 1)$ gives

$$h(\mu) = h(\Phi(\mu)) = h(\Phi^{(1,\ldots,1)}(\mu)) = \Psi^{h'(1,\ldots,1)}(h(\mu)) =$$

$$= \Psi^{(1,\ldots,1)}(h(\mu)) = \Psi(h(\mu)).$$

b) Let be $\mu \in \mathbf{B}^n$ and $\rho \in P_n$. The hypothesis states that $\exists t' \in \mathbf{R}, \forall t \geq t'$,

$$\Phi^\rho(\mu, t) = \Phi^\rho(\mu, t + T_0)$$

and in this situation

$$\Psi^{h'(\rho)}(h(\mu), t) = h(\Phi^\rho(\mu, t)) = h(\Phi^\rho(\mu, t + T_0)) = \Psi^{h'(\rho)}(h(\mu), t + T_0).$$

c) Let $\mu, \mu' \in \mathbf{B}^n$ be arbitrary and fixed. The hypothesis (12) states that

$$\exists \rho \in P_n, \exists t \in \mathbf{R}, \Phi^\rho(h^{-1}(\mu), t) = h^{-1}(\mu'),$$

wherefrom

$$\Psi^{h'(\rho)}(\mu, t) = \Psi^{h'(\rho)}(h(h^{-1}(\mu)), t) = h(\Phi^\rho(h^{-1}(\mu), t) = h(h^{-1}(\mu')) = \mu'.$$

The situation with (13) is similar. ∎

11. DYNAMICAL BIFURCATIONS

Definition 11.1. *We consider the case when the generator function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, $\mathbf{B}^n \times \mathbf{B}^m \ni (\mu, \lambda) \to \Phi(\mu, \lambda) \in \mathbf{B}^n$ of $\Xi_{\Phi(\cdot,\lambda)}$ depends on the parameter $\lambda \in \mathbf{B}^m$. We fix λ and let be $\lambda' \in \mathbf{B}^m$. If $\Phi(\cdot, \lambda)$ and $\Phi(\cdot, \lambda')$ are conjugated, then $\Phi(\cdot, \lambda')$ is called an* ***admissible*** *(or* ***allowable****)* ***perturbation of*** *$\Phi(\cdot, \lambda)$.*

Remark 11.1. *Intuitively speaking (Ott, [2]) a dynamical bifurcation is a qualitative change in the dynamic of the system $\Xi_{\Phi(\cdot,\lambda)}$ that occurs at the variation of the parameter λ.*

Definition 11.2. *If for any parameters $\lambda, \lambda' \in \mathbf{B}^m$ the systems $\Xi_{\Phi(\cdot,\lambda)}$ and $\Xi_{\Phi(\cdot,\lambda')}$ are equivalent, then Φ is called* ***structurally stable****; the existence of λ, λ' such that $\Xi_{\Phi(\cdot,\lambda)}$ and $\Xi_{\Phi(\cdot,\lambda')}$ are not equivalent is called a* ***dynamical bifurcation****.*

Equivalently, let us fix an arbitrary $\lambda \in \mathbf{B}^m$. If $\forall \lambda' \in \mathbf{B}^m$, $\Phi(\cdot, \lambda')$ is an admissible perturbation of $\Phi(\cdot, \lambda)$, then Φ is said to be ***structurally stable****, otherwise we say that Φ has a* ***dynamical bifurcation****.*

Remark 11.2. *If $\forall \lambda \in \mathbf{B}^m, \forall \lambda' \in \mathbf{B}^m$ the bijections $h : \mathbf{B}^n \to \mathbf{B}^n, h' \in \Omega_n$ exist such that $\forall \nu \in \mathbf{B}^n$, the diagram*

$$\begin{array}{ccc} \mathbf{B}^n & \overset{\Phi^{\nu}(\cdot,\lambda)}{\rightarrow} & \mathbf{B}^n \\ h \downarrow & & \downarrow h \\ \mathbf{B}^n & \overset{\Phi^{h'(\nu)}(\cdot,\lambda')}{\rightarrow} & \mathbf{B}^n \end{array}$$

commutes, then Φ is structurally stable, otherwise we have a dynamical bifurcation.

Example 11.1. *In Figure 9 ($n = 2, m = 1$),*

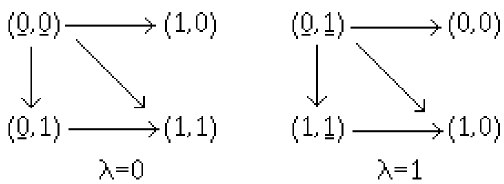

Fig. 9. Structural stability.

Φ is structurally stable and the bijections h, h' are defined accordingly to the following table:

(μ_1, μ_2)	$h(\mu_1, \mu_2)$	$h'(\mu_1, \mu_2)$
$(0,0)$	$(0,1)$	$(0,0)$
$(0,1)$	$(1,1)$	$(1,0)$
$(1,0)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,0)$	$(1,1)$

Example 11.2. *In Figure 10 ($n = 2, m = 1$),*

Φ has a dynamical bifurcation.

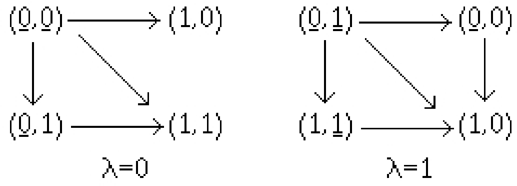

Fig. 10. Dynamical bifurcation.

Definition 11.3. *The* ***bifurcation diagram*** *is a partition of the set of systems* $\{\Xi_{\Phi(\cdot,\lambda)}|\lambda \in \mathbf{B}^m\}$ *in classes of equivalence given by the equivalence of the systems, together with representative state portraits for each class of equivalence.*

Example 11.3. *Figure 10 is a bifurcation diagram.*

Definition 11.4. *The* ***bifurcation diagram*** *([2], page 5) is the graph that gives the position of the fixed points depending on a parameter, such that a bifurcation exists.*

Remark 11.3. *Such a(n informal) definition works for calling Figure 10 a bifurcation diagram, since there fixed points exist. However for Figure 11*

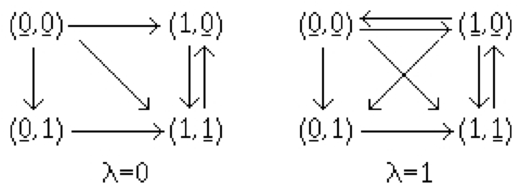

Fig. 11. Dynamical bifurcation.

this definition does not work, because a bifurcation exists there, but no fixed points.

Definition 11.5. *Let be* $\Phi, \Psi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$. *The families of systems* $(\Xi_{\Phi(\cdot,\lambda)})_{\lambda\in\mathbf{B}^m}$ *and* $(\Xi_{\Psi(\cdot,\lambda)})_{\lambda\in\mathbf{B}^m}$ *are called* ***equivalent*** *if there exists a bijection* $h'' : \mathbf{B}^m \to \mathbf{B}^m$ *such that* $\forall\lambda \in \mathbf{B}^m, \Xi_{\Phi(\cdot,\lambda)}$ *and* $\Xi_{\Psi(\cdot,h''(\lambda))}$ *are equivalent in the sense of Definition 10.1.*

References

[1] C.-D. Constantinescu, *Chaos, fractals and applications*, Flower Power, Piteşti, 2003.(in Romanian)

[2] M.-F. Danca, *Logistic map: dynamics, bifurcation and chaos*, University of Piteşti , 2001 (in Romanian).

[3] A. Georgescu, M. Moroianu, I. Oprea, *Bifurcation theory, principles and applications*, University of Piteşti, 1999.(in Romanian)

[4] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, Second Edition, Springer, 1997.

[5] Gr. C. Moisil, *The algebraic theory of switching circuits*, Pergamon Press and Editura Tehnica, Bucharest, 1969.

[6] M., *Dynamic and bifurcation for two generalized van der Pol models*, University of Piteşti, 2001 (in Romanian).

[7] M. P. Trifan, *Dynamic and bifurcation in the mathematical study of the cancer*, Pământul, Piteşti, 2006. (in Romanian)

[8] S. E. Vlad, *Boolean dynamical systems*, Romai Journal, **3**, 2(2007), 277-324 .

ROMAI J., **5**, 1(2009), 155–162

BOGDANOV-TAKENS SINGULARITIES IN AN EPIDEMIC MODEL

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Abstract This paper represents a continuation of our previous study [1] on the epidemic model of Kermack and McKendrick involving two individuals populations, namely the *susceptible* and the *infective*. The normal form for the double zero singularity when some parameter vanishes is deduced by using the method in [2]. The phase portrait for this case is sketched.

Keywords: epidemic model, normal form, degenerated singularity.

2000 MSC: 37G05.

1. INTRODUCTION

Some of the first written references about infectious diseases (considered in those times as natural phenomena) are back dated in the fifth century B.C. At the beginning of the XVII's century, as a result of the population migration, the humankind was faced with a large number of epidemic diseases. Even now, despite the progress made in medicine, infectious diseases are still a major health problem throughout the world. Each year, infectious diseases, such as tuberculosis, hepatitis, malaria, HIV, causes the death of millions of people (e.g. tuberculosis - 2 mil./year, HIV - 3 mil./year). For this reason a better knowledge of symptoms, steady states and treatments is necessary. Tracking down the disease in its initial phase leads to a larger number of possible applied treatments and also their efficiency. Since there exist many complicated factors, only the mathematical models may be able to provide information about the dynamics of epidemics. The number of mathematical models developed to describe and interpret patterns of transmission of infectious diseases "explodes" in recent years. Some of them are described by ordinary differential equations (ode's) or by partial differential equations (pde's).

2. MATHEMATICAL MODEL

In this paper we consider the Cauchy problem $x(0) = x_0, y(0) = y_0$ for the following ode's governing the evolution of two classes of the populations [3]

$$\begin{cases} \dot{x} &= -\mu xy + \rho, \\ \dot{y} &= \mu xy - \nu y, \end{cases} \tag{1}$$

where x, y are the state functions, x are *susceptible individuals*, y are *infected individuals*; μ, ρ, ν are nonnegative parameters; μxy represent new infected individuals when the populations x and y are combined, ρ are new suspected individuals, νy being some individuals from y which *die* or are in quarantine or are *immune*, t is the independent variable and the dot over quantities stands for the derivative with respect to t.

In [2] are presented the equilibria for the ode's (1) when some parameters are vanishing. Namely, in the cases 1) $\rho = 0, \mu, \nu \neq 0, \mu x_0 - \nu \neq 0$; 2) $\rho = \mu = 0, \nu \neq 0$; 3) $\rho = \nu = 0, \mu \neq 0, x_0, y_0 \neq 0$ the nonhyperbolic equilibria are degenerated saddle-nodes, while for 4) $\rho = 0, \mu, \nu \neq 0, \mu x_0 - \nu = 0$; 5) $\rho = \nu = 0, \mu \neq 0, x_0 = 0$ and/or $y_0 = 0$ and 6)$\rho = \mu = \nu = 0$ they are double zero. In Section 3 we deduce the normal form of the governing equations only at these singularities. This is the first step in deducing the corresponding miniversal unfolding about the singularities.

3. NORMAL FORM "AT THE POINT"

In the cases 5) and 6) the linearized system around $\mathbf{0}$ has the matrix $\mathbf{A} = \mathbf{0_2}$. Accordingly, we can not speak about degeneracy or nondegeneracy of the equilibria with double zero eigenvalues. For case 4) the following theorem takes place.

Theorem 3.1. *The ode's (1) are topologically equivalent, for* $\rho = 0, \mu, \nu \neq 0, \nu = \mu x_0$, *to the ode's*

$$\begin{cases} \dot{r}_1 &= \quad r_2 + r_1^2\left(\frac{\mu_0}{2} + \frac{\mu_0}{6x_0}r_1 - \frac{\mu_0^2}{12x_0}r_1^2\right) + O(r_1^3), \\ \dot{r}_2 &= \quad O(r_1^4) \end{cases} \tag{2}$$

and, so, the equilibrium point $\mathbf{e} = (x_0, 0)$ *is a degenerated Bogdanov-Takens singularity.*

Proof. For $\rho = 0, \mu, \nu \neq 0, \nu = \mu x_0$, system (1) becomes

$$\begin{cases} \dot{x} &= -\mu xy, \\ \dot{y} &= \quad \mu xy - \mu x_0 y. \end{cases} \tag{3}$$

System (3) possesses two equilibria: $\mathbf{e}_0 = (0,0)$ and $\mathbf{e} = (x_0, 0)$, where x_0 is given and μ is the parameter. As these equilibria exist for an infinity of values of the parameter μ, we choose one of them. Let it be μ_0. Then "at the point" μ_0 system (3) has the form

$$\begin{cases} \dot{x} &= -\mu_0 xy, \\ \dot{y} &= \mu_0 xy - \mu_0 x_0 y. \end{cases} \tag{3'}$$

By the change of coordinates $u_1 = x - x_0, u_2 = y$, the equilibrium point $\mathbf{e} = (x_0, 0)$ is carried at the origin of coordinates $\mathbf{u_0} = (u_{01}, u_{02}) = (0,0)$. In the (u_1, u_2)-plane ode's (3') read

$$\begin{cases} \dot{u}_1 &= -\mu_0 x_0 u_2 - \mu_0 u_1 u_2, \\ \dot{u}_2 &= \quad \mu_0 u_1 u_2. \end{cases} \tag{4}$$

The matrix $\mathbf{A}_1(\mathbf{0})= \begin{pmatrix} 0 & -\mu_0 x_0 \\ 0 & 0 \end{pmatrix}$, associated with the linearized system around $(u_{01}, u_{02}) = \mathbf{0}$, has the eigenvalues $\lambda_1 = 0, \lambda_2 = 0$, hence the equilibrium $\mathbf{u}_0$ is a double zero singularity. ∎

Bringing of matrix $\mathbf{A}_1(\mathbf{0})$ to canonical form implies the transformation of the canonical base $(\mathbf{e}_1, \mathbf{e}_2)$ of $\mathbf{R}^2$ to the base $\{\mathbf{v}_+, \mathbf{v}_-\}$ where $\mathbf{v}_+ = (1, 0)$,

$\mathbf{v}_- = (0, -\mu_0 x_0)$ and $\mathbf{A}_1\mathbf{v}_+ = 0$, $\mathbf{A}_1\mathbf{v}_- = \mathbf{v}_+, < \mathbf{v}_+, \mathbf{v}_- >= 0$ (hence $\mathbf{v}_+$ is an eigenvector while $\mathbf{v}_-$ is an associated eigenvector of $\mathbf{A}_1$). By the change of coordinates $\mathbf{u} \to P\mathbf{n}$, where $P = \{\mathbf{v}_+, \mathbf{v}_-\}$, the system (4) in $(\mathbf{0}, \mathbf{0})$ becomes

$$\begin{cases} \dot{n}_1 &= \quad n_2 + n_1 n_2 / x_0, \\ \dot{n}_2 &= \mu_0 n_1 n_2. \end{cases} \tag{5}$$

In (5), $\mathbf{X}(\mathbf{n}) = \begin{pmatrix} n_1 n_2 / x_0 \\ \mu_0 n_1 n_2 \end{pmatrix}$ (in [4] notation) and represents the nonlinear part of (5) while the matrix of the first order terms reads $\mathbf{An} = \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix}$.

Elimination of the second order nonresonant terms from (5)

Proposition 3.1. *The dynamical system associated with (5) is topologically equivalent to the dynamical system associated with*

$$\begin{cases} \dot{q}_1 &= q_2 + \mu_0 q_1^2/2 + q_1^2 q_2/(2x_0^2) + O\left(|q_1, q_2|^4\right), \\ \dot{q}_2 &= \mu_0 q_1^2 q_2/(2x_0^2) + O\left(|q_1, q_2|^4\right). \end{cases} \tag{6}$$

Proof. Denote $\mathbf{n} = (n_1, n_2), \mathbf{q} = (q_1, q_2)$. In order to determine the transformation which carries (5) in (6) we apply the method described in [1], [4], briefly presented in the following.

Let $\mathcal{H}_n^k$ be the Hilbert space of n-dimensional vector homogeneous polynomials of degree k. Then $\dim\mathcal{H}_n^k = n(k+1)$.

Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ be a base for $\mathcal{H}$, where $\mathbf{u}_1 = n_1^2\mathbf{e}_2$, $\mathbf{u}_2 = n_1 n_2 \mathbf{e}_2$, $\mathbf{u}_3 = n_2^2\mathbf{e}_2$, $\mathbf{u}_4 = n_1^2\mathbf{e}_1$, $\mathbf{u}_5 = n_1 n_2 \mathbf{e}_1$, $\mathbf{u}_6 = n_2^2\mathbf{e}_1$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Obviously, $\dim\mathcal{H}_2^2 = 6$. Denote by L_A^2 the Lie parenthesis of the operator defined by A.

Let ImL_A^2 be the range of L_A^2 and denote by $KerL_A^2$ the null space of L_A^2. Then the splitting $\mathcal{H}_2^2 = ImL_A^2 \bigoplus KerL_A^2$ holds. Since

$$L_A^k(\mathbf{u}_i) = (k - i + 1)\mathbf{u}_{i+1} - \mathbf{u}_{i+k+1}, \ 1 \le i \le k+1$$

and

$$L_A^2(\mathbf{u}_1) = 2\mathbf{u}_2 - \mathbf{u}_4, L_A^2(\mathbf{u}_2) = \mathbf{u}_3 - \mathbf{u}_5, L_A^2(\mathbf{u}_3) = \mathbf{u}_6 = L_A^2(\mathbf{u}_5),$$

$$L_A^2(\mathbf{u}_4) = 2\mathbf{u}_5, L_A^2(\mathbf{u}_6) = 0,$$

a basis for ImL_A^2 is $\mathcal{B}_1 = \{L_A^2(\mathbf{u}_1), L_A^2(\mathbf{u}_2), L_A^2(\mathbf{u}_3),\ L_A^2(\mathbf{u}_4)\}$.

By the above algebra it reads $\mathcal{B}_1 = \{2\mathbf{u}_2 - \mathbf{u}_4, \mathbf{u}_3 - \mathbf{u}_5, \mathbf{u}_6, 2\mathbf{u}_5\}$. Choose $\mathcal{B}_2 = \{\mathbf{u}_1, \mathbf{u}_4\}$ as a basis of $KerL_A^2$. Thus, every vector from basis $\mathcal{B}$ can be expressed in terms of the vectors from $\mathcal{B}_1$ and $\mathcal{B}_2$.

Let $\boldsymbol{\xi}_1 = \{\xi_1, \xi_2\}^T = \sum_{i=1}^{6} \alpha_i \mathbf{u}_i$ be an arbitrary element of $\mathcal{H}_2^2$ and take into account that

$$2\mathbf{u}_2 - \mathbf{u}_4 = L_A^2(\mathbf{u}_1), \mathbf{u}_3 - \mathbf{u}_5 = L_A^2(\mathbf{u}_2), \mathbf{u}_6 = L_A^2(\mathbf{u}_3), 2\mathbf{u}_5 = L_A^2(\mathbf{u}_4),$$

implying

$$\mathbf{u}_2 = \frac{\mathbf{u}_4 + L_A^2(u_1)}{2},\ \mathbf{u}_3 = L_A^2(\mathbf{u}_2) + \frac{1}{2}L_A^2(\mathbf{u}_4),\ \mathbf{u}_5 = \frac{1}{2}L_A^2(\mathbf{u}_4).$$

Then $\boldsymbol{\xi}_1$ can be written in the form

$$\boldsymbol{\xi}_1 = L_A^2\left(\frac{\alpha_2}{2}\mathbf{u}_1 + \alpha_3\mathbf{u}_2 + \frac{\alpha_3 + \alpha_5}{2}\mathbf{u}_4 + \alpha_6\mathbf{u}_3\right) + [\alpha_1\mathbf{u}_1 + (\alpha_4 + \frac{\alpha_2}{2})\mathbf{u}_4],$$

i.e. as a sum of a vector from $\mathcal{B}_1$ and a resonant term from $\mathcal{B}_2$.

Let us take as $\boldsymbol{\xi}_1$ the nonlinear term from (5), i.e.

$$\boldsymbol{\xi}_1 = \begin{pmatrix} n_1 n_2/x_0 \\ \mu_0 n_1 n_2 \end{pmatrix} = \frac{1}{x_0}\mathbf{n}_5 + \mu_0\mathbf{n}_2.$$

Hence $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_6 = 0, \alpha_5 = \frac{1}{x_0}, \alpha_2 = \mu_0$, therefore the resonant term is $\frac{\mu_0}{2}\mathbf{n}_4$ while the preimage of $\boldsymbol{\xi}_1$ through the operator $\widetilde{L}_A^2 : \mathcal{H}_2^2 \to ImL_A^2, \widetilde{L}_A^2(\mathbf{n}) = L_A^2(\mathbf{n})$, reads $\frac{\mu_0}{2}\mathbf{n}_1 + \frac{1}{2x_0}\mathbf{n}_4$. In order to eliminate the second order nonresonant terms we use the transformation $\mathbf{n} = \mathbf{q} + \mathbf{h}(\mathbf{q})$, where $\mathbf{h}_2(\mathbf{q}) = \frac{\mu_0}{2}\mathbf{n}_1 + \frac{1}{2x_0}\mathbf{n}_4 = \begin{pmatrix} \frac{1}{2x_0}n_1^2 \\ \frac{\mu_0}{2}n_1^2 \end{pmatrix}$, $\boldsymbol{\xi}_1 = \widetilde{L}_A^2(\mathbf{h}(\mathbf{q}))$. In other words, this transformation has the form

$$n_1 = q_1 + \frac{1}{2x_0}q_1^2, n_2 = q_2 + \frac{\mu_0}{2}q_1^2 \tag{7}$$

implying $\dot{n}_1 = \dot{q}_1(1 + \frac{q_1}{x_0}), \dot{n}_2 = \dot{q}_2 + \mu_0 q_1 \dot{q}_1$. Taking into account (5) and (7) and using the asymptotic expansion $(1 + \frac{q_1}{x_0})^{-1} \sim 1 - \frac{q_1}{x_0} + \frac{q_1^2}{x_0^2} - ...$, we obtain (6). ∎

Elimination of the third order nonresonant terms from (6)

Proposition 3.2. *The system (6) is topologically equivalent to the system (2).*

Proof. In this case the Hilbert space $\mathcal{H}_2^3$ has the dimension $dim\mathcal{H}_2^3 = 8$.

The basis of $\mathcal{H}_2^3$ is

$$\mathcal{B} = \{\mathbf{q}_i \in \mathcal{H}_2^3 | \mathbf{q}_i = n_1^{4-i} n_2^{i-1} \mathbf{e}_2, i = \overline{1,4}; \mathbf{q}_i = n_1^{8-i} n_2^{i-5} \mathbf{e}_1, i = \overline{5,8}\}.$$

Because $\mathcal{H}_2^3 = ImL_A^3 \bigoplus KerL_A^3$ a basis of ImL_A^3 is

$$\mathcal{B}_1 = \{3\mathbf{u}_2 - \mathbf{u}_5, 2\mathbf{u}_3 - \mathbf{u}_6, \mathbf{u}_4 - \mathbf{u}_7, -\mathbf{u}_8, 3\mathbf{u}_6, 2\mathbf{u}_7\}$$

and a basis of $KerL_A^3$ is $\mathcal{B}_2 = \{\mathbf{u}_1, \mathbf{u}_5\}$.

Thus, the decompositions along $\mathcal{B}_1$ and $\mathcal{B}_2$ of vectors from $\mathcal{B}$ are

$$\mathbf{u}_1 = 0 + \mathbf{u}_1, \mathbf{u}_2 = (3\mathbf{u}_2 - \mathbf{u}_5)/3 + \mathbf{u}_5/3, \mathbf{u}_3 = (2\mathbf{u}_3 - \mathbf{u}_6)/2 + 3\mathbf{u}_6/6,$$

$$\mathbf{u}_4 = [(\mathbf{u}_4 - \mathbf{u}_7) + 2\mathbf{u}_7/2] + 0, \mathbf{u}_5 = 0 + \mathbf{u}_5, \mathbf{u}_6 = 3\mathbf{u}_6/3 + 0,$$

$$\mathbf{u}_7 = 2\mathbf{u}_7/2 + 0, \mathbf{u}_8 = -(-\mathbf{u}_8) + 0.$$

Similarly, for an arbitrary element $\boldsymbol{\xi}_2$ of $\mathcal{H}_2^3$, the succession of equalities

$$\boldsymbol{\xi}_2 = L_A^3\left(\frac{\alpha_2}{2}\mathbf{u}_1 + \frac{\alpha_3}{2}\mathbf{u}_2 + \alpha_4\mathbf{u}_3 - \alpha_8\mathbf{u}_4 + \frac{\alpha_3 + 2\alpha_6}{6}\mathbf{u}_5 + \frac{\alpha_4 + \alpha_7}{2}\mathbf{u}_6\right) +$$

$$[\alpha_1\mathbf{u}_2 + \frac{\alpha_2 + 3\alpha_5}{3}\mathbf{u}_5] = L_A^3(\mathcal{H}_2^3) + [\alpha_1\mathbf{u}_2 + \frac{\alpha_2 + 3\alpha_5}{3}\mathbf{u}_5]$$

holds. For the ode's (5) we have $\xi_2 = \begin{pmatrix} q_1^2 q_2/2x_0^2 \\ \mu_0 q_1^2 q_2/2x_0 \end{pmatrix}$, leading to $\mathbf{h}_3(\mathbf{r}) = \begin{pmatrix} r_1^3/6x_0^2 \\ \mu_0 r_1^3/6x_0 \end{pmatrix}$. Therefore, by the transformation $\mathbf{q} = \mathbf{r} + \mathbf{h}_3(\mathbf{r})$, applied to the (5), we obtain (2). ∎

Remark 3.1. *The ode's (2) represent the normal form up to order three for ode's (4).*

Remark 3.2. *If, in Proposition 1,* $\mathcal{B}_1 = \{2\boldsymbol{u}_2 - \boldsymbol{u}_4, \boldsymbol{u}_3 - \boldsymbol{u}_5, \boldsymbol{u}_6, 2\boldsymbol{u}_5\}$, $\mathcal{B}_2 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ *and, in Proposition 2,* $\mathcal{B}_1 = \{3\boldsymbol{u}_2 - \boldsymbol{u}_5, 2\boldsymbol{u}_4 - \boldsymbol{u}_7, -\boldsymbol{u}_8, 3\boldsymbol{u}_6, 2\boldsymbol{u}_7\}$ *and* $\mathcal{B}_2 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ *then we obtain the normal form, equivalent to (2),*

$$\begin{cases} \dot{r_1} &= r_2 + O\left(|r|^3\right), \\ \dot{r_2} &= \mu_0 r_1 r_2 + O\left(|r|^3\right). \end{cases} \tag{8}$$

In the literature it is this form which defines the equilibrium **e** *as a degenerated Bogdanov-Takens singularity.*

4. CONCLUSIONS

The topological type of one nonhyperbolic singularity was investigated. In the linear case the phase trajectories (fig. 1) are straight lines parallel to Ox-axis without the attractive or repulsive directions while in the nonlinear case they are curves limited by the Ox-axis.

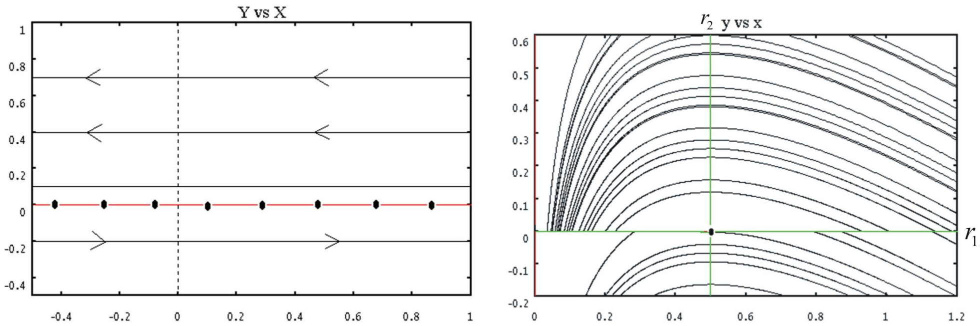

Fig. 1. The phase portraits in the linear and nonlinear cases, for $\mu = 0.3, \nu = 0.15$ and $x_0 = 0.5$.

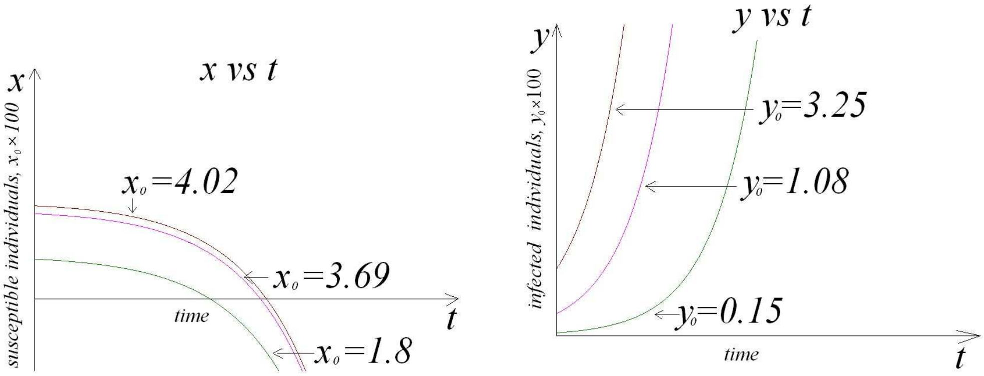

Fig. 2. The evolution in time of susceptible and infected populations for some parameter values.

The evolution of the x, y populations (fig. 2) shows the parameter variations leads to decreasing number of susceptible individuals and one increasing of infected individuals. This phenomenon is caused by the term y, i.e. by the death or immunity of the individuals.

References

[1] Chow, S. N., Hale, J. K., *Methods of bifurcation theory*, Springer, Berlin, 1982.

[2] Chow, S. N., Li, C., Wang, D. *Normal form and bifurcation of planar vector fields*, Cambridge University Press, 1994.

[3] Jones, D. S., Sleeman, B. D., *Differential equations and mathematical biology*, Chapman & Hall, 2003.

[4] Trifan, M. P., Cîrlig, G. V., *Saddle-node bifurcation in an epidemic model*, The 12-th Conf. on Applied and Industrial Mathematics, Pitesti, 2004.