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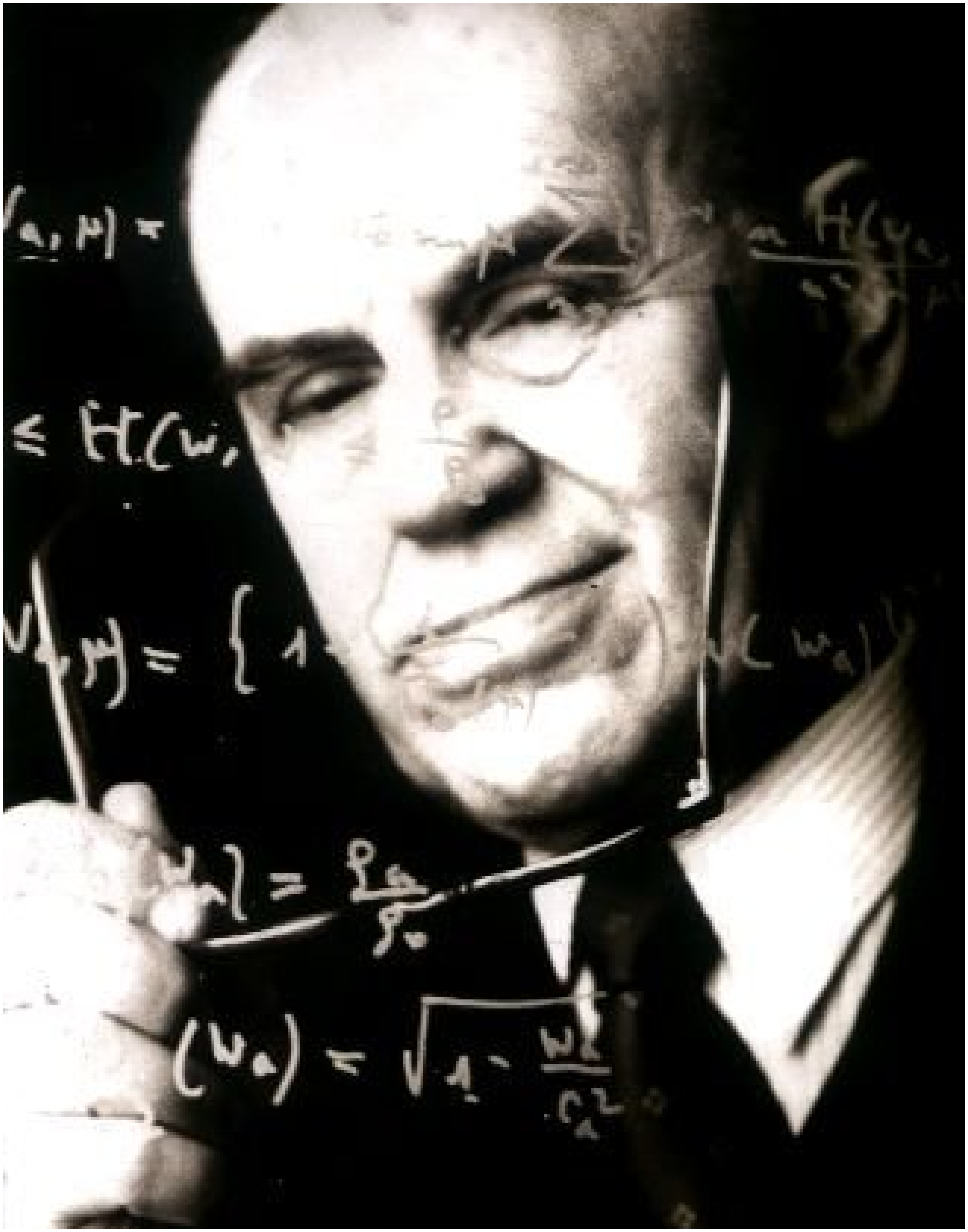

Acad. Caius Iacob 1912-1992

Academician Caius Iacob –

100 years from birth

Caius Iacob was born on 29 martie 1912 in Arad. His parents were Lazăr and Cornelia Iacob. Lazăr Iacob, a well known personality of his époque, was Professor of Canon Law at university level (in Arad and in Oradea), and contributed to the realization of the Great Union from 1918, event that gave birth to the modern Romanian State.

Caius Iacob began attending the primary school at the age of only 5 years, in 1917. Then, from 1921 to 1924 he was pupil of the middle school hosted by the High School Moise Nicoară. Between 1924 and 1928 he followed the classes of High School Emanoil Gojdu, Oradea.

When he was only 19 years old, he graduated the Faculty of Mathematics of Bucharest. His mathematical thinking was formed by his remarkable Romanian Professors: Gheorghe Țițeica, Anton Davidoglu, David Emmanuel, Dimitrie Pompeiu, Nicolae Coculescu, Victor Vâlcovici.

Then he performed doctoral studies at the Faculty of Sciences of University of Paris, under the guidance of Prof. Henri Villat. Here he had the privilege of receiving courses from great professors such as Ellie Cartan, Henri Lebesgue, Jean Leray, Emil Goursat, Paul Montel, Gaston Julia, and, obviously, Henri Villat.

In 1935, at the age of only 23, he defended his PhD Thesis, *Sur la détermination des fonctions harmoniques conjugées par certaines conditions aux limites. Applications à l'Hydrodynamique,* confirming thus the precocity of his intelligence and the talent for exact sciences shown in his infancy.

Returning in Romania, he began his academic activity as Professor's Assistant at the Polytechnical School of Timișoara. This was the first step on an academic pathway that lead him to the position of Professor at the University of Bucharest.

More specific, Caius Iacob was:

- Professor's Assistant at the Polytechnic School of Timișoara, between 1935-1938;
- Professor's Assistant at the Section of Mathematics of the Faculty of Sciences, between 1938-1939;
- Professor's Assistant at the Laboratory of Mechanics of the University of Bucharest, between 1939-1942;
- Lecturer (Conferențiar) at the Department of General Mathematics of the University of Cluj from 1942 to December 1943;
- Professor at the Department of Mechanics of University of Cluj between December 1943 and October 1950;
- Since October 1950, Professor at the Department of Mathematics and Physics of the University of Bucharest. He occupied this position until his retirement, in 1982.

Between 1952-1953 he was Vice-Rector of University of Bucharest.

His pedagogical activity was doubled by a prodigious scientific activity.

We make below a selective presentation of his domains of interest, in which he gained very important results [1]:

- problems of the theory of complex potential that occur in the study of Fluid Mechanics;
- Dirichlet problem for plane multiple-connected domains;
- modified Dirichlet problem (he introduced the modified Green function in the study of this problem);
- Riemann and Hilbert problems with given singularities and Dirichlet problem (in the classical sense or modified) with given singularities, for important particular cases;
- in the mechanics of perfect fluids: the theory of jets, in particular, the deviation of jets in the presence of solid obstacles; the movement of a fluid in the presence of some rotating body; the complex potential of a movement with given singularities of a incompressible fluid, in a multiple connected domain; the theory of thin wing;
- for viscous fluids, the stationary Poiseuille flows in interesting particular cases;
- in aerodynamics and gas dynamics, the exact solutions for some subsonic movements;
- approximation methods – movements without circulation around obstacles; hodographic approximation relying on successive approximation ;
- in the field of supersonic aerodynamics – the theory of the airfoil and the theory of conical movements in the case of conical low profile obstacles;
- in the theory of elasticity – studies of the torsion of an elastic rod in certain specific conditions.

(see also the List of Papers, taken from [2], at page vii).

As a result of this impressing scientific activity, Professor Caius Iacob became in 1955 a Corresponding Member of the Romanian Academy and, in 1963, a Full Member of the Romanian Academy.

Academician Caius Iacob gained recognition of his work not only within our country, but also abroad. As Prof. St. I. Gheorghiță shows in [1], his works „were cited by H. Villat, Th. Karman, D. Riabouchinsky, J. Leray, U. Cisotti, B. Demtchenko, Robert Sauer, J. Kravtchenko, L. I. Sedov, A. Weinstein, P. Germain, R. Bader, H. Cabannes, M. V. Keldis, D. Gaier, G. I. Dombrovski, N. I. Mushelisvili, A. Bunimovici, M. A. Lavrentiev, B. V. Savat, M. I. Gurevici, D. Gilbarg, M. Borelli, M. Schiffer, R. von Mises, K. Friedrichs, R. Finn, L. Bers, etc".

All the aspects of his activity were developed at highest level. Concerning the teaching activities, his former students keep in their memory the high quality of his courses, in which not only he presented the subject of the lessons but also he used to place the subject in the framework of the history of science, opening large horizons of knowledge to his students. He encouraged students to perform research work and generously lead many PhD stages.

The list of the mathematicians and/or mechaniciens that were guided by Acad. Caius Iacob during doctoral studies is posted on the webpage [2].

As a Head of Department of Mechanics at Faculty of Mathematics in Bucharest, Acad. Caius Iacob stimulated the study of many branches of Mechanics such as Mechanics of Fluids, Theory of Elasticity, Theory of Plasticity, Rock Mechanics, Rheology, so on.

Moreover, in 1977 with his contribution, the Section of Mechanics was created at the Faculty of Mathematics in Bucharest. Students that attended the courses of this section studied five years (a cycle of studies equivalent to university plus master studies) and the best of them could chose workplaces in research institutes. This situation lead to a strong development of the study of mechanics in our country, many specialists in this field being formed after 1977.

We must add that Acad. Caius Iacob was not a member of the communist party – as most of the persons in the Academic medium were before 1989. Only his remarkable talent and achievements allowed him to have such an academic career without accepting to be a member of the party that tried to control everyone and everything. More than that, he had the courage of facing the communist authorities in specific problems concerning university level education. As an example, while he was a Vice-Rector of the University of Bucharest, he fought for the right to bring documentation from Western countries in our universities. That happened in 1952, in a period when the policy of the leading (communist) party was that of isolationism (only scientific literature from Soviet Russia was allowed) [3].

After the fall of the communism, he became member of the historical party "Partidul Național Țărănesc – Creștin și Democrat" (National Christian and Democrat Party of Paysans) and he represented this Party in the Romanian Parliament from 1990.

In 1991, Acad. Caius Iacob, Professor Adelina Georgescu and some other enthusiasts of Mathematics and Mechanics, founded The Institute of Applied Mathematics of the Romanian Academy, with Prof. Adelina Georgescu as Director. By the union of this Institute and the Center of Mathematical Statistics, in 2001, the actual „G. Mihoc – C. Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy was born.

Acad. Caius Iacob left this world on February 6, 1992.

His scientific heritage will remain as a value of the culture of the world. In our country he will be always remembered as a prominent Professor, Scientist and Citizen. He left behind a school of Mechanics that was very strong in the last two decades of the last century. Unhappily the evolution of economy and that of policies in Education in our country after 1989 lead to a weakening of this school. Since the local industry was awakened and many research institutes disappeared, many young researchers in the field left the country while the young graduates of the High Schools did not feel encouraged to study Mechanics in University.

Academician's Caius Iacob activity in the field of teaching and guiding scientific life in University of Bucharest must remain as a example of what should be done in order to strengthen again the school of Mechanics in our country. It is obvious that such an action must be supported by responsible policies, based on the national interest, in Education, Research, Economy.

Romanian Academy celebrated 100 years from Academician's Caius Iacob birth on March 29, 2012.

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LIST OF PUBLISHED WORKS OF ACAD. CAIUS IACOB [2]

Boundary Value Problems, Theory of Complex Potential, Armonic functions

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DIRAC STRUCTURES IN BANACH COURANT ALGEBROIDS

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Dedicated to Acad. Prof. Dr. Mitrofan Choban on the occasion of his 70th birthday.

Abstract We introduce the notion of Courant algebroid $(E, h, [.,.], \rho)$, in the category of Banach vector bundles. A Dirac structure is defined as a subbundle of a Courant algebroid E that equals its orthocomplement with respect to the metric h and the set of its sections is closed with respect to the operation [.,.]. Our main result is that a Dirac structure in a Courant algebroid is a Banach Lie algebroid.

Keywords: Banach Courant algebroids, Dirac structures.
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1. INTRODUCTION

The notion of Lie algebroid was extended to the category of Banach vector bundles by the present author [2], and independently by F. Pelletier [14]. In [2] it was shown that the Lie algebroids form a category. In a different direction, C. Ida [8] considers the coomology of Banach Lie algebroids and proves that if $(M, \pi_)$ is a Banach Poisson manifold, the Banach Lie algebroid cohomology of $(T^*M, \{.,\}, \sharp_\pi)$ is the Lichnerowicz-Poisson cohomology of (M, π). Next steps are done by M.Anastasiei and A. Sandovici [3], who introduced the Dirac structures on Banach manifolds and related them to Lie algebroids.

Dirac structures on finite dimensional manifolds were introduced by T. Courant and A. Weinstein (see [6]) and were systematically studied by T. Courant in [5]. They became an important tool in generalized geometry by studies of I. Vaisman (see [18] and the references therein). Following the direction opened by S. Vacaru in [24], certain applications of Dirac structures could appear in Theoretical Physics.

Dirac structures were used in the study of the mechanical systems described by constraint Hamiltonian systems or implicit Lagrangian systems (see the consistent work done by H. Yoshimura and J. E. Marsden in [19, 20]), while the reduction of nonholonomic systems in terms of Dirac structures was formulated by M. Jotz and T.S. Ratiu ([11]).

Another field where Dirac structures are useful is the study of the integrability of the nonlinear evolution equations. In the monograph [7], I. Dorfman emphasized the necessity of considering Dirac structures on infinite dimensional spaces; she also made a first step in this direction by developing an algebraic formalism independent of dimension.

A first study of the concept of a Dirac structure within the framework offered by infinite–dimensional smooth manifolds is due to the present author and A. Sandovici, [3]. Here the well known result that an integrable Dirac structure defines a Lie algebroid is extended to Banach manifold category. Our main reference for the geometry of infinite dimensional manifolds is a book by S. Lang, [12].

In Section 2 we define the Banach Lie algebroids and we show they form a category. In Section 3 we introduce the notion of Banach Courant algebroid $(E, h, [.,.], \rho)$ using three axioms and define a Dirac structure as a subbundle of E which is totally isotropic with respect to h and its set of sections is closed with respect to $[.,.]$. Then we prove that any Dirac structure has a Lie algebroid structure.

2. ANCHORED VECTOR BUNDLES

Let M be a smooth, i.e. C^∞, Banach manifold modeled on Banach space $\mathbb{M}$ and let $\pi : E \to M$ be a Banach vector bundle whose type fiber is a Banach space $\mathbb{E}$. We denote by $\tau : TM \to M$ the tangent bundle of M.

Definition 1.1 *We say that the vector bundle $\pi : E \to M$ is an anchored vector bundle if there exists a vector bundle morphism $\rho : E \to TM$. The morphism ρ will be called the anchor map.*

Let $\mathcal{F}(\mathcal{M})$ be the ring of smooth real functions on M.

We denote by $\Gamma(E)$ the $\mathcal{F}(\mathcal{M})$-module of smooth sections in the vector bundle (E, π, M) and by $\mathcal{X}(\mathcal{M})$ the module of smooth sections in the tangent bundle of M (vector fields on M).

The vector bundle morphism ρ induces an $\mathcal{F}(\mathcal{M})$-module morphism which will be denoted also by $\rho : \Gamma(E) \to \mathcal{X}(\mathcal{M})$, $\rho(s)(x) = \rho(s(x))$, $x \in M, s \in \Gamma(E)$.

Examples.

1. The tangent bundle of M is trivially anchored vector bundle with $\rho = I$ (identity).
2. Let A be a tensor field of type $(1, 1)$ on M. It is regarded as a section of the bundle of linear mappings $L(TM, TM) \to M$ and also as a morphism $A : TM \to TM$. In the other words, A may be thought as an anchor map.
3. Any subbundle of TM is an anchored vector bundle with the anchor the inclusion map in TM.
4. Let now $\pi : E \to M$ be any submersion and π_* be the differential (tangent map) of π. The union of subspaces $(\pi_*)^{-1}(u), u \in E$ provides a subbundle of the tangent bundle of E, $\tau_E : TE \mapsto E$, denoted by VE and called the vertical subbundle. As a subbundle, by the Example 3), this is an anchored vector bundle.

5. Let $\tau^* : T^*M \mapsto M$ be the cotangent bundle of M and the Whitney sum $\tau \oplus \tau^* : TM \oplus T^*M \mapsto M$ called sometimes the big tangent bundle. This is an anchored bundle with the anchor given by the projection $pr_1 : TM \oplus T^*M \mapsto TM$.

Theorem 1.1. *The anchored vector bundles form a category.*
Proof. The objects are the pairs (E, π_E, M, ρ_E) with ρ_E the anchor of E and the category morphism $(f, \phi) : (E, \pi_E, M, \rho_E) \to (F, \pi_H, N, \rho_F)$ is a vector bundle morphism $(f, \phi) : E \to F$ which verifies the condition $\rho_F \circ f = \phi_* \circ \rho_E$, where ϕ_* is the tangent map of $\phi : M \mapsto N$.

Let $\pi : E \to M$ be an anchored Banach vector bundle with the anchor $\rho_E : E \to TM$ and the induced morphism $\rho_E : \Gamma(E) \to \mathfrak{X}(M)$.

Assume there exists defined a bracket $[,]_E$ on the space $\Gamma(E)$ that provides a structure of real Lie algebra on $\Gamma(E)$.

Definition 2.1. *The triplet $(E, \rho_E, [,]_E)$ is called a Banach Lie algebroid if*

(i) $\rho : (\Gamma(E), [,]_E) \to (\mathfrak{X}(M), [,])$ is a Lie algebra homomorphism and

(ii) $[fs_1, s_2]_E = f[s_1, s_2]_E - \rho_E(s_2)(f)s_1$, for every $f \in \mathcal{F}(M)$ and $s_1, s_2 \in \Gamma(E)$.

Examples:

1. The tangent bundle $\tau : TM \to M$ is a Banach Lie algebroid with the anchor the identity map and the usual Lie bracket of vector fields on M.

2. For any submersion $\zeta : F \to M$, where F is a Banach manifold, the vertical bundle VF over F is an anchored Banach vector bundle. As the Lie bracket of two vertical vector fields is again a vertical vector field it follows that $(VF, i, [,]_{VF})$, where $i : VF \to TF$ is the inclusion map is a Banach Lie algebroid. This applies, in particular, to any Banach vector bundle $\pi : E \to M$.

Let $\Omega^q(E) := \Gamma(L^a_q(E))$ be the $\mathcal{F}(M)-$ module of differential forms of degree q. Its elements are sections of the vector bundle of alternating multilinear forms on E, see [12], p.61. In particular, $\Omega^q(TM)$ will be denoted by $\Omega^q(M)$. The differential operator $d_E : \Omega^q(E) \to \Omega^{q+1}(E)$ is given by the formula

$$\begin{aligned}(d_E\o)(s_0, \ldots, s_q) &= \textstyle\sum_{i=0,\ldots,n}(-1)^i\rho_E(s_i)\o(s_0, \ldots, \widehat{s_i}, \ldots, s_q)\\ &+ \textstyle\sum_{0\le i<j\le q}(-1)^{i+j}(\o([s_i, s_j]_E), s_0, \ldots \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_q)\end{aligned}$$

for $s_1, \ldots, s_q \in \Gamma(E)$, where hat over a symbol shows that symbol must deleted.

Definition 1.3. *A vector bundle morphism $f : E \to E'$ over $f_0 : M \to M'$ is a morphism of the Banach Lie algebroids $(E, \rho_E, [,]_E$ and $(E', \rho_{E'}, [,]_{E'})$ if the map induced on forms $f^* : \O^q(E') \to \O^q(E)$ defined by $(f^*\o')_x(s_1, \ldots, s_q) = \o'_{f0(x)}(fs_1, \ldots, fs_q)$, $s_1, \ldots, s_2 \in \Gamma(E)$ commutes with the differential i.e.*

$$d_E \circ f^* = f^* \circ d_E. \tag{1}$$

Using this definition it is easy to prove

Theorem 1.2. *The Banach Lie algebroids with the morphisms defined in the above form a category.*

For applications of Lie algebroids we refer to [1].

3. BANACH COURANT ALGEBROIDS

The first definition of a Courant algebroid was given by Liu Z., Weinstein A. and Xu P. in [10] using five axioms. An alternative definition, again with five axioms was given by D. Roytenberg, [15].

In the paper [16], K. Uchino shows that in the both cases, two from those five axioms are consequences of the other three. Thus he arrives at a definiton of a Courant algebroid with three axioms, which definition was used in the seminal paper by Keller F. and Waldman S., [9].

Definition 3.1. *A Courant algebroid is an anchored vector bundle* (E, ρ) *together with a nondegenerate symmetric bilinear form h, a bracket* $[\cdot,\cdot] : \Gamma(E)\times\Gamma(E) \to \Gamma(E)$ *on the sections of the bundle such that for all* $e_1, e_2, e_3 \in \Gamma(E)$ *and* $f \in \mathcal{F}(M)$ *the following conditions hold:*

(i) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,

(ii) $[e_1, e_2] + [e_2, e_1] = \mathcal{D}h(e_1, e_2)$, *where* $\mathcal{D} : \mathcal{F}(M) \to \Gamma(E)$ *is defined by* $h(\mathcal{D}f, e) = \rho(e)f$,

(iii) $\rho(e_1)h(e_2, e_3) = h([e_1, e_2], e_3) + h(e_2, [e_1, e_3])$.

We notice that the bracket $[\cdot,\cdot]$ is not skew-symmetric.

An easy consequence of $(i) - (iii)$ is

Proposition 3.1. *Let* $(E, [\cdot,\cdot])$ *be a Courant algebroid. Then we have*

(i) $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$ *where in the right side we have the usual bracket of vector fields on M.*

(ii) $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2$ *(Leibniz's rule).*

The standard example of Courant algebroid is given by

Proposition 3.2. *Let* $E = TM \oplus T^*M$ *be the big tangent bundle over M. We take* $h((X, \alpha), (Y, \beta)) = \alpha(Y) + \beta(X)$, *where* $X, Y \in \mathcal{X}(M)$ *and* $\alpha, \beta \in \Lambda^1(M)$ *together with the bracket* $[(X, \alpha), (Y, \beta)] = ([X, Y], \underset{X}{L}\beta - \underset{Y}{i}d\alpha)$ *and the anchor* ρ *defined by* $\rho(X, \alpha) = X$. *All these endow E with the structure of a Courant algebroid.*

Proof. One verifies the axioms $(i) - (iii)$ from the Definition 3.1.

Definition 3.2. *A Dirac structure is a vector subbundle* $\mathcal{L}$ *of a Courant algebroid* $(E, [\cdot,\cdot], h, \rho)$ *which coincides with its orthocomplement with respect to h i.e.* $\mathcal{L} = \mathcal{L}^\perp$ *and is closed with respect to* $[\cdot,\cdot]$, *i.e.* $[\Gamma(E), \Gamma(E)] \subseteq \Gamma(E)$.

Now we show that any Dirac structure can be endowed with a structure of Lie algebroid. To this aim we replace the operation $[\cdot,\cdot]$ with the following one:

$$\widetilde{[e_1, e_2]} = [e_1, e_2] + \frac{1}{2}\mathcal{D}h(e_1, e_2)$$

and we prove

Theorem 3.1. *Let $\mathcal{L}$ be a Dirac structure. Then the triple $(\mathcal{L}, \widetilde{[\ ,\ ]}_{|\mathcal{L}}, \rho|_L)$ is a Banach Lie algebroid.*

Proof. The condition $\mathcal{L} = \mathcal{L}^{\perp}$ is equivalent with $h(e_1, e_2) = 0$ for $e_1, e_2 \in \Gamma(\mathcal{L})$. Thus $\widetilde{[\cdot,\cdot]}_{|\mathcal{L}}$ reduces to $[e_1, e_2]_{|\mathcal{L}}$ and the bracket $[\cdot,\cdot]_{|\mathcal{L}}$ is skew-symmetric. It follows that (i) from the Definition 3.1 becomes the usual Jacobi identity. Then, by the Proposition 3.1, $\rho_{|\mathcal{L}}$ is a Lie algebras homomorphism and the Leibniz identity holds. The proof is complete.

Example 3.1. In [4] one proves that if A is a Lie algebroid and A^* is its dual (in general not a Lie algebroid) then $E = A \oplus A^*$ has a structure of a Courant algebroid.

Let ω be a nondegenerate 2-form in A. It defines a map $\omega : A \to A^*$ by $e \to \omega(e) : \Gamma(A) \to \mathbb{R}$ with $\omega(e)f = \omega(e, f)$, $e, f \in \Gamma(A)$. We take $\mathcal{L}$ =graphω = $\{(e, \sigma)|\ e \in \Gamma(A), \sigma \in \Gamma(A^*)$ with $\sigma = \omega(e)\}$. If $\mathcal{L}$ is a vector subbundle and ω is closed, then $\mathcal{L}$ is a Dirac structure. For details see [4].

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ON THE EIGENVALUES OF A PROBLEM FOR THE LAPLACE OPERATOR ON AN S-DIMENSIONAL SPHERE

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Abstract Boundary eigenvalue problem for the Laplace operator on an s-dimensional sphere with the boundary condition on derivatives is considered. For this problem and adjoint to it the eigenvalues with relevant eigen- and associated elements are found. It is proved that the length of Jordan chains is not greater than 3.

Keywords: Laplace operator, eigenvalues, eigenfunctions, Jordan chains, Bessel equation.
2010 MSC: 35J05.

1. INTRODUCTION

In recent years the theory of nonlocal boundary value problems is developed intensively. The present work is associated with research in this area [1]. The eigen- and associated functions of the problems with boundary conditions on derivatives in areas with spherical symmetry are found. In the present work, we consider the problem

$$(\Delta + \lambda)\Phi(x) = 0, \quad \frac{\partial \Phi}{\partial r} \big|_{|x|=r_0} = \frac{\partial \Phi}{\partial r} \big|_{|x|=1},$$

where $x \in \Omega = \{x \in \mathbb{R}^s \mid \ |x| < 1\}, \ r_0 < 1$.

2. CASE $S = 2$

First, we consider the case $s = 2$ of the above problem in cylindrical coordinates:

$$\begin{gathered} (\Delta + \lambda)\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \lambda\Phi = 0, \\ \Phi \in C^{2+\alpha}(\Omega), \quad \Omega = \{r \mid r < 1\}, \\ \frac{\partial \Phi}{\partial r} \big|_{r=r_0} = \frac{\partial \Phi}{\partial r} \big|_{r=1} . \end{gathered} \tag{2.1}$$

By separating variables $\Phi(r, \varphi) = X(r)Y(\varphi)$ and by using the periodicity condition $\Phi(r, \varphi + 2\pi) = \Phi(r, \varphi)$ and boundedness at zero, we find

$$Y(\varphi) = C_1 \cos n\varphi + C_2 \sin n\varphi,$$

$$X'' + \frac{1}{r}X' + \left(\lambda - \frac{n^2}{r^2}\right)X = 0, \quad \text{(the Bessel equation)}$$

$$\|X(0)\| < \infty, \quad X'_r(r_0) = X'_r(1).$$

The eigenfunctions, determined by $X^{(1)}(r) = J_n(\alpha r)$, $\alpha^2 = \lambda$, (where $J_n(\cdot)$ are the Bessel functions), correspond to the eigenvalues that are the roots of the equation

$$f(\alpha) \equiv -J'_n(\alpha r_0) + J'_n(\alpha) = 0. \tag{$*$}$$

According to the standard procedure of constructing of adjoint (in the Lagrange sense) equation, using integration by parts of the bilinear form $\int\limits_{\Omega_1 \cup \Omega_2} (\Delta\Phi)\Psi r dr d\varphi$, where $\Omega_1 = \{r, \varphi \mid 0 \le r < r_0\}$, $\Omega_2 = \{r, \varphi \mid r_0 < r \le 1\}$, the second and the third conditions in the system (2.1), we obtain the adjoint to (2.1) problem:

$$\begin{gathered}
(\Delta + \lambda)\Psi = 0, \quad \Psi \in C^{2+\alpha}(\Omega_1) \cup C^{2+\alpha}(\Omega_2), \\
{\Psi_r}'(r_0 - 0, \varphi) = {\Psi_r}'(r_0 + 0, \varphi), \quad {\Psi_r}'(1, \varphi) = 0, \\
\Psi(1, \varphi) + r_0 \left[\Psi(r_0 - 0, \varphi) - \Psi(r_0 + 0, \varphi)\right] = 0.
\end{gathered} \tag{2.2}$$

Since Ψ is bounded, the solution to (2.2) $\Psi(r, \varphi) = \mathcal{X}^{(1)}(r)[C_1 \cos n\varphi + C_2 \sin n\varphi]$ has the form

$$\mathcal{X}^{(1)}(r) = \begin{cases} C_{11}J_n(\alpha r), \quad 0 \le r < r_0, \\ C_{21}J_n(\alpha r) + C_{22}N_n(\alpha r), \quad r_0 \le r \le 1 \end{cases}$$

($N_n(\alpha r)$ is the Neumann function). From the conditions (2.2) we obtain the system for determining C_{jk}:

$$\left\{\begin{array}{lllll}
C_{11}J'_n(\alpha r_0) & - & C_{21}J'_n(\alpha r_0) & - & C_{22}N'_n(\alpha r_0) \;= 0, \\
 & & C_{21}J_n(\alpha r) & + & C_{22}N_n(\alpha r) \;= 0, \\
C_{11}r_0J_n(\alpha r_0) & + & C_{21}[J_n(\alpha) - r_0J_n(\alpha r_0)] & & \\
 & + & C_{22}[N_n(\alpha) - r_0N_n(\alpha r_0)] & = 0. &
\end{array}\right. \tag{2.3}$$

Its determinant is

$$\Delta_0 = J'_n(\alpha r_0)[J'_n(\alpha)N_n(\alpha) - J_n(\alpha)N'_n(\alpha)] + r_0J'_n(\alpha) \times$$
$$\times [J_n(\alpha r_0)N'_n(\alpha r_0) - J'_n(\alpha r_0)N_n(\alpha r_0)].$$

According to [3], $J_\nu(z)N'_\nu(z) - N_\nu(z)J'_\nu(z) = \dfrac{2}{\pi z}$, we obtain the equation (*) for determining the eigenvalues $\lambda = \alpha^2$: $\dfrac{2}{\pi\alpha}f(\alpha) = \dfrac{2}{\pi\alpha}[J'_n(\alpha) - J'_n(\alpha r_0)] = 0$. By virtue of the inequality $J'_n(\alpha)J'_n(\alpha r_0) \ne 0$ and from the first equation of the system (2.3) we get

$$C_{11} = \frac{1}{J'_n(\alpha)}[C_{21}J'_n(\alpha r_0) + C_{22}N'_n(\alpha r_0)].$$

Then

$$\mathfrak{X}^{(1)}(r) = C\begin{cases} [N_n'(\alpha r_0) - N_n'(\alpha)]J_n(\alpha r), & 0 \le r < r_0, \\ J_n'(\alpha)N_n(\alpha r) - N_n'(\alpha)J_n(\alpha r), & r_0 \le r \le 1. \end{cases} \tag{2.4}$$

Theorem 2.1. *Problem (2.1) has the eigenvalues $\lambda = \alpha^2(n)$, determined by the equation (*) with the eigenfunctions $\Phi_n = J_n(\alpha r)[C_{n1}\cos n\varphi + C_{n2}\sin n\varphi]$. It corresponds to the adjoint problem (2.2) with the same eigenvalues and eigenfunctions (2.4) corresponding to them.*

We define now the conditions for absence of the Jordan chains. The condition for presence of Jordan chains is ([6])

$$\langle X^{(1)}, \mathfrak{X}^{(1)}\rangle = \int_0^1 rX^{(1)}(r)\mathfrak{X}^{(1)}(r)dr = I_n^1(\alpha) = 0.$$

Thus, the condition for absence of associated elements $\Phi^{(2)}$ has the form

$$I_n^1(\alpha) = \frac{1}{\pi r_0\alpha^3}[(n^2-\alpha^2)r_0J_n(\alpha) + (r_0^2\alpha^2 - n^2)J_n(\alpha r_0)] = \frac{1}{\alpha\pi}f'(\alpha) \ne 0. \tag{2.5}$$

Actually, $I_n(\alpha) = \int_0^1 X^{(1)}(r)\mathfrak{X}^{(1)}(r)rdr$ gives the first equality of (2.5). Since

$$J_n''(\alpha) = -\frac{1}{\alpha}J_n'(\alpha) - \left(1 - \frac{n^2}{\alpha^2}\right)J_n(\alpha),$$

$$J_n''(\alpha r_0) = -\frac{1}{\alpha r_0}J_n'(\alpha r_0) - \left(1 - \frac{n^2}{\alpha^2 r_0^2}\right)J_n(\alpha r_0),$$

then $f'(\alpha) = \dfrac{1}{r_0\alpha^2}[(n^2-\alpha^2)r_0J_n(\alpha) + (r_0^2\alpha^2 - n^2)J_n(\alpha r_0)]$.

The inhomogeneous problem (the Bessel equation) with the right-hand side $J_n(\alpha r)$ has the solution at $0 < r < 1$

$$X^{(2)}(r) = C_1J_n(\alpha r) + \frac{r}{2\alpha}J_n'(\alpha r) + \frac{N_n(\alpha r)}{2\alpha(N_n'(\alpha r_0) - N_n'(\alpha))}\times$$

$$\times\left[r_0\left(1 - \frac{n^2}{\alpha^2 r_0^2}\right)J_n(\alpha r_0) - \left(1 - \frac{n^2}{\alpha^2}\right)J_n(\alpha)\right].$$

The condition for absence of associated elements $\Phi^{(3)}$ is determined by the integral $I_n^2(\alpha) = \int_0^1 X^{(2)}(r)\mathfrak{X}^{(1)}(r)rdr \ne 0$, it can also be expressed by the relation $f''(\alpha) \ne 0$.

3. CASE $S > 2$

Let us now consider the general case $s > 2$, i.e. the problem

$$(\Delta + \lambda)\Phi = \frac{1}{r^{s-1}}\frac{\partial}{\partial r}\left(r^{s-1}\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\Delta_\theta \Phi + \lambda\Phi = 0,$$

$$\Phi \in C^{2+\alpha}(\Omega), \quad \Omega = \{r, \theta \mid r < 1, \theta = (\theta_1, \dots, \theta_{n-1})\}, \tag{3.1}$$

$$\frac{\partial \Phi(r_0, \theta)}{\partial r} = \frac{\partial \Phi(1, \theta)}{\partial r},$$

where Δ_θ is the Laplace operator on the unit sphere $S^{s-1} \subset \mathbb{R}^s$. Separating variables $\Phi = X(r)Y(\theta)$, we obtain the equation for the polyspherical functions

$$\Delta_\theta Y_{s,n} - n(n + s - 2)Y_{s,n} = 0$$

and, after substitution $X(r) = r^{-\frac{s}{2}+1}x(r)$, the Bessel equation

$$x'' + \frac{1}{r}x' + \left[\lambda - \frac{\left(n + \frac{s}{2} - 1\right)^2}{r^2}\right]x = 0 \tag{3.2}$$

At this, the last relation in (3.1) gives (under assumption that the solution is bounded) the condition that determines the eigenvalues $\lambda = \alpha^2$ as the roots of the equation

$f(\alpha) \equiv$

$$\equiv \alpha[r_0^{-\frac{s}{2}+1} J'_{n+\frac{s}{2}-1}(\alpha r_0) - J'_{n+\frac{s}{2}-1}] + \left(1 - \frac{s}{2}\right) \times [r_0^{-\frac{s}{2}} J_{n+\frac{s}{2}-1}(\alpha r_0) - J_{n+\frac{s}{2}-1}(\alpha)] = 0. \ (**)$$

The adjoint in the Lagrange sense problem:

$$(\Delta + \lambda)\Psi = 0, \quad \Omega_1 = \{r \mid r < r_0\} \cup \Omega_2 = \{r \mid r_0 < r < 1\},$$

$$\Psi'_r(r_0 - 0, \theta) = \Psi'_r(r_0 + 0, \theta), \quad \Psi'_r(1, \theta) = 0, \tag{3.3}$$

$$r_0^{s-1}[\Psi(r_0 - 0, \theta) - \Psi(r_0 + 0, \theta)] + \Psi(r_0 - 0, \theta) = 0,$$

$$\Psi(r, \theta) = \mathcal{X}_{s,n}(r)Y_{s,n}(\theta).$$

$$\mathcal{X}_{s,n}(r) = r^{-\frac{s}{2}+1}\begin{cases} C_{11}J_{n+\frac{s}{2}-1}(\alpha r), & 0 \le r < r_0, \\ C_{21}J_{n+\frac{s}{2}-1}(\alpha r) + C_{22}N_{n+\frac{s}{2}-1}(\alpha r), & r_0 \le r \le 1. \end{cases}$$

For determining the constants in the solution, we use the system

$$\begin{cases} C_{21}\left[\left(1-\dfrac{s}{2}\right)J_{n+\frac{s}{2}-1}(\alpha)+\alpha J'_{n+\frac{s}{2}-1}(\alpha)\right]+ \\ +C_{22}\left[\left(1-\dfrac{s}{2}\right)N_{n+\frac{s}{2}-1}(\alpha)+\alpha N'_{n+\frac{s}{2}-1}(\alpha)\right]=0, \\ -C_{11}\left[\left(1-\dfrac{s}{2}\right)J_{n+\frac{s}{2}-1}(\alpha r_0)+\alpha J'_{n+\frac{s}{2}-1}(\alpha r_0)\right]+ \\ +C_{21}\left[\left(1-\dfrac{s}{2}\right)J_{n+\frac{s}{2}-1}(\alpha r_0)+\alpha J'_{n+\frac{s}{2}-1}(\alpha r_0)\right]+ \\ +C_{22}\left[\left(1-\dfrac{s}{2}\right)N_{n+\frac{s}{2}-1}(\alpha r_0)+\alpha N'_{n+\frac{s}{2}-1}(\alpha r_0)\right]=0, \\ C_{11}r_0^{s/2}J_{n+\frac{s}{2}-1}(\alpha r_0)+C_{21}\left[-r_0^{s/2}J_{n+\frac{s}{2}-1}(\alpha r_0)+J_{n+\frac{s}{2}-1}(\alpha)\right]+ \\ +C_{22}\left[-r_0^{s/2}N_{n+\frac{s}{2}-1}(\alpha r_0)+N_{n+\frac{s}{2}-1}(\alpha)\right]=0 \end{cases} \tag{3.4}$$

with the determinant $-\dfrac{2}{\pi}r_0^{s/2}f(\alpha)$. The eigenfunctions of the adjoint problem are

$$\mathcal{X}^{(1)}_{s,n}(r)=\begin{cases} \left\{\left[\left(1-\dfrac{s}{2}\right)N_{n+\frac{s}{2}-1}(\alpha r_0)+\alpha r_0 N'_{n+\frac{s}{2}-1}(\alpha r_0)\right]r_0^{-s/2}- \right.\\ \left. -\left(1-\dfrac{s}{2}\right)J_{n+\frac{s}{2}-1}(\alpha r_0)+\alpha r_0 J'_{n+\frac{s}{2}-1}(\alpha r_0)\right\}\times \\ \times J_{n+\frac{s}{2}-1}(\alpha r), \quad 0\le r<r_0, \\ -\left[\left(1-\dfrac{s}{2}\right)N_{n+\frac{s}{2}-1}(\alpha)+\alpha N'_{n+\frac{s}{2}-1}(\alpha)\right]J_{n+\frac{s}{2}-1}(\alpha r)+ \\ +N_{n+\frac{s}{2}-1}(\alpha r) \quad r_0<r\le 1. \end{cases}$$

Theorem 3.1. *The problem (3.1) has the eigenvalues $\lambda=\alpha^2(n)$, determined by the equality (**) with the eigenfunctions $\Phi_n(r,\theta)=J_n(\alpha r)Y_{s,n}(\theta)$. It corresponds to the adjoint problem (3.3) with the same eigenvalues and eigenfunctions $\Psi(r,\theta)=\mathcal{X}^{(1)}_{s,n}(r)Y_{s,n}(\theta)$, corresponding to them.*

The condition for absence of associated elements $\Phi^{(2)}$ one can find by calculating the integral $I_{s,n}(\alpha)=\int_0^1 X^{(1)}_{s,n}(r)\mathcal{X}^{(1)}_{s,n}(r)r^{s-1}dr$. As before, it is related with the condition $f'(\alpha)=0$. The conditions for absence of associated elements of successively higher orders are $f''(\alpha)\neq 0$, $f'''(\alpha)\neq 0$, and on the associated elements of the third order the Jordan chains break.

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PROBLEMS OF VISUALIZATION OF CITATION NETWORKS FOR LARGE SCIENCE PORTALS

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Abstract A generally accepted way to facilitate understanding of large and complex data sets is graph visualization. In this paper we present three different methods of visualization for the citation networks, one based on hierarchical edge bundles algorithm, one implementing dynamic layered drawing, and one utilizing a geometry-based edge bundling. As test sets, we make extensive use of citation networks designed from the data sets of Open Linked Data portals.

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1. INTRODUCTION

Due to the fast progress of Semantic Web and its new branch of Linked Open Data, large amounts of structured information on various scientific fields are getting available. The main part of the content of scientific digital libraries and specialized portals constitute research publications, the most reliable source of information dedicated to any research area. The most active and influential researchers, organizations in which they work, and geographic locations of the research units – all this information is currently available in the rdf / xml format. This information evolves over time and rapidly grows in volume. To optimize the science management, new tools for investigation and analysis of these data are needed. A generally accepted way to facilitate understanding of large and complex data sets is graph visualization. The topic of our paper consists in several visualization methods for citation networks. Previously, we considered methods of visualization of information on scientific cooperation, represented by co-authorship networks derived from small information portals [1-2]. Our current work is a further development of this research. The data under consideration has significantly greater volume, and newly developed algorithms are presented to analyze and visualize this data.

A citation network is a network in which the vertices represent documents and the edges between them represent reference of one document to another. Citation networks are directed: citations go from one document to another. Citation networks evolve over time as new documents are created. The citation network analysis started with the paper of Garfield et al. [10] and has been studied by many authors [9, 12]. Force-directed methods of visualization used to be the main tool of investigation for these networks.

In this paper we present three different methods of visualization for the citation networks, one based on hierarchical edge bundles algorithm, one implementing dynamic layered drawing, and one utilizing a geometry-based edge bundling. As test sets, we make extensive use of citation networks designed from the data sets of Open Linked Data portals. The paper is organized as follows. Section II discusses extracting citation networks from the content of Linked Open Data portals. Section III demonstrates some problems of the citation networks visualization by the hierarchical edge bundles method. Section IV describes some results of visualization of the citation networks by a layered dynamic method. Section V demonstrates the citation networks visualization with a geometry-based edge bundling method. Finally, section VI presents conclusion and perspectives for further work.

2. OPEN LINKED DATA AND CITATION NETWORKS GENERATION

The datasets of Linked Open Data (LOD) portals such as DBLP, Citeseer, CORDIS, NSF, EPSRC, ACM, IEEE, [4-7]. etc. have been used as a test data. These datasets are described in RDF format and have a very impressive size. For example, the data provided by the Citeseer portal consists of 8,146,852 triples, ACM portal data comprises 12,402,336 triples, and DBLP portal has granted 28,384,790 triples. A user can either download the files in RDF format, or generate data using a sparql query. All datasets of these portals are described according to a single ontology AKT Reference Ontology [5], which is the union of several ontologies (Support Ontology, Portal Ontology, Extensions Ontology and RDF Compatibility Ontology).

Portal Ontology is the main one among these ontologies, it describes such concepts as organizations, persons, projects, publications, geographic data, etc. AKT Ontology has a rather deep hierarchical structure (Fig.1). For example, to describe the publications, there exist two root classes ”Information-Bearing-Object” and ”Abstract-Information”. Subclasses of ”Information-Bearing-Object” are the classes ”Recorded-Audio”, ”Recorded-Video”, ”Publication”, ”Edited-Book”, ”Composite-Publication”, ”Serial-Publication”, ”Periodical-Publication ”and ”Book”. All individuals of the class ”Information-Bearing-Object” have a relationship ”has-publication-reference”, pointing to an object of the class ”Publication-Reference”, which is a subclass of the class ”Abstract-Information”. In turn, the class ”Publication-Reference” has as subclasses the classes ”Web-Reference”, ”Book-Reference”, ”Edited-Book-Reference”, ”Conference-Proceedings-Reference”, ”Workshop-Proceedings-Reference”, ”Book-Section-Reference”, ”Article-Reference”, ”Proceedings-Paper-Reference”, ”Thesis-

Reference" and "Technical-Report-Reference". The individuals of the class "Publication-Reference" have such relationships as: "has-date", "has-title ", "has-place-of-publication", "cites-publication-reference", etc. There exists the class "Organization", which is a subclass of the class "Legal-Agent", and the class "Legal-Agent" is a subclass of the class "Generic-Agent". The class "Person" is a subclass of the class "Generic-Agent".

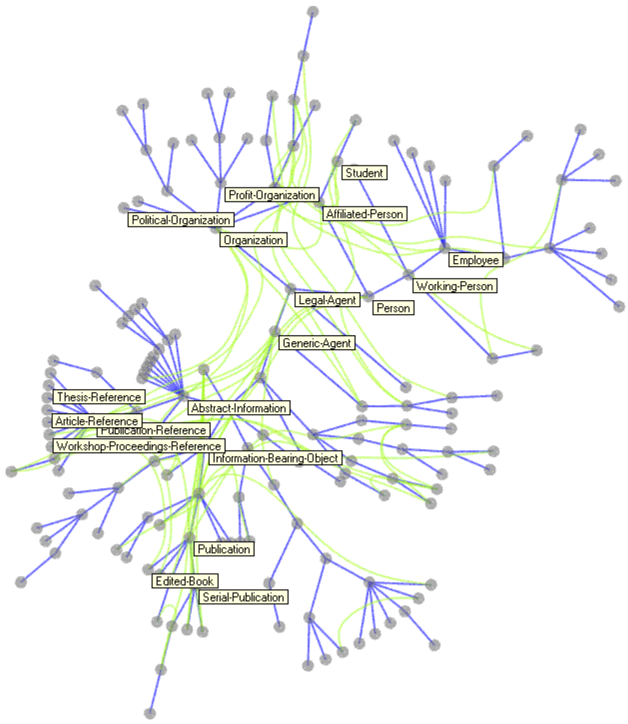


Fig. 1. AKT Reference ontology.

There are several problems of using the LOD datasets. Although all bibliographic datasets of the LOD cloud use as common vocabulary AKT Ontology, the contents of these sets are very heterogeneous and are based on very narrow subsets of this vocabulary. To describe real objects, classes of the highest level of hierarchy are normally used. For example, the classes "Publication-Reference" and "Article-Reference" are used for the description of publications while such classes as "Proceedings-Paper-Reference" are not used at all. This feature makes difficult generation of the hierarchical structure needed for applying the hierarchical edge bundles method. Also, the data sets are not complete and many attributes remain to be filled. Besides, the citation relationship (*akt: cites-publication-reference*) existing in AKT Reference Ontology, is described explicitly only for several datasets such as Citeseer and ACM [6]. However, the common mechanism of access simplifies working with these data. It is easy enough to generate a simple citation network for any storage of the LOD cloud if the publications described in these datasets have the relationship "cites-publication-reference". An example of user interface and SPARQL 1.0 query intended for citation networks generation from ACM dataset is shown in Fig. 2.

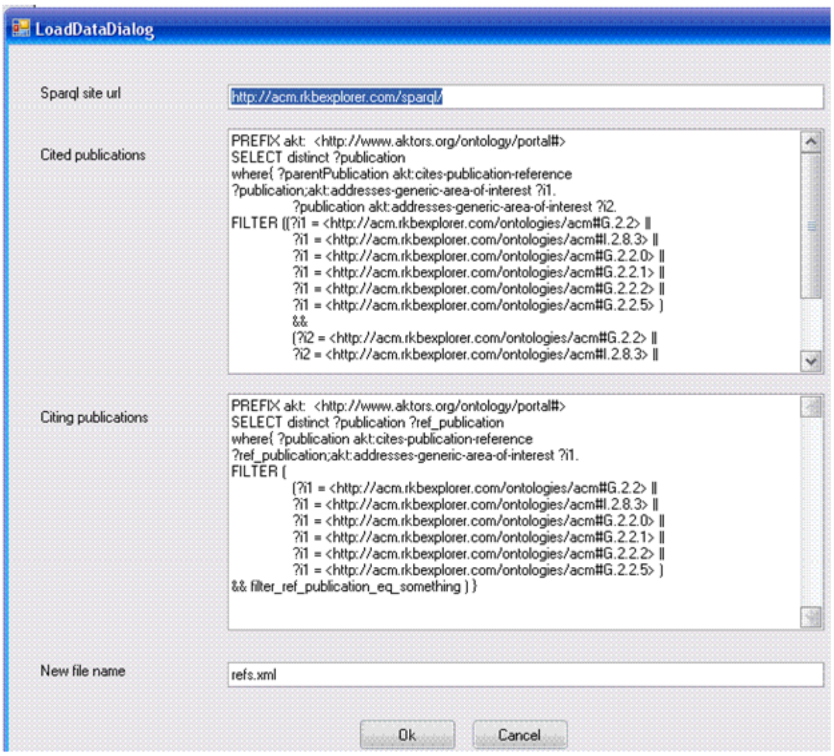


Fig. 2. An example of user interface and SPARQL 1.0 query intended for citation networks generation.

To select the desired volume of data, the query modifier LIMIT N was used. We could relatively easy extract citation networks of 20-30 thousand vertices.

3. VISUALIZATION OF CITATION NETWORKS USING THE HIERARCHICAL EDGES BUNDLES

We have started our experiments by applying already implemented hierarchical edge bundles method [11] for the citation networks visualization. This method allows a drawing of a citation network to be combined with drawings of other elements of the portal content. It is implemented as follows. Some predefined hierarchical structure is drawn as a tree whose leafs are research papers. Then each link of the citation network is modeled as a single B-spline [14] using the control points along the shortest path in the tree layout from one leaf point to another. A test set of 561 publications on information visualization for 10 years is shown in Fig. 3. A three-level hierarchy consisting of years, conferences, and publications is depicted with balloon tree method (Fig 3(a)), and the citation links are drawn with hierarchical edge bundles method. Research papers are shown as black circles. Scientific conferences and periodical issues are shown as yellow circles. The paper's publishing years constitute the upper level of hierarchy and are shown as purple circles. The edges of the tree are shown in blue (a year includes conferences, a conference includes publications). The direction of a link from a citing publication to a cited publication is shown by progression of color from purple to green. When looking at this drawing we can easily identify the years with the largest number of publications (the years 1995 and 1996). We have slightly improved the drawing comprehensibility by depicting the citation index of papers by the radius of nodes. Since we do not want the area of drawing to grow up due to the node size enlargement, the nodes overlap is permitted. Hence the nodes visibility also depends on the citation index, as it is shown in Fig. 3(b), where the number of visible nodes and the number of the node overlaps has been reduced. Further on, users can also change the width of reference links and their opacity as a function of the citation index of the incident nodes.

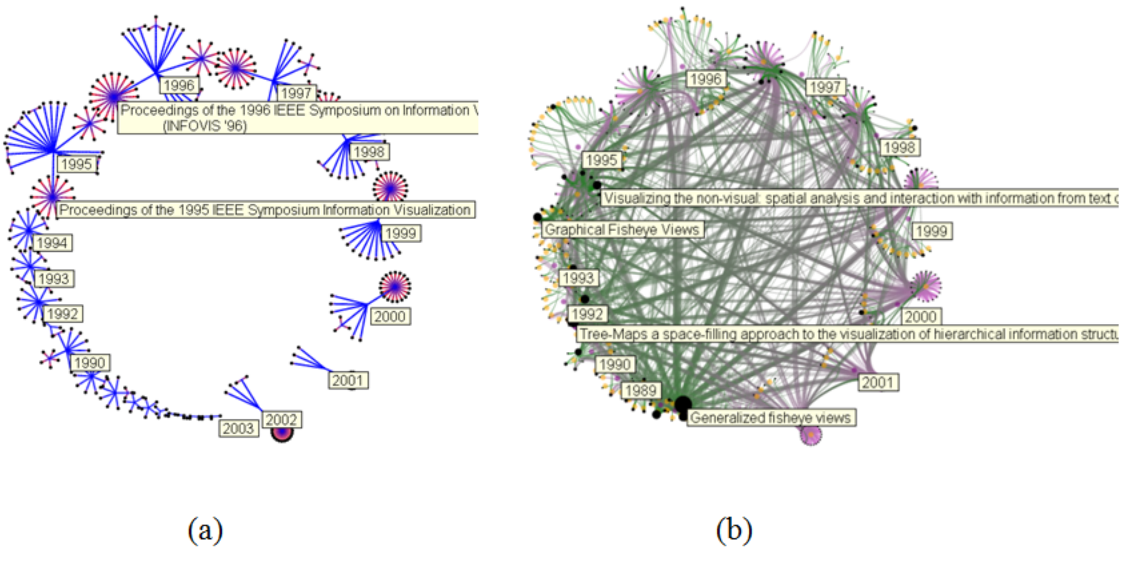


Fig. 3. Hierarchical structure and citation network. (a) A three-level hierarchy consisting of years, conferences, and publications. (b) Hierarchical edge bundles drawing of a citation network.

Some possible functions for these parameters calculation are:

$$y = (o_{\max} - o_{\min})\frac{I - I_{\min}}{I_{\max} - I_{\min}} + o_{\min} \tag{1}$$

$$y = (o_{\max} - o_{\min}) \cdot \left(1 - \sqrt{\frac{I_{\max} - I}{I_{\max} - I_{\min}}}\right) \tag{2}$$

Where I – citation index, I_{max} and I_{min} – the largest and the smallest citation index in the citation network under consideration, o_{max} and o_{min} – upper and lower bounds of values for y.

The formula (1) helps to identify the group of the most cited publications, since the node sizes are proportional to their citation indexes. The formula (2) helps to find the most cited publication since it assigns a much larger radius to the node with the highest citation index.

After the most cited papers are identified, user can choose such a node with a mouse pointer and examine its name, list of its authors and all the papers citing it as is shown in Fig. 4.

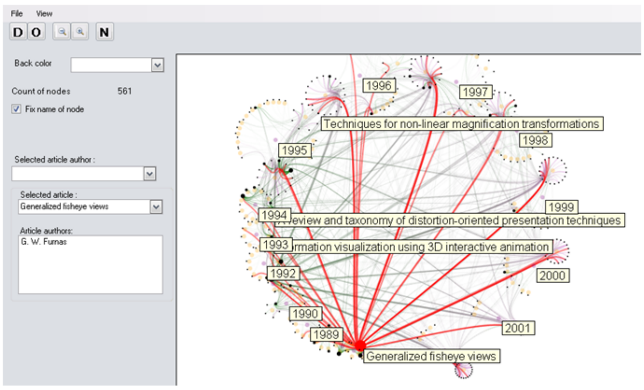
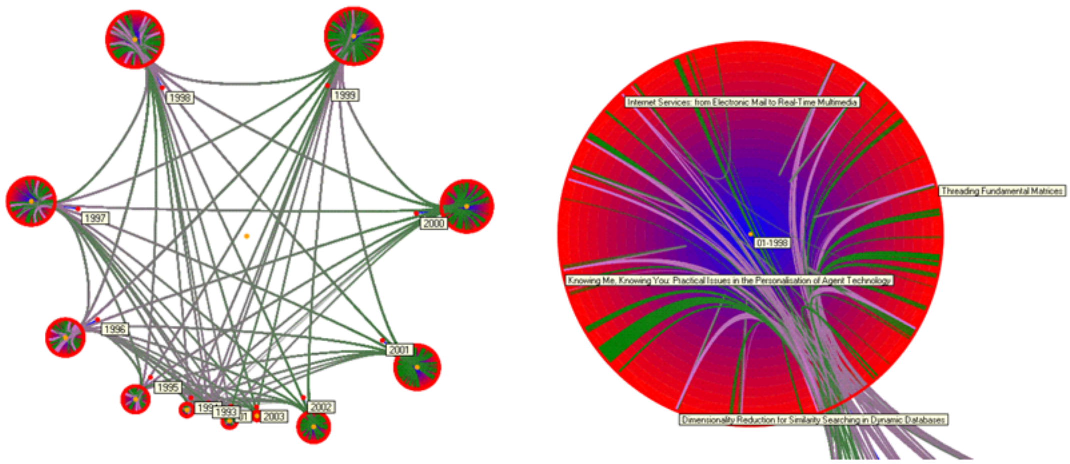


Fig. 4. The most cited paper and links citing it (shown in red).

When the size of citation network increases, the hierarchical edge bundles method gets difficult to use. For example, a drawing of a citation network of 20 000 vertices, retrieved from the Citeseer database is shown in Fig. 5. We have only managed to create a two-level hierarchy for the Citeseer dataset: the year of publication – the month of publication. That results in a drawing, rather sparse in the center (Fig. 5(a)) and very dense at the periphery (Fig. 5(b)). The time interval of these publications dataset covers the period from 1993 to 2003. The drawing permits to compare the number of publications by year: the largest number of publications of the test set falls on the years 1998 and 1989 while publications of 2003 are not numerous. Unfortunately, it is not possible to get any detailed information from this drawing. The central part of the drawing is complete graph stating that there exist citing links from any year to any posterior year in this network. And the publications of every year are that numerous, that it is very difficult to select a vertex by the mouse pointer for further investigation.



(a) (b)

Fig. 5. A citation network of 20 000 vertices retrieved from Citeseerportal. (a) A global view, (b) one-month publications of the 1998 year.

Since it is not always possible to extract a deep hierarchy allowing the hierarchical edge bundles to be applied, we have implemented two alternative strategies for citation networks visualization:

1 To emphasize the directed nature of links in the citation networks, a dynamic layered method of visualization was implemented.

2 To reduce the visual density of drawings, a geometry-based edge bundling method was developed.

4. DYNAMIC LAYERED DRAWING OF THE CITATION NETWORKS

A citation network is a directed graph, so it is desirable that all edges are directed to one side. The direction of the edges corresponds to the chronological order of publications. Also, the citation networks are assumed to be acyclic, even if it is not always the case. For example, if a scientific paper sometimes cite work that is forthcoming but not yet published, the resulting network will have a closed loop. However, such loops are rare and short.

The construction of a layered graph drawing [13] proceeds in a sequence of standard steps:

1 **Layer assignment**. The vertices of the directed acyclic graph are assigned to layers, such that each edge goes from the left to the right. In the current implementation each layer corresponds to a publishing year, i.e. the papers, published in the same year are assigned to the same layer. We are going to parameterize the length of the time intervals in the nearest future. Edges that span multiple layers are replaced by paths of dummy vertices so that, after this step, each edge in the expanded graph connects two vertices on adjacent layers of the drawing.

2 **Crossing minimization.** The vertices within each layer are permuted in an attempt to reduce the number of crossings among the edges connecting it to the previous layer. Since finding the minimum number of crossings is NP-complete, we place each vertex at a position determined by the average of the positions of its neighbors on the previous level and then permuting adjacent pairs as long as that improves the number of crossings.

3 **Coordinate assignment.** To each vertex is assigned a coordinate within its layer, consistent with the permutation calculated in the previous step. The dummy vertices are removed from the graph and the vertices and edges are drawn.

Figure 5 shows the drawing of a citation network generated by the layered method of placement. Publishing years of papers in the citation network are shown as rectangles of different colors at the top of the image. All papers published in the same year are placed in a vertical column corresponding to this year. The edges of the network correspond to the citations. The color of each edge is identical to the color of label of the year of the citing publication. The more citation links has some publication, the more input edges has the corresponding vertex, and the greater is its radius. As a result, the citation links of publications form highly visible bundles. Four buttons at the top of the screen are used to track the dynamics of the citation network year by year. The buttons ”<” and ”>” are designed to move through the drawing and observe the evolution of the citation network over time. Technically, this feature is implemented by filtering vertices and edges of the citation network. Pressing the ”>>” button displays the entire citation network, and the ”<<” button is used to clean the drawing.

The evolution over time of a citation network of papers devoted to the graph theory is shown in Fig. 5. The four fragments of this figure show different intervals of time between 1965 and 2005. In the period from 1965 to 1989 (Fig. 6(a)) the test set of publications is dominated by the “Linear-time algorithm for isomorphism of planar graphs” paper. The corresponding vertex has the largest radius and a large brown tail of input edges. In 1993 (Fig. 6 (b)) the number of references to the papers “A data

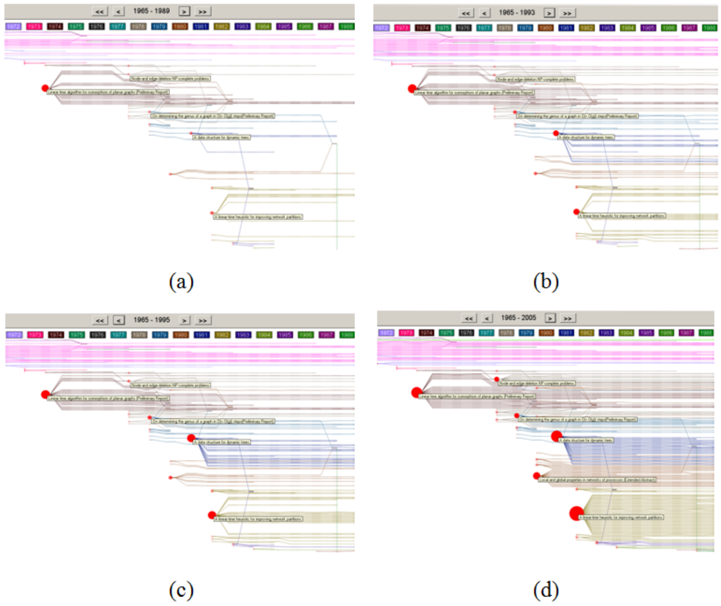
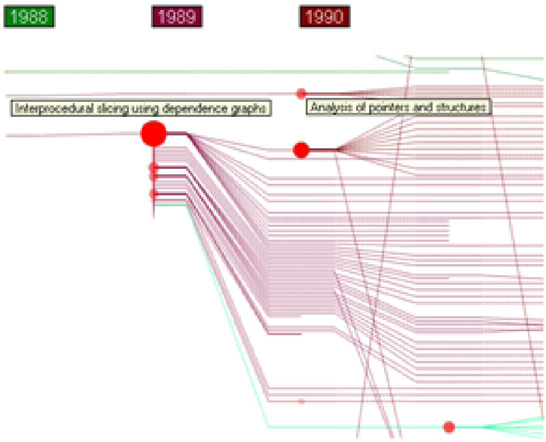


Fig. 6. The evolution of a citation network over time.

structure for dynamic trees" and "A linear-time heuristic for improving network partition" increases. In 1995 (Figure 6 (c)), these two papers have the same level of citation index as the paper "Linear-time algorithm for isomorphism of planar graphs". Finally, in 2005 (Fig. 6 (d)) the paper "A linear-time heuristic for improving network partition" gets the most cited. Hence, data sets are better comprehended due to the dynamic visualization.

It is also possible to observe growing interest to the paper "Node-and-edge-deletion NP-complete problems" that refers to the previously dominating paper "Linear-time algorithm for isomorphism of planar graphs", i.e. a chain of highly cited related publications arises.

Besides, this visualization method helps to detect errors and inaccuracies in bibliographic data. Fig. 6(a) shows a fragment of a citation network generated for the ACM dataset on the time interval from 1988 to 1990. A brown link connects the node of the "Analysis of pointers and structures" paper published in 1990 and the "Interprocedural slicing using dependence graphs" paper published in 1988. Since the color of the link corresponds to the year 1990 it should mean that the arc is oriented backward and a paper published in 1988 cites a paper published in 1990. By checking the ACM dataset (Fig. 6(b)) we have discovered that the paper "Interprocedural slicing using dependence graphs" has several dates of publication and the corresponding node is placed in the layer of the earliest date of publication.



(a)

Analysis of pointers and structures	akt:has-date	1990-01-01
Interprocedural slicing using dependence graphs	akt:cites-publication-reference	Analysis of pointers and structures
Interprocedural slicing using dependence graphs	akt:has-date	1988-01-01
Interprocedural slicing using dependence graphs	akt:has-date	1988-07-01
Interprocedural slicing using dependence graphs	akt:has-date	1990-01-01

(b)

Fig. 7. Datasets inaccuracies. (a) Backward link representing a paper published in 1988 citing a paper published in 1990. (b) Multiple dates of publishing for a paper.

The main problem with the conventional layered method is that drawings get overloaded very quickly and using the filtering removes irrelevant papers but distorts reality: irrelevant publications are the major contributors in determining the significance of other publications. The hierarchical edge bundles method is not applicable in the absence of external hierarchical structure. Therefore we have implemented an algorithm , which can reduce the drawing density by forming bundles of edges based on their own geometry, and not introduced from outside.

5. GEOMETRY-BASED EDGE BUNDLING METHOD.

The main idea of the geometry based edge bundling method [8] is to reduce the visual clutter of the image by bending the edges through a special control grid without changing the original locations of graph vertices. This method proceeds as follows:

1 Generate a rectangular NxN grid and put it over a graph drawing constructed in any way.

2 For each grid cell, calculate the main direction of the edges crossing the cell.

3 Merge into zones the adjacent cells having directions that differ by no more than the threshold value .

4 Calculate the basic direction and normal vector to the main direction of each zone.

5 Calculate the points of intersection of the normal segments with zones' boundaries .

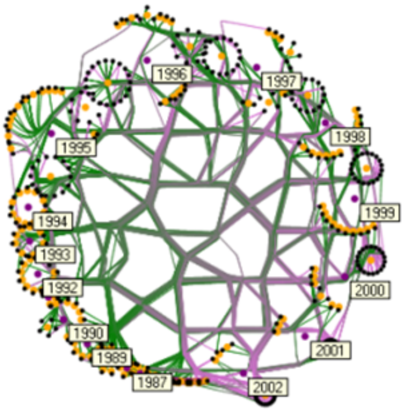
6 Use the resulting points to construct a triangulation.

7 Find for each edge of the constructed triangulation the point of intersection with the edges of the original graph drawing. Calculate the centers of these points.

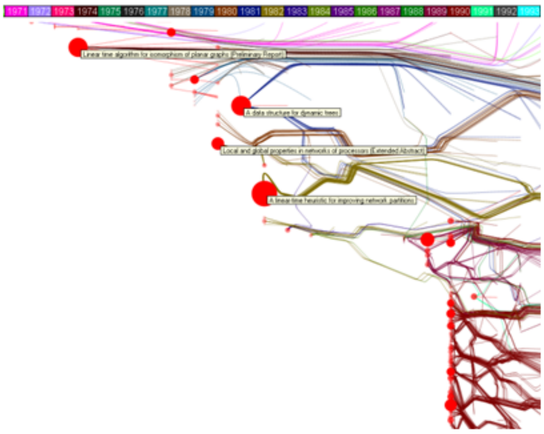
8 Use the resulting points as control points of b-splines.

Fig. 8(a) demonstrates applying the geometry based edge bundling strategy to the drawing obtained by circular drawing method from Fig.2(b). Fig. 8(b) shows applying the geometry based edge bundling strategy to the drawing obtained by layered visualization method from Fig.5(d).

No doubt, due to this methodology the drawing congestion is reduced. But at this stage, there are more questions with this method than answers. How to choose the best direction for a rectangular grid? How does the direction of the edge bundles depend on the size of the grid? How to choose the best edge direction within each zone in function of the underlying visualization method? Nevertheless, we hope to develop this method to the point where it can be used to examine trends in a research field.



(a)



(b)

Fig. 8. The application of the geometry based edge bundling strategy to the drawing obtained by layered visualization method. (a) Applying the geometry based edge bundling strategy to the circular drawing. (b) Applying the geometry based edge bundling strategy to the layered drawing.

6. CONCLUSION

In this paper we have demonstrated three visualization methods of citation networks generated for datasets of Linked Open Data portals. These drawings are rather helpful for understanding of datasets of large volumes. Also they enable users to observe the evolution of datasets over time. In the nearest future we are going to apply the previously developed clustering methods for the citation networks analysis and to compare the results obtained by the two groups of methods.

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FACTORIZATION STRUCTURES WITH NONHEREDITARY CLASSE OF PROJECTIONS

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Abstract In the category of the local convex topological vector Hausdorff spaces, we build a proper class of the factorization structures for which the classes of projections are not hereditary with respect to the class of universal monomorphisms.

Keywords: reflective and coreflective subcategories, local convex space, factorization structure.
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1. INTRODUCTION

In the category $\mathcal{C}_2\mathcal{V}$ [1], [5] of the local convex topological vector Hausdorff spaces [7], [8] the following factorization structures are examined:

$(\mathcal{E}pi, \mathcal{M}_f)$ - the class of epimorphisms, the class of kernels = the class of morphisms with dense image, the class of topological inclusions with closed image;
$(\mathcal{E}_u, \mathcal{M}_p)$ = the class of universal epimorphisms, the class of precise (exact) monomorphisms=the class of surjective morphisms, the class of topological embedding;
$(\mathcal{E}_p, \mathcal{M}_u)$ = the class of precise epimorphisms, the class of universal mono-morphisms;
$(\mathcal{E}_p^{'}, \mathcal{M}_u^{'})$ = the class of precise epimorphisms, the class of universal monomorphisms with the closed image;
$(\mathcal{E}_f, \mathcal{M}ono)$ = the class of cokernels, the class of monomorphisms = the class of factorial morphisms, the class of injective morphisms.

We will examine the following subcategories of the categorie $\mathcal{C}_2\mathcal{V}$:

$\mathcal{S}$ - the subcategory of spaces endowed with a weak topology,
Γ_0 - the subcategory of complete spaces,
Π - the subcategory of complete spaces with a weak topology,
$\widetilde{\mathcal{M}}$ - the subcategory of spaces endowed with Mackey topology.
The first subcategory is reflective while the last one is coreflective.

Definition 1.1. [2]. *Monomorphism m is called universal monomorphism if for every pushout square $u' \cdot m = m' \cdot u$ the morphism m' is a monomorphism.*

Theorem 1.1. ([2], Theorem 1.7). *A monomorphism $m : X \longrightarrow Y$ is a universal monomorphism in the category $\mathcal{C}_2\mathcal{V}$ if any continuous functional defined on X extends through m.*

Let $\mathcal{K}$ be a coreflective subcategory with the coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ (see [2] p. 2.11). We denote

$$\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}.$$

Dual, let $\mathcal{R}$ be a reflective subcategory with the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$. We denote by

$$\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}.$$

Definition 1.2. [2], [5], [9]. *Let $\mathcal{A}$ be a class of morphisms of the category $\mathcal{C}$. It is said that the morphism $f \in \mathcal{A}^{\perp}$ if for some commutative square (see diagram below)*

$$f \cdot s = t \cdot g$$

with $g \in \mathcal{A}$ there exists a unique morphism diagonal d so that

$$f \cdot d = t$$

$$d \cdot g = s.$$

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "s"'] & \cdot \arrow[d, "t"] \arrow[dl, "d"'] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

In this case we denote $f \perp g$. The class $\mathcal{A}^{\perp}$ is called the class of down orthogonal morphisms to the morphisms of class $\mathcal{A}$.

In an obvious way is defined the class $\mathcal{A}^{\top}$ of the above orthogonal morphisms. In [9], there are used the notations: $\wedge(\mathcal{A})$ instead of the $\mathcal{A}^{\perp}$ and $\tau(\mathcal{A})$ instead of the $\mathcal{A}^{\top}$.

The next notations are also used :

$$\mathcal{A}^{\perp} \cap \mathcal{M}ono = \mathcal{A}^{\llcorner},$$

$$\mathcal{A}^{\top} \cap \mathcal{E}pi = \mathcal{A}^{\urcorner}.$$

For a factorization structure $(\mathcal{P}, \mathcal{I})$ we have $\mathcal{P} = \mathcal{I}^{\urcorner}$ and $\mathcal{I} = \mathcal{P}^{\llcorner}$. For a right factorization structure (respectively left factorization structure) $(\mathcal{P}, \mathcal{I})$ we have $\mathcal{P} = \mathcal{I}^{\urcorner}$ and $\mathcal{I} = \mathcal{P}^{\perp}$ (respectively $\mathcal{P} = \mathcal{I}^{\top}$ and $\mathcal{I} = \mathcal{P}^{\llcorner}$).

Theorem 1.2. ([2], Theorem 2.12). *Let $\mathcal{K}$ be a nonzero coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. Then $((\mu\mathcal{K})^{\top}, \mu\mathcal{K})$ is a left factorization structure in the category $\mathcal{C}_2\mathcal{V}$.*

Theorem 1.3. ([2], Theorem 4.4). *Let $\mathcal{K}$ and $\mathcal{L}$ be a coreflective and reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ with the respective functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$. The pair $(\mathcal{K}, \mathcal{L})$ is called a conjugate pair of subcategories if $\mu\mathcal{K} = \varepsilon\mathcal{L}$.*

Definition 1.3. ([2], Def. 2.1). *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of category $\mathcal{C}$. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary ($\mathcal{B}$-cohereditary) if from the fact that $fg \in \mathcal{A}$ and $f \in \mathcal{B}$ (respectively $g \in \mathcal{B}$), it follows that $g \in \mathcal{A}$ (respectively $f \in \mathcal{A}$).*

Definition 1.4. ([2], Def. 3.1). *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The composition of the classes $\mathcal{A}$ şi $\mathcal{B}$ is called the class $\mathcal{A} \circ \mathcal{B}$ of all morphisms of the category $\mathcal{C}$ of ab form with the elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ for which this composition exists.*

We denote by $\mathbb{R}$ (respectively $\mathbb{K}$) the lattice of the nonzero reflective subcategories (respectively coreflective) of the category $\mathcal{C}_2\mathcal{V}$. Any factorization structure $(\mathcal{P}, \mathcal{I})$ divides the lattice $\mathbb{R}$ into three classes:

- the class $\mathbb{R}(\mathcal{P})$ of $\mathcal{P}$ - reflective subcategories;
- the class $\mathbb{R}(\mathcal{I})$ of $\mathcal{I}$ - reflective subcategories;
- $\mathbb{R}(\mathcal{P}, \mathcal{I}) = (\mathbb{R} \setminus (\mathbb{R}(\mathcal{P}) \cup \mathbb{R}(\mathcal{I}))) \cup \{\mathcal{C}_2\mathcal{V}\}$.

Theorem 1.4. 1. ([10], Theorem 1.2). *The class $\mathbb{R}(\mathcal{P})$ contains the smallest element $\overline{\mathcal{S}}$.*

2. ([10], Theorem 1.3). *If the class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary then*

$$\mathbb{R}(\mathcal{P}) = \{\mathcal{L} \in \mathbb{R} \mid \overline{\mathcal{S}} \subset \mathcal{L}\}.$$

2. FACTORIZATION STRUCTURES ASSOCIATED WITH A COREFLECTIVE SUBCATEGORY

Theorem 2.1. [2]. *Let $\mathcal{K}$ be a nonzero coreflective subcategory in the category $\mathcal{C}_2\mathcal{V}$, and $(\mathcal{P}, \mathcal{I})$ - a factorization structure with the class of projections which is $(\mu\mathcal{K})$-hereditary.*

1. The pair $(\mathcal{E}, \mathcal{M}) = ((\mathcal{I} \cdot (\mu\mathcal{K}))^{\urcorner}, \mathcal{I} \cdot (\mu\mathcal{K}))$ is a factorization structure in the category $\mathcal{C}_2\mathcal{V}$. 2. If $\widetilde{\mathcal{M}} \subset \mathcal{K}$, then the class $\mathcal{I}$ of injections is $\mathcal{P}$-cohereditary.

3.Let $p : X \longrightarrow Y \in \mathcal{P}$. The morphism p belongs to the class $\mathcal{E}$ iff the square

$$p \cdot k^X = k^Y \cdot k(p)$$

is pushout, where $k^X : kX \longrightarrow X$ and $k^Y : kY \longrightarrow Y$ are $\mathcal{K}$-coreplicas of the respective objects, and $k(p)$ is that morphism, which make the square commutative.

4. Let $f \in \mathcal{C}_2\mathcal{V}$, $f = p \cdot i$ the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism f, and $p = t \cdot u$ - the $((\mu\mathcal{K})^{\top}, \mu\mathcal{K})$-factorization of the morphism p. Then

$$f = (i \cdot t) \cdot u$$

is the $(\mathcal{E}, \mathcal{M})$*-factorization of the morphism* f.

Proof. . 1. The fact that $(\mathcal{E}, \mathcal{M})$ is a factorization structure in the category $\mathcal{C}_2\mathcal{V}$ follows from the Theorem 3.2*[2].

2. We prove that the class $\mathcal{M}$ is $\mathcal{P}$-cohereditary. Let $f \cdot g \in \mathcal{M}$ and $g \in \mathcal{P}$, and

$$f = i_1 \cdot p_1 \tag{1}$$

be the $(\mathcal{P}, \mathcal{I})$-factorization of the morphism f. Since $f \cdot g \in \mathcal{M}$, it follows that this morphism can be expressed

$$f \cdot g = i \cdot m, \tag{2}$$

where $i \in \mathcal{I}$, and $m \in \mu\mathcal{K}$.

$$\begin{tikzcd}
 & \bullet \arrow[d, dashed, "p_1"] \arrow[dr, "f"] & \\
\bullet \arrow[ur, "g"] \arrow[dr, dashed, "m"'] & \bullet \arrow[r, dashed, "i_1"] \arrow[d, dashed, "h"] & \bullet \\
 & \bullet \arrow[ur, dashed, "i"'] &
\end{tikzcd}$$

From the (1) and (2) it follows the equality

$$i \cdot m = i_1 \cdot (p_1 \cdot g) \tag{3}$$

with $i \in \mathcal{I}$ and $p_1 \cdot g \in \mathcal{P}$. So, there exists a morphism h which satisfies the equalities

$$m = h \cdot p_1 \cdot g \tag{4}$$

$$i_1 = i \cdot h \tag{5}$$

From (5) it results that h is a monomorphism. Then, from the equality (4), we get that $m \in \mu\mathcal{K}$; also it follows that h and $p_1 \cdot g$ belong to the class $\mu\mathcal{K}$. Since $\widetilde{\mathcal{M}} \subset \mathcal{K}$ it follows that $\mu\mathcal{K} \subset \mu\widetilde{\mathcal{M}}$. Therefore $p_1 \cdot g \in \mu\mathcal{K} \subset \mu\widetilde{\mathcal{M}} = \mathcal{M}_u \cap \mathcal{E}_u$, and g is an epimorphism. Since the class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary, we deduce that $p_1 \in \mathcal{M}_u$. So $p_1 \in \mu\mathcal{K}$. Therefore, in the equality (1), $i_1 \in \mathcal{I}$, and $p_1 \in \mu\mathcal{K}$, i.e. $f \in \mathcal{M}$.

3. Let $p : X \longrightarrow Y \in \mathcal{P}$. We construct the corresponding square

$$p \cdot k^X = k^Y \cdot k(p) \tag{6}$$

By using the morphisms k^X and $k(p)$ we construct the pushout square

$$\begin{tikzcd}
kX \arrow[rr, dashed, "k(p)"] \arrow[dd, "k^X"'] & & kY \arrow[dl, dashed, "v"] \arrow[dd, "k^Y"] \\
 & T \arrow[dr, dashed, "t"] & \\
X \arrow[ur, dashed, "u"] \arrow[rr, "p"] & & Y
\end{tikzcd}$$

$$u \cdot k^X = v \cdot k(p) \tag{7}$$

Then, there exist a morphism t such that

$$p = t \cdot u \tag{8}$$

$$k^Y = t \cdot v \tag{9}$$

Consider $p \in \mathcal{E}$. From the equality (8) we deduce that $t \in \mathcal{E}$. Let us prove that $t \in \mathcal{M}$. Since $k^X \in \mathcal{E}_u$, it follows that $v \in \mathcal{E}_u$. In the equality (9) we have $k^Y \in \mathcal{M}ono$ and $v \in \mathcal{E}_u$. Since the class $\mathcal{M}ono$ is $\mathcal{E}_u$-cohereditary (lemma 2.6*[2]), we deduce that $t \in \mu\mathcal{K} \subset \mathcal{M}$. Thus t is an *iso*, and the square (6) is push out.

Conversely. Let $p \in \mathcal{P}$ and the square (1) be pushout. We will prove that $p \perp \mathcal{M}$. Since $\mathcal{M} = \mathcal{I} \cdot (\mu\mathcal{K})$ and $p \in \mathcal{P}$, it results $p \perp \mathcal{I}$. It remains to prove that $p \perp (\mu\mathcal{M})$.

Let $m : A \longrightarrow B \in \mu\mathcal{K}$, and

$$m \cdot f = g \cdot p \tag{10}$$

$$\begin{tikzcd}
 & kX \arrow[r, "k(p)"] \arrow[d, "k^X"'] & kY \arrow[d, "k^Y"] \arrow[ddll, dashed, "h"'] \\
 & X \arrow[r, "p"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "w"'] \\
kA \arrow[r, "k^A"] & A \arrow[r, "m"] & B
\end{tikzcd}$$

If $k^A : kA \longrightarrow A$ is the $\mathcal{K}$-coreplica of the object A, then $m \cdot k^A$ is the $\mathcal{K}$-coreplica of the object B. Therefore for the morphism $f \cdot k^Y$ there exists a morphism h such hat

$$g \cdot k^Y = m \cdot k^A \cdot h. \tag{11}$$

We have

$$m \cdot k^A \cdot h \cdot k(p) \overset{(11)}{=\!=} g \cdot k^Y \cdot k(p) \overset{(6)}{=\!=} g \cdot p \cdot k^X \overset{(10)}{=\!=} m \cdot f \cdot k^X,$$

i.e.

$$m \cdot k^A \cdot h \cdot k(p) = m \cdot f \cdot k^X \tag{12}$$

and since $m \in \mathcal{M}ono$ it follows that

$$k^A \cdot h \cdot k(p) = f \cdot k^X. \tag{13}$$

Considering that (6) is an push out square, we deduce that

$$k^A \cdot h = w \cdot k^Y, \tag{14}$$

$$f = w \cdot p \tag{15}$$

for a morphism w. The morphism w is the diagonal in the square (10). Its uniqueness follows from the fact that $m \in \mathcal{M}ono$, and $p \in \mathcal{E}pi$.

4. Since $i \in \mathcal{I}$, and $t \in \mu\mathcal{K}$, it follows that $i \cdot t \in \mathcal{M}$. It remains to prove that $u \in \mathcal{E}$. Let us return to the morphism $p : X \longrightarrow Y$ and see how its $((\mu\mathcal{K})^\top, \mu\mathcal{K})$-factorization is performed. We examine the commutative square

$$p \cdot k^X = k^Y \cdot k(p). \tag{16}$$

On the morphisms k^X and $k(p)$ we construct the pushout square

$$u \cdot k^X = v \cdot k(p). \tag{17}$$

Then

$$p = t \cdot u, \tag{18}$$

$$k^Y = t \cdot v \tag{19}$$

for any morphism t. It is obvious that $t \in \mu\mathcal{K}$, and v is the $\mathcal{K}$-coreplique of the respective object. It is easy to check that $u \perp \mu\mathcal{K}$, i.e. the equality (18) is the $((\mu\mathcal{K})^\top, \mu\mathcal{K})$-factorization of the morphism p. Since (17) is a pushout square, it follows that $u \in \mathcal{E}$. ∎

We denote by $\mathbb{K}(\mathcal{M}_u)$ the class of the $\mathcal{M}_u$-coreflective subcategories. It is clear that

$$\mathbb{K}(\mathcal{M}_u) = \{\mathcal{K} \in \mathbb{K} \mid \widetilde{\mathcal{M}} \in \mathcal{K}\}$$

Instead of the factorization structure $(\mathcal{P}, \mathcal{I})$ we examine the pair $(\mathcal{E}_u, \mathcal{M}_p)$. Since the class $\mathcal{E}_u$ is $\mathcal{M}_u$-hereditary we deduce that it is $\mu\mathcal{K}$-hereditary for any $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$. Further on, let $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{K}(\mathcal{M}_u)$ and $\mathcal{K}_1 \neq \mathcal{K}_2$. Obviously $\mu\mathcal{K}_1 \neq \mu\mathcal{K}_2$ and $\mathcal{M}_p \cdot (\mu\mathcal{K}_1) \neq \mathcal{M}_p \cdot (\mu\mathcal{K}_2)$. Taking into account for $\mathbb{K}(\mathcal{M}_u)$ is a proper class, we get:

Corollary 2.1. *The factorization structures $(\mathcal{M}_p \cdot (\mu\mathcal{K})^{\urcorner}, \mathcal{M}_p \cdot (\mu\mathcal{K}))$, for $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$, form a proper class of factorization structures with the $\mathcal{E}_u$-cohereditary injection classes.*

Consider the case when $(\mathcal{P}, \mathcal{I}) = (\mathcal{E}pi, \mathcal{M}_f)$. Since the class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary (Lemma 2.6 [2]), we get:

Corollary 2.2. ([2], lemma2.6). *The factorization structures $(\mathcal{M}_f \cdot (\mu\mathcal{K})^{\urcorner}, \mathcal{M}_f \cdot (\mu\mathcal{K}))$, for $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$, form a proper class of factorization structure with the $\mathcal{E}pi$-cohereditary injection classes.*

Theorem 2.2. *If $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$, then the class $\mathcal{M}_f \cdot (\mu\mathcal{K})$ consists of those elements of class $\mathcal{M}_p \cdot (\mu\mathcal{K})$ which have a closed image.*

In particular, it results

$$(\mathcal{E}_p, \mathcal{M}_u) = ((\mathcal{M}_p \cdot (\mu\widetilde{\mathcal{M}}))^{\urcorner}, \mathcal{M}_p \cdot (\widetilde{\mu\mathcal{M}})),$$

$$(\mathcal{E}_p^{'}, \mathcal{M}_u^{'}) = ((\mathcal{M}_f \cdot (\mu\widetilde{\mathcal{M}}))^{\urcorner}, \mathcal{M}_f \cdot (\mu\widetilde{\mathcal{M}}))$$

.

Proof. Let $f : X \longrightarrow Y \in \mathcal{M}_p \cdot (\mu\mathcal{K})$ and the morphism f have a closed image. We examined the $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of the morphism $f = i \cdot p$

$$X \xrightarrow{\quad p \quad} Z \xrightarrow{\quad i \quad} Y$$

Since f is a bijective mapping it follows that the image of morphism i is closed. Consequently $i \in \mathcal{M}_f$, $p \in \mu\mathcal{K}$, $i \in \mathcal{M}_f$ and $f = i \cdot p \in \mathcal{M}_f \cdot (\mu\mathcal{K})$.

Conversely. Let $f : X \longrightarrow Y \in \mathcal{M}_f \cdot (\mu\mathcal{K})$. Then the morphism f can be written under the form

$$f = i \cdot b,$$

where $i \in \mathcal{M}_f$, and $b \in \mu\mathcal{K} \subset \mathcal{E}_u$. Therefore b is a bijective mapping. Thus the morphism f has a closed image and $f : X \longrightarrow Y \in \mathcal{M}_p \cdot (\mu\mathcal{K})$. ∎

3. A PARTITION OF THE LATTICE $\mathbb{R}$

Let us fix the factorization structures

$$(\mathcal{E}, \mathcal{M}) = ((\mathcal{M}_f \cdot (\mu\mathcal{K}))^{\urcorner}, \mathcal{M}_f \cdot (\mu\mathcal{K}))$$

where $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$. We examine the partition of the class $\mathbb{R}$ according to the structure sets: $\mathbb{R}(\mathcal{E})$, $\mathbb{R}(\mathcal{M})$, $\mathbb{R}(\mathcal{E}, \mathcal{M})$.

Lemma 3.1. *The full spaces with weak topology subcategory is included in spaces with Mackey topology subcategory:* $\Pi \subset \widetilde{\mathcal{M}}$.

Proof. Every object X of the subcategory Π is isomophic to an object of the from K^{τ}, where $K = \mathbb{R}$ or $K = \mathbb{C}$ ([6] ch 4, section I, Proposition 13). And K is an object of subcategory $\widetilde{\mathcal{M}}$ and this subcategory is closed respect to the products ([7], Theorem 4.3). ∎

Theorem 3.1. *The subcategory* Π *is the smallest element in class* $\mathbb{R}(\mathcal{E})$.

Proof. Let X be an object of category $\mathcal{C}_2\mathcal{V}$, and $\pi^X : X \longrightarrow \pi X$ be the Π-replica of this object. We prove that $\pi^X \in \mathcal{E}$. Since $\Pi \subset \widetilde{\mathcal{M}}$, and $\widetilde{\mathcal{M}} \subset \mathcal{K}$, it follows that $1 : \pi X \longrightarrow \pi X$ is the $\mathcal{K}$-coreplica of this object. Let $k^X : kX \longrightarrow X$ be the $\mathcal{K}$-coreplica of the object X. We have the following commutative square

$$\pi^X \cdot k^X = 1 \cdot k(\pi^X),$$

to be proved to is pushout. It is obvious that $k(\pi^X) = \pi^X \cdot k^X$ and, since, $\pi^X \cdot k^X$ is an epimorphism, it follows that the respective square is pushout.

$$\begin{tikzcd} kX \arrow[r, "k(\pi^X) = \pi^X \cdot k^X"] \arrow[d, "k^X"'] & k\pi X = \pi X \arrow[d, "1"] \\ X \arrow[r, "\pi^X"] & \pi X \end{tikzcd}$$

Let us consider that the class $\mathcal{E}$ is $\mathcal{M}_u$-hereditary. By using the Theorem 1.3 we get that $\mathbb{R}(\mathcal{E}) = \mathbb{R}$. This is possible when $\mathcal{K} = \mathcal{C}_2\mathcal{V}$. In this case we have $\mathcal{M} = \mathcal{M}_f \cdot (\mu\mathcal{C}_2\mathcal{V}) = \mathcal{M}_f \cdot \mathcal{I}so = \mathcal{M}_f$, and $(\mathcal{E}, \mathcal{M}) = (\mathcal{E}pi, \mathcal{M}_f)$. Indeed, the class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary. If $\mathcal{K} \neq \mathcal{C}_2\mathcal{V}$, since $\mathcal{K} \in \mathbb{R}(\mathcal{M})$, it follows that the classes $\mathbb{R}(\mathcal{P})$ and $\mathbb{R}$ do not coincide. ∎

Corollary 3.1. *Let* $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$ *and* $\mathcal{K} \neq \mathcal{C}_2\mathcal{V}$. *Then the factorization structures* $((\mathcal{M}_f \cdot (\mu\mathcal{K}))^{\urcorner}, \mathcal{M}_f \cdot (\mu\mathcal{K}))$ *have:*

1. *the class of injections* $\mathcal{M}_f \cdot (\mu\mathcal{K})$ *is* $\mathcal{E}pi$*-cohereditary,*
2. *the class of projections* $(\mathcal{M}_f \cdot (\mu\mathcal{K}))^{\urcorner}$ *is not* $\mathcal{M}_u$*-hereditary,*
3. *these factorization structures form a proper class.*

Theorem 3.2. *Let* $(\mathcal{K}, \mathcal{L})$ *be a conjugated pair of subcategories in the category* $\mathcal{C}_2\mathcal{V}$, *and*

$$(\mathcal{E}, \mathcal{M}) = ((\mathcal{M}_f \cdot (\mu\mathcal{K}))^{\urcorner}, \mathcal{M}_f \cdot (\mu\mathcal{K})).$$

Then $\mathcal{L}$ *is the smallest element in class* $\mathbb{R}(\mathcal{M})$.

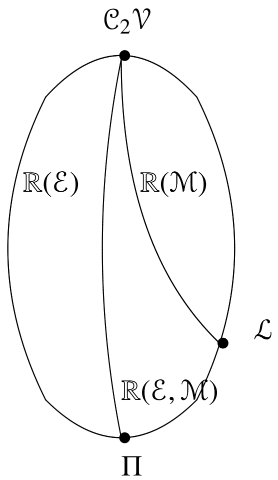
Proof. Since $\mu\mathcal{K} = \varepsilon\mathcal{L}$ it follows that $\mathcal{L} \in \mathbb{R}(\mathcal{M})$. Let $\mathcal{R} \in \mathbb{R}(\mathcal{M})$ and let us prove that $\mathcal{L} \subset \mathcal{R}$. For it let X be any object of the category $\mathcal{C}_2\mathcal{V}$, and $r^X : X \longrightarrow rX$ $\mathcal{R}$ be its replica. Since $r^X \in \mathcal{M} = \mathcal{M}_f \cdot (\mu\mathcal{K})$, it follows that it can be written under the form

$$f^X = i \cdot t,$$

where $i \in \mathcal{M}_f$, and $t \in \mu\mathcal{K}$. Since r^X is an epimorphism, it results that i is an isomorphism. Thus we have shown that $\mathcal{R}$ is $(\varepsilon\mathcal{L})$-reflective. Let $l : X \longrightarrow lX$ be the $\mathcal{L}$-replica of the object X. Then, there exist a morphism u such that

$$l^X = u \cdot r^X.$$

This relationship shows that $\mathcal{L} \subset \mathcal{R}$.



∎

Example 3.1. *Let us examine the dividing of lattice $\mathbb{R}$ from the factorization structure $(\mathcal{E}_p^{'}, \mathcal{M}_u^{'})$ to check the following equalities*

$$(\mathcal{E}_p^{'}, \mathcal{M}_u^{'}) = ((\mathcal{M}_f \cdot (\mu\mathcal{M}))^{\urcorner}, \mathcal{M}_f \cdot (\mu\mathcal{M})) =$$

$$= ((\mathcal{M}_f \cdot (\varepsilon\mathcal{S}))^{\urcorner}, \quad \mathcal{M}_f \cdot (\varepsilon\mathcal{S})) = ((\mathcal{M}_f \cdot (\mathcal{E}_u \cap \mathcal{M}_u))^{\urcorner}, \mathcal{M}_f \cdot (\mathcal{E}_u \cap \mathcal{M}_u)).$$

We recall that a locally convex space is called semireflexive if it is quasicomplet in the weak topology.

Let $q\Gamma_0$ be the subcategory of the quasicomplet spaces and

$$\mathcal{L} = \mathcal{S} \cap q\Gamma_0.$$

We proved that $\mathcal{L} \in \mathbb{R}(\mathcal{E}'_p, \mathcal{M}'_u)$ in the dividing by structure $(\mathcal{E}_p^{'}, \mathcal{M}_u^{'})$.

Let X be a semireflexive space, for which the topology is not weak, and $s^X : X \longrightarrow sX$ is its S-replica. Then sX belong of the subcategory $\mathcal{L}$. Therefore there exists, in the category $\mathcal{C}_2\mathcal{V}$, an object for which $\mathcal{L}$-replica belong of the class $\mathcal{M}_u^{'}$.

Let now X be a locally convex space which is not semireflexive, $s^X : X \longrightarrow sX$ - its S-replica, and $q^X : sX \longrightarrow qsX$ the $q\Gamma_0$-replica of the object sX. Then qsX is a space with weak topology, i.e. $qsX \in |\mathcal{L}|$. It is obvious that q^X is the $\mathcal{L}$-replica of the object sX.

Since the morphism q^X is un epimorphism, then $q^X \perp \mathcal{M}_f$. Further , q^X is a topological inclusion with dense image. We proved that $q^X \perp \varepsilon S$, i.e. $q^X \perp (\mathcal{E}_u \cap \mathcal{M}_u)$. Let $b \in \mathcal{E}_u \cap \mathcal{M}_u$ and

$$b \cdot u = v \cdot q^X \tag{20}$$

$$\begin{tikzcd}
\bullet \arrow[rr, "q^X"] \arrow[dd, "u"'] \arrow[dr, "t"] & & \bullet \arrow[dd, "v"] \\
& \bullet \arrow[ur, dashed, "b'"'] \arrow[dl, dashed, "v'"] & \\
\bullet \arrow[rr, "b"] & & \bullet
\end{tikzcd}$$

We construct, based on morphisms b and v, the pull-back square

$$b \cdot v' = v \cdot b'. \tag{21}$$

Then there exists a morphism t such that

$$u = v' \cdot t, \tag{22}$$

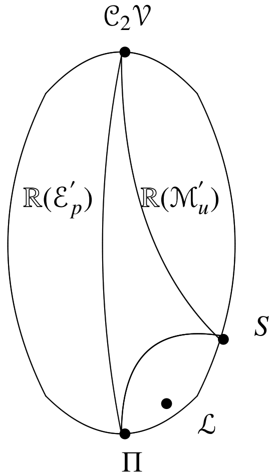
$$q^X = b' \cdot t. \tag{23}$$

and $b' \in \mathcal{E}_u \cap \mathcal{M}_u$. Since $b' \in \mathcal{M}_u$, $q^X \in \mathcal{E}pi$ and the class $\mathcal{E}pi$ is $\mathcal{E}_u$-hereditary ([2] Lemma 2.6) it follows that $t \in \mathcal{E}pi$. Therefore $q^X \in \mathcal{M}_p$, $t \in \mathcal{E}pi$, and the class $\mathcal{M}_p$ is $\mathcal{E}pi$-cohereditary, so that $b' \in \mathcal{M}_p$, and $b' \in \mathcal{E}_u \cap \mathcal{M}_p = \mathcal{I}so$.

The diagonal of square (20) is the morphism $v' \cdot (b')^{-1}$.

Thus we have proved that in the category $\mathcal{C}_2\mathcal{V}$ there exists the object $\mathcal{L}$-replica which belongs to the class $\mathcal{M}'_u$ and not belongs to class $\mathcal{E}'_p$ and there exists the object $\mathcal{L}$-replica which belongs to class $\mathcal{E}'_p$ and not belongs to class $\mathcal{M}'_u$.

Consequently $\mathcal{L} \in \mathbb{R}(\mathcal{E}'_p, \mathcal{M}'_u)$.



By the Theorem 5.4 [3] the subcategory $s\mathcal{R}$ of the semireflexive spaces is equal with the right product of the subcategory $\widetilde{\mathcal{M}}$ and subcategory $\mathcal{L} = \mathcal{S} \cap q\Gamma_0$ and also equal with the semireflexive product of subcategory $\mathcal{S}$ and subcategory $q\Gamma_0$:

$$s\mathcal{R} = \widetilde{\mathcal{M}} *_d (\mathcal{S} \cap q\Gamma_0) = S *_{sr} q\Gamma_0.$$

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MAPPING PROPERTIES OF SOME SUBCLASSES OF ANALYTIC FUNCTIONS UNDER GENERAL INTEGRAL OPERATORS

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Abstract Considering certain subclasses of analytic functions, we study the mapping properties of these classes under two general integral operators.

Keywords: analytic function, spirallike function, integral operator, Al-Oboudi differential operator.
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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

The following definition of fractional derivative given by Owa [17] (also by Srivastava and Owa [22]) will be required in our investigation.

The fractional derivative of order γ is defined, for a function f, by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\gamma}} d\xi \quad (0 \le \gamma < 1), \tag{2}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

It readily follows from (2) that

$$D_z^{\gamma} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \le \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^\gamma f$, Owa and Srivastava [18] introduced the operator $\Omega^\gamma : \mathcal{A} \to \mathcal{A}$, which is known as an extension of the fractional derivative and fractional integral, as follows:

$$\Omega^\gamma f(z) = \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k. \tag{3}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [3], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator (namely, generalized Al-Oboudi differential operator) $D_\lambda^{n,\gamma}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\lambda^{1,\gamma} f(z) &= (1-\lambda)\Omega^\gamma f(z) + \lambda z (\Omega^\gamma f(z))' := D_\lambda^\gamma (f(z)), \quad \lambda \geq 0,\ 0 \leq \gamma < 1, \end{aligned} \tag{4}$$

$$\begin{aligned} D_\lambda^{2,\gamma} f(z) &= D_\lambda^\gamma \left(D_\lambda^{1,\gamma} f(z)\right), \\ &\vdots \\ D_\lambda^{n,\gamma} f(z) &= D_\lambda^\gamma \left(D_\lambda^{n-1,\gamma} f(z)\right), \quad n \in \mathbb{N}. \end{aligned} \tag{5}$$

If f is given by (1), then by (3), (4) and (5), we see that

$$D_\lambda^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma,\lambda)\, a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6}$$

where

$$\Psi_{k,n}(\gamma,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} (1+(k-1)\lambda)\right]^n.$$

Remark 1.1. (**i**) *When $\gamma = 0$, we get Al-Oboudi differential operator [2].*

(**ii**) *When $\gamma = 0$ and $\lambda = 1$, we get Sălăgean differential operator [21].*

(**iii**) *When $n = 1$ and $\lambda = 0$, we get Owa-Srivastava fractional differential operator [18].*

In [9], the author defined the classes $\mathcal{S}_{\gamma,\lambda}^n(\beta, b)$ and $\mathcal{K}_{\gamma,\lambda}^n(\beta, b)$ as follows:

Definition 1.1. *Let $\mathcal{S}_{\gamma,\lambda}^n(\beta, b)$ be the class of functions $f \in \mathcal{A}$ satisfying*

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z\left(D_\lambda^{n,\gamma} f(z)\right)'}{D_\lambda^{n,\gamma} f(z)} - 1\right)\right\} > \beta \tag{7}$$

for all $z \in \mathbb{U}$, where $b \in \mathbb{C} - \{0\}$ and $0 \leq \beta < 1$.

Definition 1.2. *Let $\mathcal{K}_{\gamma,\lambda}^n(\beta, b)$ be the class of functions $f \in \mathcal{A}$ satisfying*

$$\Re\left\{1 + \frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma} f(z)\right)''}{\left(D_\lambda^{n,\gamma} f(z)\right)'}\right\} > \beta \tag{8}$$

for all $z \in \mathbb{U}$*, where* $b \in \mathbb{C} - \{0\}$ *and* $0 \leq \beta < 1$.

We note that $f \in \mathcal{K}^n_{\gamma,\lambda}(\beta, b)$ if and only if $zf' \in \mathcal{S}^n_{\gamma,\lambda}(\beta, b)$.

Remark 1.2. *We have the following classes for special values of the parameters* $n, \gamma, \lambda, \beta$ *and* b.

(i) $\mathcal{S}^0_{\gamma,\lambda}(\beta, b) \equiv \mathcal{S}^1_{0,0}(\beta, b) \equiv \mathcal{S}^*_\beta(b)$ *and* $\mathcal{K}^0_{\gamma,\lambda}(\beta, b) \equiv \mathcal{K}^1_{0,0}(\beta, b) \equiv \mathcal{C}_\beta(b)$*. These classes were introduced by Frasin [13],*

(ii) $\mathcal{S}^0_{\gamma,\lambda}(\beta, 1) \equiv \mathcal{S}^1_{0,0}(\beta, 1) \equiv \mathcal{S}^*(\beta)$ *and* $\mathcal{K}^0_{\gamma,\lambda}(\beta, 1) \equiv \mathcal{K}^1_{0,0}(\beta, 1) \equiv \mathcal{K}(\beta)$ *which are the classes of starlike functions of order* β *and convex functions of order* β *in* $\mathbb{U}$*, respectively,*

(iii) $\mathcal{S}^0_{\gamma,\lambda}(0, 1) \equiv \mathcal{S}^1_{0,0}(0, 1) \equiv \mathcal{S}^*$ *and* $\mathcal{K}^0_{\gamma,\lambda}(0, 1) \equiv \mathcal{K}^1_{0,0}(0, 1) \equiv \mathcal{K}$ *which are familiar classes of starlike and convex functions in* $\mathbb{U}$*, respectively,*

(iv) $\mathcal{S}^n_{0,1}(\beta, 1) \equiv \mathcal{S}_n(\beta)$ *which is the class of n-starlike functions of order* β *defined by Sălăgean [21].*

Observe that if $f \in \mathcal{S}^n_{\gamma,\lambda}(\beta, b)$ (or $\mathcal{K}^n_{\gamma,\lambda}(\beta, b)$), then $D^{n,\gamma}_\lambda f \in \mathcal{S}^*_\beta(b)$ (or $\mathcal{C}_\beta(b)$).

Now we define new classes of functions by using the generalized Al-Oboudi differential operator $D^{n,\gamma}_\lambda$ as follows:

Definition 1.3. *Let* $\mathcal{S}^n_{\gamma,\lambda}(\alpha; \beta, b)$ *be the class of functions* $f \in \mathcal{A}$ *satisfying*

$$\Re\left\{e^{i\alpha}\left[1 + \frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)} - 1\right)\right]\right\} > \beta\cos\alpha \tag{9}$$

for a real number α $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$ *and for all* $z \in \mathbb{U}$*, where* $b \in \mathbb{C} - \{0\}$ *and* $0 \leq \beta < 1$.

Definition 1.4. *Let* $\mathcal{K}^n_{\gamma,\lambda}(\alpha; \beta, b)$ *be the class of functions* $f \in \mathcal{A}$ *satisfying*

$$\Re\left\{e^{i\alpha}\left(1 + \frac{1}{b}\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)''}{\left(D^{n,\gamma}_\lambda f(z)\right)'}\right)\right\} > \beta\cos\alpha \tag{10}$$

for a real number α $(-\frac{\pi}{2} < \alpha < \frac{\pi}{2})$ *and for all* $z \in \mathbb{U}$*, where* $b \in \mathbb{C} - \{0\}$ *and* $0 \leq \beta < 1$.

It is clear that $f \in \mathcal{K}^n_{\gamma,\lambda}(\alpha; \beta, b)$ if and only if $zf' \in \mathcal{S}^n_{\gamma,\lambda}(\alpha; \beta, b)$.

Remark 1.3. *We note that*

(i) $\mathcal{S}^n_{\gamma,\lambda}(0; \beta, b) \equiv \mathcal{S}^n_{\gamma,\lambda}(\beta, b)$ *and* $\mathcal{K}^n_{\gamma,\lambda}(0; \beta, b) \equiv \mathcal{K}^n_{\gamma,\lambda}(\beta, b)$*,*

(ii) $\mathcal{S}^0_{\gamma,\lambda}(0; \beta, b) \equiv \mathcal{S}^1_{0,0}(0; \beta, b) \equiv \mathcal{S}^*_\beta(b)$ *and* $\mathcal{K}^0_{\gamma,\lambda}(0; \beta, b) \equiv \mathcal{K}^1_{0,0}(0; \beta, b) \equiv \mathcal{C}_\beta(b)$ *studied in [5] and [12],*

(iii) $\mathcal{S}^0_{\gamma,\lambda}(0; \beta, 1) \equiv \mathcal{S}^1_{0,0}(0; \beta, 1) \equiv \mathcal{S}^*(\beta)$ *and* $\mathcal{K}^0_{\gamma,\lambda}(0; \beta, 1) \equiv \mathcal{K}^1_{0,0}(0; \beta, 1) \equiv \mathcal{K}(\beta)$*,*

(iv) $\mathcal{S}^0_{\gamma,\lambda}(0; 0, 1) \equiv \mathcal{S}^1_{0,0}(0; 0, 1) \equiv \mathcal{S}^*$ *and* $\mathcal{K}^0_{\gamma,\lambda}(0; 0, 1) \equiv \mathcal{K}^1_{0,0}(0; 0, 1) \equiv \mathcal{K}$*,*

(v) $\mathcal{S}^0_{\gamma,\lambda}(\alpha; 0, 1) \equiv \mathcal{S}^1_{0,0}(\alpha; 0, 1) \equiv \mathcal{S}_p(\alpha)$ *which is the class of* α*-spirallike functions recently studied in [14] and [15].*

From the general classes $\mathcal{S}^{n}_{\gamma,\lambda}(\alpha;\beta,b)$ and $\mathcal{K}^{n}_{\gamma,\lambda}(\alpha;\beta,b)$, we obtain the following new classes for special parameters n, γ, λ such as

1. $\mathcal{S}^{n}_{0,\lambda}(\alpha;\beta,b) \equiv \mathcal{S}^{n}_{\lambda}(\alpha;\beta,b)$ and $\mathcal{K}^{n}_{0,\lambda}(\alpha;\beta,b) \equiv \mathcal{K}^{n}_{\lambda}(\alpha;\beta,b)$,

2. $\mathcal{S}^{n}_{0,1}(\alpha;\beta,b) \equiv \mathcal{S}^{n}(\alpha;\beta,b)$ and $\mathcal{K}^{n}_{0,1}(\alpha;\beta,b) \equiv \mathcal{K}^{n}(\alpha;\beta,b)$,

3. $\mathcal{S}^{0}_{\gamma,\lambda}(\alpha;\beta,b) \equiv \mathcal{S}^{1}_{0,0}(\alpha;\beta,b) \equiv \mathcal{S}_{p}(\alpha;\beta,b)$ and $\mathcal{K}^{0}_{\gamma,\lambda}(\alpha;\beta,b) \equiv \mathcal{K}^{1}_{0,0}(\alpha;\beta,b) \equiv \mathcal{K}_{p}(\alpha;\beta,b)$.

By using the generalized Al-Oboudi differential operator, the author introduced the following integral operators $D^{n,\gamma}_{\lambda}F$ and $D^{n,\gamma}_{\lambda}G$, [8].

Definition 1.5. *Let $n \in \mathbb{N}_0$, $l = (l_1, \ldots, l_m) \in \mathbb{N}^m_0$, and $k_j > 0$ $(1 \leq j \leq m)$. One defines the integral operator $I_{n,m,l,k} : \mathcal{A}^m \to \mathcal{A}$,*

$$I_{n,m,l,k}(f_1, \ldots, f_m) = F, \tag{11}$$

such that

$$D^{n,\gamma}_{\lambda}F(z) = \int_0^z \left(\frac{D^{l_1,\gamma}_{\lambda} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^{l_m,\gamma}_{\lambda} f_m(t)}{t}\right)^{k_m} dt \quad (z \in \mathbb{U}),$$

where $f_1, \ldots, f_m \in \mathcal{A}$ and D is the generalized Al-Oboudi differential operator.

Remark 1.4. *The integral operator $D^{n,\gamma}_{\lambda}F$ generalizes many operators which were introduced and studied recently.*

(**i**) *For $\gamma = 0$, we get the integral operator*

$$D^{n}_{\lambda}F(z) = \int_0^z \left(\frac{D^{l_1}_{\lambda} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^{l_m}_{\lambda} f_m(t)}{t}\right)^{k_m} dt$$

introduced by Bulut [7]; here D is the Al-Oboudi differential operator.

(**ii**) *For $n = 0$, $\gamma = 0$ and $l_1 = \ldots = l_m = l \in \mathbb{N}_0$, we get the integral operator*

$$F(z) = \int_0^z \left(\frac{D^{l}_{\lambda} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^{l}_{\lambda} f_m(t)}{t}\right)^{k_m} dt$$

introduced by Bulut [11]; here D is the Al-Oboudi differential operator.

(**iii**) *For $\gamma = 0$ and $\lambda = 1$, we get the integral operator*

$$D^{n}F(z) = \int_0^z \left(\frac{D^{l_1} f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{D^{l_m} f_m(t)}{t}\right)^{k_m} dt$$

introduced by Breaz et al. [6]; here D is the Sălăgean differential operator.

(**iv**) *For $n = 0$ and $D^{0,\gamma}_{\lambda} f_j = D^{1,0}_{0} f_j = f_j$ $(j \in \{1, \ldots, m\})$, we get the integral operator*

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{k_1} \cdots \left(\frac{f_m(t)}{t}\right)^{k_m} dt$$

introduced by D. Breaz and N. Breaz [4].

(**v**) *For $n = 0$, $m = 1$, $k_1 = k \in [0, 1]$, $k_2 = \cdots = k_m = 0$ and $D_{\lambda}^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{S}$, we get the integral operator*

$$F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^k dt$$

studied by Miller et al. [16].

(**vi**) *For $n = 0$, $m = 1$, $k_1 = 1$, $k_2 = \cdots = k_m = 0$ and $D_{\lambda}^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{A}$, we get the integral operator of Alexander*

$$F(z) = \int_0^z \frac{f(t)}{t} dt$$

introduced by Alexander [1].

Definition 1.6. *Let $n \in \mathbb{N}_0$, $l = (l_1, \ldots, l_m) \in \mathbb{N}_0^m$, and $k_j > 0$ $(1 \leq j \leq m)$. One defines the integral operator $J_{n,m,l,k} : \mathcal{A}^m \to \mathcal{A}$,*

$$J_{n,m,l,k}(f_1, \ldots, f_m) = G, \tag{12}$$

such that

$$D_{\lambda}^{n,\gamma} G(z) = \int_0^z \left[\left(D_{\lambda}^{l_1,\gamma} f_1(t)\right)'\right]^{k_1} \ldots \left[\left(D_{\lambda}^{l_m,\gamma} f_m(t)\right)'\right]^{k_m} dt \quad (z \in \mathbb{U}),$$

where $f_1, \ldots, f_m \in \mathcal{A}$ and D is the generalized Al-Oboudi differential operator.

Remark 1.5. *The integral operator $D_{\delta}^{n,\gamma} G$ many operators which were introduced and studied recently.*

(**i**) *For $n = 0$ and $D_{\delta}^{0,\gamma} f_j = D_0^{1,0} f_j = f_j \in \mathcal{A}$ $(1 \leq j \leq m)$, we have the integral operator*

$$G_{k_1,\ldots,k_m}(z) = \int_0^z \left(f_1'(t)\right)^{k_1} \ldots (f_m'(t))^{k_m} dt \tag{13}$$

introduced by Breaz et al. [7].

(**ii**) *For $n = 0$, $m = 1$, $k_1 = k \in \mathbb{C}$, $k_2 = \cdots = k_m = 0$ and $D_{\delta}^{0,\gamma} f_1 = D_0^{1,0} f_1 = f \in \mathcal{A}$, we have the integral operator*

$$G_k(z) = \int_0^z (f'(t))^k dt$$

introduced by Pfaltzgraff [20] (see also Pascu and Pescar [19]).

In this paper, we investigate some properties of the above integral operators $D_{\delta}^{n,\gamma} F$ and $D_{\delta}^{n,\gamma} G$ for the classes

$$\mathcal{S}_{\gamma,\lambda}^n(\alpha;\beta,b) \qquad \text{and} \qquad \mathcal{K}_{\gamma,\lambda}^n(\alpha;\beta,b)\,.$$

2. MAIN RESULTS

Theorem 2.1. *Let $l_j \in \mathbb{N}_0, 0 \le \beta_j < 1, k_j > 0 \;\; (1 \le j \le m)$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, b \in \mathbb{C} - \{0\}$. Also suppose that*

$$0 \le 1 + \sum_{j=1}^{m} k_j(\beta_j - 1) < 1. \tag{14}$$

If $f_j \in \mathcal{S}_{\gamma,\lambda}^{l_j}\left(\alpha;\beta_j,b\right) \;\; (1 \le j \le m)$, then $I_{n,m,l,k}(f_1,\ldots,f_m) = F \in \mathcal{K}_{\gamma,\lambda}^{n}(\alpha;\delta,b)$ with

$$\delta = 1 + \sum_{j=1}^{m} k_j(\beta_j - 1). \tag{15}$$

Proof. Since $f_j \in \mathcal{A} \;\; (1 \le j \le m)$, by (6), we have

$$\frac{D_\lambda^{l_j,\gamma} f_j(z)}{z} = 1 + \sum_{k=2}^{\infty} \Psi_{k,l_j}(\gamma,\lambda)\, a_{k,j} z^{k-1}$$

and

$$\frac{D_\lambda^{l_j,\gamma} f_j(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$. By (11), we get

$$\left(D_\lambda^{n,\gamma} F(z)\right)' = \left(\frac{D_\lambda^{l_1,\gamma} f_1(z)}{z}\right)^{k_1} \cdots \left(\frac{D_\lambda^{l_m,\gamma} f_m(z)}{z}\right)^{k_m}.$$

This equality implies that

$$\ln\left(D_\lambda^{n,\gamma} F(z)\right)' = k_1 \ln \frac{D_\lambda^{l_1,\gamma} f_1(z)}{z} + \cdots + k_m \ln \frac{D_\lambda^{l_m,\gamma} f_m(z)}{z}$$

or equivalently

$$\ln\left(D_\lambda^{n,\gamma} F(z)\right)' = k_1\left[\ln D_\lambda^{l_1,\gamma} f_1(z) - \ln z\right] + \cdots + k_m\left[\ln D_\lambda^{l_m,\gamma} f_m(z) - \ln z\right].$$

By differentiating above equality, we get

$$\frac{\left(D_\lambda^{n,\gamma} F(z)\right)''}{\left(D_\lambda^{n,\gamma} F(z)\right)'} = \sum_{j=1}^{m} k_j \left[\frac{\left(D_\lambda^{l_j,\gamma} f_j(z)\right)'}{D_\lambda^{l_j,\gamma} f_j(z)} - \frac{1}{z}\right].$$

Then by multiplying the above relation with z/b, we have

$$\begin{aligned}\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}F(z)\right)''}{\left(D_\lambda^{n,\gamma}F(z)\right)'} &= \sum_{j=1}^{m}k_j\frac{1}{b}\left(\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}{D_\lambda^{l_j,\gamma}f_j(z)}-1\right)\\ &= \sum_{j=1}^{m}k_j\left[1+\frac{1}{b}\left(\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}{D_\lambda^{l_j,\gamma}f_j(z)}-1\right)\right]-\sum_{j=1}^{m}k_j\end{aligned}$$

or equivalently

$$e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}F(z)\right)''}{\left(D_\lambda^{n,\gamma}F(z)\right)'}\right)=\left(1-\sum_{j=1}^{m}k_j\right)e^{i\alpha}+\sum_{j=1}^{m}k_je^{i\alpha}\left[1+\frac{1}{b}\left(\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}{D_\lambda^{l_j,\gamma}f_j(z)}-1\right)\right].$$

Since $f_j\in\mathcal{S}_{\gamma,\lambda}^{l_j}\left(\alpha;\beta_j,b\right)$ $(1\le j\le m)$, we get

$$\begin{aligned}\Re\left\{e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}F(z)\right)''}{\left(D_\lambda^{n,\gamma}F(z)\right)'}\right)\right\} &= \left(1-\sum_{j=1}^{m}k_j\right)\Re\left\{e^{i\alpha}\right\}\\ &\quad+\sum_{j=1}^{m}k_j\Re\left\{e^{i\alpha}\left[1+\frac{1}{b}\left(\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}{D_\lambda^{l_j,\gamma}f_j(z)}-1\right)\right]\right\}\\ &> \left(1-\sum_{j=1}^{m}k_j\right)\cos\alpha+\sum_{j=1}^{m}k_j\beta_j\cos\alpha\\ &= \left(1+\sum_{j=1}^{m}k_j(\beta_j-1)\right)\cos\alpha\end{aligned}$$

for all $z\in\mathbb{U}$. Hence, we obtain $F\in\mathcal{K}_{\gamma,\lambda}^{n}(\alpha;\delta,b)$ with δ given by (2). ∎

By setting $\beta_1=\cdots=\beta_m=\beta$ in Theorem 2.1, we obtain the following.

Corollary 2.1. *Let* $l_j\in\mathbb{N}_0, 0\le\beta<1, k_j>0\ (1\le j\le m)$, *and* $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}, b\in\mathbb{C}-\{0\}$. *Also suppose that*

$$0\le 1+(\beta-1)\sum_{j=1}^{m}k_j<1.$$

If $f_j\in\mathcal{S}_{\gamma,\lambda}^{l_j}(\alpha;\beta,b)$ $(1\le j\le m)$, *then* $I_{n,m,l,k}(f_1,\ldots,f_m)=F\in\mathcal{K}_{\gamma,\lambda}^{n}(\alpha;\mu,b)$ *with*

$$\mu=1+(\beta-1)\sum_{j=1}^{m}k_j. \tag{16}$$

Putting $m = 1$ and $k_1 = k$ in Corollary 2.1, we obtain the following.

Corollary 2.2. *Let $l \in \mathbb{N}_0, 0 \leq \beta < 1, k > 0$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, b \in \mathbb{C} - \{0\}$. Also suppose that*

$$0 \leq 1 + (\beta - 1)k < 1.$$

If $f \in \mathcal{S}^l_{\gamma,\lambda}(\alpha;\beta, b)$, then

$$D^{n,\gamma}_\lambda F(z) = \int_0^z \left(\frac{D^{l,\gamma}_\lambda f(t)}{t} \right)^k dt \in \mathcal{K}^n_{\gamma,\lambda}(\alpha;\rho, b)$$

with

$$\rho = 1 + (\beta - 1)k. \tag{17}$$

Putting $k = 1$ in Corollary 2.2, we obtain the following.

Corollary 2.3. *Let $l \in \mathbb{N}_0, 0 \leq \beta < 1$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, b \in \mathbb{C} - \{0\}$. If $f \in \mathcal{S}^l_{\gamma,\lambda}(\alpha;\beta, b)$, then the integral operator*

$$D^{n,\gamma}_\lambda F(z) = \int_0^z \frac{D^{l,\gamma}_\lambda f(t)}{t} dt \in \mathcal{K}^n_{\gamma,\lambda}(\alpha;\beta, b).$$

Corollary 2.4. *Let $f \in \mathcal{S}_p(\alpha;\beta, b)$. Then the image of f by the integral operator of Alexander belongs to $\mathcal{K}_p(\alpha;\beta, b)$, i.e.*

$$F(z) = \int_0^z \frac{f(t)}{t} dt \in \mathcal{K}_p(\alpha;\beta, b).$$

Theorem 2.2. *Let $l_j \in \mathbb{N}_0, 0 \leq \beta_j < 1, k_j > 0$ $(1 \leq j \leq m)$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, b \in \mathbb{C} - \{0\}$. Also suppose that*

$$0 \leq 1 + \sum_{j=1}^m k_j(\beta_j - 1) < 1.$$

If $f_j \in \mathcal{K}^{l_j}_{\gamma,\lambda}\left(\alpha;\beta_j, b\right)$ $(1 \leq j \leq m)$, then $J_{n,m,l,k}(f_1, \ldots, f_m) = G \in \mathcal{K}^n_{\gamma,\lambda}(\alpha;\delta, b)$ with δ is given by (2).

Proof. By (12), we get

$$\left(D^{n,\gamma}_\lambda G(z)\right)' = \left[\left(D^{l_1,\gamma}_\lambda f_1(z)\right)'\right]^{k_1} \ldots \left[\left(D^{l_m,\gamma}_\lambda f_1(z)\right)'\right]^{k_m}. \tag{18}$$

This equality implies that

$$\left(D^{n,\gamma}_\lambda G(z)\right)'' = \sum_{j=1}^m k_j \left[\left(D^{l_j,\gamma}_\lambda f_j(z)\right)'\right]^{k_j} \frac{\left(D^{l_j,\gamma}_\lambda f_j(z)\right)''}{\left(D^{l_j,\gamma}_\lambda f_j(z)\right)'} \prod_{\substack{r=1 \\ (r\neq j)}}^m \left[\left(D^{l_r,\gamma}_\lambda f_r(z)\right)'\right]^{k_r}. \tag{19}$$

Thus by using (5) and (6), we obtain

$$\frac{z\left(D_\lambda^{n,\gamma}G(z)\right)''}{\left(D_\lambda^{n,\gamma}G(z)\right)'} = \sum_{j=1}^{m} k_j \frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)''}{\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}.$$

Then by multiplying the above relation with $1/b$, we have

$$\begin{aligned}\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}G(z)\right)''}{\left(D_\lambda^{n,\gamma}G(z)\right)'} &= \sum_{j=1}^{m} k_j \frac{1}{b}\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)''}{\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}\\ &= \sum_{j=1}^{m} k_j \left(1+\frac{1}{b}\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)''}{\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}\right) - \sum_{j=1}^{n} k_j\end{aligned}$$

or equivalently

$$e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}G(z)\right)''}{\left(D_\lambda^{n,\gamma}G(z)\right)'}\right) = \left(1-\sum_{j=1}^{m} k_j\right)e^{i\alpha} + \sum_{j=1}^{m} k_j e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)''}{\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}\right).$$

Since $f_j \in \mathcal{K}_{\gamma,\lambda}^{l_j}\left(\alpha;\beta_j,b\right)$ $(1 \le j \le m)$, we get

$$\begin{aligned}\Re\left\{e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{n,\gamma}G(z)\right)''}{\left(D_\lambda^{n,\gamma}G(z)\right)'}\right)\right\} &= \left(1-\sum_{j=1}^{m} k_j\right)\Re\left\{e^{i\alpha}\right\}\\ &\quad + \sum_{j=1}^{m} k_j \Re\left\{e^{i\alpha}\left(1+\frac{1}{b}\frac{z\left(D_\lambda^{l_j,\gamma}f_j(z)\right)''}{\left(D_\lambda^{l_j,\gamma}f_j(z)\right)'}\right)\right\}\\ &> \left(1-\sum_{j=1}^{m} k_j\right)\cos\alpha + \sum_{j=1}^{m} k_j\beta_j\cos\alpha\\ &= \left(1+\sum_{j=1}^{m} k_j(\beta_j-1)\right)\cos\alpha\end{aligned}$$

for all $z \in \mathbb{U}$. Hence, we obtain $G \in \mathcal{K}_{\gamma,\lambda}^{n}(\alpha;\delta,b)$ with δ is given by (2). ∎

Corollary 2.5. *Let* $l_j = 0, 0 \le \beta_j < 1, k_j > 0$ $(1 \le j \le m)$, *and* $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, b \in \mathbb{C} - \{0\}$. *Also suppose that*

$$0 \le 1 + \sum_{j=1}^{m} k_j(\beta_j - 1) < 1.$$

If $f_j \in \mathcal{K}_p\left(\alpha;\beta_j,b\right)$ $(1 \leq j \leq m)$*, then* $G_{k_1,\ldots,k_m} \in \mathcal{K}_p(\alpha;\delta,b)$ *with* δ *is given by* (2).

Other interesting corollaries of Theorem 2.2 can be obtained considering the same particular cases as for Corollaries 2.1 – 2.4.

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SOME REMARKS ON THE POMPEIU-HAUSDORFF DISTANCE BETWEEN ORDER INTERVALS

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Abstract In this paper we obtain some estimations for the Pompeiu-Hausdorff distance between order intervals of a normed lattice and of an ordered normed space.

Keywords: Pompeiu-Hausdorff distance, normed lattices, ordered normed spaces, order intervals.
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1. INTRODUCTION

The distance between the subsets of $\mathbb{R}^n$ was introduced by the Romanian mathematician Dimitrie Pompeiu in 1905 in his Ph. D. Thesis [10] (see also [3]). Is worth noting that the concept of metric space appeared in the Ph. D. thesis of Maurice Fréchet [6] published a little bit later in 1906. Felix Hausdorff studied the notion of set distance in the setting of metric spaces in his book from 1914 [7]. Hausdorff quoted Pompeiu as the author of the notion of distance between sets, but not in the main text of the book, only in the final notes, as was the procedure at that time to indicate the references ([8], p. 343). And thus the distance between sets came to be called the Hausdorff distance. We will call this distance Pompeiu-Hausdorff distance like in the book of R. Tyrrell Rockafellar and Roger J.-B. Wets "Variational Analysis" ([12], p. 144).

In this section we recall some basic definitions and results about Pompeiu-Hausdorff distance. First we recall the notion of quasi-metric. A function $q : X \times X \longrightarrow \mathbb{R}_+$ is said to be a *quasi-metric* on the set X if for each $x, y, z \in X$, q satisfies the following conditions:

(1) $q(x, x) = 0$;

(2) $q(x, y) = q(y, x) = 0 \Rightarrow x = y$;

(3) $q(x, y) \leq q(x, z) + q(z, y)$.

For every quasi-metric q it is always possible to define another quasi-metric, called the *conjugated quasi-metric* of q, defined by $\tilde{q}(x, y) = q(y, x)$, and a metric, called the *metric associated* to q, defined by

$$d_q(x, y) = \max\{q(x, y), \tilde{q}(x, y)\}, \quad x, y \in X. \tag{1}$$

The associated metric d_q is the smallest metric majorizing q.

Let (X, d) be a metric space and $\mathcal{P}_{c,b}(X)$ the collection of all nonempty subsets A of X which are closed and bounded. The function $q : \mathcal{P}_{c,b}(X) \times \mathcal{P}_{c,b}(X) \longrightarrow \mathbb{R}_+$ defined by

$$q(A, B) = \sup_{a \in A} \operatorname{dist}(a, B), \tag{2}$$

where $\operatorname{dist}(a, B) = \inf_{b \in B} d(a, b)$, is a quasi-metric on the set $\mathcal{P}_{c,b}(X)$. The requirement that the subsets A and B are closed is necessary in order to obtain the equality $A = B$ and not $\overline{A} = \overline{B}$. The requirement that A and B are bounded is necessary to obtain $q(A, B) < \infty$. It is worth to remark that using the quasi-metric (2) we can characterize the order relation given by the inclusion between the sets of $\mathcal{P}_{c,b}(X)$ as follows

$$A \subset B \Leftrightarrow q(A, B) = 0.$$

The metric associated by formula (1) to the quasi-metric q defined in (2), that is,

$$d_{PH}(A, B) = \max\{q(A, B), q(B, A)\}, \tag{3}$$

where $A, B \in \mathcal{P}_{c,b}(X)$, is called the *Pompeiu-Hausdorff distance*, PH-distance for short, between the sets A and B relative to the metric d. (By convention, $d_{PH}(\emptyset, \emptyset) = 0$ and $d_{PH}(A, \emptyset) = \infty$ for $A \neq \emptyset$.) It is well known that d_{PH} is a metric on $\mathcal{P}_{c,b}(X)$ ([1], Lemma 3.72, p.110).

For a nonempty subset A of a metric space (X, d) we denote by $N_\varepsilon[A] = \{x \in X \mid d(x, A) \leq \varepsilon\}$ the closed ε-neighborhood of A. A useful formula for computing the quasi-metric (2) is

$$q(A, B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon[B]\}, \tag{4}$$

where, by convention, $\inf \emptyset = +\infty$. Then for the Pompeiu-Hausdorff distance we have the following formula ([1], Lemma 3.71, p.110),

$$d_{PH}(A, B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$

In general, the computation of the PH-distance between sets is not an easy operation. The aim of this paper is to obtain some useful estimations for PH-distance in the particular case in which the sets are order intervals of a normed lattice or of an ordered normed space.

For the unexplained terminology, especially for ordered normed spaces, see [1] and [2].

2. POMPEIU-HAUSDORFF DISTANCE BETWEEN ORDER INTERVALS OF A NORMED LATTICE

In this section X is a normed lattice. This means that X is a normed vector lattice endowed with a lattice norm, that is, a norm with the property that $x, y \in X$, $|x| \leq |y|$

implies $\|x\| \leq \|y\|$. We denote an order interval of X by $\overline{\underline{x}} = [\underline{x}, \overline{x}] = \{a \in X \mid \underline{x} \leq a \leq \overline{x}\}$ and the set of all order intervals of X by $\mathbb{I}X$. Since in a normed lattice X the order intervals are closed and bounded subsets of X, the quasi-metric (2) restricted to the set $\mathbb{I}X$, that is, $q : \mathbb{I}X \times \mathbb{I}X \to \mathbb{R}_+$, is given by the formula

$$q(\overline{\underline{x}}, \overline{\underline{y}}) = \sup_{a \in \overline{\underline{x}}} \operatorname{dist}(a, \overline{\underline{y}}) = \sup_{a \in \overline{\underline{x}}} \inf_{b \in \overline{\underline{y}}} \|b - a\|. \tag{5}$$

The first result gives a double estimation for the quasi-metric q. It is the main result of this section because all others are obtained from it.

Proposition 2.1. *For any order intervals $\overline{\underline{x}} = [\underline{x}, \overline{x}]$ and $\overline{\underline{y}} = [\underline{y}, \overline{y}]$ of the normed lattice X we have*

$$\max\{\left\|(\underline{x} - \underline{y})^-\right\|, \left\|(\overline{x} - \overline{y})^+\right\|\} \leq q(\overline{\underline{x}}, \overline{\underline{y}}) \leq \left\|(\underline{x} - \underline{y})^- \vee (\overline{x} - \overline{y})^+\right\|.$$

Proof. Let a be any element in $\overline{\underline{x}}$. Define $b_a = (\overline{y} \wedge a) \vee \underline{y} = \overline{y} \wedge (a \vee \underline{y})$. Then $b_a \in \overline{\underline{y}}$ and

$$\begin{aligned} a - b_a &= a - (\overline{y} \wedge a) \vee \underline{y} = a + [-(\overline{y} \wedge a)] \wedge (-\underline{y}) = \\ &= [a + (-\overline{y}) \vee (-a)] \wedge (a - \underline{y}) = \\ &= [(a - \overline{y}) \vee 0] \wedge (a - \underline{y}) = (a - \overline{y})^+ \wedge (a - \underline{y}). \end{aligned}$$

Consequentely

$$-(\underline{x} - \underline{y})^- = 0 \wedge (\underline{x} - \underline{y}) \leq a - b_a \leq (a - \overline{y})^+ \leq (\overline{x} - \overline{y})^+$$

and

$$|a - b_a| \leq (\underline{x} - \underline{y})^- \vee (\overline{x} - \overline{y})^+.$$

Then we have

$$q(\overline{\underline{x}}, \overline{\underline{y}}) = \sup_{a \in \overline{\underline{x}}} \inf_{b \in \overline{\underline{y}}} \|b - a\| \leq \sup_{a \in \overline{\underline{x}}} \|a - b_a\| \leq \left\|(\underline{x} - \underline{y})^- \vee (\overline{x} - \overline{y})^+\right\|.$$

On the other hand,

$$q(\overline{\underline{x}}, \overline{\underline{y}}) = \sup_{a \in \overline{\underline{x}}} \inf_{b \in \overline{\underline{y}}} \|b - a\| \geq \inf_{b \in \overline{\underline{y}}} \|\overline{x} - b\| \geq \left\|(\overline{x} - \overline{y})^+\right\|,$$

and analogously $q(\overline{\underline{x}}, \overline{\underline{y}}) \geq \left\|(\underline{x} - \underline{y})^-\right\|$. ∎

We recall that (see [1], p.357) a normed lattice X is called an M-space if its norm has the property

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}, \quad x, y \geq 0,$$

and an L-space if the norm has the property

$$\|x + y\| = \|x\| + \|y\|, \quad x, y \geq 0.$$

The following corollary gives the expression of the quasi-metric (5) and of the PH-distance (3) between two order intervals of an M-space X.

Corollary 2.1. *If X is an M-space, then for all order intervals $\overline{x}$, $\overline{y}$ of X we have the formulae:*

(i) $q(\overline{x}, \overline{y}) = \max\{\left\|(\underline{x} - \underline{y})^-\right\|, \left\|(\overline{x} - \overline{y})^+\right\|\}$.

(ii) $d_{PH}(\overline{x}, \overline{y}) = \max\{\left\|\underline{x} - \underline{y}\right\|, \|\overline{x} - \overline{y}\|\}$.

In particular, for $X = \mathbb{R}$ (the real axis) we obtain that the PH-distance $d_{PH}(\overline{x}, \overline{y})$ coincides with Moore's distance ([9], p. 52) used in interval analysis.

Next corollary gives a very useful estimation for the PH-distance between two intervals of the form $[0, x]$ and $[0, y]$, with $x, y \in X_+$.

Corollary 2.2. *Let X be a normed lattice.*

(i) For any $x, y \in X_+$ we have,

$$q([0, x], [0, y]) = \left\|(x - y)^+\right\| \leq \|x - y\|. \tag{6}$$

The equality holds if $0 \leq y \leq x$.

(ii) For any $x, y \in X_+$ we have,

$$d_{PH}([0, x], [0, y]) = \max\{\left\|(x - y)^+\right\|, \left\|(x - y)^-\right\|\} \leq \|x - y\|. \tag{7}$$

The equality holds if $0 \leq y \leq x$ or $0 \leq x \leq y$.

Remark 2.1. (a) If X is an L-space and in (6) we have equality, then $0 \leq y \leq x$. Indeed, in this case we have $\left\|(x - y)^+\right\| = \|x - y\|$. Since $|x - y| = (x - y)^+ + (x - y)^-$ and X is a L-space we obtain $\left\|(x - y)^-\right\| = 0$, hence $(x - y)^- = 0$, and therefore $y \leq x$.

(b) If X is an L-space and in (7) we have equality, then $0 \leq y \leq x$ or $0 \leq x \leq y$. □

Finally we show what happens when we compute the PH-distance between two intervals $[0, x]$ and $[0, y]$ with $x \wedge y = 0$.

Corollary 2.3. *If X is a normed lattice and $x, y \in X$ are such that $x \wedge y = 0$, then we have:*

(i) $q([0, x], [0, y]) = \|x\|$.

(ii) $d_{PH}([0, x], [0, y]) = \max\{\|x\|, \|y\|\}$.

Proof. (i) results by using the equality $(x - y)^+ = x - x \wedge y$ in Corollary 2.2. ∎

In the theory of normed lattices a notion which is extensively used is that of almost order bounded set. We recall that a subset A of a normed vector lattice X is said to be *almost order bounded* if for every $\varepsilon > 0$ there exists a positive element x such that

$$A \subseteq [-x, x] + \varepsilon B_X,$$

where $B_X = \{x \in X \mid \|x\| \leq 1\}$ ([13], p. 501).

The following proposition gives some useful equivalent formulations for the notion of almost order bounded set by using the quasi-metric (5) .

Proposition 2.2. *With the above notation we have the following equivalences:*

(i) $A \subseteq [-x, x] + \varepsilon B_X$.

(ii) $q(A, [-x, x]) \leq \varepsilon$.

(iii) $\sup\limits_{a \in A} \left\| (|a| - x)^+ \right\| \leq \varepsilon$.

(iv) $\sup\limits_{a \in A} q([0, |a|], [0, x]) \leq \varepsilon$.

Proof. $(i) \Leftrightarrow (ii)$ is straightforward by definitions if we use for the computation of the quasi-norm q the formula (4).

$(i) \Leftrightarrow (iii)$ is proved in [13], Theorem 122.1, p. 501.

$(iii) \Leftrightarrow (iv)$ results by using Corollary 2.2. ∎

Obviously, every order bounded set is almost order bounded, and every almost order bounded set is norm bounded. Also every totally bounded set is almost order bounded ([13] Theorem 122.2, p.502).

3. POMPEIU-HAUSDORFF DISTANCE BETWEEN ORDER INTERVALS OF AN ORDERED NORMED SPACE

In this section X is an ordered normed space with the positive cone X_+. This means that X is a normed space equipped with a cone denoted X_+ which gives the order relation on X. An ordered Banach space is a Banach space endowed with a cone, not necessarily closed ([2], p. 85).

The cone X_+ is called *normal* if there exists a constant $C > 0$ such that

$$0 \leq x \leq y \Rightarrow \|x\| \leq C \|y\| . \tag{8}$$

Actually the constant $C \geq 1$ and the infimum of all C for which the inequality (8) holds is called the *normal constant of the cone* X_+.

If X_+ is a normal cone, then *every order bounded set is norm bounded.* (The converse is true if $\text{int}(X_+) \neq \emptyset$.) So in an ordered normed space X endowed with a normal cone X_+ the order intervals of X are norm bounded. We will assume also that the cone X_+ is closed to have all order intervals closed.

The quasi-metric (5) between two order intervals $[0, x]$ and $[0, y]$, with $x, y \in X_+$, has the following form:

$$q([0, x], [0, y]) = \sup_{0 \le a \le x} \operatorname{dist}(a, [0, y]) = \sup_{0 \le a \le x} \inf_{0 \le b \le y} \|a - b\| . \tag{9}$$

In the case when the positive cone X_+ is closed we can characterize the order relation on X by using this quasi-metric.

Proposition 3.1. *Let X be an ordered normed space such that its positive cone X_+ is closed and normal. Then*

$$0 \le x \le y \Leftrightarrow q([0, x], [0, y]) = 0.$$

Proof. If $0 \le x \le y$, then $[0, x] \subset [0, y]$ and $q([0, x], [0, y]) = 0$.

Conversely, if $q([0, x], [0, y]) = 0$, then for every $a \in [0, x]$ we have $\inf_{0 \le b \le y} \|a - b\| = 0$. Then there exists a sequence (b_n), with $b_n \in [0, y]$, such that $\|a - b_n\| < 1/n$. Since $0 \le b_n \le y$, $\|b_n - a\| \to 0$, and X_+ is closed, all these imply that $0 \le a \le y$. Hence, $[0, x] \subset [0, y]$. ∎

In the theory of ordered vector spaces is well known that those results that are true in vector lattices (where exist positive and negative parts of an element and its modulus) can be obtained in an ordered vector space if this space has the Riesz decomposition property. We recall that an ordered vector space X is said to have the *Riesz decomposition property* if for three positive elements $x, u, v \in X$ such that $x \le u + v$ there exist $x_1, x_2 \in X$ with $x = x_1 + x_2$ and $0 \le x_1 \le u$, $0 \le x_2 \le v$.

Proposition 3.2. *Let X be an ordered normed space such that the positive cone X_+ is closed and normal with the normal constant C. Then we have:*

(i) $C^{-1} \|y - x\| \le q([0, y], [0, x])$, *for all $x, y \in X$ such that* $0 \le x \le y$.

(ii) If X has the Riesz decomposition property and $0 \le x \le y$, *then*

$$q([0, y], [0, x]) \le C \|y - x\| .$$

Proof. (i) Let us remark first that, for every $a \in [0, x]$, we have

$$0 \le y - x \le y - a \Rightarrow \|y - x\| \le C \|y - a\| \Rightarrow \|y - a\| \ge C^{-1} \|y - x\| .$$

Then

$$q([0, y], [0, x]) = \sup_{0 \le b \le y} \inf_{0 \le a \le x} \|b - a\| \ge \inf_{0 \le a \le x} \|y - a\| \ge C^{-1} \|y - x\| .$$

(ii) Assume now that X has the Riesz decomposition property. For every $b \in [0, y]$, we write

$$0 \leq b \leq y = (y - x) + x.$$

Since X has the Riesz decomposition property there exist $b_1, b_2 \geq 0$ such that $b = b_1 + b_2$ and $0 \leq b_1 \leq y - x$, $0 \leq b_2 \leq x$. Denote $a_b = b_2$. Then $a_b \in [0, x]$ and $0 \leq b - a_b = b - b_2 = b_1 \leq y - x$. Hence $\|b - a_b\| \leq C \|y - x\|$, from where we have

$$\inf_{0 \leq a \leq x} \|b - a\| \leq C \|y - x\|.$$

Therefore

$$q([0, y], [0, x]) = \sup_{0 \leq b \leq y} \inf_{0 \leq a \leq x} \|b - a\| \leq C \|y - x\|.$$

∎

Corollary 3.1. *Let X be an ordered normed space such that the positive cone X_+ is closed and normal with the normal constant C. If X has the Riesz decomposition property, then for any $x, y \in X$ such that* $0 \leq x \leq y$ *we have*

$$C^{-1} \|y - x\| \leq d_{PH}([0, y], [0, x]) \leq C \|y - x\|.$$

Let us remark that in the above proposition the cone X_+ is not generating. (The cone X_+ is called *generating* if $X = X_+ - X_+$.) In order to obtain some estimation for the PH-distance between the order intervals $[0, x]$ and $[0, y]$ we assume that $0 \leq x \leq y$. If x and y are two arbitrary positive elements of X, the following results of E. Yu. Emel'yanov is known ([5], p. 67). In this case, in addition to Proposition 3.2, is assumed that the space X is a Banach space and the cone X_+ is generating.

Proposition 3.3. *Let X be an ordered Banach space such that the positive cone X_+ is closed, normal and generating. If X has the Riesz decomposition property, then there exists a constant $K >$* 0 *such that for every $x, y \in X_+$ we have:*

(i) $q([0, x], [0, y]) \leq K \|x - y\|$.

(ii) $d_{PH}([0, x], [0, y]) \leq K \|x - y\|$.

The next proposition contains some characterizations of a normal cone with the aid of the quasi-norm q defined in (9) or with the PH-distance associated to q. Before giving the statement of the proposition we make some convention about the notation used in it. If in formula (2) the set $A = \{a\}$, then we will write $q(a, B)$ instead of $q(\{a\}, B)$, and similarly $q(A, b)$ if $B = \{b\}$.

Proposition 3.4. *Let X be an ordered normed space such that the positive cone X_+ is closed. The following conditions are equivalent.*

(i) The positive cone X_+ is normal with the normal constant $C \geq$ 1, *that is,*

$$0 \leq x \leq y \Rightarrow \|x\| \leq C \|y\|.$$

(ii) There exists a constant $C \geq 1$ such that for any $x, y \geq 0$ we have

$$d_{PH}([0,x],[0,y]) \leq C \max\{\|x\|, \|y\|\}.$$

(iii) The function $X_+ \times X_+ \longrightarrow \mathbb{R}_+ : (x,y) \to d_{PH}([0,x],[0,y])$ is continuous at $(0,0)$.

(iv) The function $X_+ \longrightarrow \mathbb{R}_+ : x \to q([0,x],0)$ is continuous at 0.

(v) There exists a constant $C \geq 1$ such that for every $x \in X$ we have

$$\|x\| \leq C\, q(-x, X_+) + (1+C)\, q(x, X_+).$$

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in X_+$. Then we have

$$q([0,x],[0,y]) = \sup_{0 \leq a \leq x} \inf_{0 \leq b \leq y} \|a-b\| \leq \sup_{0 \leq a \leq x} \|a\| \leq C \|x\|.$$

Similarly, we get $q([0,y],[0,x]) \leq C \|y\|$. Therefore it follows

$$d_{PH}([0,x],[0,y]) \leq C \max\{\|x\|, \|y\|\}.$$

$(ii) \Rightarrow (iii)$ For every $\varepsilon > 0$ there exists $\delta_\varepsilon = \dfrac{\varepsilon}{C} > 0$ such that if $\max\{\|x\|, \|y\|\} < \delta_\varepsilon$ we have $d_{PH}([0,x],[0,y]) < C \cdot \dfrac{\varepsilon}{C} = \varepsilon$.

$(iii) \Rightarrow (iv)$ Obvious, if we write the definition of continuity for the function $(x,0) \to d_{PH}([0,x],0)$ and remark that in this case $d_{PH}([0,x],0) = q([0,x],0)$ since $q(0,[0,x]) = 0$.

$(iv) \Rightarrow (i)$ Since the function $X_+ \longrightarrow \mathbb{R}_+ : x \to q([0,x],0)$ is continuous at 0, for $\varepsilon = 1$ there exists $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow q([0,x],0) < 1.$$

Let $0 \leq z \leq x$, with $x \neq 0$, and let η such that $0 < \eta < \delta$. Then $0 \leq \dfrac{z}{\|x\|}\eta \leq \dfrac{x}{\|x\|}\eta$. Since $\left\|\dfrac{x}{\|x\|}\eta\right\| = \eta < \delta$ we have

$$q\left(\left[0, \frac{x}{\|x\|}\eta\right], 0\right) < 1,$$

that is

$$\sup\left\{\|b\| : 0 \leq b \leq \frac{x}{\|x\|}\eta\right\} < 1.$$

In particular, for $b = \dfrac{z}{\|x\|}\eta$ we have $\left\|\dfrac{z}{\|x\|}\eta\right\| < 1$, or $\|z\| < \dfrac{1}{\eta}\|x\|$. Finally, for $\eta \to \delta$ we obtain $\|z\| < \dfrac{1}{\delta}\|x\|$. This shows that the positive cone X_+ is normal with the normal constant $C = \dfrac{1}{\delta}$.

$(i) \Rightarrow (v)$ First we remark that

$$q(x, X_+) = \mathrm{dist}(x, X_+) = \inf\{\|x - y\| \mid y \geq 0\},$$

$$q(-x, X_+) = \mathrm{dist}(-x, X_+) = \inf\{\|x + z\| \mid z \geq 0\}.$$

Then there exist two sequences (y_n) and (z_n) in X_+ such that

$$\|x - y_n\| < q(x, X_+) + \frac{1}{n}, \text{ and } \|x + z_n\| < q(-x, X_+) + \frac{1}{n}.$$

Therefore

$$\|y_n + z_n\| < q(x, X_+) + q(-x, X_+) + \frac{2}{n},$$

and

$$\begin{aligned} \|x\| &= \|-x\| \leq \|-x + y_n\| + \|y_n\| < q(x, X_+) + \frac{1}{n} + C\,\|y_n + z_n\| < \\ &< q(x, X_+) + \frac{1}{n} + C\left(q(x, X_+) + q(-x, X_+) + \frac{2}{n}\right) < \\ &< C\,q(-x, X_+) + (1 + C)\,q(x, X_+) + \frac{1 + 2C}{n}, \end{aligned}$$

from where we obtain the required inequality.

$(v) \Rightarrow (i)$ Suppose that (v) holds and consider $0 \leq x \leq y$. Then

$$q(-x, X_+) \leq \inf_{z \geq 0} \|x + z\| \leq \|x + (y - x)\| = \|y\|,$$

and $q(x, X_+) = 0$. By (v) we obtain $\|x\| \leq C\,\|y\|$. ∎

Remark 3.1. (a) The equivalence between the normality of the cone X_+ and the continuity of the function $X_+ \times X_+ \longrightarrow R_+ : (x, y) \to d_{PH}([0, x], [0, y])$ at $(0, 0)$ is affirmed in [4] without proof.

(b) The function $q(-x, X_+) = \inf\{\|x + z\| \mid z \geq 0\}$ used in Proposition 3.4 is called a half-norm. For the study of half-norms and their duals, see [11].

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ON SOME LACUNARY σ– STRONG ZWEIER CONVERGENT SEQUENCE SPACES

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Abstract In this paper we define three classes of new sequence spaces. We give some relations related to these sequence spaces. We also introduce the concept of $\left[S_{\sigma}^{\theta}\right]_{Z}$ −statistically convergence and obtain some inclusion relations related to these new sequence spaces.

Keywords: σ -convergence, lacunary sequence, Zweier space, statistical convergence.
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1. INTRODUCTION

Let l_{∞}, c and c_o be the linear spaces of bounded, convergent and convergent to zero sequences with complex terms, respectively. Note that l_{∞}, c and c_o are Banach spaces with the sup-norm

$$\|(x_k)\|_{\infty} = \sup_k |x_k| .$$

A sequence space X with a linear topology is called a K-space if each of maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called FK-space if X is a complete linear metric space and a BK-space is a normed FK-space.

Shaefer [13] defined the $\sigma-$ *convergence* as follows: Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma\left(\sigma^{m-1}(n)\right), m = 1, 2, 3, ...$ A continuous linear functional ϕ on l_{∞} (the set of all bounded sequences) is said to be an invariant mean or a $\sigma-$ *mean* if and only if

$$(i)\ \ \phi((x_k)) \geq 0 \text{ when the sequence } (x_k) \text{ has } x_k \geq 0 \text{ for all k,}$$

$$(ii)\ \ \phi(e) = 1, \text{ where } e = (1, 1, 1, ...)$$

and

$$(iii)\ \ \phi\left(\{x_{\sigma(k)}\}\right) = \phi(\{x_k\}) \text{ for all } (x_k) \in l_{\infty}.$$

For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space c, the set of all convergent sequences, in the sense that $\phi(x) = \lim x_k$ for all $(x_k) \in c$. Consequently, $c \subset V_{\sigma}$, where V_{σ} is the set of bounded sequences all of whose $\sigma-$ *means* are equal. It was natural to expect that invariant mean must rise to a new type of convergence, namely, strong invariant convergence, just as almost convergence gives rise to concept of strong almost convergence and this concept was introduced and discussed by Mursaleen [10] . If $[V_\sigma]$ denotes the set of all strongly $\sigma-$ *convergent* sequences, then Mursaleen defined

$$[V_\sigma] = \left\{ x = (x_k) \in l_\infty : \lim_m \frac{1}{m} \sum_{k=1}^{m} \left| x_{\sigma^k(n)} - L \right| = 0, \text{ uniformly in n} \right\}.$$

$[V_\sigma]^o$ denotes the subset of these sequences in $[V_\sigma]$ for which $L = 0$. Taking $\sigma(n) = n + 1$, we obtain $[V_\sigma] = [\widehat{c}]$ so that strong $\sigma-$ *convergence* generalizes the concept of strong almost convergence. Note that $[V_\sigma] \subset V_\sigma \subset l_\infty$.

By a lacunary sequence $\theta = (k_r; r = 0, 1, 2, ...)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$.The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [3] as follows:

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some L} \right\}.$$

The space N_θ is a BK-space with the norm

$$\|(x_k)\|_\theta = \sup_r \frac{1}{h_r} \sum_{k\in I_r} |x_k| .$$

N_θ^o denotes the subset of these sequences in N_θ for which $L = 0$ and $\left(N_\theta^o, \|(x_k)\|_\theta\right)$ is also a BK-space.

Now we define the following sequence spaces as follows:

$$\left[V_\sigma^\theta\right]^o = \left\{ (x_k) \in l_\infty : \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left| x_{\sigma^k(n)} \right| = 0, \text{ uniformly in n} \right\},$$

$$\left[V_\sigma^\theta\right] = \left\{ (x_k) \in l_\infty : \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left| x_{\sigma^k(n)} - L \right| = 0, \text{ uniformly in n, for some } L \right\}$$

and

$$\left[V_\sigma^\theta\right]^\infty = \left\{ (x_k) \in l_\infty : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} \left| x_{\sigma^k(n)} \right| < \infty \right\}.$$

It is easy to see that the space $\left[V_\sigma^\theta\right]$ is a BK space with the norm

$$\|(x_k)\|_\theta = \sup_{r,m} \frac{1}{h_r} \sum_{k\in I_r} \left| x_{\sigma^k(n)} \right| .$$

It is clear that if $\theta = (2^r)$ then $\left[V_\sigma^\theta\right]$-summability reduces to ordinary $[V_\sigma]$ $-$summability.

For a sequence space μ, the matrix domain μ_A of an infinite matrix A is defined by

$$\mu_A = \{(x_k) : \ Ax \in \mu\}$$

where $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix of real or complex numbers and $Ax = (A_n(x))_{n=1}^{\infty}$ if $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$.

In [11] , Şengönül introduced sequence spaces Z^w and Z_o^w as the set of all sequences such that Z-transforms of them are in the spaces c and c_o, respectively, i.e.,

$$Z^w = \{(x_k) : \ Z(x_k) \in c\} \text{ and } Z_o^w = \{(x_k) : \ Z(x_k) \in c_o\}$$

where $Z = (z_{nk})_{n,k=0}^{\infty}$ denotes by the matrix

$$z_{nk} = \begin{cases} \frac{1}{2}, k \le n \le k+1 \\ 0, \qquad otherwise \end{cases} \quad (n, k \in \mathbb{N}) .$$

This matrix is called Zweier matrix. Note that Z is a regular matrix [8] .

The purpose of this paper is to introduce and study the concept of $\left[V_\sigma^\theta\right]_Z$ $-$strong Zweier convergence and $\left[S_\sigma^\theta\right]_Z$ $-$statistical Zweier convergence.

2. $\left[V_\sigma^\theta\right]_Z$- STRONG ZWEIER CONVERGENCE

We introduce the sequence spaces $\left[V_\sigma^\theta\right]_Z^o$, $\left[V_\sigma^\theta\right]_Z$ and $\left[V_\sigma^\theta\right]_Z^\infty$ as the set of all sequences such that Z-transforms are in $\left[V_\sigma^\theta\right]^o$, $\left[V_\sigma^\theta\right]$ and $\left[V_\sigma^\theta\right]^\infty$, respectively, that is

$$\left[V_\sigma^\theta\right]_Z^o = \left\{(x_k) : \ \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) \right| = 0, \text{ uniformly in n} \right\},$$

$$\left[V_\sigma^\theta\right]_Z = \left\{(x_k) : \ \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L \right| = 0, \text{ uniformly in n, for some } L \right\}$$

and

$$\left[V_\sigma^\theta\right]_Z^\infty = \left\{(x_k) : \ \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) \right| < \infty \right\}$$

Define the sequence $y = \left(y_k^n\right)$ which will be frequently used throughout the paper, as Z-transform of a sequence $x = (x_k)$, i.e.,

$$y_k^n = \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) \ \ (k \in \mathbb{N}) \tag{2.1}$$

Theorem 2.1. *The sequence spaces* $\left[V_\sigma^\theta\right]_Z^o$, $\left[V_\sigma^\theta\right]_Z$ *and* $\left[V_\sigma^\theta\right]_Z^\infty$ *are linear spaces over the complex field* $\mathbb{C}$*; moreover, these become BK-spaces with respect to the norm*

$$\|(x_k)\|_{\left[V_\sigma^\theta\right]_Z^o} = \|(x_k)\|_{\left[V_\sigma^\theta\right]_Z} = \|(x_k)\|_{\left[V_\sigma^\theta\right]_Z^\infty} = \|Z(x_k)\|_{\left[V_\sigma^\theta\right]}.$$

Proof. The first part of the theorem is a routine verification and so we omit it. Since the sequence spaces $\left[V_\sigma^\theta\right]_Z^o$ and $\left[V_\sigma^\theta\right]_Z$ are BK-spaces with respect to the norm defined (2.1) and the matrix $Z = (z_{nk})_{n,k=0}^\infty$ is normal, i.e., $z_{nk} \neq 0$ for $0 \le k \le n$ and $z_{nk} = 0$ for $k > n$ for all $n, k \in \mathbb{N}$ and also from Theorem 4.3.2 of Wilansky [4] gives the fact that $\left[V_\sigma^\theta\right]_Z^o$, $\left[V_\sigma^\theta\right]_Z$ and $\left[V_\sigma^\theta\right]_Z^\infty$ are the BK-spaces. ∎

Theorem 2.2. *The sequence spaces* $\left[V_\sigma^\theta\right]_Z^o$, $\left[V_\sigma^\theta\right]_Z^o$ *and* $\left[V_\sigma^\theta\right]_Z^\infty$ *are linearly isomorphic to the sequence spaces* N_θ, N_θ^o *and* N_θ^∞*, respectively.*

Proof. We should show the existence of a linear bijection between the spaces $\left[V_\sigma^\theta\right]_Z^o$,$\left[V_\sigma^\theta\right]_Z^o$ and $\left[V_\sigma^\theta\right]_Z^\infty$ and $\left[V_\sigma^\theta\right]^o$, $\left[V_\sigma^\theta\right]$ and $\left[V_\sigma^\theta\right]^\infty$. Consider the transformation Z define, with the notation (2.1), from $\left[V_\sigma^\theta\right]_Z^o$ to N_θ^o by

$$Z : \left[V_\sigma^\theta\right]_Z^o \to \left[V_\sigma^\theta\right]^o$$
$$(x_k) \to Z(x_k) = (y_k)$$

where the sequence (y_k) is given by (2.1). The linearity of transformation Z is clear. Further, it is trivial that $(x_k) = (0)$ whenever $Z(x_k) = 0$ and hence Z is injective. Let $\left(y_k^n\right) \in \left[V_\sigma^\theta\right]^o$ and the sequence (x_k) by

$$x_{\sigma^k(n)} = 2\sum_{i=0}^{k} (-1)^{i-k} y_i^n \quad (i \in \mathbb{N})$$

Then

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} \left| \frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right)\right|$$
$$= \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left| \frac{1}{2}\left(2\sum_{i=0}^{k} (-1)^{i-k} y_i^n + 2\sum_{i=0}^{k-1} (-1)^{(i-1)-k} y_i^n\right)\right|$$
$$= \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left|y_k^n\right|$$

which says us that $(x_k) \in \left[V_\sigma^\theta\right]_Z^o$. Additionally, we observe that

$$\|(x_k)\|_{\left[V_\sigma^\theta\right]_Z^o} = \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left| \frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right)\right|$$

$$= \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(2 \sum_{i=0}^{k} (-1)^{i-k} y_i^n + 2 \sum_{i=0}^{k} (-1)^{(i-1)-k} y_i^n \right) \right|$$

$$= \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left| y_k^m \right| = \|(y_k)\|_{[V_\sigma^\theta]^o}.$$

Thus, we have $(x_k) \in \left[V_\sigma^\theta\right]^o$ and consequently Z is surjective. Hence, Z is linear bijection which therefore says us that the sequence spaces $\left[V_\sigma^\theta\right]_Z^o$ and $\left[V_\sigma^\theta\right]^o$ are linearly isomorphic as was desired. The others can be proved similarly. This completes the proof. ∎

There is a relation between the sequence space $\left[V_\sigma^\theta\right]$ and the sequence space $[V_\sigma]$ of strongly invariant Cesaro summable sequences defined by

$$[V_\sigma] = \left\{ (x_k) \in l_\infty : \lim_m \frac{1}{m} \sum_{k=1}^{m} \left| x_{\sigma^k(n)} - L \right| = 0, \text{ uniformly in n, for some L} \right\}.$$

Clearly, in the special case $\theta = (2^r)$, we have $\left[V_\sigma^\theta\right] = [V_\sigma]$.

Also, we see that, there are strong connection between the sequence space $\left[V_\sigma^\theta\right]_Z$ and the sequence space $[w_\sigma]_Z$, which is defined by

$$[w_\sigma]_Z = \left\{ (x_k) : \lim_m \frac{1}{m} \sum_{k=1}^{m} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| = 0, \text{uniformly in n, for some } L \right\}.$$

Clearly, in the special case $\theta = (2^r)$, we have $\left[V_\sigma^\theta\right]_Z = [w_\sigma]_Z$.

Theorem 2.3. *Let $\theta = (k_r)$ be a lacunary sequence with* $\liminf q_r > 1$*, then* $[w_\sigma]_Z \subset \left[V_\sigma^\theta\right]_Z$.

Proof. Let $(x_k) \in [w_\sigma]_Z$. Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$$

Then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r} \sum_{k=1}^{k_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \geq \frac{1}{k_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right|$$

$$\geq \frac{\delta}{1+\delta} \cdot \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right|.$$

This completes the proof. ▮

Theorem 2.4. *Let* $\theta = (k_r)$ *be lacunary sequence with* $\limsup_r q_r < \infty$, *then* $\left[V_\sigma^\theta\right]_Z \subset [w_\sigma]_Z$.

Proof. If $\limsup_r q_r < \infty$, then there exists $B > 0$ such that $q_r < C$ for all $r \geq 1$. Let $x = (x_k) \in \left[V_\sigma^\theta\right]_Z$ and $\varepsilon > 0$.There exists $B > 0$ such that for every $j \geq B$

$$A_j = \frac{1}{h_j} \sum_{k \in I_j} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| < \varepsilon.$$

we can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, 3, ...$ Now let m be any integer with $k_{r-1} < m < k_r$,where $r \geq B$. Then

$$\begin{aligned}
\frac{1}{m} \sum_{k=1}^{m} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| &\leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
= \frac{1}{k_{r-1}} \sum_{k \in I_1} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| &+ \frac{1}{k_{r-1}} \sum_{k \in I_2} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
&+ ... + \frac{1}{k_{r-1}} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
= \frac{k_1}{k_{r-1} k_1} \sum_{k \in I_1} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| &+ \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)} \sum_{k \in I_2} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
&+ ... + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})} \sum_{k \in I_B} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
&+ ... + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})} \sum_{k \in I_r} \left| \frac{1}{2} \left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)} \right) - L \right| \\
&= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + ... + \frac{k_B - k_{B-1}}{k_{r-1}} A_B + ... + \frac{k_r - k_{r-1}}{k_{r-1}} A_r \\
&\leq \left\{ \sup_{j \geq 1} A_j \right\} \frac{k_B}{k_{r-1}} + \left\{ \sup_{j \geq B} A_j \right\} \frac{k_r - k_B}{k_{r-1}} \\
&\leq K . \frac{k_B}{k_{r-1}} + \varepsilon C.
\end{aligned}$$

This completes the proof. ▮

Corollary 2.1. *Let* $\theta = (k_r)$ *be lacunary sequence* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, *then* $\left[V_\sigma^\theta\right]_Z = [w_\sigma]_Z$.

Proof. The result follows from Theorem 2.3 and Theorem 2.4. ▮

3. $\left[S_{\sigma}^{\sigma}\right]_Z$-STATISTICAL ZWEIER CONVERGENCE

In this section we introduce the concept of $\left[S_{\sigma}^{\theta}\right]_Z$-statistical convergence and give some inclusion relations related to this sequence space.

The notion on statistical convergence was introduced by Fast [6] and studied by various authors (see $[1-2], [5], [7], [9], [11], [14])$.

Definition 3.1. [6] *A sequence (x_k) is said to be statistically convergent to a number L if for every $\varepsilon > 0$,*

$$\lim_m \frac{1}{m} |\{k \leq m : \ |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write

$$S - \lim x_k = L \text{ or } x_k \to L(S) \text{ and } S = \{(x_k) : \ \text{ for some } L, S - \lim x_k = L\}.$$

Definition 3.2. [9] *A sequence (x_k) is said to be lacunary-statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S^{\theta} - \lim x_k = L$ or $x_k \to l\left(S^{\theta}\right)$ and
$S = \left\{(x_k) : \ \text{ for some } L, S^{\theta} - \lim x_k = L\right\}$.

Definition 3.3. [11] *A sequence (x_k) is said to be S_{σ}^{θ}–statisticaly convergent to L if for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : \ \left|x_{\sigma^k(n)} - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } n.$$

In this case we write $S_{\sigma}^{\theta} - \lim x_k = L$ or $x_k \to L\left(S_{\sigma}^{\theta}\right)$ and
$S_{\sigma}^{\theta} = \left\{(x_k) : \ \text{ for some } l, S_{\sigma}^{\theta} - \lim x = L\right\}$.

Definition 3.4. *A sequence (x_k) is said to be S_Z-statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_m \frac{1}{m} \left|\left\{k \leq m : \ \left|\frac{1}{2}(x_k + x_{k-1}) - L\right| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $[S_{\sigma}]_Z - \lim x_k = L$ or $x_k \to L([S_{\sigma}]_Z)$ and
$[S_{\sigma}]_Z = \{(x_k) : \ \text{ for some } L, \ [S_{\sigma}]_Z - \lim x = L\}$.

Definition 3.5. *A sequence (x_k) is said to be $\left[S_{\sigma}^{\theta}\right]_Z$-statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : \ \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } n.$$

In this case we write $\left[S_{\sigma}^{\theta}\right]_Z - \lim x_k = L$ or $x_k \to L\left(\left[S_{\sigma}^{\theta}\right]_Z\right)$ and $\left[S_{\sigma}^{\theta}\right]_Z = \left\{x = (x_k) : \text{ for some } L, \left[S_{\sigma}^{\theta}\right]_Z - \lim x = L\right\}$.

In the cases $\theta = (2^r)$ we shall write $[S_{\sigma}]_Z$ instead of $\left[S_{\sigma}^{\theta}\right]_Z$ and $\theta = (2^r)$, $\sigma(n) = n+1$; we shall write S_Z instead of $\left[S_{\sigma}^{\theta}\right]_Z$, respectively.

Theorem 3.1. *Let $\theta = (k_r)$ be a lacunary sequence. Then $\left[V_{\sigma}^{\theta}\right]_Z \subset \left[S_{\sigma}^{\theta}\right]_Z$.*

Proof. Let $(x_k) \in \left[V_{\sigma}^{\theta}\right]_Z$. Then

$$\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|\frac{1}{2}\left(x_{\sigma^k(n)}+x_{\sigma^{k-1}(n)}\right)-L\right|\geq\varepsilon}}\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$
$$+\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|\frac{1}{2}\left(x_{\sigma^k(n)}+x_{\sigma^{k-1}(n)}\right)-L\right|<\varepsilon}}\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$\geq \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|\frac{1}{2}\left(x_{\sigma^k(n)}+x_{\sigma^{k-1}(n)}\right)-L\right|\geq\varepsilon}}\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$\geq \frac{1}{h_r}\sum_{k\in I_n}\varepsilon \geq \frac{\varepsilon}{h_r}\left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right|.$$

It follows that $x_k \to l\left(\left[S_{\sigma}^{\theta}\right]_Z\right)$. This completes the proof. ∎

Theorem 3.2. *Let $\theta = (k_r)$ be a lacunary sequence. If $x = (x_k) \in l_\infty$ and $x_k \to L\left(\left[V_{\sigma}^{\theta}\right]_Z\right)$, then $x_k \to L\left(\left[S_{\sigma}^{\theta}\right]_Z\right)$.*

Proof. Suppose that $(x_k) \in l_\infty$ and $x_k \to L\left(\left[V_{\sigma}^{\theta}\right]_Z\right)$. Since $\sup\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right)\right| < \infty$, there is a constant $A > 0$ such that $\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right)\right| < A$ for all $k, n \in \mathbb{N}$. Therefore we have, for $\varepsilon > 0$

$$\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$= \frac{1}{h_r} \sum_{\substack{k\in I_r \\ \left|\frac{1}{2}\left(x_{\sigma^k(n)}+x_{\sigma^{k-1}(n)}\right)-L\right|\geq\varepsilon}} \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$+\frac{1}{h_r} \sum_{\substack{k\in I_r \\ \left|\frac{1}{2}\left(x_{\sigma^k(n)}+x_{\sigma^{k-1}(n)}\right)-L\right|<\varepsilon}} \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right|$$

$$\leq \frac{A}{h_r}\left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right| + \frac{1}{h_r}\sum_{k\in I_r}\varepsilon$$

$$= \frac{A}{h_r}\left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

Taking limit as $\varepsilon \to 0$, the desired result follows. ∎

Corollary 3.1. *Let $\theta = (k_r)$ be a lacunary sequence. Then $l_\infty \cap \left[V^\theta_\sigma\right]_Z = l_\infty \cap \left[S^\theta_\sigma\right]_Z$.*

Proof. It follows from Theorem 3.1. and Theorem 3.2. ∎

Theorem 3.3. *Let $\theta = (k_r)$ be a lacunary sequence. Then $[S_\sigma]_Z \subset \left[S^\theta_\sigma\right]_Z$.*

Proof. Given $\varepsilon > 0$, we have

$$\left|\left\{k \leq m : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right| \supset \left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right|.$$

Therefore

$$\frac{1}{m}\left|\left\{k \leq m : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{m}\left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{h_r}{m}.\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{1}{2}\left(x_{\sigma^k(n)} + x_{\sigma^{k-1}(n)}\right) - l\right| \geq \varepsilon\right\}\right|.$$

Taking limit as $m \to \infty$ uniformly in n, we get that $x_k \to l\left(\left[S^\theta_\sigma\right]_Z\right)$.This completes the proof. ∎

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FREE VIBRATIONS IN A THIN RETICULATED STRUCTURE

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Abstract This paper presents the homogenization of the wave problem for a reticulated structure composed of thin slashes in two dimensions ox_1 and ox_2. The result of this paper continues the study of the homogenization of the wave problem for such a periodic structure, started in [2]. In this article homogenization is done after the small thickness of the bars and the result is a two-dimensional problem of wave on a fixed domain.

Keywords: homogenization , ε - periodicity, reticulated structure.
2010 MSC: 35J20.

1. INTRODUCTION

Our paper continues the study of asymptotic behavior of oblique plates at free vibrations. The structure for which we study the wave problem is a three-dimensional reticulated structure where the ε periodicity goes in two directions. The material of the structure has small thickness, namely bars – distributed ε - periodically - which make up have thickness $\varepsilon\delta$, except in edge thickness $\varepsilon\delta/2$. Thickness or height of the structure is considered small and denoted by e. The perforated domain that we study the waves problem depends on three small parameters ε, δ and e. In our case the period ε and the thickness e, we take them comparable.

This article contains five sections. The first section is a short introduction. In the second section we present the geometry of the domain. In the third section we present the problem of wave on reticulated structure previously considered, and the section four is the main result obtained in article [2] which means homogenization after ε - the period of the wave problem and obtain a two-dimensional boundary problems.

Section five is novel in relation to article [2] and consists in homogenization of wave problem obtained in [2] after δ - thickness of the material from the cell period that is covered the domain ω from $\mathbb{R}^2$. The result is obtained by dilatation method introduced in [1], while the homogenized coefficients are those obtained in [3].

The obtained limit problem is a two-dimensional problem of waves, namely the oscillations problem for the rectangular membrane to cover the whole structure (the domain ω of the plan), including boundary. Note, homogenized coefficients depend on the characteristic constants of the material.

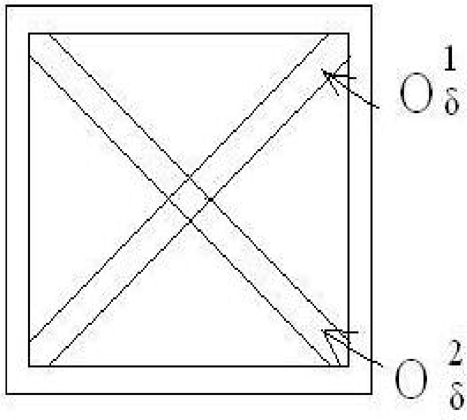

Fig. 1. The periodicity cell Y_δ.

2. THE GEOMETRY OF THE DOMAIN

Let $\omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ and $\Omega^e = \omega \times \left(-\frac{e}{2}, \frac{e}{2}\right)$, and ω is periodically covered with the reference cell $Y = (0, 1) \times (0, 1)$. Mathematically means $\omega = \omega \cap \varepsilon Y$. We consider the periodicity cell $Y_\delta = H_\delta \cup V_\delta \cup O^1_\delta \cup O^2_\delta$ and represented in figure 1, where Y_δ is occupied by material of cell Y, $T_\delta = Y \backslash Y_\delta$ called the hole and consider $\omega_{\varepsilon\delta}$ perforated domain from ω after distribution of the periodicity cell Y_δ with period ε after two dimensions ox_1 and ox_2.

Consider the three-dimensional perforated domain $\Omega^e_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left(-\frac{e}{2}, \frac{e}{2}\right)$ which is a reticulated structure type plates that depends on three small parameters: ε period, e the plate thickness and δ the thickness of the slashes which forms the covers. The domain $\Omega^e_{\varepsilon\delta}$ is represented in figure 2.

3. THE STATEMENT OF THE WAVE PROBLEM

We are in the case $e = k\varepsilon$, k a strictly positive constant. In this situation the period ε and the thickness of structure e is the same power.

We have the structure $\Omega^k_{\varepsilon\delta}$ instead of $\Omega^e_{\varepsilon\delta}$. We introduce the following notations: $\Gamma^{k\varepsilon}_0 = \partial\omega \times \left\{+\frac{k\varepsilon}{2}\right\}$ which represents the outer border of the top cover in which is embedded crosslinked structure $\Omega^k_{\varepsilon\delta}$; $\gamma^k_{\varepsilon\delta} = \partial\Omega^k_{\varepsilon\delta} \backslash \Gamma^{k\varepsilon}_0 = \partial T^k_{\varepsilon\delta} \cup \Gamma^{k\pm}_{\varepsilon\delta} \cup \left(\partial\omega \times \left(-\frac{k\varepsilon}{2}, \frac{k\varepsilon}{2}\right)\right)$, where $T^k_{\varepsilon\delta} = (\omega \cap \varepsilon T_\delta) \times \left(-\frac{k\varepsilon}{2}, \frac{k\varepsilon}{2}\right)$ are holes in $\Omega^k_{\varepsilon\delta}$, $\Gamma^{k\pm}_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left\{\pm\frac{k\varepsilon}{2}\right\}$ are the two covers (upper and lower) of the structure $\Omega^k_{\varepsilon\delta}$, $\partial\omega \times \left(-\frac{k\varepsilon}{2}, \frac{k\varepsilon}{2}\right)$ is the lateral border of the structure.

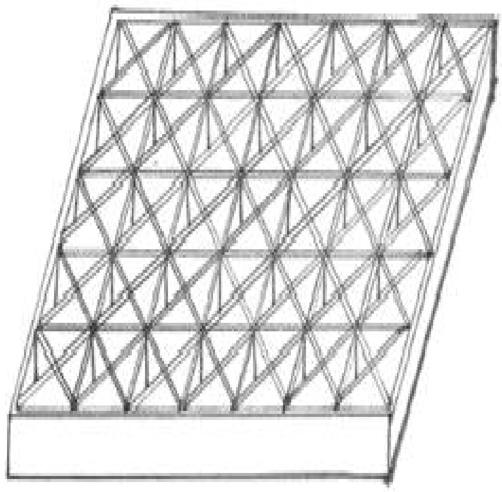

Fig. 2. Reticulated structure $\Omega^{e}_{\varepsilon\delta}$.

The wave problem on $\Omega^{k}_{\varepsilon\delta}$ is

$$\begin{array}{c} v_k^{\varepsilon\delta''} - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_k^{\varepsilon\delta}}{\partial x_j}\right) = 0 \ \text{ in } \ \Omega^k_{\varepsilon\delta} \times (0,T) \\ a_{ij}\frac{\partial u_k^{\varepsilon\delta}}{\partial x_j} n_i = 0 \ \text{ on } \ \gamma^k_{\varepsilon\delta} \times (0,T) \\ u_k^{\varepsilon\delta} = 0 \ \text{ on } \ \Gamma_0^{k\varepsilon} \times (0,T) \\ u_k^{\varepsilon\delta}(0) = u^0_{k\varepsilon\delta} \ \text{ in } \ \Omega^k_{\varepsilon\delta} \\ u_k^{\varepsilon\delta'}(0) = u^1_{k\varepsilon\delta} \ \text{ in } \ \Omega^k_{\varepsilon\delta}. \end{array} \tag{1}$$

The coefficients $\left(a_{ij}\right)_{i,j=1,3}$ are elliptical and symmetric, that is $\exists \ \alpha > 0$ a constant so that $a_{ij}\xi_i\xi_j \geq \alpha\xi_i\xi_i$, $\forall \xi \in \mathbb{R}^3$ and $a_{ij} = a_{ji}$.

We make change in the variable and the function:

$$z_1 = x_1, \ \ z_2 = x_2, \ \ z_3 = \frac{x_3}{k\varepsilon}, \ z_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right);$$

$$u_k^{\varepsilon\delta}(x_1, x_2, x_3, t) = v_{k\varepsilon\delta}(z_1, z_2, z_3, t)\,.$$

The problem (1) becomes:

$$\begin{array}{c} v''_{k\varepsilon\delta} + A_{k\varepsilon\delta} v_{k\varepsilon\delta} = 0 \ \text{ in } \ \Omega_{\varepsilon\delta} \times (0,T) \\ \left(a_{i\beta} \frac{\partial v_{k\varepsilon\delta}}{\partial z_\beta} + (k\varepsilon)^{-1} a_{i3} \frac{\partial v_{k\varepsilon\delta}}{\partial z_3} \right) n_i = 0 \ \text{ on } \ \Gamma_{\varepsilon\delta} \times (0,T) \\ v_{k\varepsilon\delta} = 0 \ \text{ on } \ \gamma_0 \times (0,T) \\ v_{k\varepsilon\delta}(0) = v^0_{\varepsilon\delta} \ \text{ in } \ \Omega_{\varepsilon\delta} \\ v'_{k\varepsilon\delta}(0) = v^1_{\varepsilon\delta} \ \text{ in } \ \Omega_{\varepsilon\delta}. \end{array} \tag{2}$$

where: $\Omega_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\Gamma_{\varepsilon\delta} = \partial\Omega_{\varepsilon\delta} \backslash \gamma_0$ and $\gamma_0 = \partial\omega \times \left\{+\frac{1}{2}\right\}$, and the covers are $\Gamma^{\pm}_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left\{\pm\frac{1}{2}\right\}$.

The elliptical operator $A_{k\varepsilon\delta}$ is:

$A_{k\varepsilon\delta} = -\frac{\partial}{\partial z_\alpha}\left(a_{\alpha\beta}\frac{\partial}{\partial z_\beta}\right) - (k\varepsilon)^{-1}\frac{\partial}{\partial z_\alpha}\left(a_{\alpha 3}\frac{\partial}{\partial z_3}\right) - (k\varepsilon)^{-1}\frac{\partial}{\partial z_3}\left(a_{3\beta}\frac{\partial}{\partial z_\beta}\right) - (k\varepsilon)^{-2}\frac{\partial}{\partial z_3}\left(a_{33}\frac{\partial}{\partial z_3}\right).$

Here we introduce the space $V_{\varepsilon\delta}$, and the norm $\|\cdot\|_{\varepsilon\delta}$ on the space $V_{\varepsilon\delta}$.

4. THE HOMOGENIZATION OF THE WAVE PROBLEM AFTER $\varepsilon \to 0$

In [2] we obtained the following result:

Theorem 4.1. *We consider that the initial data* (2) *satisfies conditions:*

a) $\left\|v^0_{\varepsilon\delta}\right\|_{\varepsilon\delta} \le c\delta$ and $\tilde{v}^0_{\varepsilon\delta} \underset{\varepsilon}{\rightarrow} v^0_\delta$ *weak in* $H^1_0(\omega)$

b) $\left\|v^1_{\varepsilon\delta}\right\|_{L^2(\omega_{\varepsilon\delta})} \le c\delta$ and $\tilde{v}^1_{\varepsilon\delta} \underset{\varepsilon}{\rightarrow} v^1_\delta$ *weak in* $L^2(\omega)$

Then there is an extending operator $P^{\varepsilon\delta} \in \mathbf{L}\left(L^\infty\left(0,T;H^1_0(\omega)\right); L^\infty\left(0,T;L^2(\omega)\right)\right)$ *so that we have the convergences:*

c) $P^{\varepsilon\delta} v_{k\varepsilon\delta} \underset{\varepsilon}{\rightarrow} v_{k\delta}$ *weak* * $L^\infty\left(0,T;H^1_0(\omega)\right)$

d) $P^{\varepsilon\delta} v'_{k\varepsilon\delta} \underset{\varepsilon}{\rightarrow} v'_{k\delta}$ *weak* * $L^\infty\left(0,T;L^2(\omega)\right)$

where $v_{k\delta}$ *satisfy the limit problem:*

$$\begin{array}{c} (\text{meas } Y_\delta)\, v''_{k\delta} - q^{\delta k}_{\alpha\beta} \frac{\partial^2 v_{k\delta}}{\partial z_\alpha \partial z_\beta} = 0, (0,T) \times \Omega \\ v_{k\delta} = 0\, (0,T) \times \delta\omega \\ v_{k\delta}(0) = \frac{v^0_\delta}{\text{meas } Y_\delta} \ \text{ in } \ \omega \\ v'_{k\delta}(0) = \frac{v^{1,*}_\delta}{\text{meas } Y_\delta} \ \text{ in } \ \omega \end{array} \tag{3}$$

where:

$v^{1,*}_\delta = \int_{-\frac{1}{2}}^{\frac{1}{2}} v^1_\delta dz_3$, *and the coefficients* $q^{\delta k}_{\alpha\beta}$ *are given by:*

$$q^{\delta k}_{\alpha\beta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Y_\delta} \left(a_{\gamma\beta} \frac{\partial w^{\delta k}_\alpha}{\partial y_\gamma} + k^{-1} a_{3\beta} \frac{\partial w^{\delta k}_\alpha}{\partial y_3} \right) dy, \quad \alpha, \beta = 1, 2$$

where the correction functions $w_\alpha^{\delta k}$ satisfy the problem:

$$-\frac{\partial}{\partial y_\beta}\left(a_{\gamma\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma}\right)-k^{-1}\frac{\partial}{\partial y_\beta}\left(a_{3\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right)-k^{-1}\frac{\partial}{\partial y_3}\left(a_{\gamma 3}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma}\right)-k^{-2}\frac{\partial}{\partial y_3}\left(a_{33}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right)=0$$

$$\left(a_{\gamma j}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma}+k^{-1}a_{3j}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right)n_j=0$$

on

$$\left[\partial T_\delta\times\left(-\frac{1}{2},\frac{1}{2}\right)\right]\cup\left[Y_\delta\times\left\{\pm\frac{1}{2}\right\}\right]$$

5. THE HOMOGENIZATION OF THE WAVE PROBLEM AFTER $\delta \to 0$

Theorem 5.1. *Assuming the hypothesis of the above theorem satisfied, we have:*

a)$\delta^{-1}v_\delta^0 \underset{\delta}{\rightarrow} v^0$ *weak in* $H_0^1(\omega)$,

b)$\delta^{-1}v_\delta^{1,*} \underset{\delta}{\rightarrow} v^1$ *weak in* $L^2(\omega)$,
respective
c)$v_{k\delta} \underset{\delta}{\overset{v}{\rightarrow}}$ *weak* $*\ L^\infty\left(0,T;H_0^1(\omega)\right)$,

d)$v'_{k\delta} \underset{\delta}{\overset{v'}{\rightarrow}}$ *weak* $*\ L^\infty\left(0,T;L^2(\omega)\right)$,
where v_δ *satisfies the boundary problem:*

$$\begin{array}{c}\left(2+2\sqrt{2}\right)v''-q^*_{\alpha\beta}\frac{\partial^2 v}{\partial z_\alpha\partial z_\beta}=0\quad\text{in}\quad\omega\times(0,T)\\ v=0\quad\text{on}\quad\partial\omega\times(0,T)\\ v(0)=\frac{v^0}{2+2\sqrt{2}\ \text{ in }\ \omega}\\ v'(0)=\frac{v^1}{2+2\sqrt{2}}\quad\text{in}\quad\omega.\end{array}\tag{4}$$

The coefficients $q^*_{\alpha\beta}$ are the main results of the article [3] and have the form

$$\begin{array}{l}q^*_{11}=D\left[\frac{1}{A_{22}}+\frac{\sqrt{2}}{a_{11}-a_{13}-a_{31}+a_{33}}+\frac{\sqrt{2}}{a_{11}+a_{22}+a_{33}+a_{13}+a_{22}+a_{31}}\right]\\ q^*_{22}=D\left[\frac{1}{A_{11}}+\frac{\sqrt{2}}{a_{11}-a_{13}-a_{31}+a_{33}}+\frac{\sqrt{2}}{a_{11}+a_{22}+a_{33}+a_{13}+a_{22}+a_{31}}\right]\\ q^*_{12}=q^*_{21}=\sqrt{2}D\left[\frac{1}{a_{11}-a_{13}-a_{31}+a_{33}}-\frac{1}{a_{11}+2a_{22}+a_{33}+a_{13}+a_{31}}\right].\end{array}\tag{5}$$

where:
$D=\det A,\quad \left(A_{ij}\right)_{i,j}=\left(\left(a_{ij}\right)_{i,j}\right)^{-1}$*, and* A_{11},A_{22} *are algebraic complements.*

Proof. Multiply the first equation of the system (3) with $v'_{k\delta}$ and integrate by parts on $\omega\times(0,T)$

$$\frac{1}{2}\int_0^T \frac{d}{dt}\left[\int_\omega (\text{meas } Y_\delta)\left(v'_{k\delta}\right)^2 dz\right]dt + \frac{1}{2}\int_0^T \frac{d}{dt}\left[\int_\omega q_{\alpha\beta}^{\delta k}\frac{\partial v_{k\delta}}{\partial z_\alpha}\frac{\partial v_{k\delta}}{\partial z_\beta}dz\right]dt = 0. \quad (6)$$

Noting the energy of the system (3) by

$$E_\delta(t) = \frac{1}{2}\int_\omega (\text{meas } Y_\delta)\left(v'_{k\delta}\right)^2 dz + \frac{1}{2}\int_\omega q_{\alpha\beta}^{\delta k}\frac{\partial v_{k\delta}}{\partial z_\alpha}\frac{\partial v_{k\delta}}{\partial z_\beta}dz. \quad (7)$$

From the relation (6) obtain

$$E_\delta(T) = E_\delta(0)$$

which implies based on conservation of energy

$$E_\delta(t) \le E_\delta(0) = \frac{1}{2(\text{meas } Y_\delta)}\left\|v_\delta^{1,*}\right\|_{L^2(\omega)}^2 + \frac{1}{2(\text{meas } Y_\delta)^2}\int_\omega q_{\alpha\beta}^{\delta k}\frac{\partial v_{k\delta}^0}{\partial z_\alpha}\frac{\partial v_{k\delta}^0}{\partial z_\beta}dz$$

and applying Holder's inequality we have

$$E_\delta(t) \le \frac{1}{2(\text{meas } Y_\delta)}\left\|v_\delta^{1,*}\right\|_{L^2(\omega)}^2 + \frac{1}{2(\text{meas } Y_\delta)^2}q_{\alpha\beta}^{\delta k}\left\|\frac{\partial v_\delta^0}{\partial z_\alpha}\right\|_{L^2(\omega)}\cdot\left\|\frac{\partial v_\delta^0}{\partial z_\beta}\right\|_{L^2(\omega)}. \quad (8)$$

From $\left\|v_{\varepsilon\delta}^0\right\|_{\varepsilon\delta} \le c\delta$ and $\tilde{v}_{\varepsilon\delta}^0 \underset{\varepsilon}{\rightarrow} v_\delta^0$ weak in $H_0^1(\Omega)$ we have

$$\left\|\frac{\partial v_\delta^0}{\partial z_\alpha}\right\|_{L^2(\omega)} \le c\delta, \quad (9)$$

and from $\left\|v_{\varepsilon\delta}^1\right\|_{L^2(\Omega_{\varepsilon\delta})} \le c\delta$ and $\tilde{v}_{\varepsilon\delta}^1 \underset{\varepsilon}{\rightarrow} v_\delta^1$ weak in $L^2(\Omega)$ we have

$$\left\|v_\delta^1\right\|_{L^2(\Omega)} \le c\delta \Longrightarrow \left\|v_\delta^{1,*}\right\|_{L^2(\omega)} \le c\delta. \quad (10)$$

Using the estimations (9) and (10) in (8) we find

$$E_\delta(t) \le \frac{c\delta}{2+2\sqrt{2}} + \frac{\delta^{-1}q_{\alpha\beta}^{\delta k}}{\left(2+2\sqrt{2}\right)^2}c\delta, \quad (11)$$

and using the method of dilatation have

$$\begin{array}{c} \delta^{-1}q_{\alpha\beta}^{\delta k} \rightarrow q_{\alpha\beta}^* \\ c_1 \le \delta^{-1}q_{\alpha\beta}^{\delta k} \le c_2 \end{array} \quad (12)$$

where c_1 and c_2 are independent of δ.

From the second equation of the relation (12) obtain in the estimation (11)

$$E_\delta(t) \leq c\delta$$

and considering the energy expression we obtain

$$\delta^{-1}(\text{meas } Y_\delta)\left\|v'_{k\delta}\right\|^2_{L^2(\omega)} + \delta^{-1}q^{\delta k}_{\alpha\beta}\int_\omega \frac{\partial v_{k\delta}}{\partial z_\alpha}\frac{\partial v_{k\delta}}{\partial z_\beta}dz \leq c. \tag{13}$$

From the first equation of the relation (12)

$$\delta^{-1}q^{\delta k}_{\alpha\beta} = q^*_{\alpha\beta} + \theta^{\delta k}_{\alpha\beta} \tag{14}$$

with

$$\theta^{\delta k}_{\alpha\beta} \xrightarrow[\delta\to 0]{} 0.$$

We choose a δ small enough

$$\theta^{\delta k}_{\alpha\beta} > -\frac{1}{2}q^*_{\alpha\beta}$$

and considering (14) and the elliptic coefficients $q^*_{\alpha\beta}$, the estimation (13) becomes

$$\left(2+2\sqrt{2}\right)(1-\delta)\left\|v'_{k\delta}\right\|^2_{L^2(\omega)} + \frac{c_0}{2}\left\|v_{k\delta}\right\|^2_{H^1(\omega)} \leq c$$

from where

$$\left\|v_{k\delta}\right\|_{H^1(\omega)} \leq c$$
$$\left\|v'_{k\delta}\right\|_{L^2(\omega)} \leq c$$

estimates that provides convergences (c) and (d).

From $\left\|v^0_{\varepsilon\delta}\right\|_{\varepsilon\delta} \leq c\delta$ and $\tilde{v}^0_{\varepsilon\delta} \underset{\varepsilon}{\rightarrow} v^0_\delta$ weak in $H^1_0(\Omega)$ we have estimation

$$\left\|v^0_\delta\right\|_{H^1(\omega)} \leq c\delta$$

which together with (10) give us convergences (a) and (b).

Now, we establish the limit problem.

Multiply the first equation of the system (3) with ϕw, $\phi \in \mathrm{D}(\omega)$, $w \in \mathrm{D}(0,T)$. Integrate by parts on $\omega \times (0,T)$, pass to the limit as $\delta \to 0$ with (c) and the first equation of the relation (12), and then integrate by parts

$$\int_0^T\int_\omega \left(2+2\sqrt{2}\right)v''\phi w dz dt - \int_0^T\int_\omega q^*_{\alpha\beta}\frac{\partial^2 v}{\partial z_\alpha \partial z_\beta}dz dt = 0.$$

Meaning the first equation of system (4):

$$\left(2+2\sqrt{2}\right)v'' - q^*_{\alpha\beta}\frac{\partial^2 v}{\partial z_\alpha \partial z_\beta} = 0 \quad \text{in} \quad \omega\times(0,T).$$

Passing to the limit in the third equation and the fourth equation of the system (3), with the convergences (a), (b), (c) and (d) we obtain the initial conditions. ∎

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EXISTENCE RESULTS FOR A SEMILINEAR EVOLUTION SYSTEM INVOLVING MEASURES

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Abstract In this paper we prove some existence results for $\mathcal{L}^\infty$-solutions to a semilinear evolution system of the form :

$$\begin{cases} du = (Au + F(u,v))\,dt + df, \ t \in [0,+\infty[\\ dv = (Bv + G(u,v))\,dt + dg, \ t \in [0,+\infty[\\ u(0) = u_0 \\ v(0) = v_0. \end{cases}$$

where $A : D(A) \subseteq X \to X$ generates a C_0-semigroup of contractions $\{S_A(t)\,;t \geq 0\}$ in a real Banach space $(X, \|\cdot\|_X)$, $B : D(B) \subseteq Y \to Y$ generates a C_0-semigroup of contractions $\{S_B(t)\,;t \geq 0\}$ in a real Banach space $(Y, \|\cdot\|_Y)$, $F : X \times Y \to X$, $G : X \times Y \to Y$, $f \in BV([0,+\infty[\,;X)$, $g \in BV([0,+\infty[\,;Y)$, $u_0 \in X$ and $v_0 \in Y$. The proofs are essentially based on an interplay between compactness and Lipschitz type arguments.

Keywords: reaction-diffusion system, compact semigroup, function of bounded variation.
2010 MSC: Primary 47*J*35, 35*K*37, 35*K*45, Secondary 47*D*03.

1. INTRODUCTION

The purpose of this paper is to prove some local and global existence results concerning $\mathcal{L}^\infty$-solutions to semilinear evolution system of the type

$$\begin{cases} du = (Au + F(u,v))\,dt + df, \ t \in [0,+\infty[\\ dv = (Bv + G(u,v))\,dt + dg, \ t \in [0,+\infty[\\ u(0) = u_0 \\ v(0) = v_0, \end{cases} \tag{1}$$

where $A : D(A) \subseteq X \to X$ generates a C_0-semigroup of contractions $\{S_A(t)\,;t \geq 0\}$ in the real Banach space $(X, \|\cdot\|_X)$, $B : D(B) \subseteq Y \to Y$ generates a C_0-semigroup of contractions $\{S_B(t)\,;t \geq 0\}$ in the real Banach space $(Y, \|\cdot\|_Y)$, $F : X \times Y \to X$, $G : X \times Y \to Y$, $f \in BV([0,+\infty[\,;X)$, $g \in BV([0,+\infty[\,;Y)$, $u_0 \in X$ and $v_0 \in Y$. An example of a reaction-diffusion system with measures, in which the Dirac measure is concentrated at point, is also included. A special case is that when $A \equiv 0$, it is the case when the diffusion process in the first equation is absent. In this situation we shall say that the corresponding system for (1) is *semidiffusive*.

Recall that Ahmed [1]-[4] had considered the question of existence of measure valued solutions for semilinear systems, but not necessarily of reaction-diffusion type, in the case when A, B generates analytic semigroups. Also, Ahmed [1]-[4] had considered some applications of the existence results in optimal control theory and minimax problems for certain dynamical systems.

Bidaut-Véron, García-Huidobro, Yarur [11] studied a semilinear parabolic system with absorption terms in a bounded domain of $\mathbb{R}^n$, with Dirichlet or Neumann condition, and proved the existence and uniqueness of the Cauchy problem when the initial data are L^1-function or bounded measures.

An abstract reaction-diffusion system with measures has been considered in Amann, Quittner [5], in the case when A, B generates analytic semigroups. For weakly or strongly coupled parabolic systems, they have defined the solution using the technique of interpolation-extrapolation spaces and the Riesz representation theorem for a bounded Banach-space-valued Radon measure, obtained an existence theorem for global solutions and show that the solution depends Lipschitz continuously on the data. In Amann, Quittner [6] these results were applied in the optimal control theory for parabolic systems involving measures .

In the case when $f \equiv g \equiv 0$ and A, B are m-dissipative (possible nonlinear) operators, the problem (1) was recently studied by Burlică, Roşu [12]. For de case when $f \equiv g \equiv 0$, $A = \Delta\varphi$, $B = \Delta\psi$ (Δ is the Laplace operator), where $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing functions with $\varphi(0) = 0, \psi(0) = 0$ and F, G satisfy various conditions, the corresponding parabolic reaction-diffusion system has been considered by Díaz, Vrabie [18]. Moreover, Díaz, Vrabie [18], [25] follow a different strategy, based upon the compactness of the generalized Green operator associated to the nonlinear diffusion equation in [17]. After that, a similar strategy was used in many papers. We note also that a similar strategy was used in Burlică, Roşu [12] and is used in our article as well.

For $\varphi(u) = au$, $\psi(v) = bv$, where $a > 0$, $b > 0$ (a special case for our system (1)), Kouachi [22] obtained the existence of global solutions to a reaction-diffusion systems, with Neumann condition on the boundary, via a Lyapunov functional. Badraoui [7] proved existence of global solutions for the thermal-diffusive combustion system on unbounded domains, using an abstract theory with analytic semigroups. In Badraoui [8] the asymptotic behavior of solutions to a reaction-diffusion system with unbounded domains was studied.

We notice that special reaction-diffusion systems, not necessarily with measures, arise as mathematical models in Climatology (see Díaz, Muñoz, Schiavi [14], Díaz, Schiavi [16]), Medicine (see Díaz, Tello [15], Maddalena [23]), Biology, Population Dynamics, and the list can be continued.

The paper is divided into four sections, the second section being concerned with the introduction of the $\mathcal{L}^\infty$-solution for linear Cauchy problem involving measures and such basic properties of the $\mathcal{L}^\infty$-solution as regularity and compactness in $L^p(a, b; X)$. The results are taken up from Vrabie [26], [27]. Also, in the second section we introduce the $\mathcal{L}^\infty$-solution and present some existence results for a semillinear Cauchy problem involving measures. The results are from Grosu [19], [21]. Section 3 contains the statement and proof of our main results. Section 4 presents a significant example of a reaction-diffusion system in which the Dirac measure is concentrated at point.

2. PRELIMINARIES

We assume familiarity with the basic concepts and results concerning C_0 -semigroups and infinite-dimensional vector-valued functions of bounded variation and we refer to Barbu and Precupanu [10], Pazy [24] and Vrabie [27] for details. First, we recall for easy reference some results established in Vrabie [26], [27] and Grosu [19], [21].

Let $\mathcal{P}([a,b])$ be the set of all partitions of the interval $[a,b]$. We recall that, if $g : [a,b] \to X$ then, for each $\mathcal{P} \in \mathcal{P}([a,b])$, $\mathcal{P} : a = t_0 < t_1 < ... < t_k = b$, the number

$$Var_{\mathcal{P}}(g,[a,b]) = \sum_{i=0}^{k-1} \|g(t_{i+1}) - g(t_i)\|$$

is called the *variation of the function g relatively to the partition* $\mathcal{P}$. If

$$\sup_{\mathcal{P}\in\mathcal{P}([a,b])} Var_{\mathcal{P}}(g,[a,b]) < +\infty,$$

then g is said to be *of bounded variation*, and the number

$$Var(g,[a,b]) = \sup_{\mathcal{P}\in\mathcal{P}([a,b])} Var_{\mathcal{P}}(g,[a,b])$$

is called *the variation* of the function g on the interval $[a,b]$. We denote by $BV([a,b];X)$ the vector space of all function of bounded variation from $[a,b]$ to X. Also, we denote by $BV(\mathbb{R};X)$ the space of all functions $g : \mathbb{R} \to X$ whose restrictions to any interval $[a,b]$ belong to $BV([a,b];X)$.

Proposition 2.1. *If $g \in BV([a,b];X)$, then g is piecewise continuous on $[a,b]$, i.e. there exists an at most countable subset E of $[a,b]$, such that g is continuous on $[a,b] \setminus E$ and, at each $t \in E \cap [a,b[$ and each $s \in E \cap]a,b]$, there exists $g(t+0)$ and $g(s-0)$.*

See Vrabie [27], Proposition 1.4.2*, p.* 14.

Definition 2.1. *A family $\mathcal{G}$ in $BV([a,b];X)$ is of equibounded variation on $[a,b]$ if there exists $m_{\mathcal{G}} > 0$ such that, for each $g \in \mathcal{G}$, we have*

$$Var(g,[a,b]) \le m_{\mathcal{G}}.$$

However, for simplicity reasons, we preferred to consider only C_0-semigroups of contractions. Note that all the results which will follow hold true also for the general case of C_0-semigroups not necessarily of contractions. Let $g \in BV([a,b];X)$ and let $\{S(t);t \geq 0\}$ be a C_0-semigroup of contractions in a Banach space X such that the semigroup is continuous from $]0,\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology. Then, for each $t \in]a,b]$, there exists a unique element $\int_a^t S(t-s)\,dg(s) \in X$ such that

$$\int_a^t S(t-s)\,dg(s) = \lim_{\mu(\mathcal{P})\downarrow 0} \sum_{i=0}^{k-1} S(t-\tau_i)(g(t_{i+1}) - g(t_i))$$

and it is called *the Riemann-Stieltjes integral* on $[a,b]$ of the operator-valued function $\tau \mapsto S(t-\tau)$ with respect to the vector-valued function g. See Vrabie [27], p. 205, p. 206 and Theorem 9.1.1, p. 208.

Remark 2.1. *Since $\{S(t);t \geq 0\}$ is a C_0-semigroup of contractions, whenever $\int_a^t S(t-s)\,dg(s) \in X$, we have*

$$\left\| \int_a^t S(t-s)\,dg(s) \right\| \leq Var(g,[a,t]) .$$

for each $t \in]a,b]$.

Remark 2.2. *For each $c \in [a,b[$, and each $\delta > 0$ such that $c+\delta \in [a,b]$, and each $t \in [c+\delta,b]$, we have*

$$\int_c^{c+\delta} S(t-s)\,dg(s) = \int_c^{c+\delta} \chi_{]c,c+\delta]} S(t-s)\,dg(s) + S(t-c)(g(c+0) - g(c)),$$

where $\chi_{]c,c+\delta]}$ denotes the characteristic function of $]c,c+\delta]$.

See Vrabie [27], Remark 9.1.1, p. 207.

Next, let us consider the nonhomogeneous Cauchy problem

$$\begin{cases} du = (Au)\,dt + dg \\ u(a) = \xi, \end{cases} \tag{2}$$

where $A : D(A) \subseteq X \to X$ generates a C_0-semigroup of contractions $\{S_A(t);t \geq 0\}$ in the real Banach space X, $\xi \in X$ and $g \in BV([a,b];X)$.

Definition 2.2. *Let A be the infinitesimal generator of a C_0-semigroup of contractions $\{S(t);t \geq 0\}$, which is continuous from $]0,\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology. The function $u : [a,b] \to X$ given by*

$$u(t) = S(t-a)\xi + \int_a^t S(t-s)\,dg(s) \tag{3}$$

for each $t \in [a,b]$ is called an $\mathcal{L}^\infty$-solution on $[a,b]$ of the problem (2).

Remark 2.3. *We notice that each $\mathcal{L}^{\infty}$-solution u satisfies*

$$\|u(t)\| \leq \|\xi\| + Var(g, [a,b]),$$

for each $t \in [a,b]$.

Theorem 2.1. (*Regularity of $\mathcal{L}^{\infty}$-solutions*) *Let $g \in BV([a,b];X)$ and $(a,\xi) \in \mathbb{R} \times X$. Let A be the infinitesimal generator of a C_0-semigroup of contractions which is continuous from $]0,\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology and let u be the $\mathcal{L}^{\infty}$-solution of* (2) *corresponding to ξ and g. Then, for each $t \in [a,b[$ and each $s \in]a,b]$, there exists $u(t+0)$ and $u(s-0)$ and*

$$\begin{cases} u(t+0) - u(t) = g(t+0) - g(t) \\ u(s) - u(s-0) = g(s) - g(s-0). \end{cases} \tag{4}$$

So, u is continuous from the right (*left*) *at $t \in [a,b[$* ($t \in]a,b]$) *if and only if g is continuous from the right* (*left*) *at t. In particular, u is continuous at any point at which g is continuous and thus u is piecewise continuous on $[a,b]$.*

See Vrabie [27], Theorem 9.2.1*, p.* 210.

In that follows, we assume that A is the infinitesimal generator of a C_0-semigroup of contractions which is continuous from $]0,\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology and then, for each $(\xi, g) \in X \times BV([a,b];X)$, the Cauchy problem (2) has a unique $\mathcal{L}^{\infty}$-solution u. Furthermore, for $p \in [1,+\infty[$, we denote by $Q : X \times BV([a,b];X) \to L^p(a,b;X)$,

$$Q(\xi, g) = u$$

the $\mathcal{L}^{\infty}$-solution operator.

Theorem 2.2. *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a compact C_0-semigroup of contractions $\{S_A(t); t \geq 0\}$, let $\mathcal{D}$ be a bounded subset in X and $\mathcal{G}$ a subset in $BV([a,b];X)$ of equibounded variation. Then, for each $p \in [1,+\infty[$, $Q(\mathcal{D},\mathcal{G}) = \{Q(\xi,g); (\xi,g) \in \mathcal{D} \times \mathcal{G}\}$ is relatively compact subset in $L^p(a,b;X)$.*

See Vrabie [27], Theorem 9.4.2*, p.* 219.

In order to formulate our existence results, we introduce the notion of $\mathcal{L}^{\infty}$-solution of a semilinear equation involving measures, as in Grosu [20]. Next, let A be as above, let $D \subset \mathbb{R} \times X$ be a nonempty and open subset and $f : D \to X$ a continuous function. Let $g \in BV(\mathbb{R};X)$ and $(a,\xi) \in \mathbb{R} \times X$. Let us consider the Cauchy problem :

$$\begin{cases} du = (Au + f(t,u))\,dt + dg \\ u(a) = \xi \end{cases} \tag{5}$$

Definition 2.3. *A function $u : [a,c] \to X$ is called an $\mathcal{L}^{\infty}$-solution on $[a,c]$ of the problem* (5) *if:*

(i) *for each $t \in [a,c[$ there exists $u(t+0)$;*

(ii) *for each* $t \in [a, c[$, $(t, u(t+0)) \in D$;

(iii) $t \to f(t, u(t+0))$ *is in* $L^1(a, c; X)$ *and* u *is an* $\mathcal{L}^\infty$*-solution on* $[a, c]$ *in the sense of Definition* 2.1 *for the following Cauchy problem*

$$\begin{cases} du = (Au)\,dt + dh \\ u(a) = \xi, \end{cases}$$

where $h : [a, c] \to X$ *is defined by*

$$h(t) = \int_a^t f(s, u(s+0))\,ds + g(t), \tag{6}$$

for all $t \in [a, c]$.

We define the L^∞*-solution of* (1) *only on a semi-open interval* $[a, c[$ *by requiring* (*i*), (*ii*), (*iii*) *as above, except for the condition "*$t \to f(t, u(t+0))$ *is in* $L^1([a, c[\,; X)$*" which should be relaxed to "*$t \to f(t, u(t+0))$ *is in* $L^1_{\text{loc}}([a, c[\,; X)$*".*

Remark 2.4. *By Theorem* 2.1 *we observe that, if* $A : D(A) \subseteq X \to X$ *is the infinitesimal generator of a* C_0*-semigroup of contractions which is continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology and* $u : [a, c] \to X$ *is an* $\mathcal{L}^\infty$*-solution of the problem* (5) *on* $[a, c]$*, then* u *is piecewise continuous on* $[a, c]$. *Since* $f : D \to X$ *is continuous, then* $f(t, u(t+0)) = f(t, u(t))$ *a.e. on* $[a, c]$. *Thus, in* (6)*,* h *is given in fact by*

$$h(t) = \int_a^t f(s, u(s))\,ds + g(t),$$

for each $t \in [a, c]$.

Let us remark also that, in Definition 2.3*, we ask* $(t, u(t+0)) \in D$*, for all* $t \in [a, c[$ *instead of* $(t, u(t)) \in D$*, for all* $t \in [a, c]$ (*or* $[a, c[$)*, as in the definition of a* C^0*-solution. We have to impose that condition because, for certain* L^∞*-solutions, it may happen that, at some point of discontinuity* $t \in]a, c]$ *for* u, $(t, u(t-0)) \notin D$. *Since, by Theorem* 2.1*,* $(t, u(t))$ *is uniquely determined by the jump condition* $u(t) - u(t-0) = g(t) - g(t-0)$*, then* $(t, u(t))$ *might be, not only outside* D*, but even outside its closure.*

Lemma 2.1. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contractions,* $\{S_A(t)\,; t \geq 0\}$*, continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology and let* $g \in BV([a, b]\,; X)$. *If* $f : [a, b] \times X \to X$ *is continuous, bounded and globally Lipschitz with respect to its second argument then, for each* $\xi \in X$*, the problem* (1) *has a unique* $\mathcal{L}^\infty$*-solution defined on* $[a, b]$.

See Grosu [21], Lemma 2.1.

Theorem 2.3. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contraction* $\{S_A(t)\,; t \geq 0\}$ *which is continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology. Let us assume that* $D \subset \mathbb{R} \times X$ *is a nonempty and open* *subset in* $\mathbb{R} \times X$ *and* $f : D \to X$ *is a continuous function which is locally Lipschitz with respect to the second variable. Then, for each* $(a, \xi) \in \mathbb{R} \times X$ *with*

$$(a, g(a+0) - g(a) + \xi) \in D,$$

there exists $c > a$ *such that the Cauchy problem* (5) *has a unique* $\mathcal{L}^\infty$*-solution on* $[a, c]$ (*or on* $[a, c[$) *in the sense of Definition* 2.3.

See Grosu [19], Theorem 2.2 *and Vrabie [27], Theorem* 12.1.2*, p.* 272.

Theorem 2.4. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a compact* C_0*-semigroup of contraction* $\{S_A(t)\,; t \geq 0\}$*. Let* $D \subset \mathbb{R} \times X$ *be nonempty and open and* $f : D \to X$ *a continuous function. Then for each* $(a, \xi) \in \mathbb{R} \times X$ *with*

$$(a, g(a+0) - g(a) + \xi) \in D$$

there exists $c > a$ *such that the Cauchy problem* (5) *has at least one* $\mathcal{L}^\infty$*-solution on* $[a, c]$ (*or on* $[a, c[$) *in the sense of Definition* 2.3.

See Grosu [19], Theorem 3.2 *and Vrabie [27], Theorem* 12.2.2*, p.* 275.

Definition 2.4. *An* $\mathcal{L}^\infty$*-solution* $u : \mathbb{I} \to X$ *of* (5)*, with* $\mathbb{I} = [a, c[$ $(\mathbb{I} = [a, c])$ *is continuable if there exists another* $\mathcal{L}^\infty$*-solution of* (5)*,* $v : [a, b] \to X$*, with* $b \geq c$ $(b > c)$*, such that* $u(t) = v(t)$*, for each* $t \in \mathbb{I}$*. If* $b > c$*, the* $\mathcal{L}^\infty$*-solution* u *is called strictly continuable. A* $\mathcal{L}^\infty$*-solution is called saturated* (*non-continuable*) *if it is not continuable. If the projection of* D *on* $\mathbb{R}$ *contained* $\mathbb{R}_+$*, a* $\mathcal{L}^\infty$*-solution* u *is called global if it is defined on* $[a, +\infty[$.

Lemma 2.2. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contractions, let* $f : D \to X$ *be continuous and* $g \in BV(\mathbb{R}; X)$*. Let us assume that either the semigroup is compact, or* f *is locally Lipschitz with respect to the second variable and the semigroup is continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology. An* $\mathcal{L}^\infty$*-solution,* $u : [a, c[\to X$*, of* (5) *is*

(i) continuable with $b = c$ *if and only if there exists* $u(c-0) = \lim\limits_{t \uparrow c} u(t)$ *and*

$$(c, g(c+0) - g(c-0) + u(c-0)) \notin D;$$

(ii) strictly continuable if and only if there exists $u(c-0) = \lim\limits_{t \uparrow c} u(t)$ *and*

$$(c, g(c+0) - g(c-0) + u(c-0)) \in D.$$

See Grosu [19], Lemma 4.1 *and Vrabie [27], Lemma* 12.3.1*, p.* 277.

Theorem 2.5. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contractions, let* $f : D \to X$ *be continuous and* $g \in BV(\mathbb{R}; X)$*. Let us assume that either the semigroup is compact, or* f *is locally Lipschitz with respect* *to its second argument and the semigroup is continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology. If* $u : \mathbb{J} \to X$ *is an* $\mathcal{L}^\infty$*-solution of* (5)*, with* $\mathbb{J} = [a, c[$ *or* $\mathbb{J} = [a, c]$*, then either* u *is saturated, or* u *can be continued up to a saturated one.*

See Grosu [19], Theorem 4.1 *and Vrabie [27], Theorem* 12.3.1*, p.* 278.

Theorem 2.6. *Let* $A : D(A) \subseteq X \to X$ *be the infinitesimal generator of a* C_0*-semigroup of contractions, let* $f : \mathbb{R}_+ \times X \to X$ *be continuous and let* $g \in BV(\mathbb{R}; X)$*. Let us assume that either the semigroup is compact, or* f *is locally Lipschitz with respect to its last argument and the semigroup is continuous from* $]0, +\infty[$ *to* $\mathcal{L}(X)$ *in the uniform operator topology. Further, let us assume that there exist two continuous functions* $h, k : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|f(t, u)\| \leq k(t)\|u\| + h(t) \tag{7}$$

for each $(t, u) \in \mathbb{R}_+ \times X$*. Then, for each* $(a, \xi) \in \mathbb{R}_+ \times X$*,* (5) *has at least one global* $\mathcal{L}^\infty$*-solution* $u : [a, +\infty[\to X$ *(or, for each* $(a, \xi) \in \mathbb{R}_+ \times X$*, then each saturated* $\mathcal{L}^\infty$*-solution* $u : [a, c[\to X$ *of* (5) *is global, i.e.* $c = +\infty$*).*

See Grosu [19], Theorem 4.3 *and Vrabie [27], Theorem* 12.3.3*, p.* 281.

3. EXISTENCE RESULTS FOR SEMILINEAR SYSTEMS

We define the $\mathcal{L}^\infty$-solution of the problem (1).

Definition 3.1. *A function* $(u, v) : [0, T] \to X \times Y$ *is called an* $\mathcal{L}^\infty$*-solution on* $[0, T]$ *of the problem* (1) *if :*

(i) *for each* $t \in [0, T[$*, there exists* $u(t + 0)$ *and* $v(t + 0)$ *;*

(ii) $t \mapsto F(u(t + 0), v(t + 0))$ *is in* $L^1(0, T; X)$*,* $t \mapsto G(u(t + 0), v(t + 0))$ *is in* $L^1(0, T; Y)$*, and* u *is an* $\mathcal{L}^\infty$*-solution on* $[0, T]$ *in the sense of Definition* 2.3 *for the following Cauchy problem*

$$\begin{cases} du = (Au + F(u, v))\,dt + df \\ u(0) = u_0, \end{cases}$$

while v *is an* $\mathcal{L}^\infty$*-solution on* $[0, T]$ *in the sense of Definition* 2.3 *for the Cauchy problem*

$$\begin{cases} dv = (Bv + G(u, v))\,dt + dg \\ v(0) = v_0. \end{cases}$$

We define the $\mathcal{L}^\infty$*-solution of the problem* (1) *on a semi-open interval* $[0, T[$ *by requiring* (*i*)*,* (*ii*) *in Definition* 3.1 *except for the condition "*$t \mapsto F(u(t + 0), v(t + 0))$ *is in* $L^1(0, T; X)$ *,* $t \mapsto G(u(t + 0), v(t + 0))$ *is in* $L^1(0, T; Y)$*" which should be relaxed to "*$t \mapsto F(u(t + 0), v(t + 0))$ *is in* $L^1_{\text{loc}}([0, T[\,; X)$*,* $t \mapsto G(u(t + 0), v(t + 0))$ *is in* $L^1_{\text{loc}}([0, T[\,; Y)$*".*

The purpose of this section is to prove an existence result concerning $\mathcal{L}^\infty$-solutions to a reaction-diffusion system of the form (1), where f, g, A, B, F, G satisfy the hypotheses :

(H_0) $f \in BV([0,+\infty[\,;X)$ and $g \in BV([0,+\infty[\,;Y)$;

(H_1) $A : D(A) \subseteq X \to X$ generates a C_0-semigroup of contractions $\{S_A(t)\,;t \geq 0\}$ in a real Banach space $(X, \|\cdot\|_X)$, which is continuous from $]0,+\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology ;

(H_2) $B : D(B) \subseteq Y \to Y$ generates a compact C_0-semigroup of contractions $\{S_B(t)\,;t \geq 0\}$ in a real Banach space $(Y, \|\cdot\|_Y)$;

(H_3) $F : X \times Y \to X$ is continuous and locally Lipschitz with respect to its first variable, i.e. for each (η_1, η_2) in $X \times Y$, there are $\rho_F = \rho_F(\eta_1, \eta_2) > 0$ and $l_F = l_F(\eta_1, \eta_2) > 0$ such that

$$\|F(u,v) - F(\widetilde{u}, v)\|_X \leq l_F \|u - \widetilde{u}\|_X \,,$$

for each $u, \widetilde{u} \in D(\eta_1, \rho_F)$ and $v \in D(\eta_2, \rho_F)$ ($D(\eta, \rho)$ is the open ball of radius ρ, centered at η);

(H_4) $G : X \times Y \to Y$ is a continuous mapping ;

(H_5) There exist $a_i > 0$, $b_i > 0$, $c_i > 0$, $i \in \{1,2\}$ such that

$$\|F(u,v)\|_X \leq a_1 \|u\|_X + b_1 \|v\|_Y + c_1,$$

for each $(u,v) \in X \times Y$, and

$$\|G(u,v)\|_Y \leq a_2 \|u\|_X + b_2 \|v\|_Y + c_2,$$

for each $(u,v) \in X \times Y$.

Namely, we will prove

Theorem 3.1. *Assume that* (H_0), (H_1), (H_2), (H_3) *and* (H_4) *are satisfied. Then, for each* $(u_0, v_0) \in X \times Y$, *there exists* $T_0 > 0$ *such that* (1) *has at least one* $\mathcal{L}^\infty$*-solution* $(u,v) : [0, T_0] \to X \times Y$. *If, in addition,* (H_5) *is satisfied, then* (1) *has at least one global* $\mathcal{L}^\infty$*-solution* $(u,v) : [0,+\infty[\to X \times Y$.

We mention here that the proof of Theorem 3.1 and of the other theorems which we need for this proof (Theorem 3.2, Theorem 3.3 bellow) is essentially based on an interplay between compactness and Lipschitz type arguments.

First, let us consider the specific case when F and G satisfy :

(H_3') $F : X \times Y \to X$ is continuous, bounded, i.e. there exists $m_F > 0$ such that

$$\|F(u,v)\|_X \leq m_F,$$

for each $u \in X, v \in Y$, and globally Lipschitz with respect to its first variable, i.e. there exists $L_F > 0$ such that

$$\|F(u,v) - F(\widetilde{u},v)\|_X \le L_F \|u - \widetilde{u}\|_X ,$$

for each $u, \widetilde{u} \in X$ and $v \in Y$;

(H_4') $G : X \times Y \to Y$ is continuous, bounded, i.e. there exists $m_G > 0$ such that

$$\|G(u,v)\|_Y \le m_G,$$

for each $u \in X, v \in Y$, and globally Lipschitz with respect to its second variable, i.e. there exist $L_G > 0$ such that

$$\|G(u,v) - G(u,\widetilde{v})\|_Y \le L_G \|v - \widetilde{v}\|_Y ,$$

for each $u \in X$ and $v, \widetilde{v} \in Y$.

Theorem 3.2. *Assume that* (H_0), (H_1), (H_2), $\left(H_3'\right)$ *and* $\left(H_4'\right)$ *are satisfied. Then, for each* $(u_0, v_0) \in X \times Y$, *and each* $T > 0$ *there exists at least one* $\mathcal{L}^\infty$*-solution* $(u,v) : [0,T] \to X \times Y$ *of* (1).

Proof. The idea of the proof consists in showing that a suitable defined operator has at least one fixed point, whose existence is equivalent with the existence of at least one $\mathcal{L}^\infty$-solution of (1) .

To begin with, let us fix $(u_0, v_0) \in X \times Y$ and $T > 0$.

Since $g \in BV([0,+\infty[\,;Y)$, there exist $m_g > 0$ such that, for each $t \in [0,T]$, we have

$$Var(g,[0,t]) \le m_g.$$

Let us define

$$r = \left[(1 + L_G T)\|v_0\|_Y + m_G T + m_g\right] e^{L_G T}$$

and

$$\mathcal{K} = \{v \in L^\infty(0,T;Y) \,;\, \|v(t)\|_Y \le r, \text{ a.e in } [0,T]\}.$$

Obviously, $\mathcal{K}$ is nonempty, bounded, closed and convex, in $L^\infty(0,T;Y)$, and thus in $L^1(0,T;Y)$, as well. In what follows, we consider $\mathcal{K}$ as a subset in $L^1(0,T;Y)$. Now, let us define $\mathcal{R} : \mathcal{K} \subset L^1(0,T;Y) \to L^1(0,T;X)$ by

$$\mathcal{R}\widetilde{v} = u \tag{8}$$

for each $\widetilde{v} \in \mathcal{K}$, where u is the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of the Cauchy problem

$$\begin{cases} du = (Au + F(u,\widetilde{v}))\,dt + df \\ u(0) = u_0, \end{cases} \tag{9}$$

in the sense of Definition 2.3. Since the C_0-semigroup of contractions $\{S_A(t)\,; t \geq 0\}$ is continuous from $]0,+\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology, F is bounded, by (3), (6) and by Remark 2.4 we obtain that $\mathcal{R}$ is well-defined and

$$(\mathcal{R}\widetilde{v})(t) = S_A(t)\,u_0 + \int_0^t S_A(t-s)\,F(u(s),\widetilde{v}(s))\,ds + \int_0^t S_A(t-s)\,df(s), \tag{10}$$

for each $t \in [0,T]$. Next, let us define the operator $\mathcal{T}{:}\mathcal{K} \subset L^1(0,T;Y) \to L^1(0,T;Y)$ by

$$\mathcal{T}\widetilde{v} = v \tag{11}$$

for each $\widetilde{v} \in \mathcal{K}$, where v is the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of the Cauchy problem

$$\begin{cases} dv = (Bv + G(u,v))\,dt + dg \\ v(0) = v_0, \end{cases} \tag{12}$$

in the sense of Definition 2.3, with u the $\mathcal{L}^\infty$-solution on $[0,T]$ of (9). Since the C_0 -semigroup of contractions $\{S_B(t)\,; t \geq 0\}$ is continuous from $]0,+\infty[$ to $\mathcal{L}(Y)$ in the uniform operator topology, G is bounded, by (3), (6) and by Remark 2.4 we obtain that $\mathcal{T}$ is well-defined and

$$(\mathcal{T}\widetilde{v})(t) = S_B(t)\,v_0 + \int_0^t S_B(t-s)\,G(u(s),v(s))\,ds + \int_0^t S_B(t-s)\,dg(s), \tag{13}$$

for each $t \in [0,T]$. We mention that, if we assume that (H_0), (H_1), (H_2), $\left(H_3'\right)$ and $\left(H_4'\right)$ are satisfied, then each one of the problems (9) and (12) has a unique $\mathcal{L}^\infty$-solution defined on $[0,T]$. See Lemma 2.1.

At this point let us observe that $\mathcal{T}$ has a unique fixed point in $L^1(0,T;Y)$ if and only if $(u,v) : [0,T] \to X \times Y$ defined by (8) and (11) is an $\mathcal{L}^\infty$-solution of the problem (1) on $[0,T]$. Thus, to complete the proof, it suffices to show that $\mathcal{T}$ satisfies the hypotheses of Schauder's Fixed Point Theorem. To this aim, we have to show that:

(i) $\mathcal{T}$ maps $\mathcal{K}$ into $\mathcal{K}$;

(ii) $\mathcal{T}$ is continuous;

(iii) $\mathcal{T}$ is compact.

(i) Let $\widetilde{v} \in \mathcal{K}$ be arbitrary. A computational argument involving (13) and Remark 2.1 shows that, for each $t \in [0,T]$, we have

$$\|(\mathcal{T}\widetilde{v})(t)\|_Y \leq \|S_B(t)\,v_0\|_Y$$

$$+\int_0^t \|S_B(t-s)\|_{\mathcal{L}(Y)}\|G(u(s),v(s))\|_Y\,ds+\left\|\int_0^t S_B(t-s)\,dg(s)\right\|_Y$$

$$\leq \|S_B(t)\|_{\mathcal{L}(Y)}\|v_0\|_Y+\int_0^t \|G(\mathcal{R}\widetilde{v}(s),v_0)\|_Y\,ds$$

$$+\int_0^t \|G(\mathcal{R}\widetilde{v}(s),\ \mathcal{T}\widetilde{v}(s))-G(\mathcal{R}\widetilde{v}(s),v_0)\|_Y\,ds+Var(g,[0,t]),$$

for each $\widetilde{v}\in\mathcal{K}$ and $t\in[0,T]$. Using $\left(H_4'\right)$ to increase the right hand side, we obtain that

$$\|(\mathcal{T}\widetilde{v})(t)\|_Y$$

$$\leq \|v_0\|_Y+m_GT+L_G\int_0^t \|(\mathcal{T}\widetilde{v})(s)-v_0\|_Y\,ds+m_g$$

$$\leq \|v_0\|_Y+m_GT+L_G\int_0^t \|(\mathcal{T}\widetilde{v})(s)\|_Y\,ds+L_G\|v_0\|_Y T+m_g,$$

for each $\widetilde{v}\in\mathcal{K}$ and $t\in[0,T]$. From Gronwall's Inequality, it follows that

$$\|(\mathcal{T}\widetilde{v})(t)\|_Y\leq\left[(1+L_GT)\|v_0\|_Y+m_GT+m_g\right]e^{L_GT}=r$$

for each $\widetilde{v}\in\mathcal{K}$ and $t\in[0,T]$. Therefore $\mathcal{T}$ maps $\mathcal{K}$ into itself.

(ii) Next, we will prove that $\mathcal{T}$ is continuous from $\mathcal{K}$ into itself, both domain and range being endowed with the induced strong topology of $L^1(0,T;Y)$. Indeed, let $(\widetilde{v}_n)_{n\in\mathbb{N}}\subset\mathcal{K}$ be such that $\widetilde{v}_n\to\widetilde{v}$ in $L^1(0,T;Y)$. For each $n\in\mathbb{N}^*$, let $\mathcal{R}\widetilde{v}_n=u_n$ be the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of

$$\begin{cases} du_n=(Au_n+F(u_n,\widetilde{v}_n))\,dt+df\\ u_n(0)=u_0,\end{cases}\tag{14}$$

and let $\mathcal{T}\widetilde{v}_n=v_n$ be the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of

$$\begin{cases} dv_n=(Bv_n+G(u_n,v_n))\,dt+dg\\ v_n(0)=v_0,\end{cases}\tag{15}$$

where u_n is the $\mathcal{L}^\infty$-solution on $[0,T]$ of (14). Moreover, let $\mathcal{R}\widetilde{v}=u$ be the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of (9) and let $\mathcal{T}\widetilde{v}=v$ be the unique $\mathcal{L}^\infty$-solution on $[0,T]$ of (12). We mention that, if we assume that (H_0), (H_1), (H_2), $\left(H_3'\right)$ and $\left(H_4'\right)$ are satisfied, then each one of the problems (9), (12), (14) and (15) has a unique $\mathcal{L}^\infty$-solution defined on $[0,T]$. See Lemma 2.1. We will prove that $v_n\to v$ in $L^1(0,T;Y)$.

First, from (10) and the corresponding relation to (14), we have that

$$\|u_n(t)-u(t)\|_X=$$

$$\left\|\int_0^t S_A(t-s)F(u_n(s),\widetilde{v}_n(s))\,ds-\int_0^t S_A(t-s)F(u(s),\widetilde{v}(s))\,ds\right\|_X\leq$$

$$\int_0^t \|S_A(t-s)\|_{\mathcal{L}(X)} \|F(u_n(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds \leq$$

$$\int_0^t \|F(u_n(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}_n(s))\|_X \, ds +$$

$$\int_0^t \|F(u(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds,$$

for each $t \in [0, T]$ and each $n \in \mathbb{N}^*$. Then, by $\left(H_3'\right)$, we obtain that

$$\|u_n(t) - u(t)\|_X \leq L_F \int_0^t \|u_n(s) - u(s)\|_X \, ds +$$

$$\int_0^T \|F(u(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds,$$

for each $t \in [0, T]$ and each $n \in \mathbb{N}^*$. From Gronwall's Inequality, we get

$$\|u_n(t) - u(t)\|_X \leq e^{L_F T} \int_0^T \|F(u(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds, \tag{16}$$

for each $t \in [0, T]$ and each $n \in \mathbb{N}^*$. We prove that

$$\lim_{n\to\infty} e^{L_F T} \int_0^T \|F(\ u(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds = 0. \tag{17}$$

Let us assume by contradiction that this is not the case. Then there would exist $\varepsilon_0 > 0$ such that, for each $n \in \mathbb{N}^*$,we have

$$e^{L_F T} \int_0^T \|F(u(s), \widetilde{v}_n(s)) - F(u(s), \widetilde{v}(s))\|_X \, ds \geq \varepsilon_0.$$

But F is continuous, $(\widetilde{v}_n)_n$ is bounded and $\widetilde{v}_n \to \widetilde{v}$ in $L^1(0, T; Y)$. There exists a subsequence $(\widetilde{v}_{n_k})_{k\in\mathbb{N}^*}$ of $(\widetilde{v}_n)_{n\in\mathbb{N}^*}$,such that $\lim\limits_{k\to\infty} \widetilde{v}_{n_k}(t) = \widetilde{v}(t)$ a.e. for $t \in [0, T]$. Therefore, we get that

$$\lim_{k\to\infty} e^{L_F T} \int_0^T \left\|F(\ u(s), \widetilde{v}_{n_k}(s)) - F(\ u(s), \widetilde{v}(s))\right\|_X \, ds = 0,$$

which contradicts the inductive hypothesis. This contradiction can be eliminated only if (17) holds. Then, by (16) we obtain that $u_n \to u$ in $L^1(0, T; X)$. Similarly, we prove that $v_n \to v$ in $L^1(0, T; Y)$.

(iii) In order to apply Schauder's fixed point Theorem, we have merely to check that $\mathcal{T}(\mathcal{K})$ is relatively compact in $L^1(0, T; Y)$. But this happens by virtue of Theorem

2.2. For each $\widetilde{v} \in \mathcal{K}$, let $\mathcal{R}\widetilde{v} = u$ be the unique $\mathcal{L}^{\infty}$ -solution of (9) and let $\mathcal{T}\widetilde{v} = v$ be the unique $\mathcal{L}^{\infty}$-solution of (12). Let us define

$$\mathcal{G} = \left\{ t \mapsto \int_0^t G\left(u\left(s\right), v\left(s\right)\right) ds + g\left(t\right); \widetilde{v} \in \mathcal{K} \right\}$$

From (H_0) and $\left(H_4'\right)$ it follows that $\mathcal{G}$ is of equibounded variation. Let us consider the problem

$$\begin{cases} dv = (Bv)\, dt + dg \\ v\left(0\right) = v_0. \end{cases} \tag{18}$$

where $g \in \mathcal{G}$. Since B generates a compact C_0-semigroup of contractions, from Theorem 2.2, it follows that $Q\left(\{0\}, \mathcal{G}\right)$ is relatively compact in $L^1\left(0, T; Y\right)$. But $\mathcal{T}(\mathcal{K}) \subseteq Q\left(\{0\}, \mathcal{G}\right)$, and so $\mathcal{T}(\mathcal{K})$ is a relatively compact set, too.

Consequently, by Schauder's fixed point Theorem, $\mathcal{T}$ has at least one fixed point $v \in \mathcal{K}$ which by means of (u, v), where $u = \mathcal{R}v$ defines an $\mathcal{L}^{\infty}$ -solution of the problem (1) on $[0, T]$, and this completes the proof. ∎

Next, we suppose that :

$\left(H_4''\right)$ $G : X \times Y \to Y$ is continuous and locally Lipschitz with respect to its second variable, i.e. for each $(\eta_1, \eta_2) \in X \times Y$, there exist $\rho_G = \rho_G\left(\eta_1, \eta_2\right) > 0$ and $l_G = l_G\left(\eta_1, \eta_2\right) > 0$ such that

$$\|G\left(u, v\right) - G(u, \widetilde{v})\|_Y \leq l_G \|v - \widetilde{v}\|_Y,$$

for each $u \in D\left(\eta_1, \rho_G\right)$ and $v, \widetilde{v} \in D\left(\eta_2, \rho_G\right)$.

Theorem 3.3. *Assume that (H_0), (H_1), (H_2), (H_3) and $\left(H_4''\right)$ are satisfied. Then, for each $(u_0, v_0) \in X \times Y$, there exists $T_0 > 0$ such that (1) has at least one $\mathcal{L}^{\infty}$-solution $(u, v) : [0, T_0] \to X \times Y$ (or $(u, v) : [0, T_0[\to X \times Y$).*

Proof. We will rewrite (1) for a suitable chosen functions and we will apply Theorem 2.2.

First, let $(u_0, v_0) \in X \times Y$ be arbitrary and let us denote $\eta_1 = f\left(0 + 0\right) - f\left(0\right) + u_0$, $\eta_2 = g\left(0 + 0\right) - g\left(0\right) + v_0$. Let $\rho_F = \rho_F\left(\eta_1, \eta_2\right) > 0$, $\rho_G = \rho_G\left(\eta_1, \eta_2\right) > 0$, $l_F = l_F\left(\eta_1, \eta_2\right) > 0$ and $l_G = l_G\left(\eta_1, \eta_2\right) > 0$ be given by (H_3) and $\left(H_4''\right)$. Let us define

$$r = \min\left\{\rho_F, \rho_G\right\}.$$

As F and G are continuous, they are locally bounded and, diminishing r if necessary, we may assume that there exists $m_F = m_F\left(\eta_1, \eta_2\right) > 0$ and $m_G = m_G\left(\eta_1, \eta_2\right) > 0$ such that

$$\|F\left(u, v\right)\|_X \leq m_F \text{ and } \|G\left(u, v\right)\|_Y \leq m_G,$$

for each $(u,v) \in D(\eta_1, r) \times D(\eta_2, r)$. Let us define $\Pi_X : X \to D(\eta_1, r)$ by

$$\Pi_X(u) = \begin{cases} u, & \text{for } u \in D(\eta_1, r) \\ \dfrac{r}{\|u-u_0\|_X}(u-u_0)+u_0, & \text{for } u \in X \setminus D(\eta_1, r) \end{cases}$$

and $\Pi_Y : Y \to D(\eta_2, r)$ by

$$\Pi_Y(v) = \begin{cases} v, & \text{for } v \in D(\eta_2, r) \\ \dfrac{r}{\|v-v_0\|_Y}(v-v_0)+v_0, & \text{for } v \in Y \setminus D(\eta_2, r). \end{cases}$$

It is known that Π_X, Π_Y are Lipschitz continuous with Lipschitz constant 2. Moreover, they map the space X and Y into $D(\eta_1, r)$ and $D(\eta_2, r)$ respectively. Next, let us define $F_r : X \times Y \to X$ and $G_r : X \times Y \to Y$ by

$$F_r(u,v) = F(\Pi_X(u), \Pi_Y(v)) \text{ and } G_r(u,v) = G(\Pi_X(u), \Pi_Y(v))$$

for each $(u,v) \in X \times Y$.

We consider the reaction-diffusion system

$$\begin{cases} du = (Au + F_r(u,v))\,dt + df, \; t \in [0,+\infty[\\ dv = (Bv + G_r(u,v))\,dt + dg, \; t \in [0,+\infty[\\ u(0) = u_0 \\ v(0) = v_0. \end{cases} \tag{19}$$

We will prove that F_r and G_r satisfy $\left(H_3'\right)$ and $\left(H_4'\right)$ and we will use the Theorem 3.2. Since both F and Π_X are continuous, it follows that F_r is continuous. Moreover F_r satisfies

$$\|F_r(u,v) - F_r(\widetilde{u},v)\|_X = \|F(\Pi_X(u), \Pi_Y(v)) - F(\Pi_X(\widetilde{u}), \Pi_Y(v))\|_X,$$

for each $u, \widetilde{u} \in X$ and each $v \in Y$. But $\Pi_X(u)$, $\Pi_X(\widetilde{u}) \in D(\eta_1, r) \subseteq D(\eta_1, \rho_F)$ and $\Pi_Y(v) \in D(\eta_2, r) \subseteq D(\eta_2, \rho_F)$. So, by (H_3), we obtain that

$$\|F_r(u,v) - F_r(\widetilde{u},v)\|_X \le l_F \|\Pi_X(u) - \Pi_X(\widetilde{u})\|_X \le 2l_F \|u - \widetilde{u}\|_X,$$

for each $u, \widetilde{u} \in X$ and each $v \in Y$. Also, F_r satisfies

$$\|F_r(u,v)\|_Y = \|F(\Pi_X(u), \Pi_Y(v))\|_X \le m_F,$$

for each $(u,v) \in X \times Y$. Since both G and Π_Y are continuous, it follows that G_r is continuous. Moreover, G_r satisfies

$$\|G_r(u,v) - G_r(u,\widetilde{v})\|_Y = \|G(\Pi_X(u), \Pi_Y(v)) - G(\Pi_X(u), \Pi_Y(\widetilde{v}))\|_Y,$$

for each $u \in X$ and each $v, \widetilde{v} \in Y$. But $\Pi_X(u) \in D(\eta_1, r) \subseteq D(\eta_1, \rho_G)$ and $\Pi_Y(v)$, $\Pi_Y(\widetilde{v}) \in D(\eta_2, r) \subseteq D(\eta_2, \rho_G)$. So, by $\left(H_4''\right)$, we obtain that

$$\|G_r(u,v) - G_r(u,\widetilde{v})\|_Y \le l_G \|\Pi_Y(v) - \Pi_Y(\widetilde{v})\|_Y \le 2l_G \|v - \widetilde{v}\|_Y,$$

for each $u, \widetilde{u} \in X$ and each $v \in Y$. Also, G_r satisfies

$$\|G_r(u,v)\|_Y = \|G(\Pi_X(u), \Pi_Y(v))\|_Y \le m_G,$$

for each $(u,v) \in X \times Y$.

From Theorem 2.2 we know that for each $(u_0, v_0) \in X \times Y$ and for each $T > 0$, there exists at least one $\mathcal{L}^\infty$-solution $(u,v) : [0,T] \to X \times Y$ of (19). We will prove that this $\mathcal{L}^\infty$-solution is in fact an $\mathcal{L}^\infty$-solution on $[0,T_0]$, with $T_0 < T$ small enough, of the problem (1) in the sense of Definition 3.1. Indeed, by virtue of Theorem 2.1 , it follows that u, v are piecewise continuous on $[0,T]$ and so $u(t+0) = u(t)$ a.e. on $[0,T]$ and $v(t+0) = v(t)$ a.e. on $[0,T]$. But F_r, G_r are continuous functions and accordingly

$$\begin{aligned}
\int_0^t F_r(u(s+0), v(s+0))\,ds &= \int_0^t F_r(u(s), v(s))\,ds,\\
\int_0^t G_r(u(s+0), v(s+0))\,ds &= \int_0^t G_r(u(s), v(s))\,ds,
\end{aligned}$$

for each $t \in [0,T]$. Since $u(0) = u_0$, by Theorem 2.1 we have

$$u(t) - \eta_1 = u(t) - f(0+0) + f(0) - u_0 = u(t) - u(0+0).$$

Then, by taking into account that $\lim_{t\downarrow 0} u(t) = u(0+0)$, it follows that there exists $T_{01} \in [0,T]$ such that, for each $t \in]0, T_{01}[$, we have that

$$\left\|u(t) - \eta_1\right\| < r,$$

i.e. $(t, u(t)) \in]0, T_{01}[\times D(\eta_1, r)$. Since u is piecewise continuous on $[0, T_{01}]$ it follows that $u(t+0) \in D(\eta_1, r)$ for each $t \in [0, T_{01}[$. Analogously, there exists $T_{02} \in [0,T]$ such that $v(t+0) \in D(\eta_2, r)$ for each $t \in [0, T_{02}[$. Let $T_0 = \min\{T_{01}, T_{02}\}$. But in this case

$$\Pi_X(u(t+0)) = u(t+0) \text{ and } \Pi_Y(v(t+0)) = v(t+0)$$

for each $t \in [0, T_0[$.

Then $F_r(u(s+0), v(s+0))$ must coincide with $F(u(s+0), v(s+0))$ and $G_r(u(s+0), v(s+0))$ must coincide with $G(u(s+0), v(s+0))$, for each $s \in [0, T_0[$. Hence the function $(u,v) : [0,T_0] \to X \times Y$ is an $\mathcal{L}^\infty$-solution of the problem (1) in the sense of Definition 3.1, as is claimed. ∎

We continue with the proof of Theorem 3.1.

Proof. The idea of the proof is to approximate the continuous mapping G with a sequence of locally Lipschitz mappings $(G_n)_{n\in\mathbb{N}^*}$ such that F and G_n satisfy the hypotheses of Theorem 3.3, to obtain a sequence of $\mathcal{L}^\infty$-solutions $((u_n, v_n))_{n\in\mathbb{N}^*}$, and then to show that, on a subsequence at least, $((u_n, v_n))_{n\in\mathbb{N}^*}$ converges to a $\mathcal{L}^\infty$-solution of (1).

To this aim, as $G : X \times Y \to Y$ is continuous, for each $n \in \mathbb{N}^*$, there exists $G_n : X \times Y \to Y$ a locally Lipschitz mapping, which satisfies

$$\|G_n(u,v) - G(u,v)\|_Y \leq \frac{1}{n}, \tag{20}$$

for each $(u,v) \in X \times Y$. See Theorem 1.13, p. 16 in Cârjă [13]. That means that, for each $n \in \mathbb{N}^*$, G_n satisfies $\left(H_4''\right)$. Let us $(u_0, v_0) \in X \times Y$ and let us consider, for each $n \in \mathbb{N}^*$, the problem

$$\begin{cases} du_n = (Au_n + F(u_n, v_n))\,dt + df, \ t \in [0, +\infty[\\ dv_n = (Bv_n + G_n(u_n, v_n))\,dt + dg, \ t \in [0, +\infty[\\ u_n(0) = u_0 \\ v_n(0) = v_0. \end{cases} \tag{21}$$

Thanks to Theorem 3.3, we know that, for each $n \in \mathbb{N}^*$ and each $(u_0, v_0) \in X \times Y$, there exists $T_n > 0$ such that (21) has at least one saturated $\mathcal{L}^\infty$-solution $(u_n, v_n) : [0, T_n[\to X \times Y$. See also Theorem 2.5. We will prove next that $T_n > 0$ can be chosen independent of n. Let us denote $\eta_1 = f(0+0) - f(0) + u_0$, $\eta_2 = g(0+0) - g(0) + v_0$ and let us observe that there are $r = r(\eta_1, \eta_2) > 0$, $r < \rho_F$ and $M = M(\eta_1, \eta_2) > 0$, such that

$$\|F(u,v)\|_X \leq M \text{ and } \|G(u,v)\|_Y \leq M,$$

for each $(u,v) \in D(\eta_1, r) \times D(\eta_2, r)$. Let us consider $T > 0$, small enough, such that

$$\left\|S_A(t)\eta_1 - \eta_1\right\|_X + MT + Var(f, [0,T]) \leq r$$

and

$$\left\|S_B(t)\eta_2 - \eta_2\right\|_X + MT + Var(g, [0,T]) \leq r$$

for each $t \in [0,T]$. We will prove that $T \leq T_n$, for each $n \in \mathbb{N}^*$, and therefore all the solutions (u_n, v_n) are defined at least on $[0,T]$. Finally, we will pass to the limit for $n \to \infty$ to get the solution of (1).

We define

$$E = \left\{\tau \in]0, T_n[\,; \left\|u_n(t) - \eta_1\right\|_X \leq r, \left\|v_n(t) - \eta_2\right\|_Y \leq r, \text{ for } t \in [0,\tau]\right\} \text{ and}$$
$$T_n^* = \sup E,$$

and we will show that $T_n^* \geq T$, for each $n \in \mathbb{N}^*$. Let us assume by contradiction that there exists $n \in \mathbb{N}^*$ such that $T_n^* < T$. First, we will show that there exists

$$\lim_{t \uparrow T_n^*} u_n(t) = u_n(T_n^* - 0).$$

Let $l > 0$ be arbitrary. As $\sup E = T_n^*$, then there exists $\tau_l \in E$ (hence $\tau_l < T_n$) such that $T_n^* - l < \tau_l \leq T_n^*$. Since

$$u_n(t) = S_A(t)u_0 + \int_0^t S_A(t-s)F(u_n(s+0), v_n(s+0))\,ds + \int_0^t S_A(t-s)\,df(s),$$

for $t = T_n^* - l \in [0, T_n[$, with $l > 0$ arbitrary but fixed, then the conclusion follows from the continuity of $t \mapsto S_A(t)\, u_0$, the condition $t \mapsto F(u_n(t+0), v_n(t+0))$ is in $L^1_{\text{loc}}([0, T_n[\,; X)$, the assumption $T_n^* < T < +\infty$ and Theorem 2.1. In the same manner, we deduce the existence of

$$\lim_{t \uparrow T_n^*} v_n(t) = v_n(T_n^* - 0).$$

Next, let us consider again an arbitrary $l > 0$. As $\sup E = T_n^*$, then there exists $\tau_l \in E$ (hence $\tau_l < T_n$) such that $T_n^* - l < \tau_l \leq T_n^*$. Let us denote $\gamma_1^* = f(T_n^* + 0) - f(T_n^* - 0)$. Then,by Remark 2.1, Remark 2.2 and by the assumption $T_n^* < T$, we have

$$\begin{gathered}
\left\| u_n(T_n^* - l) + \gamma_1^* - \eta_1 \right\|_X = \\
\Bigg\| S_A(T_n^* - l)\, u_0 + \int_0^{T_n^* - l} S_A(T_n^* - l - s)\, F(u_n(s), v_n(s))\, ds + \\
\int_0^{T_n^* - l} \chi_{]0, T_n^* - l]} S_A(T_n^* - l - s)\, df(s) + \\
S_A(T_n^* - l)(f(0+0) - f(0)) + \gamma_1^* - \eta_1 \Big\|_X \leq \\
\left\| S_A(T_n^* - l)\, \eta_1 - \eta_1 \right\|_X + \int_0^{T_n^* - l} \| F(u_n(s), v_n(s)) \|_X\, ds + \\
Var(f,]0, T_n^* - l]) + \left\| f(T_n^* + 0) - f(T_n^* - 0) \right\|_X < \\
\left\| S_A(T_n^* - l)\, \eta_1 - \eta_1 \right\|_X + M(T_n^* - l) + Var(g, [0, T]) \leq \\
\left\| S_A(T_n^* - l)\, \eta_1 - \eta_1 \right\|_X + MT + Var(g, [0, T]) \leq r
\end{gathered}$$

Then $u_n(T_n^* - 0) + f(T_n^* + 0) - f(T_n^* - 0) \in IntD(\eta_1, r)$ which is false because, for each $n \in \mathbb{N}^*$, (u_n, v_n) is strictly noncontinuable. So, for each $n \in \mathbb{N}^*$, we have $T_n^* \geq T$, and hence $T \leq T_n$, for each $n \in \mathbb{N}^*$.

Since G is bounded on $D(\eta_1, r) \times D(\eta_2, r)$, $(u_n(s), v_n(s)) \in D(\eta_1, r) \times D(\eta_2, r)$ for each $s \in [0, T]$ and G_n satisfy (20), then, from (H_0) it follows that the set

$$\left\{ t \mapsto \int_0^t G_n(u_n(s), v_n(s))\, ds + g(t)\, ; n \in \mathbb{N}^* \right\}$$

is of equibounded variation on $[0, T]$. Moreover, B generates a compact C_0-semigroup and, by Theorem 2.2, we obtain that the set $\{v_n; n \in \mathbb{N}^*\}$ is relatively compact in $L^1(0, T; Y)$. So, on a subsequence at least, $(v_{n_k})_{k \in \mathbb{N}^*}$ converges to some v in $L^1(0, T; Y)$. With this v, let us consider the problem

$$\begin{cases} du = (Au + F(u, v))\, dt + df, t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{22}$$

From (H_3) and Theorem 2.3 we obtain that there exists $T_0 \in]0, T]$ such that the problem (22) has a unique $\mathcal{L}^\infty$-solution $u : [0, T_0] \to X$. Now, we deduce that, for each $t \in [0, T_0]$ we have

$$\left\|u_{n_k}(t) - u(t)\right\|_X =$$

$$\left\|\int_0^t S_A(t-s) F(u_{n_k}(s), v_{n_k}(s))\, ds - \int_0^t S_A(t-s) F(u(s), v(s))\, ds\right\|_X \leq$$

$$\int_0^t \|S_A(t-s)\|_{\mathcal{L}(X)} \left\|F(u_{n_k}(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds \leq$$

$$\int_0^t \left\|F(u_{n_k}(s), v_{n_k}(s)) - F(u(s), v_{n_k}(s))\right\|_X ds +$$

$$\int_0^t \left\|F(u(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds \leq$$

$$l_F \int_0^t \left\|u_{n_k}(s) - u(s)\right\|_X ds + \int_0^{T_0} \left\|F(u(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds.$$

From Gronwall's Inequality, we get

$$\left\|u_{n_k}(t) - u(t)\right\|_X \leq e^{L_F T_0} \int_0^{T_0} \left\|F(u(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds, \tag{23}$$

for each $t \in [0, T_0]$. We prove that

$$\lim_{k\to\infty} e^{L_F T_0} \int_0^{T_0} \left\|F(u(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds = 0. \tag{24}$$

Let us assume by contradiction that this is not the case. Then there exists $\varepsilon_0 > 0$ such that, for each $k \in \mathbb{N}^*$,we have

$$e^{L_F T_0} \int_0^{T_0} \left\|F(u(s), v_{n_k}(s)) - F(u(s), v(s))\right\|_X ds \geq \varepsilon_0.$$

But F is continuous, $v_{n_k} \to v$ in $L^1(0, T_0; Y)$. There exists a subsequence $\left(v_{n_{k_p}}\right)_{p\in\mathbb{N}^*}$ of $(v_{n_k})_{k\in\mathbb{N}^*}$,such that $\lim_{p\to\infty} v_{n_{k_p}}(t) = v(t)$ a.e. for $t \in [0, T_0]$. Therefore, we get that

$$\lim_{p\to\infty} e^{L_F T_0} \int_0^{T_0} \left\|F\left(u(s), v_{n_{k_p}}(s)\right) - F(u(s), v(s))\right\|_X ds = 0,$$

which contradicts the inductive hypothesis. This contradiction can be eliminated only if (24) holds, and then, by (23) we obtain that $u_{n_k} \to u$ in $L^1(0, T_0; X)$.

If, in addition, (H_5) is satisfied, then, by Theorem 2.6 , each $\mathcal{L}^\infty$-solution of (1) can be extended up to a global one and this achieves the proof. ∎

Remark 3.1. *We note that the proof of Theorem* 3.1 *follows the very same lines if the functions F and G are of the form $F = F(t, u, v)$ and $G = G(t, u, v)$, for each $(t, u, v) \in [0, +\infty[\times X \times Y$.*

4. AN EXAMPLE

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with sufficiently smooth boundary Γ. Let us consider the reaction-diffusion system with measures :

$$\begin{cases} u_t = \alpha\Delta u + f(t,x,u,v) + \delta(\cdot - t_{01}) & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega \\ v_t = \beta\Delta v + g(t,x,u,v) + \delta(\cdot - t_{02}) & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) = v(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \Gamma \\ u(0,x) = u_0(x), v(0,x) = v_0(x) & \text{for } x \in \Omega \end{cases} \tag{25}$$

where $\alpha \geq 0, \beta > 0$, $f, g : \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $\delta(\cdot - t_0)$ is the Dirac measure concentrated at $t_0 \in]0, +\infty[$, and $u_0 \in L^1(\Omega), v_0 \in L^1(\Omega)$. We notice that t is the "temporal" variable and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator with respect to a "spatial" variable.

Theorem 4.1. *Assume that f, g are continuous and that the function f is locally Lipschitz with respect to its third variable, i.e. for each $(\eta_1, \eta_2) \in \mathbb{R}^2$ there exists $\rho_f = \rho_f(\eta_1, \eta_2) > 0$ and $l_f = l_f(\eta_1, \eta_2) > 0$ such that*

$$|f(t,x,u,v) - f(t,x,\widetilde{u},v)| \leq l_f |u - \widetilde{u}|, \tag{26}$$

for all $(t,x) \in \mathbb{R}_+ \times \Omega$ and $u, \widetilde{u}, v \in \mathbb{R}$ with $|u - \eta_1| < \rho_f$, $|\widetilde{u} - \eta_1| < \rho_f$, $|v - \eta_2| < \rho_f$. Then, for each $(u_0, v_0) \in L^1(\Omega) \times L^1(\Omega)$, there exists $T_0 > 0$ such that (25) *has at least one $\mathcal{L}^\infty$-solution $(u,v) : [0, T_0] \to X \times Y$ and satisfying the initial conditions $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$ a.e. for $x \in \Omega$. If, in addition, assume that there exist $a_i > 0$, $b_i > 0$, $c_i > 0$, $i \in \{1, 2\}$ such that*

$$|f(t,x,u,v)| \leq a_1 |u| + b_1 |v| + c_1 \tag{27}$$

for each $(t,x,u,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$ and

$$|g(t,x,u,v)| \leq a_2 |u| + b_2 |v| + c_2 \tag{28}$$

for each $(t,x,u,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$, then the system (25) *has at least one global $\mathcal{L}^\infty$-solution $(u,v) : [0, +\infty[\to X \times Y$.*

Proof. In order to use Theorem 3.1 we shall rewrite (25) as a reaction-diffusion system (1). To this aim, take $X = Y = L^1(\Omega)$, and let us define the operator $A : D(A) \subseteq L^1(\Omega) \to L^1(\Omega)$ by

$$\begin{cases} D(A) = \left\{u \in L^1(\Omega); u \in W_0^{1,1}(\Omega), \alpha\Delta u \in L^1(\Omega)\right\} \\ Au = \alpha\Delta u, \text{ for each } u \in D(A) \end{cases}$$

and the operator $B : D(B) \subseteq L^1(\Omega) \to L^1(\Omega)$ by

$$\begin{cases} D(B) = \left\{v \in L^1(\Omega); v \in W_0^{1,1}(\Omega), \beta\Delta v \in L^1(\Omega)\right\} \\ Bv = \beta\Delta v, \text{ for each } v \in D(B). \end{cases}$$

It is known that A generates a compact C_0-semigroup of contractions $\{S_A(t)\,;t \geq 0\}$ on X if $\alpha > 0$, and which is only continuous from $]0,+\infty[$ to $\mathcal{L}(X)$ in the uniform operator topology if $\alpha = 0$. See for instance Vrabie [27], Theorem 7.2.7, p. 160. Also it is known that B generates a compact C_0-semigroup of contractions $\{S_B(t)\,;t \geq 0\}$ on Y. Next, let us define $f_0 : [0,+\infty[\to L^1(\Omega)$ and $g_0 : [0,+\infty[\to L^1(\Omega)$ by

$$(f_0(t))(x) = \begin{cases} -\frac{1}{2} & \text{for } 0 \leq t < t_{01} \\ 0 & \text{for } t = t_{01} \\ \frac{1}{2} & \text{for } t_{01} < t < +\infty \end{cases}$$

and

$$(g_0(t))(x) = \begin{cases} -\frac{1}{2} & \text{for } 0 \leq t < t_{02} \\ 0 & \text{for } t = t_{02} \\ \frac{1}{2} & \text{for } t_{02} < t < +\infty, \end{cases}$$

for all $t \in [0,+\infty[$ and a.e. $x \in \Omega$. Clearly $f_0 \in BV\big([0,+\infty[\,;L^1(\Omega)\big)$, $g_0 \in BV\big([0,+\infty[\,;L^1(\Omega)\big)$ and let us remark that $(df_0)(t) = \delta(t-t_{01})$ and $(dg_0)(t) = \delta(t-t_{02})$ in the sense of distributions.

Now, let us observe that (25) may be rewritten as a Cauchy problem in $L^1(\Omega)$ of the form :

$$\begin{cases} du = (Au + F(u,v))\,dt + df_0, \; t \in [0,+\infty[\\ dv = (Bv + G(u,v))\,dt + dg_0, \; t \in [0,+\infty[\\ u(0) = u_0 \\ v(0) = v_0, \end{cases} \tag{29}$$

where A, B, f_0, g_0, u_0, v_0 are as above, while $F, G : [0,T] \times L^1(\Omega) \times L^1(\Omega) \to L^1(\Omega)$ are the superposition operators on $L^1(\Omega) \times L^1(\Omega)$ associated with f and g respectively, i.e.

$$(F(t,u,v))(x) = f(t,x,u(x),v(x)) \text{ and } (G(t,u,v))(x) = g(t,x,u(x),v(x))$$

for each $u, v \in L^1(\Omega)$, for each $t \in [0,+\infty[$ and a.e. for $x \in \Omega$. See for instance Vrabie [27], Definition A.6.1, p. 313. By (iii) in Lemma A.6.1, p. 312 in Vrabie [27], it readily follows that F and G are well defined on $L^1(\Omega) \times L^1(\Omega)$. In addition, since f, g are continuous, it follows that F, G are continuous on $[0,+\infty[\times L^1(\Omega) \times L^1(\Omega)$. Moreover F satisfies (H_3). Then, by Theorem 3.1 and Remark 3.2, we conclude that there exists $T_0 \in\,]0,+\infty[$ such that the problem (27) has at least one $\mathcal{L}^\infty$-solution on $[0,T_0]$ in the sense of Definition 3.1. In addition, by virtue of (27) and (11), by Theorem 3.1, it follows that the above $\mathcal{L}^\infty$-solution for the problem (27) can be extended up to a global $\mathcal{L}^\infty$-solution. ∎

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APPROXIMATE CONTROLLABILITY OF FRACTIONAL STOCHASTIC FUNCTIONAL EVOLUTION EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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Abstract In this paper, the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert space driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ has been established and discussed by using the theory of fractional calculus, fixed point technique, stochastic analysis technique and methods adopted directly from deterministic control problems. As an application that illustrates the abstract results, an example is given.

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1. INTRODUCTION

The concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems (Bashirov and Mahmudov, 1999; Klamka, 1991, 2000; Balachandran and Sakthivel, 2001). The controllability of nonlinear deterministic systems in a finite and infinite dimensional space by using different kinds of approaches have been considered in many publications (see [1, 2, 6] and the references therein). Moreover, the exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Klamka [8] derived a set of sufficient conditions for the exact controllability of semilinear systems. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors [9, 10], in which the authors effectively used the fixed point approach. Fu and Mei [5] studied the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable.

Very recently, Sakthivel et al. [13] established a set of sufficient conditions for obtaining the approximate controllability of semilinear fractional differential systems in Hilbert spaces. Also, Sakthivel et al. in [14] investigated approximate controllability problem for nonlinear fractional stochastic systems driven by Wiener process, which are natural generalizations of the well known controllability concepts from the theory of infinite dimensional deterministic control systems. Specifically, they studied the approximate controllability of nonlinear fractional control systems under the assumption that the associated linear system is approximately controllable.

However, to the best of our knowledge, the approximate controllability problem for nonlinear fractional stochastic functional system driven by fractional Brownian motion in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we investigate the approximate controllability problem of a class of nonlinear fractional stochastic functional systems, we consider a mathematical model given by the following fractional functional equation with control:

$$\begin{aligned} {}^cD_t^q[x(t) - \varphi(t, x_t)] &= Ax(t) + Bu(t) + \phi(t, x_t) + \sigma_H(t)\frac{dB_Q^H(t)}{dt} \\ x(t) &= \psi(t), \quad t \in [-r, 0], \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in a Hilbert space U; $B_Q^H = \{B_Q^H(t), t \in [0, T]\}$ is a fBm with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$; $0 < q < 1$ and ${}^cD_t^q$ denotes the Caputo fractional derivative operator of order q. $x_t \in \mathcal{C}_r$ denote the function defined by $x_t(v) = x(t + v)$, $\forall v \in [-r, 0]$, where $\mathcal{C}_r = C([-r, 0], U)$ is the space of continuous functions f from $[-r, 0]$ to U.

We will study the approximate controllability problem for nonlinear fractional control systems of the form (1) under the assumption that the associated linear system is approximately controllable.

The paper is organized as follows. In Section 2 we e will first revise some results concerning fractional calculus including pathwise stochastic integration with respect to fractional Brownian motion and some estimates for such integrals. Second, we provide some definitions, lemmas and notations necessary to establish our main results. In Section 3 we formulate and prove conditions for approximate controllability of the fractional stochastic functional dynamical control system (1) using the contraction mapping principle. As an application that illustrates the abstract results, an example is given.

2. PRELIMINARIES

In this section we introduce some notations, definitions, a technical lemmas and preliminary fact which are used in what follows.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,T]), \mathbf{P})$ be a complete probability space with a filtration satisfying the standard conditions.

Definition 2.1. *The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t, t \in [0,T]\}$, having the properties $B_0^H = 0$, $\mathbf{E}B_t^H = 0$ and $\mathbf{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.*

Let $T > 0$ and denote by Υ the linear space of $\mathcal{R}$-valued step functions on $[0,T]$, that is, $\phi \in \Upsilon$ if

$$\phi(t) = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})}(t),$$

where $t \in [0,T]$, $x_i \in \mathbf{R}$ and $0 = t_1 < t_2 < \cdot < t_n = T$. For $\phi \in \Upsilon$ its Wiener integral with respect to B^H is

$$\int_0^T \phi(s) dB^H(s) = \sum_{i=1}^{n-1} z_i \Big(B^H(t_{i+1}) - B^H(t_i)\Big).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Υ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})} \mapsto \int_0^T \phi(s) dB^H(s)$$

is an isometry between Υ and the linear space $span\{B^H(t), t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t), t \in [0,T]\}$ (see [12]). The image of an element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of ϕ with respect to B^H.
Let us now consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$, where β denoting the Beta function, and $t > s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Upsilon \mapsto L^2([0,T])$ be the linear operator given by

$$\mathcal{K}_H\phi(s)(s) = \int_s^t \phi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Then $(\mathcal{K}_H\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s)$ and $\mathcal{K}_H$ is an isometry between Υ and $L^2([0,T])$ that can be extended to $\mathcal{H}$.

Denoting $L^2_{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H}, \mathcal{K}_H\phi \in L^2([0,T])\}$, since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]). \tag{2}$$

Moreover the following result hold:

Lemma 2.1 ([12])**.** *For $\phi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T\int_0^T |\phi(r)||\phi(u)||r-u|^{2H-2}drdu \le c_H\|\phi\|^2_{L^{1/H}([0,T])}.$$

Let us now consider two separable Hilbert spaces $(U, |\cdot|_U, <.,.>_U)$ and $(V, |\cdot|_V, <.,.>_V)$. Let $L(V,U)$ denote the space of all bounded linear operator from V to U and $Q \in L(V,V)$ be a non-negative self adjoint operator. Denote by $L^0_Q(V,U)$ the space of all $\xi \in L(V,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. the norm is given by

$$|\xi|^2_{L^0_Q(V,U)} = \left|\xi Q^{\frac{1}{2}}\right|^2_{HS} = tr(\xi Q\xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from V to U.

Let $\{B^H_n(t)\}_{n\in\mathbf{N}}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\{e_n\}_{n\in\mathbf{N}}$ be a complete orthonormal basis in V. Define the V-valued stochastic process $B^H_Q(t)$ by

$$B^H_Q(t) = \sum_{n=1}^{\infty} B^H_n(t)Q^{\frac{1}{2}e_n}, \quad t \ge 0.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space V, that is, it holds that $B^H_Q(t) \in L^2(\Omega, V)$. Then, we say that $B^H_Q(t)$ is a V-valued Q-cylindrical fBm with covariance operator Q.

Let $\psi : [0,T] \to L^0_Q(V,U)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}})e_n\|_{L^2([0,T],U)} < \infty. \tag{3}$$

Definition 2.2. *Let $\psi : [0,T] \to L^0_Q(V,U)$ satisfy (4). Then, its stochastic integral with respect to the fBm B^H_Q is defined for $t \ge 0$ as*

$$\int_0^t \psi(s)dB^H_Q(s) := \sum_{n=1}^{\infty}\int_0^t \psi(s)Q^{\frac{1}{2}}e_n dB^H_n(s) = \sum_{n=1}^{\infty}\int_0^t \left(\mathcal{K}_H(\psi Q^{\frac{1}{2}}e_n)\right)(s)dW(s),$$

where W is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T],U)} < \infty, \tag{4}$$

then in particular (4) holds, which follows immediately from (3).

The following lemma is proved in [12] and obtained as a simple application of Lemma 1.

Lemma 2.2 ([12]). *For any $\psi : [0,T] \to L_Q^0(V,U)$ such that (5) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbf{E}\Big|\int_{\beta}^{\alpha} \psi(s)dB_Q^H(s)\Big|_U^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_{\beta}^{\alpha} |\psi Q^{\frac{1}{2}} e_n|_U^2 ds,$$

where $c = c(H)$. If in addition

$$\sum_{n=1}^{\infty} |\psi Q^{\frac{1}{2}} e_n|_U \ \ \textit{is uniformly convergent for} \ \ t \in [0,T], \tag{5}$$

then

$$\mathbf{E}\Big|\int_{\beta}^{\alpha} \psi(s)dB_Q^H(s)\Big|_U^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1}\int_{\beta}^{\alpha} |\psi(s)|_{L_Q^0(V,U)}^2 ds. \tag{6}$$

Now, we recall the following known definitions on the fractional integral and derivative

Definition 2.3 ([12]). *Let $f \in L^1(0,T)$ and $\alpha > 0$. The fractional Riemann-Liouville integral of f of order α is defined for almost all $t \in (0,T)$ by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\alpha) = \int_0^{\infty} \theta^{\alpha-1}e^{-\theta}d\theta$ is the Euler function.

Definition 2.4. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0,\infty) \to \mathbf{R}$ can be written as*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds, \quad t > 0, n-1 < \alpha < n. \tag{7}$$

Definition 2.5. *The Caputo derivative of order α for a function $f : [0,\infty) \to \mathbf{R}$ can be written as*

$$^{c}D^{\alpha} f(t) = ^{L} D^{\alpha}\bigg(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^k(0)\bigg), \quad t > 0, n-1 < \alpha < n. \tag{8}$$

If $f(t) \in C^n[0,\infty)$, then

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^n(s)ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n$$

Now, we denote by $\mathcal{C}(0,T;L^2(\Omega;U)) = \mathcal{C}(0,T;L^2(\Omega,\mathcal{F},\mathbf{P};U))$ the Banach space of all continuous functions from $[0,T]$ into $L^2(\Omega;U)$ equipped with the sup norm.

Let us consider a fixed real number $r \geq 0$. If $x \in \mathcal{C}(-r,T;L^2(\Omega;U))$ for each $t \in [0,T]$ we denote by $x_t \in \mathcal{C}(-r,0;L^2(\Omega;U))$ the function defined by $x_t(v) = x(t+v)$, for $v \in [-r,0]$.

Consider the fractional functional equation with control of the form

$$\begin{cases} ^cD_t^q[x(t) - \varphi(t,x_t)] &= Ax(t) + Bu(t) + \phi(t,x_t) + \sigma_H(t)\dfrac{dB_Q^H(t)}{dt} \\ x(t) &= \psi(t), \quad t \in [-r,0], \end{cases} \tag{9}$$

where $B_Q^H(t)$ is the fractional Brownian motion which was introduced above, the initial data $\psi \in \mathcal{C}(-r,0;L^2(\Omega;U))$ and $A : Dom(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $S(.)$ on U. Here, for $0 < q < 1$, $^cD_t^q$ denote the Caputo fractional derivative operator of order q, control function $u(.)$ is given in $L^2([0,T],\tilde{U})$, a Banach space of admissible control functions with $\tilde{U}$ is a Hilbert space and $B \in L(\tilde{U},U)$. Further $\varphi,\phi : [0,T] \times \mathcal{C}(-r,0;U) \to U$ and $\sigma_H : [0,T] \to L_Q^0(V,U)$ are appropriate functions.

Definition 2.6. *A U-valued process $x(t)$ is called a mild solution of (1) if $x \in \mathcal{C}(-r,T;L^2(\Omega;U))$, $x(t) = \psi(t)$ for $t \in [-r,0]$, and, for $t \in [0,T]$, satisfies*

$$\begin{aligned} x(t) &= \\ &S(t)\Big[\psi(0) - \varphi(0,x_0)\Big] + \varphi(t,x_t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\Big[Bu(s) + \phi(s,x_s)\Big]ds \\ +\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\varphi(s,x_s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\sigma_H(s)dB_Q^H(s). \end{aligned} \tag{10}$$

We will make use of the following assumptions on data of the problem:

$(\mathbf{H}_1)$ The semigroup $(S(t))_{t\geq 0}$ is a bounded linear operator on U and satisfies for $t \geq 0$

$$|S(t)x|_U \leq Me^{\lambda t}|x|, \qquad M \geq 1, \lambda \in \mathbf{R} \text{ and } x \in U.$$

$(\mathbf{H}_2)$ The functions ϕ,φ satisfy the following Lipschitz condition: there exist constants $c_1, c_2 > 0$ for $x,y \in U$ and $t \geq 0$ such that

$$\begin{aligned} |\phi(t,x) - \phi(t,y)|_U^2 &\leq c_1|x-y|_U^2, \\ |\varphi(t,x) - \varphi(t,y)|_U^2 &\leq c_2|x-y|_U^2. \end{aligned}$$

($\mathbf{H}_3$) The functions ϕ, φ are continuous and satisfy the usual linear growth condition i.e., there exist constants $c_3, c_4 > 0$ for $x, y \in U$ and $t \geq 0$ such that

$$\begin{array}{rcl} |\phi(t,x)|_U^2 & \leq & c_3(1+|x|_U^2), \\ |\varphi(t,x)|_U^2 & \leq & c_4(1+|x|_U^2). \end{array}$$

($\mathbf{H}_4$) The function σ_H satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$ in V, we have

$$\sum_{n=1}^{\infty} \|\sigma_H Q^{1/2} e_n\|_{L^2([0,T];U)} < \infty.$$
$$\sum_{n=1}^{\infty} |\sigma_H(t, x(t)) Q^{1/2} e_n|_U \ \text{ is uniformly convergent for } \ t \in [0,T].$$

Note that, by assumption ($\mathbf{H}_4$), for every $t \in [0,T]$, $\displaystyle\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)} ds < \infty$.

($\mathbf{H}_5$) The linear stochastic system is approximately controllable on $[0,T]$.

Let B^*, $S^*(.)$ be respectively the operator adjoint of B and $S(.)$. Define the controllability Grammian operator by $\Theta_0^t = \int_0^t (t-s)^{2(q-1)} S(t-s) BB^* S^*(t-s) ds$. Then, for each $0 \leq s \leq t$, the operator $\theta\Big(\theta I + \Theta_0^t\Big)^{-1} \to 0$ in the strong operator topology as $\theta \to 0^+$.

We consider the corresponding linear fractional deterministic control system to (9)

$$\left\{ \begin{array}{rcl} {}^cD_t^q x(t) & = & Ax(t) + (Bu)(t) \quad t \in [0,T], \\ x(t) & = & \psi(t), \quad t \in [-r, 0], \end{array} \right. \tag{11}$$

Proposition 2.1. *The deterministic system (11) is approximately controllable on* $[0,T]$ *iff the operator* $\theta\Big(\theta I + \Theta_0^t\Big)^{-1} \to 0$ *as* $\theta \to 0^+$.

We note that the approximate controllability for linear fractional deterministic control system (11) is a natural generalization of approximate controllability of linear first order control system (see [11], Theorem 2).

Definition 2.7. *System* (9) *is approximately controllable on* $[0,T]$ *if* $\overline{\mathcal{R}(T)} = L^2(\Omega, \mathcal{F}, \mathbf{P}; U)$*, where* $\mathcal{R}(t) = \{x(t) = x(t,u) : u \in L^2([0,T], \tilde{U})\}$.

The following lemma is required to define the control function. The reader can refer to [10] for the proof.

Lemma 2.3. *For all adapted, U-valued process* $z_T \in L^2(\Omega; U)$*, there exists* $f \in L^2\Big(\Omega; L^2([0,T]; L_Q^0)\Big)$ *such that* $z_T = \mathbf{E} z_T + \displaystyle\int_0^T f(s) dB_Q^H(s)$.

Let $\theta > 0$ and $z_T \in L^2(\Omega; U)$. Define the control function in the following form:

$$u^\theta(t,x) =$$
$$= B^*(T-t)^{q-1}S^*(T-t)\Big[(\theta I + \Theta_0^T)^{-1}\big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0,x_0)] - \varphi(T,x(T))\big)$$
$$+ \int_0^T (\theta I + \Theta_0^T)^{-1} f(s)dB_Q^H(s) - \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\varphi(s,x(s))ds$$
$$- \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\phi(s,x(s))ds$$
$$- \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\sigma_H(s)dB_Q^H(s)\Big].$$

Now for our convenience, let us assume that the function f satisfies the condition $(\mathbf{H}_4)$. Set $c_5 = \max\{|f(s)|_U^2 : 0 \le s < t \le T\}$.

Lemma 2.4. *There exists a positive real constant N such that for all $x \in \mathcal{C}(-r,T;U)$, we have*

$$\mathbf{E}|u^\theta(t,x)|^2 \le \frac{N}{\theta^2}\Big(1 + \int_0^t \mathbf{E}|x(s)|_U^2 ds\Big). \tag{12}$$

Proof. Let $x \in \mathcal{C}(-r,T;U)$ and $T > 0$ be fixed. We have

$$\mathbf{E}|u^\theta(t,x)|^2 \le$$
$$\le 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)(\theta I + \Theta_0^T)^{-1}(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0,x_0)])\Big|^2$$
$$+ 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)(\theta I + \Theta_0^T)^{-1}\varphi(T,x(T))\Big|^2$$
$$+ 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\int_0^t (\theta I + \Theta_0^T)^{-1} f(s)dB_Q^H(s)\Big|^2$$
$$+ 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\varphi(s,x(s))ds\Big|^2$$
$$+ 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\phi(s,x(s))ds\Big|^2$$
$$+ 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\sigma_H(s)dB_Q^H(s)\Big|^2.$$

From the Hölders inequality, the assumption on the data and Lemma 2.4., we have

$$\begin{aligned}
\mathbf{E}|u^\theta(t,x)|^2 \le\ & \frac{6}{\theta^2}|B|^2 T^{2q-2}\Big(\frac{M^2e^{2\lambda t}}{q-1}\Big)^2\Big[\mathbf{E}|z_T|^2 - \mathbf{E}|\psi(0) - \varphi(0,x_0)|^2\Big] \\
&+ \frac{6}{\theta^2}|B|^2 T^{2q-2}\Big(\frac{Me^{\lambda t}}{q-1}\Big)^2\Big[\mathbf{E}|\varphi(T,x(T))|^2 - c_6 H(2H-1)T^{2H}\Big] \\
&+ \frac{6}{\theta^2}|B|^2 T^{4q-3}\Big(\frac{M^2e^{2\lambda t}}{(q-1)\Gamma(q)}\Big)^2 c_7 \int_0^t (1 + \mathbf{E}|x(s)|_U^2)ds \\
&+ \frac{6}{\theta^2}|B|^2 T^{4q-3}\Big(\frac{M^2e^{2\lambda t}}{(q-1)\Gamma(q)}\Big)^2 cH(2H-1)T^{2H-1}\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)}ds,
\end{aligned}$$

where $c_6 = cc_5$ and $c_7 = c_3 + c_4$.
Remark that condition ($\mathbf{H}_4$) ensures the existence of a positive constant c_8 such that

$$6|B|^2 T^{4q-3}\left(\frac{M^2 e^{2\lambda t}}{(q-1)\Gamma(q)}\right)^2 cH(2H-1)T^{2H-1}\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)}ds \le c_8, \quad \text{for all} \quad t \ge 0.$$

Thus it follows from the above inequalities and linear growth condition that there exists $N > 0$ such that

$$\mathbf{E}|u^\theta(t,x)|^2 \le \frac{N}{\theta^2}\left(1 + \int_0^t \mathbf{E}|x(v)|^2_U)dv\right).$$

∎

3. CONTROLLABILITY RESULTS

In this section, we formulate and prove conditions for approximate controllability of the fractional functional stochastic dynamical control system (9) using the contraction mapping principle. In particular, we establish approximate controllability of nonlinear fractional functional stochastic control system (9) under the assumptions that the corresponding linear system is approximately controllable.

We first define the operator $\mathcal{P}_\theta : \mathcal{C}(0,T;L^2(\Omega,U)) \to \mathcal{C}(0,T;L^2(\Omega,U))$, $\theta > 0$ by

$$\begin{aligned} &\mathcal{P}_\theta x(t) = \\ &S(t)\Big[\psi(0) - \varphi(0,x_0)\Big] + \varphi(t,x_t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\Big[Bu^\theta(s,x) + \phi(s,x_s)\Big]ds \\ + \quad &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\varphi(s,x_s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)\sigma_H(s)dB_Q^H(s). \end{aligned} \tag{13}$$

Lemma 3.1. *For any $x \in \mathcal{C}(0,T;L^2(\Omega,U))$, $(\mathcal{P}_\theta x)(.)$ is continuous on [0,T] in L^2 sense.*

Proof. Let $x \in \mathcal{C}(0,T;L^2(\Omega,U))$ be fixed and $0 \le t_1 < t_2 \le T$. Then from Eq. (13) we have

$$\begin{aligned} &\mathbf{E}|(\mathcal{P}_\theta x)(t_2) - (\mathcal{P}_\theta x)(t_1)|^2 \le \\ &\le 6\left[\mathbf{E}|(S(t_2) - S(t_1))[\psi(0) - \varphi(0,x_0)]|^2 + \mathbf{E}|\varphi(t_2,x_{t_2}) - \varphi(t_1,x_{t_1})|^2\right] \\ + \quad &6\Big[\sum_{i=1}^{4}\mathbf{K}|\Sigma_i^x(t_2) - \Sigma_i^x(t_1)|^2\Big]. \end{aligned}$$

From the strong continuity of $S(.)$, the first term on the R.H.S goes to zero as $t_2 - t_1 \to 0$ [5]. The Lipschitz condition on φ implies that the second term goes to zero as

$t_2 - t_1 \to 0$.

Next, it follows from Hölder's inequality and assumptions on the theorem that

$$\begin{aligned}
&\mathbf{K}|\Sigma_1^x(t_2) - \Sigma_1^x(t_1)|^2 \leq \\
&3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1 - s)^{q-1}[S(t_2 - s) - S(t_1 - s)]\phi(s, x_s)ds\right|^2 \\
+\quad &3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]S(t_2 - s)\phi(s, x_s)ds\right|^2 \\
+\quad &3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2 - s)^{q-1}\phi(s, x_s)ds\right|^2 \\
\leq\quad &3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{K}\left|[S(t_2 - s) - S(t_1 - s)]\phi(s, x_s)\right|^2 ds \\
+\quad &3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\left(\int_0^{t_1}\left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right]^2 ds\right)\left(\int_0^{t_1}\mathbf{E}|\phi(s, x_s)|^2 ds\right) \\
+\quad &3\frac{(t_2 - t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|\phi(s, x_s)|^2 ds.
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
&\mathbf{E}|\Sigma_2^x(t_2) - \Sigma_2^x(t_1)|^2 \leq \\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1 - s)^{q-1}[S(t_2 - s) - S(t_1 - s)]Bu^\alpha(s, x)ds\right|^2 \\
+\quad &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]S(t_2 - s)Bu^\alpha(s, x)ds\right|^2 \\
+\quad &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2 - s)^{q-1}Bu^\alpha(s, x)ds\right|^2 \\
\leq\quad &3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\left|[S(t_2 - s) - S(t_1 - s)]Bu^\alpha(s, x)\right|^2 ds \\
+\quad &3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2|B|^2\left(\int_0^{t_1}\left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right]^2 ds\right)\left(\int_0^{t_1}\mathbf{E}|u^\alpha(s, x)|^2 ds\right) \\
+\quad &3\frac{(t_2 - t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2|B|^2\int_{t_1}^{t_2}\mathbf{E}|u^\alpha(s, x)|^2 ds.
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E}|\Sigma_3^x(t_2) - \Sigma_3^x(t_1)|^2 \leq \\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1 - s)^{q-1}[S(t_2 - s) - S(t_1 - s)]\varphi(s, x_s)ds\right|^2 \\
+\quad &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]S(t_2 - s)\varphi(s, x_s)ds\right|^2 \\
+\quad &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2 - s)^{q-1}\varphi(s, x_s)ds\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq\ 3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\Big|[S(t_2-s)-S(t_1-s)]\varphi(s,x_s)\Big|^2ds\\
&+\ 3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2ds\right)\left(\int_0^{t_1}\mathbf{E}|\varphi(s,x_s)|^2ds\right)\\
&+\ 3\frac{(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|\varphi(s,x_s)|^2ds.
\end{aligned}
$$

Similarly, using Lemma 2.4 and assumptions on the theorem we get

$$
\begin{aligned}
&\mathbf{E}|\Sigma_4^x(t_2)-\Sigma_4^x(t_1)|^2\leq\\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}[S(t_2-s)-S(t_1-s)]\sigma_H(s)dB_Q^H(s)\right|^2\\
&+\ 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]S(t_2-s)\sigma_H(s)dB_Q^H(s)\right|^2\\
&+\ 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\sigma_H(s)dB_Q^H(s)\right|^2\\
&\leq\ \frac{3cH(2H-1)T^{2H-1}t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\Big|[S(t_2-s)-S(t_1-s)]\sigma_H(s)\Big|^2ds\\
&+\ \frac{3}{\Gamma^2(q)}cH(2H-1)T^{2H-1}\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2ds\right)\\
&\times\ \left(\int_0^{t_1}\mathbf{E}|S(t_2-s)\sigma_H(s)|^2ds\right)\\
&+\ \frac{3cH(2H-1)T^{2H-1}(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|S(t_2-s)\sigma_H(s)|^2ds.
\end{aligned}
$$

Hence using the strong continuity $S(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2-t_1\to 0$. Thus we conclude $\mathcal{P}_\theta(x)(t)$ is continuous from the right in $[0,T)$. A similar argument shows that it is also continuous from the left in $(0,T]$. This completes the proof of this lemma. ∎

Theorem 3.1. *Assume assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_4)$ *are satisfied. Then the system* (9) *has a mild solution on* $[0,T]$.

Proof. We prove the existence of a fixed point of the operator $\mathcal{P}_\theta$ by using the contraction mapping principle. Let $x\in\mathcal{C}(0,T;L^2(\Omega,U))$. From (13) we obtain

$$
\mathbf{E}\Big|\mathcal{P}_\theta x\Big|_C^2\leq 6\left[\sup_{0\leq t\leq T}\mathbf{E}|S(t)[\psi(0)-\varphi(0,x_0)]|^2+\sup_{0\leq t\leq T}\mathbf{E}|\varphi(t,x_t)|^2+\sup_{0\leq t\leq T}\sum_{i=1}^4\mathbf{E}|\Sigma_i^x(t)|^2\right].\tag{14}
$$

Using assumptions ($\mathbf{H}_1$)-($\mathbf{H}_4$) and Lemma 2.12., we get

$$\sup_{0\le t\le T}\mathbf{E}|S(t)[\psi(0)-\varphi(0,x_0)]|^2 \le M^2e^{2\lambda T}\Big[|\psi(0)|^2+|\varphi(0,x_0)|^2\Big] \tag{15}$$

and

$$\begin{aligned}\sum_{i=1}^{4}\mathbf{E}|\Sigma_i^x(t)|^2 &\le 3\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\frac{T^{2q-1}}{2q-1}c_7\Big(1+|x|_C^2\Big)\\ &+ 3\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\frac{T^{2q}}{2q-1}|B|^2\frac{N}{\theta^2}\Big(1+|x|_C^2\Big)+c_9,\end{aligned} \tag{16}$$

where c_9 is a positive constant such that

$$3cH(2H-1)T^{2H-1}\frac{T^{2q-1}}{2q-1}\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\int_0^t|\sigma_H(s)|^2_{L_Q(V,U)}ds \le c_9.$$

Inequalities (15) and (16) together imply that $\mathbf{E}\Big|\mathcal{P}_\theta x\Big|_C^2<\infty$. By Lemma 3.1., $\mathcal{P}_\theta x\in \mathcal{C}(0,T;L^2(\Omega,U))$. Thus for each $\theta>0$, the operator $\mathcal{P}_\theta$ maps $\mathcal{C}(0,T;L^2(\Omega,U))$ into itself.

Now, we are going to use the Banach fixed point theorem to prove that $\mathcal{P}_\theta$ has a unique fixed point in $\mathcal{C}(0,T;L^2(\Omega,U))$. We claim that $\mathcal{P}_\theta$ is a contraction on $\mathcal{C}(0,T;L^2(\Omega,U))$. For $x,y\in\mathcal{C}(0,T;L^2(\Omega,U))$ we have

$$\begin{aligned}&\mathbf{E}\Big|(\mathcal{P}_\theta)(x)-\mathcal{P}_\theta)(y)\Big|_C^2\\ \le\ & 4\mathbf{E}\sum_{i=1}^{4}|\Sigma_i^x(t)-\Sigma_i^y(t)|^2\\ \le\ & 4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left[\frac{T^{2q-1}}{2q-1}c_1+\frac{N}{\theta^2}|B|^2\frac{T^{2q-1}}{2q-1}T^2+\frac{T^{2q-1}}{2q-1}c_2\right]\int_0^t\mathbf{E}|x(s)-y(s)|^2ds\\ =\ & 4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left[\frac{T^{2q-1}}{2q-1}c_{10}+\frac{N}{\theta^2}|B|^2\frac{T^{2q+1}}{2q-1}\right]\int_0^t\mathbf{E}|x(s)-y(s)|^2ds.\end{aligned}$$

It results that

$$\begin{aligned}&\sup_{0\le t\le T}\mathbf{E}\Big|(\mathcal{P}_\theta)(x)-\mathcal{P}_\theta)(y)\Big|_C^2\\ \le\ & \left[4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left(\frac{c_{10}}{2q-1}+\frac{N|B|^2}{\theta^2(2q-1)}\right)\right]T^{2q+2}\sup_{0\le t\le T}\mathbf{E}|x(t)-y(t)|_C^2.\end{aligned} \tag{17}$$

Therefore we conclude that $\mathcal{P}_\theta$ is a contraction mapping on $\mathcal{C}(0,T;L^2(\Omega,U))$. Then the mapping $\mathcal{P}_\theta$ has a unique fixed point $x(.)\in\mathcal{C}(0,T;L^2(\Omega,U))$, which is a mild solution of (9). ∎

Theorem 3.2. *Assume that the assumptions* ($\mathbf{H}_1$)-($\mathbf{H}_5$) *hold. If the function* ϕ *and* φ *are uniformly bounded and* $\{S(t);t\ge 0\}$ *is compact, then the system* (9) *is approximately controllable on* $[0,T]$.

Proof. Let x_θ be a fixed point of $\mathbf{P}_\theta$. By the stochastic Fubini theorem, it can be easily seen that

$$\begin{aligned} x_\theta(T) &\leq z_T - \theta(\theta I + \Theta)^{-1}\Big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0, x_0)] - \varphi(t, (x_\theta)_t)\Big) \\ &+ \frac{\theta}{\Gamma(q)} \int_0^T (\theta I + \Theta_s^T)^{-1}(T-s)^{q-1} S(T-s)[\phi(s, x_\theta(s)) + \varphi(s, x_\theta(s))]ds \\ &+ \frac{\theta}{\Gamma(q)} \int_0^T (\theta I + \Theta_s^T)^{-1}[(T-s)^{q-1} S(T-s)\sigma_H(s) - \Gamma(q) f(s)]dB_Q^H(s). \end{aligned}$$

It follows from the assumption on ϕ and φ that there exists $c_{11} > 0$ such that

$$|\phi(s, x_\theta(s))|^2 + |\varphi(s, x_\theta(s))|^2 \leq c_{11} \tag{18}$$

Then there is a subsequence still denoted by $\{\phi(s, x_\theta(s)), \varphi(s, x_\theta(s))\}$ which converges to weakly to, say, $\{\phi(s), \varphi(s)\}$ in U.
We have

$$\begin{aligned} &\mathbf{E}\Big|x_\theta(T) - z_T\Big|^2 \\ \leq\ & 7\mathbf{E}\Big|\theta(\theta I + \Theta_0^T)^{-1}\Big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0, x_0)] - \varphi(t, (x_\theta)_t)\Big)\Big|^2 \\ +\ & 7c_6 H(2H-1)T^{2H-1}\bigg(\int_0^T |\theta(\theta I + \Theta_s^T)^{-1}|^2_{L_Q^0} ds\bigg) \\ +\ & 7\mathbf{E}\bigg(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1}|\Big|S(T-s)(\phi(s, x_\theta(s)) - \phi(s)\Big|ds\bigg)^2 \\ +\ & 7\mathbf{E}\bigg(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\phi(s)|ds\bigg)^2 \\ +\ & 7\mathbf{E}\bigg(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1}|\Big|S(T-s)(\varphi(s, x_\theta(s)) - \varphi(s)\Big|ds\bigg)^2 \\ +\ & 7\mathbf{E}\bigg(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\varphi(s)|ds\bigg)^2 \\ +\ & 7cH(2H-1)T^{2H-1}\bigg(\int_0^T |\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\sigma_H(s)|^2_{L_Q^0} ds\bigg). \end{aligned}$$

By assumption ($\mathbf{H}_5$), for all $0 \leq s \leq T$ the operator $\theta(\theta I + \Theta_s^T)^{-1} \to 0$ strongly as $\theta \to 0^+$ and moreover $|\theta(\theta I + \Theta_s^T)^{-1}| \leq 1$. Finally, by the Lebesgue dominated convergence theorem and the compactness of $S(.)$ we get $\mathbf{E}\Big|x_\theta(T) - z_T\Big|^2 \to 0$ as $\theta \to 0^+$ which implies the approximate controllability of system (9). ∎

4. AN EXAMPLE

As a specific application of the theoretical result established in the preceding Theorem, we can consider the following example.

Let $V = L^2(0,\pi)$ and $e_n = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \ldots$. Then $\{e_n\}_n$ is a complete orthonormal basis in V. Let $U = L^2(0,\pi)$ and $A = \frac{\partial^2}{\partial z^2}$ with domain $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$. Then, it is well-known that $Av = -\sum_{n=1}^{\infty} n^2\langle v, e_n\rangle e_n$ for any $v \in U$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t) : U \to U$, where $S(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle v, e_n\rangle e_n$. In order to define the operator $Q : V \to V$, we choose a sequence $\{\eta_n\}_{n\geq 1} \subset \mathbf{R}^+$ and set $Qe_n = \eta_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\eta_n} < \infty$. Define the process $B_Q^H(s)$ by $B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\eta_n} B_n^H(t) e_n$, where $H \in (1/2, 1)$ and $\{B_n^H\}_{n\geq 0}$ is a sequence of mutually independent two-sided one-dimensional fractional Brownian motions.

We consider the following fractional stochastic control system of the form

$$\begin{cases} {}^cD_t^q[x(t,z) - \varphi(t, x_t(z))] &= \dfrac{\partial^2 x(t,z)}{\partial z^2} + y(t,z) + \phi(t, x_t(z)) + \sigma_H(t)\dfrac{dB_Q^H(t)}{dt} \\ x(t,0) &= x(t,\pi) = 0, \quad t \geq 0 \\ x(t,z) &= \psi(t,z); \quad t \in [-r, 0], \quad z \in [0,\pi], \end{cases} \tag{19}$$

where $0 < q < 1$, $r \in (0,1)$ and $T > 0$. We define $x(t)(z) = x(t,z)$, $\phi(t, x_t)(z) = \phi(t, x_t(z))$ and the bounded linear operator $B : \tilde{U} \to U$ by $Bu(t)(z) = y(t,z)$, $0 \leq z \leq \pi$, $u \in \tilde{U}$. On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (18) is approximately controllable on $[0,\pi]$ (see [5]). Therefore, with the above choices, the system (18) may be written in the abstract form (9) and all conditions of Theorem 3.3 are satisfied. Thus, by its conclusion, the fractional stochastic control system (18) is approximately controllable on $[0,\pi]$.

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FEKETE-SZEGÖ TYPE INEQUALITIES FOR CERTAIN SUBCLASSES OF SAKAGUCHI TYPE FUNCTIONS

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Abstract The purpose of the present paper is to derive the coefficient inequality for the class $C(\lambda, \phi, t)$ of certain Sakaguchi type functions $f(z)$ defined on the open unit disk for which $\frac{(1-t)[\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)]}{\lambda z^2[f''(z)-t^2 f''(tz)]+z[f'(z)-tf'(tz)]}$, $(|t| \leq 1, t \neq 1, 0 \leq \lambda \leq 1)$ lies in the region starlike with respect to 1 and is symmetric with respect to real axis. As a special case of this result, coefficient inequality for a class of functions defined through fractional derivatives are obtained.

Keywords: Sakaguchi functions, analytic functions, subordination, Fekete-Szegö inequality.
2010 MSC: 30C50, 30C45, 30C80.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta := \{z \in C |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

Let $f(z)$ and $g(z)$ be analytic functions in the unit disk Δ. We say that $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote the subordination by $f(z) \prec g(z)$. In particular, if $g(z)$ is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let P be the class of all functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ which are in Δ. If $p(z) \in P$, then it satisfies Re $(p(z)) > 0$ in Δ and $p(0) = 1$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α if it satisfies Re $(\frac{zf'(z)}{f(z)}) > \alpha$. The class of starlike functions is denoted by $S^*(\alpha)$.

A function $f(z) \in A$ is said to be in the class $C(\lambda, \alpha, t)$ if it satisfies

$$Re\left\{\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\} > \alpha,$$
$$|t| \leq 1, t \neq 1, 0 \leq \lambda \leq 1, 0 \leq \alpha < 1 \tag{2}$$

and a function $f(z) \in \mathcal{A}$ is said to be in the class $S^*(\alpha, t)$ if it satisfies

$$Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right\} > \alpha, \quad |t| \leq 1, t \neq 1. \tag{3}$$

The class $S^*(\alpha, t)$ was introduced and studied by Owa et al. [9, 10] for some $\alpha \in [0, \alpha)$ and and for all $z \in \Delta$.

For $\alpha = 0$ and $t = -1$ in $S^*(\alpha, t)$ we get the class $S^*(0, -1)$ studied be Sakaguchi [11]. A function $f(z) \in S^*(\alpha, -1)$ is called Sakaguchi function of order α.

In this paper, we define below the class $C(\lambda, \phi, t)$. For earlier works see also [1, 2, 3, 4, 12].

Definition 1.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be univalent starlike function with respect to '1' which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. Then function $f \in \mathcal{A}$ is said to be in the class $C(\lambda, \phi, t)$ if*

$$\left\{\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]}\right\} \prec \phi(z),$$
$$|t| \leq 1, t \neq 1, 0 \leq \lambda \leq 1 \tag{4}$$

In the present paper, we obtain the Fekete-Szegö inequality for the functions in the subclass $C(\lambda, \phi, t)$. We also give application of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider the class $C^\delta(\lambda, \phi, t)$ defined by fractional derivatives.

To prove our main results, we need the following lemma:

Lemma 1.1. *[6] If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \ \ (1/2 < v \leq 1).$$

Lemma 1.2. *[5] If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\},$$

where μ is complex and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$

2. MAIN RESULTS

Our main result is contained in the following theorem:

Theorem 2.1. *If $f(z)$ given by (1) belongs to $C(\lambda, \phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \frac{3\mu B_1^2(1+2\lambda)(2+t)}{4(1+\lambda)^2(1-t)}\right] & \text{if } \mu \leq \sigma_1 \\ \frac{1}{3(1+2\lambda)(2+t)(1-t)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{1}{3(1+2\lambda)(2+t)(1-t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \frac{3\mu B_1^2(1+2\lambda)(2+t)}{4(1+\lambda)^2(1-t)}\right] & \text{if } \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 = \frac{4(1 + \lambda)^2(1 - t)}{3B_1(1 + 2\lambda)(2 + t)}\left\{-1 + \frac{B_2}{B_1} + \frac{B_1(1 + t)}{(1 - t)}\right\}$$

and

$$\sigma_2 = \frac{4(1 + \lambda)^2(1 - t)}{3B_1(1 + 2\lambda)(2 + t)}\left\{1 + \frac{B_2}{B_1} + \frac{B_1(1 + t)}{(1 - t)}\right\}.$$

The result is sharp.

Proof. Let $f \in C(\lambda, \phi, t)$. Then there exists a Schwarz function $w(z) \in \mathcal{A}$ such that

$$\frac{(1 - t)[\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]} = \phi(w(z))$$
$$(z \in \Delta; |t| \leq 1, t \neq 1) \tag{5}$$

Using subordination the class P can also be characterized as $p_1(z) \in P$ if and only if $p_1(z) \prec \frac{1+z}{1-z}$ in Δ, so

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \Delta). \tag{6}$$

From (6), we obtain $1 + w(z) = (1 - w(z))(1 + c_1 z + c_2 z^2 + \cdots)$,

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots . \tag{7}$$

Let

$$\begin{aligned} p(z) &= \frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z[f'(z) - tf'(tz)]} \\ &= 1 + b_1 z + b_2 z^2 + \cdots \quad (z \in \Delta), \end{aligned} \tag{8}$$

which gives

$$\begin{aligned} b_1 &= 2(1+\lambda)(1-t)a_2 \quad \text{and} \\ b_2 &= 4(1+\lambda)^2(t^2-1)a_2^2 + 3(1+2\lambda)(2-t-t^2)a_3. \end{aligned} \tag{9}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (7), we obtain

$$p(z) = \phi(w(z)) = 1 + \frac{B_1 c_1}{2}z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)B_1 + \frac{1}{4}c_1^2 B_2\right\}z^2 + \cdots \quad (z \in \Delta), \tag{10}$$

Now from (8), (9) and (10), we have

$$\begin{aligned} 2(1+\lambda)(1-t)a_2 &= \frac{B_1 c_1}{2}, \\ 4(1+\lambda)^2(t^2-1)a_2^2 + 3(1+2\lambda)(2-t-t^2)a_3 &= \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)B_1 + \frac{1}{4}c_1^2 B_2, \\ &|t| \le 1, t \ne 1, 0 \le \lambda \le 1. \end{aligned}$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{6(1+2\lambda)(2+t)(1-t)}[c_2 - v c_1^2], \tag{11}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - B_1\left(\frac{1+t}{1-t}\right) + \frac{3\mu B_1(1+2\lambda)(2+t)}{4(1+\lambda)^2(1-t)}\right].$$

Our result now follows by an application of Lemma 1.1. To shows that these bounds are sharp, we define the functions K_{ϕ_n} $(n = 2, 3, \ldots)$ by

$$\frac{(1-t)[\lambda z^3 K'''_{\phi_n}(z) + (1+2\lambda)z^2 K''_{\phi_n}(z) + zK'_{\phi_n}(z)]}{\lambda z^2[K''_{\phi_n}(z) - t^2 K''_{\phi_n}(tz)] + z[K'_{\phi_n}(z) - tK'_{\phi_n}(tz)]} = \phi(z^{n-1}),$$

$$K_{\phi_n}(0) = 0 = [K_{\phi_n}]'(0) - 1$$

and the function F_η and G_η $(0 \leq \eta \leq 1)$ by

$$\frac{(1-t)[\lambda z^3 F_\eta'''(z) + (1+2\lambda)z^2 F_\eta''(z) + zF_\eta'(z)]}{\lambda z^2[F_\eta''(z) - t^2 F_\eta''(tz)] + z[F_\eta'(z) - tF_\eta'(tz)]} = \phi\left(\frac{z(z+\eta)}{(1+\eta z)}\right),$$
$$F_\eta(0) = 0 = [F_\eta]'(0) - 1$$

and

$$\frac{(1-t)[\lambda z^3 G_\eta'''(z) + (1+2\lambda)z^2 G_\eta''(z) + zG_\eta'(z)]}{\lambda z^2[G_\eta''(z) - t^2 G_\eta''(tz)] + z[G_\eta'(z) - tG_\eta'(tz)]} = \phi\left(\frac{-z(z+\eta)}{(1+\eta z)}\right),$$
$$G_\eta(0) = 0 = [G_\eta]'(0) - 1.$$

Obviously the functions $K_{\phi_n}, F_\eta, G_\eta \in C(\lambda, \phi, t)$. Also we write $K_\phi := K_{\phi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K_{ϕ_3} or one of its rotations. $\mu = \sigma_1$ then equality holds if and only if f is F_η or one of its rotations. $\mu = \sigma_2$ then equality holds if and only if f is G_η or one of its rotations.

If $\sigma_1 \leq \mu \leq \sigma_2$, in view of Lemma 1.1, Theorem 2.1 can be improved. ∎

Theorem 2.2. *Let $f(z)$ given by (1) belongs to $C(\lambda, \phi, t)$ and σ_3 be given by*

$$\sigma_3 = \frac{\sigma_1 + \sigma_2}{2} = \frac{4(1+\lambda)^2(1-t)}{3B_1(1+2\lambda)(2+t)}\left[\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right]$$

If $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{4}{3B_1^2}\left[(B_1 - B_2)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right) - B_1^2\left(\frac{(1+\lambda)^2(1+t)}{(1+2\lambda)(2+t)}\right) + \frac{3\mu B_1^2}{4}\right]|a_2|^2$$
$$\leq \frac{B_1}{3(1+2\lambda)(2+t)(1-t)}.$$

If $\sigma_3 < \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{4}{3B_1^2}\left[(B_1 + B_2)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right) + B_1^2\left(\frac{(1+\lambda)^2(1+t)}{(1+2\lambda)(2+t)}\right) - \frac{3\mu B_1^2}{4}\right]|a_2|^2$$
$$\leq \frac{B_1}{3(1+2\lambda)(2+t)(1-t)}.$$

Theorem 2.3. *If $f(z)$ is given by (1) belongs to $C(\lambda, \phi, t)$ then*

$$|a_3 - \mu a_2^2|$$
$$\leq \frac{B_1}{3(1+2\lambda)(2+t)(1-t)} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1(1+t)}{(1-t)} - \frac{3(1+2\lambda)(2+t)}{4(1+\lambda)^2(1-t)}\mu B_1\right|\right\}.$$

The result is sharp.

Proof. By applying the Lemma 1.2 in (11) we get Theorem 2.3. The result is sharp for the functions defined by

$$\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z^2[f''(z) - t^2 f''(tz)] + z(f'(z) - tf'(tz))} = \phi(z^2)$$

and
$$\frac{(1-t)[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)]}{\lambda z[f''(z) - t^2 f''(tz)] + z(f'(z) - tf'(tz))]} = \phi(z).$$

∎

3. APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

For two analytic functions $f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=0}^{\infty} g_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z) = z + \sum_{n=0}^{\infty} a_n g_n z^n$. For a fixed $g \in \mathcal{A}$, let $C^g(\lambda, \phi, t)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in C(\lambda, \phi, t)$.

Definition 3.1. *Let $f(z)$ be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of f of order δ is defined by*

$$ {}_0D_z^{\delta} f(z) := \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z (z-\zeta)^{-\delta} f(\zeta) d\zeta \quad (0 \le \delta < 1), \tag{12}$$

where the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring that $\log(z-\zeta)$ is real for $(z-\zeta) > 0$.

Using Definition 3.1, Owa and Srivastava (see [7, 8]; see also [15, 14]) introduced a fractional derivative operator $\Omega^{\delta} : \mathcal{A} \to \mathcal{A}$, which is defined as

$$(\Omega^{\delta} f)(z) = \Gamma(2-\delta) z_0^{\delta} D_z^{\delta} f(z), \quad (\delta \ne 2, 3, 4, \dots).$$

The class $C^{\delta}(\lambda, \phi, t)$ consists of the functions $f \in \mathcal{A}$ for which $\Omega^{\delta} f \in C(\lambda, \phi, t)$. The class $C^{\delta}(\lambda, \phi, t)$ is a special case of the class $C^g(\lambda, \phi, t)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n, (z \in \Delta).$$

Now applying Theorem 2.1 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get following theorem after an obvious change of the parameter μ:

Theorem 3.1. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ ($g_n > 0$. If $f(z)$ is given by (1) belongs to $C^g(\lambda, \phi, t)$ then*

$$|a_3 - \mu a_2^2|$$
$$\le \begin{cases} \frac{1}{3g_3(1+2\lambda)(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t} \right) - \frac{3\mu g_3 B_1^2}{4g_2^2} \frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)} \right] & \text{if } \mu \le \eta_1 \\ \frac{B_1}{3g_3(1+2\lambda)(2+t)(1-t)} & \text{if } \eta_1 \le \mu \le \eta_2 \\ \frac{1}{3g_3(1+2\lambda)(2+t)(1-t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t} \right) - \frac{3\mu g_3 B_1^2}{4g_2^2} \frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)} \right] & \text{if } \mu \le \eta_2 \end{cases}$$

where

$$\eta_1 = \frac{4g_2^2(1+\lambda)^2(1-t)}{3B_1g_3(1+2\lambda)(2+t)}\left\{-1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\},$$
$$\eta_2 = \frac{4g_2^2(1+\lambda)^2(1-t)}{3B_1g_3(1+2\lambda)(2+t)}\left\{1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\}.$$

The result is sharp.

Since

$$\Omega^\delta f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}z^n.$$

We have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta} \tag{13}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}. \tag{14}$$

For g_2, g_3 given by (13) and (14) respectively, Theorem 13 reduces to the following:

Theorem 3.2. *Let $\delta < 2$. If $f(z)$ is given by (1) belongs to $C^\delta(\lambda,\phi,t)$ then*

$$|a_3-\mu a_2^2|$$
$$\leq \begin{cases} \frac{(2-\delta)(3-\delta)}{18(1+2\lambda)(2+t)(1-t)}\times \\ \left[B_2+B_1^2\left(\frac{1+t}{1-t}\right)-\frac{9}{8}\mu\left(\frac{2-\delta}{3-\delta}\right)\frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}B_1^2\right] & \text{if } \mu\leq\eta_1^* \\ \frac{(2-\delta)(3-\delta)B_1}{18(1+2\lambda)(2+t)(1-t)} & \text{if } \eta_1^*\leq\mu\leq\eta_1^* \\ -\frac{(2-\delta)(3-\delta)}{18(1+2\lambda)(2+t)(1-t)}\times \\ \left[B_2+B_1^2\left(\frac{1+t}{1-t}\right)-\frac{9}{8}\mu\left(\frac{2-\delta}{3-\delta}\right)\frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}B_1^2\right] & \text{if } \mu\leq\eta_2^* \end{cases}$$

where

$$\eta_1^* = \frac{8}{9B_1}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right)\left\{-1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\},$$
$$\eta_2^* = \frac{8}{9B_1}\left(\frac{3-\delta}{2-\delta}\right)\left(\frac{(1+\lambda)^2(1-t)}{(1+2\lambda)(2+t)}\right)\left\{1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right\}.$$

Theorem 3.3. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. If $f(z)$ is given by (1) belongs to $C^g(\lambda,\phi,t)$ then*

$$|a_3-\mu a_2^2| \leq \frac{B_1}{3g_3(1+2\lambda)(2+t)(1-t)}\times$$
$$\max\left\{1, \left|\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)-\frac{3\mu B_1g_3}{4g_2^2}\frac{(1+2\lambda)(2+t)}{(1+\lambda)^2(1-t)}\right|\right\}.$$

The result is sharp.

Theorem 3.4. *If $f(z)$ is given by (1) belongs to $C^{\delta}(\lambda, \phi, t)$ then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1(2-\delta)(3-\delta)}{18(1+2\lambda)(2+t)(1-t)} \times \max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \frac{18\mu B_1}{16}\frac{(1+2\lambda)(2+t)(2-\delta)}{(1+\lambda)^2(1-t)(3-\delta)}\right|\right\}.$$

The result is sharp.

Theorem 3.3, Theorem 3.4 were obtained by applying Lemma 1.2.

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BRANCHING EQUATIONS IN THE ROOT-SUBSPACES AND POTENTIALITY CONDITIONS FOR THEM, FOR ANDRONOV-HOPF BIFURCATION. II.

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Abstract As the prolongation of the article [1] on the base of the relevant stationary [2;3] and non-stationary [1] bifurcation results the conditions are established for Lyapunov-Schmidt branching equations and branching equations in the root-subspaces at Poincaré-Andronov-Hopf bifurcation would be of potential type, with invariant potentials under group symmetry conditions.

Keywords: Banach spaces; nonlinear differential equation; small parameter; Lyapunov-Schmidt method; Lyapunov-Schmidt branching equation and branching equation in the root-subspaces; potentiality condition; group symmetry; potential invariance.

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1. INTRODUCTION

Since this article is the direct prolongation of the previous one [1], the contained there short presentation of the articles [2,3] basic results are omitted here. In many applications of bifurcation theory [4-6] often the following situation arises when the original nonlinear problem has not the variational structure, while the relevant Lyapunov-Schmidt branching equation (BEq) and BEq in the root-subspaces (BEqR) turn out to be potential. In the articles [2,3] for such situation in stationary problems of branching theory sufficient conditions for the potentiality of the equivalent to bifurcation problem BEq and also in [7] for the potential type BEq are established. In the article [1] such conditions are obtained for BEqRs of dynamic branching (Poincaré-Andronov-Hopf (P-A-H) bifurcation). Here sufficient conditions are established for the corresponding Lyapunov-Schmidt BEq and BEqR in dynamic branching theory to be systems of potential type. Everywhere below the terminology and notations of the works [1-7] are used.

2. BEQ AND BEQR CONSTRUCTION

As in the article [1], in real Banach spaces E_1 and E_2, for differential equation with sufficiently smooth by ε operators

$$A(\varepsilon)\frac{dx}{dt} = B(\varepsilon)x - R(x,\varepsilon),\ R(0,\varepsilon) = 0,\ R_x(0,\varepsilon) = 0,\ A_0 = A(0),\ B_0 = B(0), \quad (1)$$

P-A-H bifurcation is considered under assumption that A_0-spectrum $\sigma_{A_0}(B_0)$ of densely defined closed Fredholmian operator B_0 is decomposed on two parts: $\sigma^-_{A_0}(B_0)$ lying strictly in the left half-plane and $\sigma^0_{A_0}(B_0)$ consisting of the eigenvalues $\pm i\alpha$ of the multiplicity n with eigenelements $u^{(1)}_j = u_j = u_{1j} \pm iu_{2j}$, and eigenelements $v^{(1)}_j = v_j = v_{1j} \pm iv_{2j}$ of the conjugate operator $A^\star_0 : D_{A^\star_0} \to E^\star_1$, $B^\star_0 : D_{B^\star_0} \to E^\star_1$, i.e. $(B_0 - i\alpha A_0)u_j = 0$, $(B_0 + i\alpha A_0)\overline{u}_j = 0$, $(B^\star_0 + i\alpha A^\star_0)v_j = 0$, $(B^\star_0 - i\alpha A^\star_0)\overline{v}_j = 0$, $j = \overline{1,n}$.

H. Poincaré substitution $t = \frac{\tau}{\alpha+\mu}$, $x(t) = y(\tau)$, $\mu = \mu(\varepsilon)$ reduces the problem of $\frac{2\pi}{\alpha+\mu}$-periodic solutions construction to the determination of 2π -periodic solutions of the equation

$$\begin{aligned} \mathcal{B}y = \mu A(\varepsilon)\frac{dy}{d\tau} + \alpha(A(\varepsilon) - A_0)\frac{dy}{d\tau} - (B(\varepsilon) - B_0)y + R(y,\varepsilon) \equiv \\ \equiv \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y,\varepsilon), \quad R_y(0,\varepsilon) = 0, \end{aligned} \quad (2)$$

$$\mathcal{B}y = (\mathcal{B}y)(\tau) \equiv B_0y(\tau) - \alpha A_0\frac{dy}{d\tau}, \quad \mathcal{C}(\varepsilon)y = (\mathcal{C}(\varepsilon)y)(\tau) \equiv A(\varepsilon)\frac{dy}{d\tau},$$

where the supposed Fredholmian operator $\mathcal{B}$ and the operators in (2) are mapping the space Y of 2π-periodic continuously differentiable functions τ with values in $\mathcal{E}_1 = E_1 + iE_1$ in the space $\mathcal{E}_2 = E_2 + iE_2$ with duality between $Y, Y^\star$ $(Z, Z^\star)$ determined by the functionals

$$\ll y, f \gg = \frac{1}{2\pi}\int_0^{2\pi}\langle y(\tau), f(\tau)\rangle d\tau, y \in Y, f \in Y^\star (y \in Z, f \in Z^\star), \quad (3)$$

(in (3) $\langle\cdot,\cdot\rangle$ represents the duality between $\mathcal{E}_1, \mathcal{E}^\star_1$, $(\mathcal{E}_2, \mathcal{E}^\star_2)$). Then the zero-subspaces of the operators $\mathcal{B}$ and $\mathcal{B}^\star$ are $2n$-dimensional:

$$\mathcal{N}(\mathcal{B}) = span\left\{\varphi^{(1)}_j = \varphi_j,\ \ \varphi_j(\tau) = u_je^{i\tau}; \overline{\varphi}_j\right\}^n_1,$$

$$\mathcal{N}(\mathcal{B}^\star) = span\left\{\psi^{(1)}_j = \psi_j,\ \ \psi_j(\tau) = v_je^{i\tau}; \overline{\psi}_j\right\}^n_1.$$

Introduce the systems $\{\gamma^{(1)}_s\}^n_1 \in Y^\star$ and $\{z^{(1)}_s\}^n_1 \in Z^\star$ biorthogonal in the sense (3) to $\{\varphi^{(1)}_k\}^n_1 \in \mathcal{N}(\mathcal{B})$ and $\{\psi^{(1)}_k\}^n_1 \in \mathcal{N}(\mathcal{B}^\star)$ respectively. As such systems can be chosen the $A_0^\star$- and A_0-images of the last elements of the complete $A_0^\star$- and A_0-Jordan sets of the elements $\{\psi_k^{(1)}\}_1^n$ and $\{\varphi_k^{(1)}\}_1^n$ respectively which are always existed ($\pm i\alpha$ are the isolated eigenvalues) and are determined by the formulae for the generalized Jordan chains [8,9]

$$(B_0 - i\alpha A_0)u_j^{(k)} = A_0 u_j^{(k-1)}, (B_0 + i\alpha A_0)\overline{u}_j^{(k)} = -A_0\overline{u}_j^{(k-1)};$$

$$(B_0^\star + i\alpha A_0^\star)v_j^{(k)} = -A_0^\star v_j^{(k-1)}, (B_0^\star - i\alpha A_0^\star)\overline{v}_j^{(k)} = A_0^\star \overline{v}_j^{(k-1)},$$

$$\mathrm{z}_j^{(k)} = A_0 u_j^{(p_j+1-k)}, \vartheta_j^{(k)} = A_0^\star v_j^{(p_j+1-k)}, k = \overline{1, p_j}, j = \overline{1,n}$$

with the biorthogonality conditions

$$\langle u_j^{(k)}, \vartheta_s^{(l)} \rangle = \delta_{js}\delta_{kl}, \langle \mathrm{z}_j^{(k)}, v_s^{(l)} \rangle = \delta_{js}\delta_{kl}$$

and respectively

$$\mathcal{B}\varphi_j^{(k)} = A_0\varphi_j^{(k-1)}, \mathcal{B}\overline{\varphi}_j^{(k)} = -A_0\overline{\varphi}_j^{(k-1)},$$

$$\mathcal{B}^\star\psi_j^{(k)} = \left(B_0^\star + \alpha A_0^\star \frac{d}{d\tau}\right)\psi_j^{(k)} = -A_0^\star\psi_j^{(k-1)},$$

$$\mathcal{B}_0^\star\overline{\psi}_j^{(k)} = \left(B_0^\star + \alpha A_0^\star \frac{d}{d\tau}\right)\overline{\psi}_j^{(k)} = A_0^\star\overline{\psi}_j^{(k-1)},$$

where

$$\varphi_j^{(k)} = u_j^{(k)} e^{i\tau}, \overline{\varphi}_j^{(k)} = \overline{u}_j^{(k)} e^{-i\tau}, \psi_j^{(k)} = v_j^{(k)} e^{i\tau}, \overline{\psi}_j^{(k)} = \overline{v}_j^{(k)} e^{-i\tau}$$

$$z_j^{(k)} = \mathrm{z}_j^{(k)} e^{i\tau}, \gamma_s^{(l)} = \vartheta_s^{(l)} e^{i\tau}, k(l) = 1, p_j(p_s), j, s = \overline{1,n}$$

with the biorthogonality conditions

$$\ll \varphi_j^{(k)}, \gamma_s^{(l)} \gg = \delta_{js}\delta_{kl}, \ll z_j^{(k)}, \psi_s^{(l)} \gg = \delta_{js}\delta_{kl}, k(l) = \overline{1, p_j(p_s)}, j, s = \overline{1,n} \tag{4}$$

$K = p_1 + p_2 + ... + p_n$ is the root-number.

Introduce the following notations, available further for the writing of the projectors: $\Phi = (\varphi_1^{(1)}, ..., \varphi_1^{(p_1)}, \varphi_n^{(1)}, ..., \varphi_n^{(p_n)})$. The vectors γ, Ψ and Z are defined analogously.

Lemma 2.1. [10, 11] *Biorthogonality conditions (4) allow to introduce the projectors*

$$\mathbf{P} = \sum_{j=1}^{n}\sum_{k=1}^{p_i} \ll \cdot, \gamma_j^{(k)} \gg \varphi_j^{(k)} = \ll \cdot, \gamma \gg \Phi, \quad \overline{\mathbf{P}} = \ll \cdot, \overline{\gamma} \gg \Phi, \quad \mathbb{P} = \mathbf{P} + \overline{\mathbf{P}},$$

$$\mathbf{Q} = \sum_{j=1}^{n}\sum_{k=1}^{p_i} \ll \cdot, \psi_j^{(k)} \gg z_j^{(k)} = \ll \cdot, \psi \gg z, \overline{\mathbf{Q}} = \ll \cdot, \overline{\psi} \gg \overline{z}, \quad \mathbb{Q} = \mathbf{Q} + \overline{\mathbf{Q}},$$

generating expansions of the spaces Y and Z in direct sums $Y = Y^{2K}\dot{+}Y^{\infty-2K}$, $Z = Z_{2K}\dot{+}Z_{\infty-2K}$, $Y^{2K} = span\{\varphi_j^{(k)}, \overline{\varphi_j^{(k)}}\}_{j=\overline{1,n},k=\overline{1,p_j}}$ is the root-subspace of A-adjoint elements of the operator $\mathcal{B}$,$Z_{2K} = span\{z_j^k, \overline{z_j^k}\}_{j=\overline{1,n},k=\overline{1,p_j}}$. The operators $\mathcal{B}$ and A_0 are intertwined by the projectors $\mathbf{P}$ and $\mathbf{Q}$, $\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}$, $\mathcal{B}\mathbf{P}u = \mathbf{Q}\mathcal{B}u$ on $\mathbf{D}_{\mathcal{B}}$, $\mathcal{B}\Phi = \mathfrak{A}_0 Z$, $\mathcal{B}^\star\Psi = \mathfrak{A}\gamma$, $\mathfrak{A} = diag\{B_1, ..., B_n\}$ is cell-diagonal matrix, B_i is $p_i\times p_i$-matrix with units along secondary subdiagonal and zeros on other places; $A_0\mathbf{P} = \mathbf{Q}A_0$, $\mathcal{C}_0\mathbf{P} = \mathbf{Q}\mathcal{C}_0$ on D_{A_0}, $\mathcal{C}_0 = \mathcal{C}(0)$, $A_0\Phi = \mathfrak{A}_1 Z$, $A_0^\star\Psi = \mathfrak{A}_1\gamma$, $\mathfrak{A}_1 = diag\{B^1, ..., B^n\}$ is cell-diagonal matrix, B^i is $p_i \times p_i$-matrix with units along secondary diagonal and zeros on other places. Operators A_0 and $\mathcal{B}$ act in invariant pairs of subspaces Y^{2K}, Z_{2K} and $Y^{\infty-2K}$, $Z_{\infty-2K}$ and $\mathcal{B} : Y^{\infty-2K} \bigcap D_{\mathcal{B}} \to Z_{\infty-2K}$, $A_0 : Y^{2K} \to Z_{2K}$ are isomorphisms.

Remark 2.1. [9, 10, 4]. *Because of the invariance property of the root-number K under perturbation we can work with A_0-adjoint elements of the operator $\mathcal{B}$, the more so parameter μ enters linearly in the equation (2).*

Consider now the Lyapunov-Schmidt BEqR construction [4,11]. The usage of the E.Schmidt regularizator [4]

$$\widetilde{\mathcal{B}} = \mathcal{B} + \sum_{s=1}^{n}[\ll \cdot, \gamma_i^{(1)} \gg z_i^{(1)} + \ll \cdot, \overline{\gamma}_i^{(1)} \gg \overline{z}_i^{(1)}], \quad \widetilde{\mathcal{B}}^{-1} = \Gamma$$

allows to rewrite the equation (2) in the form of the system

$$\begin{aligned} &\widetilde{\mathcal{B}}y = \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y, \varepsilon) + \sum_{i=1}^{n}(\xi_{i1}z_i^{(1)} + \overline{\xi}_{i1}\overline{z}_i^{(1)}), \\ &\xi_{s\sigma} = \ll y, \gamma_s^{(\sigma)} \gg, \quad \overline{\xi}_{s\sigma} = \ll y, \overline{\gamma}_s^{(\sigma)} \gg, \quad \sigma = \overline{1, p_s}, s = \overline{1, n} \end{aligned} \tag{5}$$

the unique solution of the first equation of which is sought in the form

$$y = u + \xi \cdot \Phi + \overline{\xi} \cdot \overline{\Phi} = u + v(\xi, \overline{\xi}, \mu, \varepsilon), \xi = \xi(\mu(\varepsilon), \varepsilon), \overline{\xi} = \overline{\xi}(\mu(\varepsilon), \varepsilon). \tag{6}$$

Then the first equation of the system (5) gives

$$\begin{aligned} u = -(I - \mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi}_{ij}\overline{\varphi}_i^{(j)}) + \mu(I - \mu\Gamma\mathcal{C}_0)^{-1}\Gamma\mathcal{C}_0(\xi \cdot \Phi + \overline{\xi} \cdot \overline{\Phi}) + \\ +\mu(I - \mu\Gamma\mathcal{C}_0)^{-1}\Gamma(\mathcal{C}(\varepsilon) - \mathcal{C}_0)u + \mu(I - \mu\Gamma\mathcal{C}_0)^{-1}\Gamma(\mathcal{C}(\varepsilon) - \mathcal{C}_0)(\xi \cdot \Phi + \overline{\xi} \cdot \overline{\Phi}) + \\ +\mu(I - \mu\Gamma\mathcal{C}_0)^{-1}\Gamma\mathcal{R}(u + v, \varepsilon) = \\ -\sum_{i=1}^{n}\sum_{j=2}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi_{ij}\varphi_i^{(j)}}) + \mu\Gamma\mathcal{C}_0(I - \mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}(\xi_{i1}\varphi_i^{(1)} + \overline{\xi}_{i1}\overline{\varphi}_i^{(1)}) + \\ +\mu\Gamma(I - \mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon) - \mathcal{C}_0)u + \mu\Gamma(I - \mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon) - \mathcal{C}_0)(\xi \cdot \Phi + \overline{\xi} \cdot \overline{\Phi}) + \end{aligned}$$

$$+\mu\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon).$$

Taking into account the relations $\mu\Gamma\mathcal{C}_0\varphi_i^{(1)} = i\mu\varphi_i^{(2)}$, $\mu^2(\Gamma\mathcal{C}_0)^2\varphi_i^{(1)} = (i\mu)^2\varphi_i^{(3)}$,..., $\mu^{p_i-1}(\Gamma\mathcal{C}_0)^{p_i-1}\varphi_i^{(1)} = (i\mu)^{p_i-1}\varphi_i^{(p_i)}$, $\mu^{p_i}(\Gamma\mathcal{C}_0)^{p_i}\varphi_i^{(1)} = (i\mu)^{p_i}\varphi_i^{(1)}$, and generally, $\mu^l(\Gamma\mathcal{C}_0)^l\varphi_i^{(1)} = (i\mu)^l\varphi_i^{(l+1-[\frac{l+1}{p_i}]p_i)}$ according to the formulae $\Gamma^\star\gamma_s^{(1)} = \psi_s^{(1)}$, $\Gamma^\star\gamma_s^{(\sigma)} = \psi_s^{(p_s+2-\sigma)}$ [10,11] from the second equalities of the system (5) E. Schmidt BEqR follows

$$\begin{aligned}
&t_{s1}(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(1)}\gg = -\tfrac{(i\mu)^{p_s}}{1-(i\mu)^{p_s}}\xi_{s1} - \mu \ll (I-\mu\mathcal{C}_0)\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(1)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(p_s+2-\sigma)} \gg = 0,\\
&t_{s\sigma}(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(\sigma)}\gg = \xi_{s\sigma} - \tfrac{(i\mu)^{\sigma-1}}{1-(i\mu)^{p_s}}\xi_{s1} - \mu \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(p_s+2-\sigma)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(p_s+2-\sigma)} \gg = 0,\\
&\sigma = \overline{2,p_s},\ s=\overline{1,n}.
\end{aligned} \tag{7}$$

For the BEq construction, write the equation (2) in the form of the system

$$\begin{aligned}
&\widetilde{\mathcal{B}}y = \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y,\varepsilon) + \sum_{i=1}^n(\xi_{i1}z_i^{(1)} + \overline{\xi}_{i1}\overline{z}_i^{(1)}),\\
&\xi_s = \ll y,\gamma_s^{(1)}\gg, \quad \overline{\xi}_s = \ll y,\overline{\gamma}_s^{(1)}\gg .
\end{aligned} \tag{8}$$

The unique solution of the first equation (8)

$$y = \Gamma(\mu\mathcal{C}(\varepsilon)y) + \Gamma\mathcal{R}(y,\varepsilon) + \sum_{j=1}^n(\xi_j\varphi_j + \overline{\xi}_j\overline{\varphi}_j) \tag{9}$$

find in the form

$$y = \sum_{j=1}^n(\xi_j\varphi_j + \overline{\xi}_j\overline{\varphi}_j) + u(\xi,\overline{\xi},\mu,\varepsilon) \tag{10}$$

Then the second equalities (8) by using the relations $\Gamma z_j^{(1)} = \varphi_j^{(1)}$, $\Gamma\overline{z}_j^{(1)} = \overline{\varphi}_j^{(1)}$, $\Gamma^\star\gamma_j^{(1)} = \psi_j^{(1)}$, $\Gamma^\star\overline{\gamma}_j^{(1)} = \overline{\psi}_j^{(1)}$, $Y = Y^{2n} \dotplus Y^{\infty-2n}$, $Z = Z_{2n} \dotplus Z_{\infty-2n}$, $Y^{2n} = \mathcal{N}(\mathcal{B})$, $Z_{2n} = span\{z_s,\overline{z}_s\}_{s=1}^n$, $Y^{2n} = (P_n+\overline{P}_n)Y = \mathbf{P}_nY$, $P_n = \sum_{j=1}^n \ll\cdot,\gamma_j\gg\varphi_j$, $\overline{P}_n = \sum_{j=1}^n \ll\cdot,\overline{\gamma}_j^{(1)}\gg \overline{\varphi}_j^{(1)}$, $Z_{2n} = (Q_n+\overline{Q}_n) = \mathbf{Q}_nZ$, $Q_n = \sum_{j=1}^n \ll\cdot,\psi_j^{(1)}\gg z_j^{(1)}$, $\overline{Q}_n = \sum_{j=1}^n \ll\cdot,\overline{\psi}_j^{(1)}\gg \overline{z}_j^{(1)}$ give E. Schmidt BEq

$$\begin{aligned}
&t_s(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(1)}\gg = -\tfrac{(i\mu)^{p_s}}{1-(i\mu)^{p_s}}\xi_s - \mu \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(1)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(1)} \gg = 0,\\
&t_s(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\overline{\gamma}_s^{(1)}\gg = 0,\ s=\overline{1,n}.
\end{aligned} \tag{11}$$

3. CONDITIONS OF BEQ AND BEQR POTENTIALITY TYPES $A(B)$

Definition 3.1. [7] *BEq* $t(\xi,\bar{\xi},\mu,\varepsilon) = 0$, $\xi = (\xi_1,\xi_2,...,\xi_n)$ *(BEqR* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon) = 0$, $\xi = (\xi_{11},\xi_{12},\xi_{1p_1},...,\xi_{n1},\xi_{n2},...,\xi_{np_n},)$*) for dynamic branching problem of branching theory is called BEq of potential type A or BEq of potential type B if* $t(\xi,\bar{\xi},\mu,\varepsilon) = d\cdot grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon) \sim (t_1,\bar{t}_1,\dots,t_n,\bar{t}_n)^T =$
$= d\cdot\left(\frac{\partial U}{\partial\bar{\xi}_1},\frac{\partial U}{\partial\xi_1},\dots,\frac{\partial U}{\partial\bar{\xi}_n},\frac{\partial U}{\partial\xi_n}\right)$ *or* $t(\xi,\bar{\xi},\mu,\varepsilon) = grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\cdot d \sim$
$\sim (t_1,\bar{t}_1,\dots,t_n,\bar{t}_n) = \left(\frac{\partial U}{\partial\bar{\xi}_1},\frac{\partial U}{\partial\xi_1},\dots,\frac{\partial U}{\partial\bar{\xi}_n},\frac{\partial U}{\partial\xi_n}\right)\cdot d$ *with an invertible matrix d (respectively* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon) = d\cdot grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\sim$
$\sim (t_{11},\bar{t}_{11},\dots,t_{1p_1},\bar{t}_{1p_1},\dots,t_{n1},\bar{t}_{n1},\dots,t_{np_n},\bar{t}_{np_n})^T =$
$= d\cdot\left(\frac{\partial U}{\partial\bar{\xi}_{11}},\frac{\partial U}{\partial\xi_{11}},\dots,\frac{\partial U}{\partial\bar{\xi}_{1p_1}},\frac{\partial U}{\partial\xi_{1p_1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{n1}},\frac{\partial U}{\partial\xi_{n1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{np_n}},\frac{\partial U}{\partial\xi_{np_n}}\right)$ *or* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon) = grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\cdot d\sim$
$\sim (t_{11},\bar{t}_{11},\dots,t_{1p_1},\bar{t}_{1p_1},\dots,t_{n1},\bar{t}_{n1},\dots,t_{np_n},\bar{t}_{np_n}) =$
$= \left(\frac{\partial U}{\partial\bar{\xi}_{11}},\frac{\partial U}{\partial\xi_{11}},\dots,\frac{\partial U}{\partial\bar{\xi}_{1p_1}},\frac{\partial U}{\partial\xi_{1p_1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{n1}},\frac{\partial U}{\partial\xi_{n1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{np_n}},\frac{\partial U}{\partial\xi_{np_n}}\right)\cdot d$*).*

Remark 3.1. *Note here that potentiality conditions for BEq and BEqR of potentiality type* $A(B)$ *in stationary branching are obtained and proved in [7] and respectively in our communication to Int.Conf. [12].*

In the development of the article [1] results here similarly to n.2 and n.4 of [1] sufficient potentiality conditions are established for BEq (11) and BEqR (7) would be of potential type $A(B)$. Since the notions of the operators symmetrizability [2, 3] here are introduced for the equation (2) in the spaces Y, Z, elements of which are complex-valued functions in the definition 3.1 and analogs of n.2 , 4 assertions of [1] in the notion of matrices symmetricity the complex conjugation must be used as this is accepted in [1]. As there it is used for the proofs of operators $\mathcal{B}$, $\mathcal{C}(\varepsilon)$ and $\mathcal{R}_y$ symmetrizability. The finite-dimensional symmetrizators

$$J_n = \sum_{j=1}^{n}(\ll\cdot,\psi_j\gg\gamma_j + \ll\cdot,\overline{\psi_j}\gg\overline{\gamma_j})$$

for BEq and

$$J_n = \sum_{j=1}^{n}\sum_{k=1}^{p_j}[\ll\cdot,\psi_j^{(k)}\gg\gamma_j^{(k)} + \ll\cdot,\overline{\psi}_j^{(k)}\gg\overline{\gamma}_j^{(k)}]$$

for BEqR are used.

A.

Lemma 3.1. . *For the BEq (11) would be of potential type $A(B)$ it is sufficient the symmetricity of the matrices* $[\frac{\partial(d^{-1}\cdot t)}{\partial\xi\partial\bar{\xi}}]$, $[\frac{\partial(t\cdot d^{-1})}{\partial\xi\partial\bar{\xi}}]$ *,i.e.*

$$\begin{gathered}
\sum_{s=1}^{n}\left(d_{2p-1,2s-1}\frac{\partial t_s}{\partial\xi_q}+d_{2p-1,2s}\frac{\partial\bar{t}_s}{\partial\xi_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q-1,2s-1}\frac{\partial t_s}{\partial\xi_p}+d_{2q-1,2s}\frac{\partial\bar{t}_s}{\partial\xi_p}\right)},\\
\sum_{s=1}^{n}\left(d_{2p,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_q}+d_{2p,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_p}+d_{2q,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_p}\right)},\\
\sum_{s=1}^{n}\left(d_{2p-1,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_q}+d_{2p-1,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q,2s-1}\frac{\partial t_s}{\partial\xi_p}+d_{2q,2s}\frac{\partial\bar{t}_s}{\partial\xi_p}\right)},
\end{gathered}\tag{12}$$

for the type A and

$$\begin{gathered}
\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_q}d_{2s-1,2p-1}+\frac{\partial\bar{t}_s}{\partial\xi_q}d_{2s,2p-1}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_p}d_{2s-1,2q-1}+\frac{\partial\bar{t}_s}{\partial\xi_p}d_{2s,2q-1}\right)},\\
\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_q}d_{2s-1,2p}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}d_{2s,2p}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_p}d_{2s-1,2q}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_p}d_{2s,2q}\right)},\\
\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_q}d_{2s-1,2p-1}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}d_{2s,2p-1}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_p}d_{2s-1,2q}+\frac{\partial\bar{t}_s}{\partial\xi_p}d_{2s,2q}\right)}
\end{gathered}\tag{13}$$

for the type B.

The proof follows from the definition 3.1 at the usage of designation

$$d^{-1}=\begin{pmatrix} d_{2k-1,2s-1} & d_{2k-1,2s}\\ d_{2k,2s-1} & d_{2k,2s}\end{pmatrix}_{k,s=\overline{1,n}}.$$

The finding of solutions to (9) in the form (10) with the subsequent differentiation leads to relations

$$\frac{\partial y}{\partial\xi_s}=[\mu\Gamma\mathcal{C}(\varepsilon)+\Gamma\mathcal{R}_y]\frac{\partial y}{\partial\xi_s}+\varphi_s\Rightarrow\frac{\partial y}{\partial\xi_s}=[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_s,$$

$$\frac{\partial y}{\partial\xi_s}=\varphi_s+\frac{\partial u}{\partial\xi_s}\Rightarrow\varphi_s+\frac{\partial u}{\partial\xi_s}=\varphi_s+\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_s\Rightarrow$$

$$\frac{\partial t_s}{\partial\xi_q}=-\ll\frac{\partial u}{\partial\xi_q},\gamma_s^{(1)}\gg=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_q,\psi_s\gg$$

and analogously $\frac{\partial\overline{t_s}}{\partial\xi_q}=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_q,\overline{\psi_s}\gg$,

$$\frac{\partial t_s}{\partial\overline{\xi_q}}=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\overline{\varphi_q},\psi_s\gg,$$

$$\frac{\partial \overline{t_s}}{\partial \overline{\xi_q}} = - \ll \mu \mathcal{C}(\varepsilon) + \mathcal{R}_y)[I - \Gamma(\mu \mathcal{C}(\varepsilon) + \mathcal{R}_y)]^{-1}\overline{\varphi_q}, \overline{\psi_s} \gg .$$

Corolar 3.1. *When $d = I$ the usual potentiality conditions for BEq from [1] follow: $\frac{\partial t_k}{\partial \xi_s} = \overline{\frac{\partial \overline{t_s}}{\partial \overline{\xi_k}}}$, $\frac{\partial \overline{t_k}}{\partial \xi_s} = \overline{\frac{\partial \overline{t_s}}{\partial \xi_k}}$ and $\frac{\partial t_k}{\partial \overline{\xi_s}} = \overline{\frac{\partial \overline{t_s}}{\partial \overline{\xi_k}}}$, $k, s = \overline{1, n}$.*

Lemma 3.2. *Let the operator $\mathcal{B} : Y \supset D_{\mathcal{B}} \to Z$ be J–symmetrizable on $D = D_{\mathcal{B}}$ and the operator $J : Z \to Y^\star$ satisfies the requirements:*

1°. $\forall y \in Y^{\infty-2n} \Rightarrow J^\star y \in Z^\star_{\infty-2n} = \{f \in Z^\star | \ll z_s, f \gg = 0, \ll \bar{z}_s, f \gg = 0,$ $s = \overline{1, n}\}$;

2°.The matrix $\ll (\varphi, \overline{\varphi}), J(z, \bar{z}) \gg$ is symmetric, i.e. $\ll \varphi_s, Jz_k \gg = \overline{\ll \overline{\varphi_k}, J\overline{z_s} \gg}$, $\ll \overline{\varphi_s}, J\overline{z_k} \gg = \ll \overline{\overline{\varphi_k}, J\overline{z_s}} \gg$, $\ll \overline{\varphi_s}, Jz_k \gg = \ll \overline{\varphi_k, J\overline{z_s}} \gg$.

Then the operator $\Gamma = \widetilde{\mathcal{B}}^{-1}$ is J^* -symmetrizable on Z.

Now the following analog of the Theorem 4.1 [1] is true.

Theorem 3.1. *Let there exists a linear operator $J\colon Z \to Y^\star$, such that $J^\star \varphi_p = \sum_{s=1}^{n} (\overline{d}_{2p-1,2s-1}\psi_s + \overline{d}_{2p-1,2s}\overline{\psi_s})$, $J^\star \overline{\varphi_p} = \sum_{s=1}^{n} (\overline{d}_{2p,2s-1}\psi_s + \overline{d}_{2p,2s}\overline{\psi_s})$ (resp.$J^\star \varphi_p = \sum_{s=1}^{n} (\overline{d}_{2s-1,2p-1}\psi_s + \overline{d}_{2s,2p}\overline{\psi_s})$, $J^\star \overline{\varphi_p} = \sum_{s=1}^{n} (\overline{d}_{2s-1,2p}\psi_s + \overline{d}_{2s,2p}\overline{\psi_s})$ and the following requirements are realized:*

1°. Operator $\mathcal{B}$ is J-symmetrizable on D;

2°. Operator $\mathcal{C}(\varepsilon)$ and operators $B(\varepsilon) - B_0)$, $\mathcal{R}(y, \varepsilon)$ for any (y, ε) in some neighborhood of the point $(0, 0)$ are J-symmetrizable on D;

3°. For any $y \in Y^{\infty-2n} \bigcap D$ follows that $J^\star y \in Z^\star_{\infty-2n}$.

Then the BEq (10) is the system of potential type A (resp. B).

The proof follows from the analogs of assertions n.3 [1] and lemmas 3.1, 3.2.

Corollary 3.1. *When $d = I$, Theorem 3.1 coincides with Theorem 4.1 of [1].*

Remark 3.2. *In applications the matrix d often turns out to be diagonal.*

B. For the simplicity of presentation further potential BEqRs are considered.

Lemma 3.3. *For the BEqR (7) potentiality it is sufficient the symmetricity of the matrix*

$$\mathbf{D} = \mathbf{D}(\frac{\mathbf{t}, \bar{\mathbf{t}}}{\xi, \bar{\xi}}) = \frac{D(t_{11}, \overline{t_{11}}, \ldots, t_{1p_1}, \overline{t_{1p_1}} \ldots, t_{n1}, \overline{t_{n1}} \ldots, t_{np_n}, \overline{t_{np_n}})}{D(\xi_{11}, \overline{\xi_{11}}, \ldots, \xi_{1p_1}, \overline{\xi_{1p_1}}, \ldots, \xi_{n1}, \overline{\xi_{n1}}, \ldots, \xi_{np_n}, \overline{\xi_{np_n}})},$$

i.e. the equalities relations

$$\frac{\partial t_{kl}}{\partial \xi_{s\sigma}} = \overline{\frac{\partial t_{s\sigma}}{\partial \xi_{kl}}}, \quad \frac{\partial \bar{t}_{kl}}{\partial \bar{\xi}_{s\sigma}} = \overline{\frac{\partial \bar{t}_{s\sigma}}{\partial \bar{\xi}_{kl}}}, \quad \frac{\partial t_{kl}}{\partial \bar{\xi}_{s\sigma}} = \overline{\frac{\partial \bar{t}_{s\sigma}}{\partial \xi_{kl}}}. \tag{14}$$

Remark 3.3. *From (14) the reality of diagonal elements* **D** *follows (when $k = s$, $\sigma = l$).*

For the shortening of computation in (6) the following designation will be used
$u = -\sum_{i=1}^{n}\sum_{j=2}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi_{ij}\varphi_i^{(j)}}) + \mu\Gamma\mathcal{C}_0(I-\mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}(\xi_{i1}\varphi_i^{(1)} + \overline{\xi_{i1}}\overline{\varphi_i^{(1)}}))+$
$+\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}(u+v,\mu,\varepsilon)$, i.e.

$$\mathbf{R}(y,\mu,\varepsilon) = \mu(\mathcal{C}(\varepsilon)-\mathcal{C}_0)y + \mathcal{R}(y,\varepsilon), \quad v = \xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi}.$$

For the verification (14) the computation of the derivatives $\dfrac{\partial u}{\partial \xi_{sk}}, \dfrac{\partial u}{\partial \overline{\xi}_{sk}}$ is required:

$$\frac{\partial u}{\partial \xi_{s1}} = \mu\Gamma\mathcal{C}_0(I-\mu\Gamma\mathcal{C}_0)^{-1}\varphi_s^{(1)} + \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y(\frac{\partial u}{\partial \xi_{s1}} + \varphi_s^{(1)}) \Rightarrow$$

$$\Rightarrow \frac{\partial u}{\partial \xi_{s1}} = [I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}\{\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y\varphi_s^{(1)} + \Gamma(I-\mu\mathcal{C}_0)\Gamma)^{-1}\mu\mathcal{C}_0\varphi_s^{(1)}\} =$$

$$= \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0)\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y + \mu\mathcal{C}_0)\varphi_s^{(1)},$$

$$\frac{\partial u}{\partial \xi_{sk}} = -\varphi_s^{(k)} + \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y(\frac{\partial u}{\partial \xi_{sk}} + \varphi_s^{(k)}),$$

whence it follows $\dfrac{\partial u}{\partial \xi_{sk}} = -\varphi_s^{(k)}$ when $k > 1$. Now (7) and (14) give the relations of the following type

$$\begin{aligned} &\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_s^{(1)}, \psi_k^{(1)} \gg= \\ &=\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_k^{(1)}, \psi_s^{(1)} \gg, \end{aligned} \quad (15)$$

$$\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_k^{(1)}, \psi_s^{(p_s+2-\sigma)} \gg= 0, \quad (16)$$

for $\sigma \geq 2$, since $\dfrac{\partial t_{kl}}{\partial \xi_{s\sigma}} = - \ll \varphi_s^{(\sigma)}, \gamma_k^{(l)} \gg= -\delta_{sk}\delta_{\sigma l}$.

Corolar 3.2. *Formulae (15) and (16) mean that BEqR potentiality is equivalent to BEq potentiality.*

4. SYMMETRY IN P-A-H BIFURCATION PROBLEM WITH POTENTIAL BRANCHING EQUATIONS

SH(2) – symmetry. As the prolongation of the article [1] results, return now to model example of P-A-H bifurcation with $SH(2)$-symmetry generated by the following zero-subspace of the linearized operator with relevant pure imaginary eigenvalues

$$N = N(\mathcal{B}) = span\{\varphi_1 = (chx + ishx)e^{it}, \overline{\varphi_1}, \varphi_2 = (chx - ishx)e^{it}, \overline{\varphi_2}\},$$

with the following matrix representation in the definition of group invariance $\mathcal{B}_g t(\xi,\bar{\xi}) = t(\mathcal{A}_g\xi, \overline{\mathcal{A}_g\xi})$ correspondingly to the basis N

$$\mathcal{B}(\alpha_0,\alpha) = \mathcal{A}(\alpha_0,\alpha) = \\ = \begin{pmatrix} e^{i\alpha_0}ch\alpha & 0 & -ie^{i\alpha_0}sh\alpha & 0 \\ 0 & e^{-i\alpha_0}ch\alpha & 0 & ie^{-i\alpha_0}sh\alpha \\ ie^{i\alpha_0}sh\alpha & 0 & e^{i\alpha_0}ch\alpha & 0 \\ 0 & -ie^{-i\alpha_0}sh\alpha & 0 & e^{-i\alpha_0}ch\alpha \end{pmatrix}$$

Here the branching equation potentiality of the types A and B take place simultaneously. In the articles [13,14] at the usage of group analysis methods [15] the general form of C^1-BEq was constructed on allowed group symmetry (see also [1]).

$$\begin{aligned} t_1 &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_1(I_1(\xi), I_2(\xi)) + \bar{\xi}_2 F_2(I_1(\xi), I_2(\xi))\right] \\ I_1 &= \sqrt{|\xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2|},\ I_2 = \sqrt{|\xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2|}, \\ t_2 &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_2(I_1(\xi), I_2(\xi)) - \bar{\xi}_2 F_1(I_1(\xi), I_2(\xi))\right] \end{aligned} \tag{17}$$

where the functions F_1,F_2 are real-valued.

The definition 3.1 of potential type BEq (here simultaneously A and B) with $d = diag(1,1,-1,-1)$ means the symmetricity of the matrix $\mathbf{D}$, i.e. the realization of the equalities (12) or (13).

$$\begin{pmatrix} \frac{\partial t_1}{\partial \xi_1} & \frac{\partial t_1}{\partial \bar{\xi}_1} & \frac{\partial t_1}{\partial \xi_2} & \frac{\partial t_1}{\partial \bar{\xi}_2} \\ \frac{\partial \bar{t}_1}{\partial \xi_1} & \frac{\partial \bar{t}_1}{\partial \bar{\xi}_1} & \frac{\partial \bar{t}_1}{\partial \xi_2} & \frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} \\ -\frac{\partial t_2}{\partial \xi_1} & -\frac{\partial t_2}{\partial \bar{\xi}_1} & -\frac{\partial t_2}{\partial \xi_2} & -\frac{\partial t_2}{\partial \bar{\xi}_2} \\ -\frac{\partial \bar{t}_2}{\partial \xi_1} & -\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1} & -\frac{\partial \bar{t}_2}{\partial \xi_2} & -\frac{\partial \bar{t}_2}{\partial \bar{\xi}_2} \end{pmatrix} \Rightarrow \begin{array}{l} \frac{\partial t_1}{\partial \xi_1} = \overline{\frac{\partial \bar{t}_1}{\partial \bar{\xi}_1}},\quad \frac{\partial t_2}{\partial \xi_2} = \overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_2}} \Rightarrow a) \\ \frac{\partial t_1}{\partial \bar{\xi}_1} = \frac{\partial \bar{t}_1}{\partial \xi_1},\quad \frac{\partial t_2}{\partial \bar{\xi}_2} = \frac{\partial \bar{t}_2}{\partial \xi_2} \Rightarrow b) \\ \frac{\partial t_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \xi_1}},\quad \frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1}} \Rightarrow c) \\ \frac{\partial t_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}},\quad \frac{\partial \bar{t}_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \bar{\xi}_1}} \Rightarrow d) \end{array} \tag{18}$$

Here a) is the reality of diagonal elements, b) the partial potentiality, c) the symmetry along secondary subdiagonals, d) symmetry along secondary diagonal. Conditions (18) lead to the following relations, where the symbols $F_{k,1}$, $F_{k,2}$ mean the derivatives of F_k on the relevant invariants I_1, I_2:

$$\begin{aligned}\frac{\partial t_1}{\partial \xi_1} &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_{1,1}\frac{\bar{\xi}_1}{2I_1}+\bar{\xi}_1 F_{1,2}\frac{\bar{\xi}_2}{2I_2}+\bar{\xi}_2 F_{2,1}\frac{\bar{\xi}_1}{2I_1}+\bar{\xi}_2 F_{2,2}\frac{\bar{\xi}_2}{2I_2}\right],\\ \frac{\partial \bar{t}_1}{\partial \bar{\xi}_1} &= \frac{1}{\xi_1^2+\xi_2^2}\left[\xi_1 F_{1,1}\frac{\xi_1}{2I_1}+\bar{\xi}_1 F_{1,2}\frac{\xi_2}{2I_2}+\xi_2 F_{2,1}\frac{\xi_1}{2I_1}+\xi_2 F_{2,2}\frac{\xi_2}{2I_2}\right],\end{aligned} \tag{19}$$

the second equality of a) and also the equalities b) are verified analogously,

$$\begin{aligned}&\frac{\partial t_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \xi_1}} \Rightarrow (-F_{1,1})[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_2+F_{2,2}[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\\ &+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_1+F_{1,2}[\bar{\xi}_1^2(\xi_1^2+\xi_2^2)-\xi_2^2(\bar{\xi}_1^2+\bar{\xi}_2^2)]I_1+\\ &+F_{2,1}[\xi_1^2(\bar{\xi}_1^2+\bar{\xi}_2^2)-\bar{\xi}_2^2(\xi_1^2+\xi_2^2)]I_2=0,\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1}} \Rightarrow (-F_{1,1})[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_2+F_{2,2}[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\\ &+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_1+F_{1,2}[\xi_1^2(\bar{\xi}_1^2+\bar{\xi}_2^2)-\bar{\xi}_2^2(\xi_1^2+\xi_2^2)]I_1+\\ &+F_{2,1}[\bar{\xi}_1^2(\xi_1^2+\xi_2^2)-\xi_2^2(\bar{\xi}_1^2+\bar{\xi}_2^2)]I_2=0,\end{aligned} \tag{21}$$

$$\frac{\partial t_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}} \Rightarrow -F_{1,1}I_2^3+F_{2,2}I_2^2 I_1+F_{2,1}I_2 I_1^2+F_{1,2}I_1^3=0, \tag{22}$$

$$\frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}},\ \frac{\partial \bar{t}_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \bar{\xi}_1}} \Rightarrow -F_{1,1}I_2^3+F_{2,2}I_2^2 I_1+F_{2,1}I_2 I_1^2+F_{1,2}I_1^3=0. \tag{23}$$

However the relations (20) and (21) are differed from (22) only by the cofactor $|\xi_1|^2+|\xi_2|^2$. Consequently the following assertion is proved

Theorem 4.1. *C^1-BEq of potential type for P-A-H bifurcation with the symmetry $SH(2)$ on spatial variables has the form (17), where the functions F_1 and F_2 satisfy the differential equation (22).*

Corollary 4.1. *The potential is determined by the following formula [16]*

$$\sum_{k=1}^{n}[\int_0^1 t_k(\tau\xi_1,\tau\xi_2,\mu,\varepsilon)\bar{\xi}_k d\tau+\int_0^1 t_k(\tau\xi_1,\tau\xi_2,\mu,\varepsilon)\bar{\xi}_k d\tau]=U(\xi,\bar{\xi},\mu,\varepsilon).$$

In the case of analytic BEq as invariants $I_1 = \xi_1\overline{\xi_1} - \xi_2\overline{\xi_2}$ and $I_2 = \xi_1\overline{\xi_2} + \overline{\xi_1}\xi_2$ are chosen. As before the conditions a) and b) (18) are verified directly, while the conditions c) and d) (18) give

$$\begin{aligned}&[-F_{1,1}+F_{2,2}]\,(\xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2)+[F_{1,2}+F_{2,1}](\xi_1\bar{\xi}_1-\xi_2\bar{\xi}_2)=\\ &=[-F_{1,1}+F_{2,2}]I_2+[F_{1,2}+F_{2,1}]I_1=0.\end{aligned} \tag{24}$$

Theorem 4.2. *Analytic BEq of potential type for P-A-H bifurcation allowing the symmetry SH(2) has the form (17), where the functions* $F_1(I_1(\xi), I_2(\xi))$ *and* $F_2(I_1(\xi), I_2(\xi))$ *satisfy the differential equation (24) with invariants* $I_1(\xi) = \xi_1\overline{\xi_1} - \xi_2\overline{\xi_2}$ *and* $I_2(\xi) = \xi_1\overline{\xi_2} + \overline{\xi_1}\xi_2$.

Remark 4.1. *Accepted in [12,13] potentiality conditions of BEq mean the symmetry of the matrix* $\dfrac{D(t_1, \bar{t}_1, -t_2, -\bar{t}_2)}{D(\bar{\xi}_1, \xi_1, \bar{\xi}_2, \xi_2)}$, *where only its coincidence with transposed one was taken into account. Therefore in [1] we could not construct the finite dimensional symmetrizator (symmetryzing operator) and could not prove Theorem 4.1 for* C^1-*BEq. However, for analytic case (in [12] and [13] and correspondingly in [1]), it is said erroneously that* C^1-*BEq is considered) the results of [12,13] about the general form of analytic BEq and its potential turn out to be valid.*

Remark 4.2. *As in the previous our article, we assume in the future to investigate dynamic bifurcation problems with symmetries* $SO(2)$ *and* $SH(2)$ *at high order degeneration of the linearized operator.*

5. EXISTENCE OF BIFURCATION POINT

Similarly to the articles [1,3], by using the approach of Section 3, the existence theorem of P-A-H bifurcation can be proved.

Lemma 5.1. *Let be* $A(\varepsilon) \equiv A_0$ *and* $\mu\mathcal{C}_0 y + (B(\varepsilon) - B_0)y + \mathcal{R}(y, \varepsilon) = \mu\mathcal{C}_0 y + \mathbf{R}(y, \varepsilon)$, $\mathbf{R}(0, \varepsilon) \equiv 0$. *Then at the realization of the Th.3.1 conditions, potential* $U(\xi, \tilde{\xi}, \mu, \varepsilon)$ *of the potential type A(B) BEq is generated by the symmetric matrices* $d^{-1}\cdot \ll \rho(0, \mu(\varepsilon), \varepsilon)(\varphi; \overline{\varphi}), (\psi; \overline{\psi}) \gg$ *for the case A and* $\ll \rho(0, \mu(\varepsilon), \varepsilon)(\varphi; \overline{\varphi}), (\psi; \overline{\psi}) \gg \cdot d^{-1}$ *for the case B with relevant square form on* $\xi, \overline{\xi}$ *and residual term* $\omega(\xi, \overline{\xi}, \mu(\varepsilon), \varepsilon)$, $\|\omega\| = o(\sqrt{|\xi|^2 + |\overline{\xi}|^2})$ *as* $\xi \to 0$. *Components of the symmetric matrices are continuous functions in some neighborhood of the point* $\mu = 0$, $\varepsilon = 0$, *the function* ω *is continuous in the same neighborhood together with partial derivatives on* $\xi, \overline{\xi}$ *up to second order. The symmetricity of the matrix in the main part of potential is understanding in the sense of (12) for the case A ((13) for the case B), i.e. (here the symbol* $[\ldots]^\star$ *means the complex conjugation to the expression* $[\ldots]$*)*

$$\sum_{s=1}^{n}[d_{2p-1,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \psi_s^{(1)} \gg + d_{2p-1,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \overline{\psi}_s^{(1)} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q-1,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg + d_{2q-1,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg]^\star,$$

$$\sum_{s=1}^{n}[d_{2p,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi}_q^{(1)}, \psi_s^{(1)} \gg + d_{2p,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \psi_s^{(1)} \gg +d_{2q,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \overline{\psi_s^{(1)}} \gg]^{\star},$$

$$\sum_{s=1}^{n}[d_{2p-1,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi}_q^{(1)}, \psi_s^{(1)} \gg +d_{2p-1,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg +d_{2q,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg]^{\star},$$

for the case A and

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2p-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \overline{\psi_s}^{(1)} \gg d_{2s,2p-1}] =$$

$$= \sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2q-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg d_{2s,2q-1}]^{\star},$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2p}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg d_{2s,2p}] =$$

$$= \sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2q}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \overline{\psi}_s^{(1)} \gg d_{2s,2q}]^{\star},$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2p-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg d_{2s,2p-1}] =$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2q}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi_s^{(1)}} \gg d_{2s,2q}]^{\star},$$

for the case B.

Here $\rho(0,\mu(\varepsilon),\varepsilon) = (\mu\mathcal{C}(\varepsilon) + \mathcal{R}_y)[I - \Gamma(\mu\mathcal{C}(\varepsilon) + \mathcal{R}_y)]^{-1}$, in accordance with [1] and sections 2,3. The proof follows from the definition 3.1 and Lemma 3.1.

Similarly to the article [3] introduce the condition suitable also for ε belonging to some normed space Λ: α) let in some neighborhood of $\varepsilon = 0$ there exists the set S, containing the point $\varepsilon = 0$, which is continuum presented in the form $S = \overline{S}_+ \bigcup \overline{S}_-$, $0 \in \partial S_+ \bigcap \partial S_-$. Let be

$$det[d^{-1} \ll \rho(0,\mu(\varepsilon),\varepsilon),(\varphi,\overline{\varphi}),(\psi,\overline{\psi} \gg)]_{\varepsilon\in S_+\bigcup S_-} \neq 0,$$

(resp. $det[\ll \rho(0,\mu(\varepsilon),\varepsilon),(\varphi,\overline{\varphi}),(\psi,\overline{\psi} \gg)\cdot d^{-1}]_{\varepsilon\in S_+\bigcup S_-} \neq 0$) and the matrix $[d^{-1}\cdot \ll \rho(0,\mu(\varepsilon),\varepsilon),(\varphi,\overline{\varphi}),(\psi,\overline{\psi} \gg)]$ in case A (resp.the matrix $[\ll \rho(0,\mu(\varepsilon),\varepsilon),(\varphi,\overline{\varphi}),(\psi,\overline{\psi} \gg)\cdot d^{-1}]$ for the case B) has at $\varepsilon \in S_-$ ($\varepsilon \in S_+$) precisely ν_1 negative eigenvalues (ν_2 negative eigenvalues).

Lemma 5.2. [3]. *Let the condition α) with $\nu_1 \neq \nu_2$ be realized. Then for any $\delta > 0$ there exists $\varepsilon^\star$ in a neighborhood $|\varepsilon| < \delta$ such that the function $U(\xi, \overline{\xi}, \mu(\varepsilon^\star), \varepsilon^\star)$ has in it a stationary point $\xi^\star \neq 0$.*

The proof follows from homotopic invariance of Conley-Morse index [17,Th.1.4,p.67].

Theorem 5.1. *Let the branching equation of the problem (1), under Lemma 5.1 conditions, be potential type A(or B) and the condition α) be fulfilled with $\nu_1 \neq \nu_2$. Then $\varepsilon = 0 \in S$ is the bifurcation point.*

Remark 5.1. *The results [2,3] can be found in the more available collective monograph [6].*

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EXISTENCE OF POSITIVE SOLUTIONS FOR A HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract We study the existence and nonexistence of positive solutions for a system of nonlinear higher-order ordinary differential equations with multi-point boundary conditions.

Keywords: Higher-order differential system, multi-point boundary conditions, positive solutions.
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1. INTRODUCTION

We consider the system of nonlinear higher-order ordinary differential equations

$$(S) \qquad \begin{cases} u^{(n)}(t) + \alpha(t) f(v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + \beta(t) g(u(t)) = 0, & t \in (0, T), \end{cases}$$

with the multi-point boundary conditions

$$(BC) \qquad \begin{cases} u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i) + a_0, \;\; u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = 0, \\ v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i) + b_0, \;\; v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(T) = 0, \end{cases}$$

where n, m, p, $q \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $p \geq 3$, $q \geq 3$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $0 < \eta_1 < \cdots < \eta_{q-2} < T$.

By using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem $(S) - (BC)$. By a positive solution of $(S) - (BC)$ we mean a pair of functions $(u, v) \in C^n([0, T]) \times C^m([0, T])$ satisfying (S) and (BC) with $u(t) > 0$, $v(t) > 0$ for all $t \in [0, T)$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem.

The system (S) with the boundary conditions

$$(BC_1) \qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i) + a_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i) + b_0, \end{cases}$$

has been investigated in [7]. In [26], the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) where f and g are dependent of u and v, and the boundary conditions (BC_1) with $a_0 = b_0 = 0$ and $\frac{1}{2} \leq \xi_1 < \cdots < \xi_{p-2} < 1$, $\frac{1}{2} \leq \eta_1 < \cdots < \eta_{q-2} < 1$, $T = 1$. For multi-point boundary value problems for nonlinear higher-order ordinary differential equations we mention the papers [1], [19].

Multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by J. Henderson, R. Luca, S. K. Ntouyas and I. K. Purnaras, by using the Guo-Krasnosel'skii fixed point theorem. Namely, in [2], the authors give sufficient conditions for λ, μ, f and g such that the system of differential equations

$$(S_1) \qquad \begin{cases} u^{(n)}(t) + \lambda\alpha(t) f_1(u(t), v(t)) = 0, \ \ t \in (0, T), \\ v^{(m)}(t) + \mu\beta(t) g_1(u(t), v(t)) = 0, \ \ t \in (0, T), \end{cases}$$

with the boundary conditions (BC_1) with $a_0 = b_0 = 0$ has positive solutions. The system (S_1) with $f_1(u, v) = \widetilde{f}(v)$, $g_1(u, v) = \widetilde{g}(u)$ and $n = m$ (denoted by $(\widetilde{S}_1)$) with the boundary conditions (BC_1) with $a_0 = b_0 = 0$, where $n = m$, $p = q$, $a_i = b_i$, $\xi_i = \eta_i$ for $i = 1, ..., p-2$, has been studied in [22]. In [9], the authors studied the system $(\widetilde{S}_1)$ with $T = 1$ and the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$, $v(1) = \alpha v(\eta)$, where $0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$.

The systems (S) and (S_1) with $n = m = 2$ subject to various boundary conditions were studied in [3], [4], [10], [11], [13], [14], [23]. Some discrete versions of these nonlinear second-order boundary value problems have been investigated in [5], [6], [12], [15], [21], [24].

Our results obtained in this paper were inspired by the paper [20], where the authors studied the existence and nonexistence of positive solutions for the m-point boundary value problem on time scales

$$\begin{cases} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, \ \ t \in (0, \widetilde{T}), \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \ \ u(\widetilde{T}) - \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) = b, \ \ m \geq 3, \ \ b > 0, \end{cases}$$

where $(0, \widetilde{T})$ denotes a time scale interval.

Multi-point boundary value problems for ordinary differential equations or finite difference equations have applications in a variety of different areas of applied mathematics and physics. For example the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [25]); also many problems in the theory of elastic stability can be handled as multi-point problems (see [27]). The study of multi-point boundary value problems for second order differential equations was initiated by Il'in and Moiseev (see [16], [17]). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors, by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for a n-th order differential equation (problem (1) – (2) below), and in Section 3, we shall give our main results.

2. AUXILIARY RESULTS

In this section, we shall present some auxiliary results from [18] related to the following n-th order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0,T), \tag{1}$$

$$u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = 0. \tag{2}$$

We shall present these results for the interval $[0,T]$ of the t-variable. Their proofs are similar to those from [18] where $T = 1$.

Lemma 2.1. *If* $d = T^{n-1} - \sum_{i=1}^{p-2} a_i(T^{n-1} - \xi_i^{n-1}) \neq 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ *and* $y \in C([0,T])$*, then the solution of* (1)-(2) *is given by*

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\,ds + \frac{t^{n-1}}{d}\left[\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\,ds \right.\\ &\left. + \left(1 - \sum_{i=1}^{p-2} a_i\right) \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\,ds\right] + \frac{1}{d}\sum_{i=1}^{p-2} a_i \xi_i^{n-1} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\,ds \\ &- \frac{T^{n-1}}{d}\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\,ds. \end{aligned}$$

Lemma 2.2. *Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem* (1)-(2) *is*

$$G_1(t,s) = g_1(t,s) + \frac{T^{n-1} - t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s),$$

where

$$g_1(t,s) = \frac{1}{(n-1)!T^{n-1}} \begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \le s \le t \le T, \\ t^{n-1}(T-s)^{n-1}, & 0 \le t \le s \le T. \end{cases}$$

Using the above Green's function the solution of problem (1)-(2) is expressed as $u(t) = \int_0^T G_1(t,s)y(s)\,ds$.

Lemma 2.3. *The function g_1 has the properties*

a) g_1 is a continuous function on $[0,T] \times [0,T]$ and $g_1(t,s) \ge 0$ for all $(t,s) \in [0,T] \times [0,T]$;

b) $g_1(t,s) \le g_1(\theta_1(s),s)$, for all $(t,s) \in [0,T] \times [0,T]$;

c) For any $c \in \left(0, \frac{T}{2}\right)$,

$$\min_{t\in[c,T-c]} g_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s),s), \text{ for all } s \in [0,T],$$

where $\theta_1(s) = s$ if $n = 2$ and $\theta_1(s) = \begin{cases} \dfrac{s}{1-\left(1-\frac{s}{T}\right)^{\frac{n-1}{n-2}}}, & s \in (0,T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases}$ *if $n \ge 3$.*

In the case $n \ge 3$, we choose the values of θ_1 in $s = 0$ and $s = T$ such that θ_1 be a continuous function on $[0,T]$ (see also [8]).

Lemma 2.4. *Assume that $a_i \ge 0$ for all $i = 1,\ldots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $d > 0$. Then the Green's function G_1 of problem* (1)-(2) *has the properties*

a) G_1 is a continuous function on $[0,T] \times [0,T]$ and $G_1(t,s) \ge 0$ for all $(t,s) \in [0,T] \times [0,T]$;

b) $G_1(t,s) \le J_1(s)$ for all $(t,s) \in [0,T] \times [0,T]$ and for any $c \in (0,T/2)$ we have $\displaystyle\min_{t\in[c,T-c]} G_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} J_1(s)$ *for all $s \in [0,T]$,*

where $\displaystyle J_1(s) = g_1(\theta_1(s),s) + \frac{T^{n-1}}{d}\sum_{i=1}^{p-2} a_i g_1(\xi_i,s), \quad \forall\, s \in [0,T].$

Lemma 2.5. *If $a_i \ge 0$ for all $i = 1,\ldots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$ and $y(t) \ge 0$ for all $t \in [0,T]$, then the solution of problem* (1)-(2) *satisfies $u(t) \ge 0$ for all $t \in [0,T]$.*

Lemma 2.6. *Assume that $a_i \ge 0$ for all $i = 1,\ldots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$ and $y(t) \ge 0$ for all $t \in [0,T]$. Then the solution of problem* (1)-(2) *satisfies* $\displaystyle\min_{t\in[c,T-c]} u(t) \ge \frac{c^{n-1}}{T^{n-1}} \max_{t'\in[0,T]} u(t')$.

We can also formulate similar results as Lemma 2.1 - Lemma 2.6 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0,T), \tag{3}$$

$$v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = 0, \tag{4}$$

where $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i \geq 0$ for all $i = 1, \ldots, q-2$ and $h \in C([0,T])$. If $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \left(T^{m-1} - \eta_i^{m-1}\right) \neq 0$, we denote by G_2 the Green's function associated to problem (3)-(4) and defined in a similar manner as G_1. We also denote by g_2, θ_2 and J_2 the corresponding functions for (3)-(4) defined in a similar manner as g_1, θ_1 and J_1, respectively.

3. MAIN RESULTS

We present the assumptions that we shall use in the sequel:

$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $a_i \geq 0$, $i = 1, \ldots, p-2$, $b_i \geq 0$, $i = 1, \ldots, q-2$, $d = T^{n-1} - \sum_{i=1}^{p-2} a_i(T^{n-1} - \xi_i^{n-1}) > 0$, $e = T^{m-1} - \sum_{i=1}^{q-2} b_i(T^{m-1} - \eta_i^{m-1}) > 0$.

$(H2)$ The functions $\alpha, \beta : [0,T] \to [0,\infty)$ are continuous and for any $c \in (0, T/2)$ there exist t_0, $\widetilde{t_0} \in [c, T-c]$ such that $\alpha(t_0) > 0$, $\beta(\widetilde{t_0}) > 0$.

$(H3)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and there exists $r_0 > 0$ such that $f(u) < \frac{r_0}{L}$, $g(u) < \frac{r_0}{L}$ for all $u \in [0, r_0]$, where

$$L = \max\left\{\int_0^T J_1(s)\alpha(s)\,ds, \ \int_0^T J_2(s)\beta(s)\,ds\right\}.$$

$(H4)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and satisfy the conditions $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, $\lim_{u\to\infty} \frac{g(u)}{u} = \infty$.

First, we present an existence result for the positive solutions of $(S) - (BC)$.

Theorem 3.1. *Assume that the assumptions $(H1), (H2)$ and $(H3)$ hold. Then the problem $(S)-(BC)$ has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Proof. We consider the problems

$$\begin{cases} h^{(n)}(t) = 0, \quad t \in (0,T), \\ h(0) = \sum_{i=1}^{p-2} a_i h(\xi_i) + 1, \quad h'(0) = \cdots = h^{(n-2)}(0) = 0, \quad h(T) = 0, \end{cases} \tag{5}$$

$$\begin{cases} w^{(m)}(t) = 0, \quad t \in (0,T), \\ w(0) = \sum_{i=1}^{q-2} b_i w(\eta_i) + 1, \quad w'(0) = \cdots = w^{(m-2)}(0) = 0, \quad w(T) = 0. \end{cases} \tag{6}$$

The above problems (5) and (6) have the solutions

$$h(t) = \frac{T^{n-1} - t^{n-1}}{d}, \quad w(t) = \frac{T^{m-1} - t^{m-1}}{e}, \quad t \in [0, T]. \tag{7}$$

We define the functions $x(t)$ and $y(t)$, $t \in [0, T]$ by

$$x(t) = u(t) - a_0 h(t), \ \ y(t) = v(t) - b_0 w(t), \ \ t \in [0, T],$$

where (u, v) is a solution of $(S) - (BC)$. Then $(S) - (BC)$ can be equivalently written as

$$\begin{cases} x^{(n)}(t) + \alpha(t) f(y(t) + b_0 w(t)) = 0, \ \ t \in (0, T), \\ y^{(m)}(t) + \beta(t) g(x(t) + a_0 h(t)) = 0, \ \ t \in (0, T), \end{cases} \tag{8}$$

with the boundary conditions

$$\begin{cases} x(0) = \sum_{i=1}^{p-2} a_i x(\xi_i), \ \ x'(0) = \cdots = x^{(n-2)}(0) = 0, \ \ x(T) = 0, \\ y(0) = \sum_{i=1}^{q-2} b_i y(\eta_i), \ \ y'(0) = \cdots = y^{(m-2)}(0), \ \ y(T) = 0. \end{cases} \tag{9}$$

Using the Green's functions given in Section 2, a pair (x, y) is a solution of the problem (8)-(9) if and only if (x, y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \int_0^T G_1(t, s)\alpha(s) f\left(\int_0^T G_2(s, \tau)\beta(\tau) g(x(\tau) + a_0 h(\tau))\, d\tau + b_0 w(s)\right) ds, \\ y(t) = \int_0^T G_2(t, s)\beta(s) g(x(s) + a_0 h(s))\, ds, \ \ 0 \le t \le T, \end{cases} \tag{10}$$

where $h(t)$ and $w(t)$ for $t \in [0, T]$ are given by (7).

We consider the Banach space $X = C([0, T])$ with the supremum norm $\|\cdot\|$ and define the set

$$K = \{x \in C([0, T]), \ \ 0 \le x(t) \le r_0, \ \ \forall\, t \in [0, T]\} \subset X.$$

We also define the operator $\mathcal{A} : K \to X$ by

$$\mathcal{A}(x)(t) = \int_0^T G_1(t, s)\alpha(s) f\left(\int_0^T G_2(s, \tau)\beta(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 w(s)\right) ds,$$
$$0 \le t \le T, \ \ x \in K.$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(H3)$, we deduce

$$f(y(t) + b_0 w(t)) \le \frac{r_0}{L}, \ \ g(x(t) + a_0 h(t)) \le \frac{r_0}{L}, \ \ \forall\, t \in [0, T], \ \ \forall\, x, y \in K.$$

Then, by using Lemma 2.3, we obtain $\mathcal{A}(x)(t) \geq 0$ for all $t \in [0, T]$ and $x \in K$. By Lemma 2.4, for all $x \in K$, we have

$$\int_0^T G_2(s,\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau \leq \int_0^T J_2(\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau$$
$$\leq \frac{r_0}{L}\int_0^T J_2(\tau)\beta(\tau)\,d\tau \leq r_0, \quad \forall\, s \in [0, T],$$

and

$$\mathcal{A}(x)(t) \leq \int_0^T J_1(s)\alpha(s)f\left(\int_0^T G_2(s,\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau + b_0w(s)\right)ds$$
$$\leq \frac{r_0}{L}\int_0^T J_1(s)\alpha(s)\,ds \leq r_0, \quad \forall\, t \in [0, T].$$

Therefore $\mathcal{A}(K) \subset K$.

Using standard arguments, we deduce that $\mathcal{A}$ is completely continuous ($\mathcal{A}$ is compact, that is for any bounded set $B \subset K$, $\mathcal{A}(B) \subset K$ is relatively compact by Arzèla-Ascoli theorem, and $\mathcal{A}$ is continuous). By the Schauder fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $x \in K$. This element together with y given by

$$y(t) = \int_0^T G_2(t,s)\beta(s)g(x(s) + a_0h(s))\,ds, \quad t \in [0, T]$$

represents a solution for (8)-(9). This shows that our problem $(S)-(BC)$ has a positive solution $u = x + a_0h, \ v = y + b_0w$ for sufficiently small positive a_0 and b_0. ∎

In what follows, we present sufficient conditions for the nonexistence of the positive solutions of $(S) - (BC)$.

Theorem 3.2. *Let the assumptions $(H1)$, $(H2)$ and $(H4)$ be satisfied. Then the problem $(S) - (BC)$ has no positive solution for a_0 and b_0 sufficiently large.*

Proof. We suppose that (u, v) is a positive solution of $(S) - (BC)$. Then $x = u - a_0h, \ y = v - b_0w$ is a solution for (8)-(9), where h and w are the solutions of problems (5) and (6) (given by (7)). By Lemma 2.5, we have $x(t) \geq 0, \ y(t) \geq 0$ for all $t \in [0, T]$, and by $(H2)$ we deduce that $\|x\| > 0$, $\|y\| > 0$. Using Lemma 2.6, for $c \in (0, T/2)$, we also have

$$\inf_{t\in[c,T-c]} x(t) \geq \frac{c^{n-1}}{T^{n-1}}\|x\| \text{ and } \inf_{t\in[c,T-c]} y(t) \geq \frac{c^{m-1}}{T^{m-1}}\|y\|.$$

Using now (7), we deduce that $\inf_{t\in[c,T-c]} h(t) = [T^{n-1} - (T-c)^{n-1}]/d$. Therefore

$$\inf_{t\in[c,T-c]} h(t) = \frac{T^{n-1} - (T-c)^{n-1}}{T^{n-1}}\|h\| \geq \frac{c^{n-1}}{T^{n-1}}\|h\|.$$

In a similar manner we obtain $\inf_{t\in[c,T-c]} w(t) \geq c^{m-1}/T^{m-1}\|w\|$.

Therefore, we obtain

$$\begin{aligned}
\inf_{t\in[c,T-c]}(x(t)+a_0h(t)) &\geq \inf_{t\in[c,T-c]} x(t) + a_0 \inf_{t\in[c,T-c]} h(t)\\
&\geq \frac{c^{n-1}}{T^{n-1}}(\|x\|+a_0\|h\|) \geq \frac{c^{n-1}}{T^{n-1}}\|x+a_0h\|,\\
\inf_{t\in[c,T-c]}(y(t)+b_0w(t)) &\geq \inf_{t\in[c,T-c]} y(t) + b_0 \inf_{t\in[c,T-c]} w(t)\\
&\geq \frac{c^{m-1}}{T^{m-1}}(\|y\|+b_0\|w\|) \geq \frac{c^{m-1}}{T^{m-1}}\|y+b_0w\|.
\end{aligned}$$

We now consider

$$R = \frac{T^{n+m-2}}{c^{n+m-2}}\left(\min\left\{\int_c^{T-c} J_1(s)\alpha(s)\,ds,\quad \int_c^{T-c} J_2(s)\beta(s)\,ds\right\}\right)^{-1} > 0.$$

By $(H4)$, for R defined above, we deduce that there exists $M > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for all $u \geq M$.

We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that

$$\inf_{t\in[c,T-c]}(x(t)+a_0h(t)) \geq M \quad\text{and}\quad \inf_{t\in[c,T-c]}(y(t)+b_0w(t)) \geq M.$$

By using Lemma 2.4 and the above considerations, we have

$$\begin{aligned}
&y(c) = \int_0^T G_2(c,s)\beta(s)g(x(s)+a_0h(s))\,ds \geq\\
&\geq \int_c^{T-c} G_2(c,s)\beta(s)g(x(s)+a_0h(s))\,ds \geq\\
&\geq \frac{c^{m-1}}{T^{m-1}}\int_c^{T-c} J_2(s)\beta(s)g(x(s)+a_0h(s))\,ds \geq\\
&\geq \frac{2Rc^{m-1}}{T^{m-1}}\int_c^{T-c} J_2(s)\beta(s)(x(s)+a_0h(s))\,ds \geq\\
&\geq \frac{2Rc^{m-1}}{T^{m-1}}\inf_{\tau\in[c,T-c]}(x(\tau)+a_0h(\tau))\int_c^{T-c} J_2(s)\beta(s)\,ds \geq
\end{aligned}$$

$$\geq \frac{2Rc^{n+m-2}}{T^{n+m-2}}\|x+a_0h\|\int_c^{T-c} J_2(s)\beta(s)\,ds \geq 2\|x+a_0h\| \geq 2\|x\|.$$

Therefore, we obtain

$$\|x\| \leq y(c)/2 \leq \|y\|/2. \tag{11}$$

In a similar manner, we deduce

$$\begin{aligned}
x(c) &= \int_0^T G_1(c,s)\alpha(s)f(y(s)+b_0w(s))\,ds \ge \\
&\ge \int_c^{T-c} G_1(c,s)\alpha(s)f(y(s)+b_0w(s))\,ds \ge \\
&\ge \frac{c^{n-1}}{T^{n-1}}\int_c^{T-c} J_1(s)\alpha(s)f(y(s)+b_0w(s))\,ds \ge \\
&\ge \frac{2Rc^{n-1}}{T^{n-1}}\int_c^{T-c} J_1(s)\alpha(s)(y(s)+b_0w(s))\,ds \ge \\
&\ge \frac{2Rc^{n-1}}{T^{n-1}}\inf_{\tau\in[c,T-c]}(y(\tau)+b_0w(\tau))\int_c^{T-c} J_1(s)\alpha(s)\,ds \ge \\
&\ge \frac{2Rc^{n+m-2}}{T^{n+m-2}}\|y+b_0w\|\int_c^{T-c} J_1(s)\alpha(s)\,ds \ge 2\|y+b_0w\| \ge 2\|y\|.
\end{aligned}$$

So, we obtain

$$\|y\| \le x(c)/2 \le \|x\|/2. \tag{12}$$

By (11) and (12), we obtain $\|x\| \le \|y\|/2 \le \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution. ∎

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ON SURVIVAL AND RUIN PROBABILITIES IN A PERTURBED RISK MODEL

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Abstract We analyze the ruin probability in infinite and finite time horizon for some risk models. This is the probability that an insurer will face ruin when it starts with some initial reserve and is subjected to independent and identical distributed claims over time. Closed form expressions for this probability are available only in few cases, therefore actuaries dwell with approximations. In this paper, we consider a perturbed risk model in which a current premium rate will be adjusted after a claim occurs and the adjusted rate is determined by the amount of the claim. At the same time, in this risk model the surplus of the insurer is perturbed by a standard Brownian motion which is independent of the number of claims process and of claim sizes. We focus on an integro-differential equation for the survival probabilities and on a discrete-time model for the ruin probabilities. We give a numerical illustration on the latter risk model.

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1. INTRODUCTION

An actuarial risk model has two main components: one characterizing the frequency of events and another describing the claim size resulting from the occurrence of a catastrophic event. In examining the nature of the risk associated with a portfolio of policies, it is often of interest to assess how the portfolio performs over an extended period of time. One approach focuses on the use of ruin theory, which is concerned with the insurer's surplus, i.e. the excess of the income over the outgo, or claims paid, with respect to a portfolio of business. Ruin is said to occur if the insurer's surplus reaches a specified lower bound. The ruin probability is the probability of suchlike event.

There are various ways to model the surplus process of an insurance company and to define the ruin probability as well as the survival probability.

In this paper, we consider a perturbed risk model in which a current premium rate will be adjusted after a claim occurs and the adjusted rate is determined by the amount of the claim. At the same time, in this risk model the surplus of the insurer is perturbed by a standard Brownian motion which is independent of the number of claims process and of claim sizes. We focus on an integro-differential equation for the survival probabilities and on a discrete-time model for the ruin probabilities, as well as for the level of the insurer's surpluss process.

The paper is organized as follows. Section 2 discusses the Cramer-Lundberg risk model using a martingale approach. Section 3 describes the diffusion approximation for the maximum of a random walk. Section 4 is devoted to perturbed risk models: in continous time, involving survival probabilities, and in discrete time, giving the ruin probability after a certain number of periods. Section 5 contains a numerical illustration on the latter risk model.

2. MARTINGALE APPROACH IN THE CRAMER-LUNDBERG RISK MODEL

The theory of martingales provides a quick way of calculating the ruin probability. We shall assume that the evolution of the capital of a insurance company takes place in a probability space (Ω, K, P) as follows.

The initial capital (initial reserve) is $U(0) = u > 0$. Insurance premiums are cashed continuously at a constant rate $c > 0$ and claims are received at random times (moments) $T_1, T_2, ...$ $(0 = T_0 < T_1 < T_2 < ...)$ and the amounts to be paid out at these moments are described by the nonnegative random variables $X_1, X_2, ...$. Thus, taking into account receipts and claims, the capital $U(t)$ at time $t \geq 0$ is

$$U(t) = u + ct - S(t), \tag{1}$$

where $S(t) = \sum_{i\geq 1} X_i \cdot I(T_i \leq t)$, with $S(0) = 0$. Let $\Theta_i = T_i - T_{i-1}, i \geq 1$, be the inter-ocurrence times; $N(t) = \sum_{i\geq 1} I(T_i \leq t)$, $N = \{N(t); t \geq 0\}$, with $N(0) = 0$, is the claim arrival process.

The first time the insurance company's capital becomes less than zero is the time of ruin

$$\tau = \inf\{t \geq 0 : U(t) < 0\}, \tag{2}$$

and $\tau = \infty$, if $U(t) \geq 0, \forall t \geq 0$. The probability of ruin is

$$\Psi(u) = P(\tau < \infty), \tag{3}$$

and the probability of ruin before some moment T is

$$\Psi(u, T) = P(\tau \leq T). \tag{4}$$

The Cramer-Lundberg model is characterized by the following assumptions:

i) the random variables $\Theta_i, i \geq 1$ are independent and identically distributed (iid) having an exponential distribution $Exp(\lambda)$, expectation $1/\lambda$;

ii) the random variables $X_i, i \geq 1$ are iid with cumulative distribution function (cdf) $B(x)$ such that $B(0) = 0$, $\mu_B = \int_{\mathbb{R}_+} x dB(x) < \infty$;

iii) the sequences $\{X_i\}_{i\geq 1}$ and $\{T_i\}_{i\geq 1}$ are independent.

Since $\{T_k > t\} = \{N(t) < k\}$, the stochastic process N is a homogenous Poisson process with parameter rate $\lambda > 0$. Let $\hat{B}(\omega) = E\left[e^{\omega X}\right] = \int_{\mathbb{R}_+} e^{\omega x} dB(x)$ be the moment generating function (mgf), $\tilde{B}(z) = \hat{B}(-z)$ be the Laplace-Stieltjes transform, $k(\omega) = \log \hat{B}(\omega)$ be the cumulant generating function (cdf), $h(\omega) = \int_{\mathbb{R}_+} (e^{\omega x} - 1) dB(x)$ and $g(\omega) = \lambda \cdot h(\omega) - c \cdot \omega$. We have $h(z) = \tilde{B}(-z) - 1$, for all $z \geq 0$.

We consider the exponential family $\{B_\omega\}$ generated by B, i.e. $B_\omega(dx) = e^{\omega x - k(\omega)} B(dx)$ or equivalently, in terms of the cdf of B_ω, $k_\omega(\alpha) = k(\alpha + \omega) - k(\omega)$.

It is natural to consider the models which have the property that there exists a constant ρ such that $\frac{1}{t} \sum_{i=1}^{N(t)} X_i \xrightarrow{a.s.} \rho, t \to \infty$, ρ is the average amount of claim per unit of time. In this model, it is easy to see that $\rho = \lambda\mu$, i.e. on average, λ claims arrive per unit of time with μ the mean of a single claim, and also $\lim_{t\to\infty} E[\frac{1}{t} \sum_{i=1}^{N(t)} X_i] = \rho$. We consider the safety loading or the loading θ defined as the relative amount by wich the premium rate c exceeds ρ; it is necessary for the net profit condition. Indeed, from assumption (iii) we find that

$$\begin{aligned} E[U(t) - U(0)] &= ct - E[S(t)] \\ &= ct - \sum_i E[X_i] E[I(T_i \leq t)] \\ &= ct - \mu \sum_i P(T_i \leq t) \\ &= ct - \mu \sum_i P(N(t) \geq i) \\ &= ct - \mu E[N(t)] \\ &= t(c - \lambda\mu). \end{aligned}$$

Thus, a natural requirement for an insurance company to operate with a clear profit is that $c > \lambda\mu$.

Taking $X_0 = 0$, we find for $r > 0$ with $h(r) < \infty$,

$$\begin{aligned} E[e^{-r(U(t)-U(0))}] &= e^{-rct} E[e^{r\sum_{i=1}^{N(t)} X_i}] \\ &= e^{-rct} \sum_{n=0}^{\infty} E[e^{r\sum_{i=1}^{n} X_i}] P(N(t) = n) \\ &= e^{-rct} \sum_{n=0}^{\infty} (1 + h(r))^n \frac{e^{-\lambda t} (\lambda t)^n}{n!} \\ &= e^{t(\lambda h(r) - cr)} = e^{tg(r)}. \end{aligned}$$

Analogously, it can be shown that for any $s < t$, $E[e^{-r(U(t)-U(s))}] = e^{(t-s)g(r)}$.

Let $F_t = \sigma(U(s), s \le t)$. Since the stochastic process $U = \{U(t); t \ge 0\}$ is a process with independent increments, we have

$$E[e^{-r(U(t)-U(s))}|F_s] = E[e^{-r(U(t)-U(s))}] = e^{(t-s)g(r)},$$

then

$$E[e^{-rU(t)-tg(r)}|F_s] = e^{-rU(s)-sg(r)}.$$

Denoting $Z_t = e^{-rU(t)-tg(r)}, t \ge 0$, we have $E[Z_t|F_s] = Z_s, s \le t$ is a continuous analogue of the martingale property.

The stochastic process $Z = \{Z_t; t \ge 0\}$ is nonnegative with $E[Z_t] = e^{-ru} < \infty$, thus the stochastic process with continuous time is a martingale. Therefore, $E[Z_{t\wedge r}] = E[Z_0]$ for any Markov time r (in particular, for r stopping time). For time $r = \tau$, we have

$$\begin{aligned} e^{-ru} &= E[e^{-rU(t\wedge\tau)-(t\wedge\tau)g(r)}] \\ &\ge E[e^{-rU(t\wedge\tau)-(t\wedge\tau)g(r)}|\tau \le t]P(\tau \le t) \\ &= E[e^{-rU(\tau)-\tau g(r)}|\tau \le t]P(\tau \le t) \\ &\ge E[e^{-\tau g(r)}|\tau \le t]P(\tau \le t) \\ &\ge \inf_{0\le s\le t} e^{-sg(r)}P(\tau \le t). \end{aligned}$$

So,

$$\begin{aligned} P(\tau \le t) &\le \frac{e^{-ru}}{\inf\limits_{0\le s\le t} e^{-sg(r)}} \\ &= e^{-ru}\sup_{0\le s\le t} e^{sg(r)}. \end{aligned}$$

Clearly, $g(0) = 0, g'(0) = \lambda\mu - c < 0$, and $g''(r) = \lambda h''(r) \ge 0$. There exists a unique positive value $r = \gamma$ so that $g(\gamma) = 0$. Because

$$\begin{aligned} \int_0^\infty e^{rx}(1 - B(x))\,dx &= \int_0^\infty\int_x^\infty e^{rx}dB(y)\,dx \\ &= \int_0^\infty(\int_x^\infty e^{rx}dx)dB(y) \\ &= \frac{1}{r}h(r), \end{aligned}$$

γ may be asserted to be the unique root of the equation

$$\lambda\int_0^\infty e^{rx}(1 - B(x))\,dx = c.$$

Let $r = \gamma$ (γ is called the adjustment coefficient), then for any $t > 0$,

$$P(\tau \leq t) \leq e^{-\gamma u},$$

whence $P(\tau < \infty) \leq e^{-\gamma u}$. But $\Psi(u) = P(\tau < \infty)$ is the ruin probability, so the Lundberg's inequality is obtained.

Proposition 2.1. *(Lundberg's Inequality) For all $u \geq 0$,*

$$\Psi(u) \leq e^{-\gamma u}. \tag{5}$$

Proposition 2.2. *(Cramer's asymptotic ruin formula) If the adjustment coefficient γ exists, then*

$$\Psi(u) \sim \Psi_{CL}(u) = C \cdot e^{-\gamma u}, u \longrightarrow \infty, \tag{6}$$

where $C = \frac{c-\rho}{\lambda \hat{B}'(\gamma)-c}$.

Remark 1. *If $B \sim Exp(\alpha)$, then $\Psi_{CL}(u) = \Psi(u)$.*

3. DIFFUSION APPROXIMATION FOR THE MAXIMUM OF A RANDOM WALK

Closed form expressions for the ruin probability are available only in few cases, therefore actuaries are interested in approximations. There is a huge amount of research in this direction, and in this paper we focuse on the diffusion approximation.

The idea behind the diffusion approximation is first to approximate the claim surplus process by a Brownian motion with drift by matching the two first moments.

Let $\{X_n; n \geq 1\}$ be a sequence of iid random variables and let $S = \{S_n; n \geq 0\}$ be its associated random walk with drift μ. The aim is to develop high accuracy approximations for the distribution of the maximum random variable $M = \max\{S_n : n \geq 0\}$, which can be thought as the maximum of the aggregate loss or claim.

Clearly, $-\mu = E(X_1)$ must be negative in order that M is finite-valued. For $u > 0$, $\{M > u\} = \{\tau(u) < \infty\}$, where $\tau(u) = \inf\{n \geq 1 : S_n > u\}$, so that calculating the tail of M is equivalent to calculating a level crossing probability for the random walk S. In insurance risk theory, $P(\tau(u) < \infty)$ is the probability that an insurer will face ruin when the initial reserve is u and is subjected to iid claims over time. One important approximation holds as $\mu \searrow 0$. This asymptotic regime corresponds in risk theory to the setting in which the safety loading θ is small (i.e. the premium charged is close to the typical pay-out for claims). The approximation

$$P(M > u) \approx \exp\left(-2\mu u/\sigma^2\right) \tag{7}$$

is valid as $\mu \searrow 0$, where $\sigma^2 = Var(X_1)$. Because the right hand side of (7) is the exact value of the level crossing probability for the natural Brownian approximation to the random walk S, (7) is often called the diffusion approximation to the distribution of M.

There are applications for which the diffusion approximation gives poor results. Siegmund (1979) suggested a so-called "corrected diffusion approximation" (CDA) that reflects information in the increment distribution beyond the mean and variance. Blanchet and Glynn (2006) developed this method to the full asymptotic expansion initiated by Siegmund.

The first problem considered by Siegmund is to find the expected value of the maximum of a random walk with small, negative drift, and the second problem is to find the distribution of the same quantity.

The result in the first case is the following: consider an exponential family P_ω, ω belongs to a neighborhood of 0, such that under P_ω, $X_1, X_2, ...$ are independent random variables with density function $\exp(\omega x - k(\omega))$ relative to a non-arithmetic distribution F, where $k(\omega)$ is the cumulant generating function. We assume that the problem is normalized such that $E_0[X_1] = k'(0) = 0$, $Var_0[X_1] = k''(0) = 1$. Let $S_n = \sum_{i=1}^{n} X_i$, $\tau(u) = \inf\{n : S_n > u\}$, $\tau_+ = \tau(0)$ and $M = \sup_n \{S_n\}$, which is almost surely finite if $\omega < 0$. Then, as $\omega \nearrow 0$, $E_\omega[M] = \frac{1}{\Delta} - \frac{E_0[S^2_{\tau_+}]}{2E_0[S_{\tau_+}]} + o(\Delta)$, where $\Delta = \omega_1 - \omega$, and $\omega_1 > 0$ is such that $k(\omega_1) = k(\omega)$. The random walk is assumed to belong to a translation family, i.e., $P_\omega(X_1 \in A) = P_0(X_1 - \omega \in A)$, where $E_0\left|X_1^3\right| < \infty$. Then, we have $E_\omega[M] = \frac{-1}{2\omega} - \frac{E_0[S^2_{\tau_-}]}{2E_0[S_{\tau_-}]} + o(1)$, which is the result of Siegmund in the form he gave it. From the Wiener-Hopf factorization it is not hard to show that $\frac{E_0 S^2_{\tau_+}}{2E_0 S_{\tau_+}} + \frac{E_0 S^2_{\tau_-}}{2E_0 S_{\tau_-}} = \frac{E_0 X_1^3}{3}$.

The distribution of the maximum is given considering such probabilities as $P_\omega(\tau(u) < \infty) = P_\omega(M > u)$. The appropriate normalization in the exponential family case is to take $u = \frac{2\xi}{\Delta}$, in which case it was showed that as $\omega \nearrow 0$,

$$P_\omega\left(\tau\left(\frac{2\xi}{\Delta}\right) < \infty\right) = e^{-2\xi}\left(1 - \Delta\frac{E_0\left[S^2_{\tau_+}\right]}{2E_0\left[S_{\tau_+}\right]} + o(\Delta)\right). \tag{8}$$

The probability $\Psi(u)$ of ruin in a compound Poisson risk process $U = \{U(t) : t \geq 0\}$, with initial reserve u, is defined as $\Psi(u) = P(\inf_{t\geq 0} U(t) < 0)$, assuming the conditions of the Cramer-Lundberg model. Also, the net premium is considered to be received at a constant rate c over time, $c = (1+\theta)\lambda\mu_B$, where $\theta > 0$ is the relative safety loading. Thus the insurance surplus at time t is

$$U(t) = u + ct - S(t), t \geq 0. \tag{9}$$

The standard diffusion approximation (Grandell 1977) is

$$\Psi(u) \approx \Psi_D(u) = \exp\left(-2\theta u\frac{\mu_B}{\mu_B^2 + \sigma_B^2}\right), \tag{10}$$

where σ_B^2 denotes the variance of B. For light-tailed random walk problems, Siegmund (1979) derived a correction which was adapted to ruin probabilities by Asmussen and Binswanger (1997). An alternative covering also certain heavy-tailed cases was given in Hogan (1986). The result will be the corrected diffusion approximation

$$\Psi(u) \approx \Psi_{CD}(u) = \left(1 + \frac{4\theta^2 u m_1^2 m_3}{3m_2^3} - \frac{2\theta m_1 m_3}{3m_2^2}\right)\exp\left(-2\theta u \frac{m_1}{m_2}\right), \tag{11}$$

when $m_5 < \infty$, where m_i is the i-th moment of B.

We remind that if ξ is a random variable with cumulative distribution function B and cumulant generating function $k(\omega) = \ln E\left[e^{\omega\xi}\right] = \ln \hat{B}(\omega)$, the standard definition of the exponential family $\{B_\omega\}$ generated by B is

$$B_\omega(dx) = e^{\omega x - k(\omega)} B(dx) \tag{12}$$

or equivalently

$$k_\omega(\alpha) = k(\alpha + \omega) - k(\omega). \tag{13}$$

The question that naturally arises is whether k_ω is the cgf corresponding to a compound Poisson risk process in the sense that for a suitable arrival intensity λ_ω and a suitable claim-size distribution B_ω, we have $k_\omega(\alpha) = \lambda_\omega\left(\hat{B}_\omega(\omega) - 1\right) - \alpha$. The answer is yes, the solution is

$$\begin{aligned} \lambda_\omega &= \lambda \hat{B}(\omega), \\ B_\omega(dx) &= \tfrac{\exp(\omega x)}{\hat{B}(\omega)} B(dx) \end{aligned} \tag{14}$$

or equivalently

$$\hat{B}_\omega(\omega) = \frac{\hat{B}(\omega + \alpha)}{\hat{B}(\omega)}. \tag{15}$$

In the following, we formalize this for the purpose of studying the whole process. Let P be the probability measure on $D[0,\infty)$ governing a given compound Poisson risk process with arrival intensity λ and claim size distribution B, and define λ_ω, B_ω by (14). Then P_ω denotes the probability measure governing the compound Poisson risk process with arrival intensity λ_ω and claim size distribution B_ω; the corresponding expectation operator is E_ω.

Since Brownian motion is skip-free, the idea to replace the risk process by a Brownian motion ignores the presence of the overshoot and other things. The objective of the corrected diffusion approximation is to take this and other deficits into consideration. The set-up is the exponential family of compound risk processes with parameters λ_ω and B_ω. However, if we let the given risk process with safety loading $\theta > 0$ correspond to $\omega = 0$, it is more convenient here to use some value $\omega_0 < 0$ and let $\omega = 0$ correspond to $\theta = 0$ (zero drift). This is because in the regime of the diffusion approximation, θ is close to zero, and we want to consider the limit $\theta \searrow 0$ corresponding to $\omega_0 \nearrow 0$.

In terms of the given risk process with Poisson intensity λ, claim size distribution B, $k(\alpha) = \lambda\left(\hat{B}(\alpha) - 1\right) - \alpha$ and $\rho = \lambda\mu_B < 1$, $\eta = \frac{1}{\rho} - 1$, this means the following:

1) Determine $\gamma_m > 0$ such that $k'(\gamma_m) = 0$ and let $\omega_0 = -\gamma_m$.

2) Let P_0 refer to the risk process with parameters $\lambda_0 = \lambda\hat{B}(-\omega_0)$, $B_0(dx) = \frac{\exp(-\omega_0 x)}{\hat{B}(-\omega_0)}B(dx)$. Then $E_0[X^k] = \hat{B}_0^{(k)}(0) = \frac{\hat{B}^{(k)}(-\omega_0)}{\hat{B}(-\omega_0)}$ and $k_0(r) = k(r-\omega_0) - k(\omega - \omega_0)$, $k_0'(0) = 0$.

3) For each ω, let P_ω refer to the risk process with parameters $\lambda_\omega = \lambda_0\hat{B}(\omega) = \lambda\hat{B}(\omega - \omega_0)$, $B_\omega(dx) = \frac{\exp(\omega x)}{\hat{B}_0(\omega)}B_0(dx) = \frac{\exp((\omega-\omega_0)x)}{\hat{B}(\omega-\omega_0)}B(dx)$. Then $k_\omega(r) = k_0(r+\omega) - k_0(\omega) = k(r+\omega-\omega_0) - k(\omega-\omega_0)$ and the given risk process corresponds to P_{ω_0} where $\omega_0 = \gamma_m$. In this set-up we are studying $\Psi(u,T) = P_{\omega_0}(\tau(u) \le T)$ for $\omega_0 < 0, \omega_0 \nearrow 0$.

Recall that $IG(x;\zeta,u)$ (inverse Gaussian) denotes the distribution function of the passage time of the Brownian motion $\left\{W_\zeta(t)\right\}$ with unit variance and drift ζ from level 0 to level $u > 0$. One has $IG(x;\zeta,u) = IG(\frac{x}{u^2};\zeta u,1)$. The corrected diffusion approximation to be derived is

$$\Psi(u,T) \approx \Psi_{CD}(u,T) = IG(\frac{Tv_1}{u^2} + \frac{v_2}{u}; -\frac{\gamma u}{2}, 1 + \frac{v_2}{u}), \tag{16}$$

where γ is the adjustment coefficient (i.e. $k(\gamma) = 0$) and $v_1 = \lambda\hat{B}''(\gamma_m)$, $v_2 = \frac{E_0(X^3)}{2E_0(X^2)} = \frac{\hat{B}'''(\gamma_m)}{3\hat{B}''(\gamma_m)}$. The initial reserve u for the given risk process is written as $u = \zeta/\omega_0$ and, for brevity, we write $\tau = \tau(u)$, $\xi = \xi(u) = S_\tau - u$.

Note that $\mu = k_0'(\omega_0) \sim \omega_0 k_0''(\omega_0) = v_1$, $Var_{\omega_0}(S_1) \sim Var_0(S_1) = v_1$, $\omega_0 \nearrow 0$, $\left\{\frac{1}{u\sqrt{v_1}}\right\}_{t\ge 0} \xrightarrow{D} \left\{W_{\zeta\sqrt{v_1}}(t)\right\}_{t\ge 0}$,

$$\Psi\left(u, tu^2\right) \longrightarrow IG(t;\zeta\sqrt{v_1}, \frac{1}{\zeta\sqrt{v_1}}) = IG(tv_1;\zeta,1). \tag{17}$$

Let $\{W_t, t \ge 0\}$ denote the Wiener process with drift with mean μt function and variance function $\sigma^2 t$. We consider the probability of ruin in a time interval $(0,T)$. Let $\tau = \inf_{t\ge 0}\{t : u + W_t < 0\}$. The probability of ruin before T is $\Psi(u,T) = P(\tau < T) = P(\inf_{0<t<T} W_t < -u)$.

Proposition 3.1.

$$\Psi(u,T) = \Phi\left(-\frac{u+\mu T}{\sigma\sqrt{T}}\right) + e^{-2\frac{\mu}{\sigma^2}u}\Phi\left(-\frac{u-\mu T}{\sigma\sqrt{T}}\right). \tag{18}$$

Letting $T \to \infty$, the ultimate ruin probability is $\Psi(u) = e^{-2\frac{\mu}{\sigma^2}u}$ (i.e. the diffusion approximation).

Proposition 3.2. *The probability density function of the time length until ruin is given by*

$$f_T(t) = \frac{u}{\sigma\sqrt{2\pi}}t^{-\frac{3}{2}}e^{-\frac{(u-\mu t)^2}{2\sigma^2 t}}, t > 0. \tag{19}$$

4. PERTURBED RISK MODEL

4.1. INTEGRO-DIFFERENTIAL EQUATION

In the classical risk model, the premium rate c is a positive constant which does not depend on the history of claims. However, in practice, especially in car insurance, the premium rate is adjusted according to previous claims. Thereby, the constant premium rate hypothesis is quite restrictive and departs the model from reality. Lately, there have been developed risk models in which the premium rate depends on stochastic elements of the insurer's surplus process. For example, in Dufresne and Gerber (1991) a diffusion is added at the Poisson compound process, the diffusion describes the uncertainty on the total income from premiums or on the aggregate claim. Asmussen (2000) worked out a model in which the premium rates are adjusted according to the current level of the insurer's surplus. Albrecher and Asmussen (2006) studied a premium rate which is dinamically adjusted in accordance to the history of claims. Albrecher and Boxma (2004) considered (examined) a model in which the rate of the next claim arrival is induced by the size of the previous claim. They extended their model to a new one with Markov dependence such that the arrival rates as well as the claim size distributions are determined by the state of a Markov chain in continuous time. Albrecher and Teugels (2006) studied risk models with dependence between the inter-occurence times and claims size.

In the following, we will study a perturbed risk model with dependence between the premium rates and claim sizes. We will consider that the current premium rate is adjusted after the ocurrence of a claim and the adjusted rate is determined by the claime size. Moreover, the diffusion coefficient of the model will be modified adequately. Therefore, the premium rate in period $[\tau_n, \tau_{n+1})$ is a random variable depending on the random variable X_n which gives the size of the claim at the moment $\tau_n, n = 1, 2, \ldots$ .We will denote by $c(X_n)$ this premium rate suitable for the time interval $[\tau_n, \tau_{n+1})$. In addition, it is considered that the insurer's surplus is perturbed by a standard brownian motion $\{W_t, t \geq 0\}$ which is independent of $N = \{N(t); t \geq 0\}$ and $\{X_i\}_{i=1,2,\ldots}$. The diffusion coefficient for the time period $[\tau_n, \tau_{n+1})$ is denoted by $\sigma(X_n)$, as it depends also on X_n.

The insurer's surplus at moment $t \in [\tau_n, \tau_{n+1}), n \geq 1$, is $U(t)$, given by:

$$U(t) = U(\tau_n) + c(X_n)(t - \tau_n) + \sigma(X_n)(W_t - W_{\tau_n}) \tag{20}$$

and

$$U(\tau_{n+1}) = U(\tau_n) + c(X_n) T_{n+1} - X_{n+1} + \sigma(X_n)(W_{\tau_{n+1}} - W_{\tau_n}). \tag{21}$$

Regarding the claims sizes, we will consider that there exists a threshold $b > 0$, such that when $X_n > b$, the premium rate is $c(X_n) = c_2$, and when $X_n \leq b$, the premium rate is $c(X_n) = c_1$. Thus, it is considered that the base premium rate of the portfolio is c_1, and when there ocurrs a high loss, the rate premium will be increased to the level c_2. Obviously, it is reasonable to assume that $c_2 \geq c_1 > 0$. In this interpretation, c_1 can be regarded as an acceptable income rate, while c_2 is a penalty for a high demand or loss. We have:

$$c(x) = \begin{cases} c_1, x \leq b \\ c_2, x > b \end{cases},$$

and appropriately

$$\sigma(x) = \begin{cases} \sigma_1, x \leq b \\ \sigma_2, x > b \end{cases},$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are constants that describe changes of diffusion coefficients due to changes in the premium rate.

We assume that the conditions for safety loading are met, ie $c_1 > \lambda E[X_n | X_n \leq b]$ and $c_2 > \lambda E[X_n | X_n > b]$.

Let $\{U_i(t), t \geq 0\}, i = 1, 2$, be the surplus process with the initial premium rate c_i, the initial diffusion coefficient σ_i, over the first period between claims $[0, \tau_i)$. The functions $f_1(x) = f(x) \cdot I\{x \leq b\}$ and $f_2(x) = f(x) \cdot I\{x > b\}, x \geq 0$, are introduced, where f is the pdf of the distribution function B of the claim sizes.

The survival probability of the process $\{U_i(t), t \geq 0\}$ is

$$\phi_i(u) = P(U_i(t) \geq 0, \forall t \geq 0 | U_i(0) = u), i = 1, 2.$$

Let us assume that $\phi_i(u), i = 1, 2$, are twice differentiable.

Theorem 4.1. *For any $u > 0, \phi_i(u), i = 1, 2$, satisfy the following system of equations:*

$$\frac{1}{2}\sigma_1^2\phi_1''(u) + c_1\phi_1'(u) = \lambda\phi_1(u) - \lambda\int_0^u (\phi_1(u-x) f_1(x) + \phi_2(u-x) f_2(x))\,dx \quad (22)$$

and

$$\frac{1}{2}\sigma_2^2\phi_2''(u) + c_2\phi_2'(u) = \lambda\phi_2(u) - \lambda\int_0^u (\phi_1(u-x) f_1(x) + \phi_2(u-x) f_2(x))\,dx. \quad (23)$$

with the frontier conditions $\phi_i(0) = 0, \phi_i(\infty) = 1$, and $\phi_1''(0) = -2c_i\phi_i'(0)/\sigma_i^2$, $i = 1, 2$.

Proof. We pursue Zhou M. and Cai J. (2009). Considering $U_1(t)$ in a small time interval $(0, t]$ we have that

$$\begin{aligned} \Phi_1(u) &= \lambda t E[\int_0^{s(t)} (\Phi_1(s(t)-x) I(x \leq b) + \Phi_2(s(t)-x) I(x > b))\,dB(x)] \\ &\quad + (1-\lambda t) E[\Phi_1(s(t))] + o(t), \end{aligned} \quad (24)$$

where $s(t) = u + c_1 \cdot t + \sigma_1 \cdot W_t$. By Itô's formula, we obtain

$$\Phi_1(s(t)) = \Phi_1(u) + \int_0^t \left(\frac{1}{2}\sigma_1^2 \Phi_1''(s(y)) + c_1 \Phi_1'(s(y))\right) dy + \sigma_1 \int_0^t \Phi_1'(s(y))\, dW_y.$$

Because $\left\{\int_0^t \Phi_1'(s(y))\, dW_y, t \geq 0\right\}$ is a martingale, note that $E[\int_0^t \Phi_1'(s(y))\, dW_y] = 0$. Using (24) and then dividing by t on both sides, and letting $t \to 0$, we obtain

$$\frac{1}{2}\sigma_1^2 \Phi_1''(u) + c_1 \Phi_1'(u) = \lambda \Phi_1(u) - \lambda \int_0^u \Phi_1(u-x) f_1(x) + \Phi_2(u-x) f_2(x)\, dx.$$

Analogously, we obtain the other equation. The boundary condition $\Phi_i(0) = 0$ comes from the oscillation of the diffusion, and the boundary condition $\Phi_i(\infty) = 1$ holds because of the positive loading condition. ∎

4.2. DISCRETE-TIME PERTURBED RISK MODEL

Let us now consider a discrete-time insurance model, suitable for applications.

Let the increment in the surplus process in period t, usually year, be defined as

$$W_t = P_t + A_t - S_t, t \geq 1 \tag{25}$$

where: P_t is the premium collected in the t-th period, S_t stands for the losses paid in the t-th period, A_t is any cash flow other than the premium and the payment of claims. The most significant cash flow is the earning of investment income on the surplus available at the beginning of the period. The surplus at the end of the t-th period is then

$$\begin{aligned} U_t &= U_{t-1} + W_t \\ &= U_{t-1} + P_t + A_t - S_t \\ &= u + \sum_{j=1}^{t} \left(P_j + A_j - S_j\right). \end{aligned} \tag{26}$$

Let us assume that given U_{t-1}, the random variable W_t depends only upon U_{t-1} and not upon any other previous experience.

In order to calculate ruin probabilities, we consider a process defined as follows:

$$W_t^* = \begin{cases} 0, & U_{t-1}^* < 0 \\ W_t, & U_{t-1}^* \geq 0 \end{cases}, U_t^* = U_{t-1}^* + W_t^*, t \geq 1, \tag{27}$$

where $U_0^* = u$.

We evaluate the ruin probability using convolutions. The calculation is recursively, using the distribution of U_t^*. Let us suppose that we obtained the discrete probability function (pf) of U_{t-1}^*. Then the ruin probability is

$$\tilde{\Psi}(u, t-1) = P\left(U_{t-1}^* < 0\right) \tag{28}$$

and the distribution of nonnegative surplus is $f_j = P\left(U^*_{t-1} = u_j\right), j = 1, 2, ...$, where $u_j > 0, \forall j$ and u_n is the largest possible value of U^*_{t-1}. Let $g_{j,k} = P\left(W_t = w_{j,k} \left| U^*_{t-1} = u_j\right.\right)$.

Theorem 4.2. *The ruin probability after t periods of time is*

$$\tilde{\Psi}(u,t) = \tilde{\Psi}(u,t-1) + \sum_{j=1}^{n} \sum_{w_{j,k}+u_j<0} g_{j,k} \cdot f_j \tag{29}$$

and

$$P\left(U^*_{t-1} = a\right) = \sum_{j=1}^{n} \sum_{w_{j,k}+u_j=a} g_{j,k} \cdot f_j. \tag{30}$$

5. NUMERICAL ILLUSTRATION

Let us assume that annual losses can take the values 0, 2, and 10 monetary units, with probabilities 0.6, 0.3, and 0.1, respectively. Also, suppose that the initial surplus is u, and a premium of 1 monetary unit is collected at the beginning of each year, that interest is earned at 5% on any surplus available at the beginning of the year because claims are paid at the end of the year. In addition, we introduce a rebate of 0.1 m.u. which is given in any year in which there are no losses. We determine the ruin probability at the end of each of the first three years for some values of initial capital u using (29). We present our results in Table 1, in Table 2, and Figure 1.

Table 1. The ruin probability $\Psi(u,t)$.

$u\backslash t$	1	2	3
1	0.1	0.28	0.352
2	0.1	0.19	0.298
4	0.1	0.19	0.271
8	0.1	0.13	0.163
10	0.0	0.01	0.037

Table 2. The surplus process for $u = 8$ m.u. at the end of the 3rd year.

j	u_j	f_j	0	2	10
			0.6	0.3	0.1
1	0.8675	0.06	1.860875	−0.039125	−8.039125
2	6.8725	0.09	8.166125	6.266125	−1.733875
3	8.7725	0.18	10.161125	8.261125	0.261125
4	8.8675	0.18	10.260875	8.360875	0.360875
5	10.7675	0.36	12.255875	10.355875	2.355875

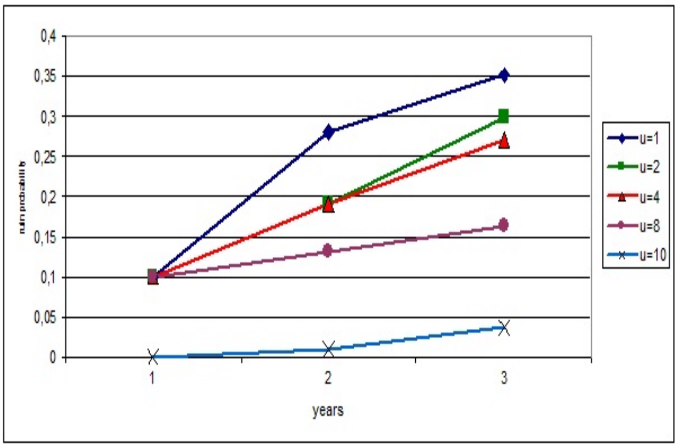

Fig. 1. The evolution of the ruin probability on the first three years, depending on the amount of the initial capital.

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ON AN EXTREMAL PROBLEM IN ANALYTIC SPACES IN TWO SIEGEL DOMAINS IN C^N

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Abstract New sharp estimates concerning distance function in certain Bergman -type spaces of analytic functions in a certain Siegel domain of first type are obtained. Related sharp new estimates for more general Siegel domains of second type are also provided. For Siegel domains of second type in C^n these are the first results of this type.

Keywords: distance estimates,analytic functions,Siegel domains of first type and second type.

2010 MSC: 42B15 (Primary), 42B30 (Secondary).

1. INTRODUCTION

In this paper we obtain sharp distance estimates in certain spaces of analytic functions in Siegel domains of first type and of second type. These types of domains are known in literature.They have been studied by many authors during last decades (see for example [10], [6], [7], [24] and references therein). In connection with the study of authomorphic functions of several complex variables, the notion of Siegel domains of the first and second type was introduced by Piatetskii-Shapiro [10], [24]. We recall basic facts that relate them to some well-known domains. The Siegel domain of first type is a particular case of a Siegel domain of second type [10]. In particular there is a Siegel domain isomorphic to unit ball of C^{m+1}, in addition, the simplest case of one dimensional Siegel domain of the first type is the upperhalfspace C_+. Note that our results below were already proved in this case in [21]. Next the Siegel domain of first type is a special type of a actively studied recently general tube domains over symmetric cones (see [19] and various references there concerning tube domains). But there are homogeneous Siegel domains of second type which are not even symmetric domains [10], [24]. Tube domains may be also viewed as special cases of Siegel domains of second type. It is known that every bounded homogeneous domain in C^n can be realized as Siegel domain of the first and the second type and that this realization is unique up to affine transformations. Siegel domains are holomorphically equivalent to a bounded domains. But there is a lot of bounded domains that are not holomorphically isomorphic to Siegel domains [10]. We will provide definitions of Siegel domains of first type and more general of second type below referring also to [24] (see also, for example, [10]).

Our line of investigation in this work can be considered as direct continuation of our previous papers on extremal problems [20], [22] and [23]. Our main two new results are contained in the second and third sections of this note. First we provide a concrete special example of a Siegel domain of first type and we obtain a sharp estimate for distance function in certain Bergman type analytic spaces on it. Next we turn in our final section to Siegel domains of the second type. We remark that here for the first time in literature we consider this extremal problem related with distance estimates in spaces of analytic functions in Siegel domains of second type. The next two sections partially also contain some required preliminaries on analysis on these domains.

In the upperhalfspace C_+ which is one dimensional tubular domain and also in general tubular domains our theorems are not new and they were obtained recently in [21], and then in general form in [19]. Moreover arguments in proofs we provided below are similar to those we have in previous cases and hence our arguments sometimes will be sketchy below. The main tool of the proof is again the so-called Bergman reproducing formula, but in Siegel domains (see, for example, [5], [6] for this integral representation and it is applications). This paper first deals with a concrete example of Siegel domain of first type and based on some results from [5] we present a sharp result in this direction. But then in the final part we turn to more general situation (see [6] for notation which will be constantly needed in this second part) and we obtain some related estimates for distances there also. Note again some results from [6] are crucial here in last section for us.

We now shortly remind the history of this extremal problem.

After the appearance of [26] various papers appeared where arguments which can be seen in [26] were extended and changed in various directions [22], [23], [20].

In particular in the mentioned papers various new results on distances for analytic function spaces in higher dimension (unit ball and polydisk) were obtained. Namely new results for large scales of analytic mixed norm spaces in higher dimension were proved.

Later several new sharp results for harmonic function spaces of several variables in the unit ball and upperhalfplane of Euclidean space were also obtained (see, for example, [20] and references therein). The classical Bergman representation formula in various domains serves as a base in all these papers in proofs of main results. Recently (see[18]) concrete analogues of our theorems were proved also in some spaces of entire functions of one and several variables. Various other extremal problems in analytic function spaces also were considered before in various papers (see for example [1], [15], [16], [14]). In those just mentioned papers other results around this topic and some applications of certain other extremal problems can be found also.

2. NEW SHARP ESTIMATES FOR DISTANCES IN ANALYTIC BERGMAN -TYPE SPACES IN SIEGEL DOMAINS OF FIRST TYPE

This section is devoted to one of the main results of this paper. We remark that our notes, namely this one and [19], are first papers with sharp results on extremal problems in higher dimension in C^n, namely in analytic function spaces in Siegel domains in C^n. We now establish some notation from [5] which will be needed for us. Let $\Omega \subset C^n$ be an open nonempty set. Let $W(\Omega)$ be the set of all weights in Ω. We mean by this a set of all Lebesgue measurable functions acting from Ω to R_+. For each such γ function let $L^2(\Omega, \gamma)$ be the Hilbert space of all f functions from Ω to C so that the quasi-norm $\int_\Omega |f(z)|^2 \gamma(z) dm(z)$ is finite,where $dm(z)$ is a Lebesgue measure on Ω. By $A^2(\Omega)$ we denote the analytic subspace of this space but for so -called special (see [5]) admissible γ weights and with the same quasinorm (see [5]). Note that the class of these weights it is a closed subspace of $L^2(\Omega)$. Next according to the well-known Riesz representation theorem there is a unique function that for all functions from this space a certain integral representation holds with a certain function called Bergman kernel which is from $L^2(\Omega)$ (see [5]and references therein). In certain cases and our case is of them in higher dimension this function called Bergman kernel can be explicitly written. This last fact alone already opens a large way for various investigations in this research area. In the present paper first we look at the family of the following admissible weights $\gamma_\alpha(\tau)$,

$$\gamma_\alpha(\tau) = (\Im\tau_1 - |\tilde{\tau}|^2)^\alpha$$

$\alpha > -1$ on the concrete Siegel domain of the first type (see [5])

$$\Omega = \left\{\tau \in C^n, \Im\tau_1 > |\tilde{\tau}|^2\right\},$$

where we denote by τ and $\tilde{\tau}$ the vectors $\tau = (\tau_1, \dots, \tau_n)$,$\tilde{\tau} = (\tau_2, \dots, \tau_n)$. Let w be a vector from C^n. Let also $dm_\beta(w) = (\Im w_1 - |\tilde{w}|^2)^\beta dm(w)$,where $dm(w)$ is a Lebesgue measure on R^{2n} and we also define a Bergman kernel as see[5]

$$B_\beta(\tau, w) = (\tau - \overline{w})^{-\beta-n-1} = (u - 2v)^{-n-1-\beta}$$

$u = i(\overline{\tau_1} - w_1)$,$v = (\tilde{w}\tilde{\tau})$, where the last expression is as usual a scalar product of two vectors in C^{n-1}. These definitions are crucial for our paper. The goal of this section to develop further some ideas from our recent already mentioned papers and to present a new sharp theorem in mentioned Siegel domain of first type .

For formulation of our result we will now need various standard definitions from the theory of these Siegel domains of first type (see [10], [5]).

Let Ω be the Siegel domain . $\mathcal{H}(\Omega)$ denotes the space of all holomorphic functions on Ω. Let further, for all positive β,

$$A^\infty_\beta(\Omega) = \left\{F \in \mathcal{H}(\Omega) : \|F\|_{A^\infty_\beta} = \sup_{x+iy\in\Omega} |F(x+iy)|\gamma_\beta(x+iy) < \infty\right\}, \tag{1}$$

(we use in this paper the following notation $w = u + iv$ and $z = x + iy$, $w \in \Omega$, $z \in \Omega$). It can be checked that this is a Banach space.

For $1 \leq p < +\infty$, $\alpha > -1$ we denote by $A^p_\alpha(\Omega)$ the weighted Bergman space consisting of analytic functions f in Ω such that

$$\|F\|_{A^p_\alpha} = \left(\int_\Omega |F(z)|^p \gamma_\alpha(z) dm(z)\right)^{1/p} < \infty.$$

This is a Banach space. Below we will restrict ourselves to $p = 2$ case following [5]. Replacing above A by L we will get as usual the corresponding larger space $L^2_\nu(\Omega)$ of all measurable functions in our domain Ω with the same quasi-norm (see [5]). The (weighted) Bergman projection P_β is the orthogonal projection from the Hilbert space $L^2_\nu(\Omega)$ onto its closed subspace $A^2_\nu(\Omega)$ and it is given by the following integral formula (see [5])

$$P_\beta f(z) = C_\beta \int_\Omega B_\beta(z, w) f(w) dm_\beta(w), \tag{2}$$

where C_β is a special constant (see [5]) and $\beta > \frac{\nu-1}{2}$. For these values of β this is a bounded linear operator from L^2_ν to A^2_ν. Hence, by using these facts, we have that for any analytic function from $A^2_\nu(\Omega)$ the following integral formula is valid for all functions from A^2_ν, for all $\beta, \beta > \frac{\nu-1}{2}$ and $\nu > -1$ (see[5])

$$f(z) = C_\beta \int_\Omega B_\beta(z, w) f(w) dm_\beta(w). \tag{3}$$

In this case sometimes below we say simply that the analytic f function allows Bergman representation via Bergman kernel with β index.

We need also the following estimate (A) of Bergman kernel from [5]. Let $t > -1$ and $\beta > 0$. Then there is a positive constant $c = c_{n,t,\beta}$ so that

$$\int_\Omega \gamma_t(\tau) |B_{t+\beta}(\tau, w)| dm(\tau) \leq c \gamma_\beta^{-1}(w),$$

$w \in \Omega$. This estimate of Bergman kernel will be used and not once below during the proof of our first theorem.

Note here also these assertions we just mentioned have direct analogues in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace C_+ and in spaces of harmonic functions in the unit ball or upperhalfspace of Euclidean space R^n. These classical facts are well- known and can be found, for example, in some items in references (see, for example, [26], [9]).

Above and throughout the paper we write C (sometimes with indexes) to denote positive constants which might be different each time we see them (and even in a chain of inequalities), but is independent of the functions or variables being discussed.

As in case of analytic functions in unit disk, polydisk, unit ball, and upperhalfspace C_+, and tubular domains over symmetric cones, and in case of spaces of harmonic functions in Euclidean space, [26], [21], [20], [22], [23], the role of the Bergman representation formula and estimates for Bergman kernel are crucial in these issues related with our extremal problem and our proof will be heavily based on them.

And as it was mentioned already above a variant of Bergman representation formula is available also in Bergman- type analytic function spaces in Siegel domains and this known fact (see [5], [24], [6]), which is crucial in various problems in analytic function spaces in Siegel domains of both types is also used in our proof below.

Moreover will also need for our proof the following additional facts on integral representation of functions on these Ω domains which follows from assertions we already formulated above. Note first that for all functions from A^∞_α the integral representations of Bergman we mentioned above with Bergman kernel

$$B_\nu(z,w)$$

(with ν index) is valid for large enough ν. This follows directly from the fact that A^∞_α for any α is a subspace of A^2_τ if τ is large enough [5]. Moreover it can be easily shown that we have a continuous embedding $A^2_\alpha \hookrightarrow A^\infty_\beta$ (see, for example, [5] where the proof can be found also) for a concrete β depending on α, $\alpha > -1$ and this naturally leads to a problem of estimating

$$\operatorname{dist}_{A^\infty_\beta}(f, A^2_\alpha)$$

for a given $f \in A^\infty_\beta$,where $\beta = \frac{\alpha+n+1}{2}$, $\alpha > -1$.

This problem on distances we just formulated will be solved in our next theorem below, which is one of the main results of this section. Let us set, for $f \in \mathcal{H}(\Omega)$, $s > 0$ and $\epsilon > 0$ and $z = x + iy \in \Omega$,

$$N_{\epsilon,s}(f) = \{z \in \Omega : |f(z)|\gamma_s(z) \geq \epsilon\}. \tag{4}$$

We denote by N_1 and by N_2 two sets- the first one is $N_{\epsilon,s}(f)$, the other one is the set of all those points, which are in tubular domain Ω, but not in N_1. Note now, to clarify the notation for readers again, by $m(z)$ or by m with only one lower index we denote in this section the Lebesgue measure on R^{2n}.

Theorem 2.1. *Let $t = \frac{\nu+1+n}{2}$. Set, for $f \in A^\infty_{\frac{n}{2}+\frac{\nu+1}{2}}$, $\nu > -1$*

$$l_1(f) = \operatorname{dist}_{A^\infty_{\frac{n}{2}+\frac{\nu+1}{2}}}(f, A^2_\nu), \tag{5}$$

$$l_2(f) = \inf\left\{\epsilon > 0 : \int_\Omega \left(\int_{N_{\epsilon,t}(f)} \frac{\gamma_{\beta-t}(w)dw}{(z-\overline{w})^{\beta+n+1}}\right)^2 \gamma_\nu(z)dm(z) < \infty\right\}. \tag{6}$$

Then there is a positive number β_0, so that for all $\beta > \beta_0$, we have $l_1(f) \asymp l_2(f)$.

Proof. We will start the proof with the following observation, which already was mentioned above. By our arguments before formulation of this theorem for all functions from $A^{\infty}_{\tau_1}$ the integral representations of Bergman with Bergman kernel

$$B_(\tau_2)(z, w)$$

is valid for large enough τ_2.

We denote below the double integral which appeared in formulation of theorem by $G(f)$ and we will show first that $l_1(f) \leq Cl_2(f)$. We assume now that $l_2(f)$ is finite.

We use the Bergman representation formula which we provided above, namely(3), and using conditions on parameters we now have the following equalities.

First we have obviously by the remark from which we started this proof that for large enough β

$$f(z) = C_\beta \int_\Omega B_\beta(z, w) f(w) dm_\beta(w) = f_1(z) + f_2(z),$$

$$f_1(z) = C_\beta \int_{N_2} B_\beta(z, w) f(w) dm_\beta(w,$$

$$f_2(z) = C_\beta \int_{N_1} B_\beta(z, w) f(w) dm_\beta(w).$$

Then we estimate both functions separately using estimate (A) provided above and following some arguments we provided in one dimensional case that is the case of upperhalfspace C_+ [21]. Here our arguments are sketchy since they are parallel to arguments from [21]. Using definitions of N_1 and N_2 above after some calculations following arguments from [21] using the estimate (A) of Bergman kernel we mentioned above we will have immediately

$$f_1 \in A^{\infty}_{\frac{\nu+n+1}{2}}$$

and

$$f_2 \in A^2_\nu.$$

We easily note the last inclusion follows directly from the fact that l_2 is finite.

Moreover it can be easily seen that the norm of f_1 can be estimated from above by $C\epsilon$, for some positive constant C ([21]), since obviously

$$\sup_{N_2} |f(w)| \gamma_t(w) \leq \epsilon.$$

Note this last fact follows directly from the definition of N_2 set and estimate (A) above which leads to the following inequality

$$\int_\Omega \gamma_{\beta-t}(\tau) |B_\beta(\tau, w)| dm(\tau) \leq C \gamma_t^{-1}(w),$$

$w \in \Omega$ for all β so that $\beta > \beta_0$, for some large enough fixed β_0 which depends on n, ν, where

$$t = (\frac{1}{2})(\nu + 1 + n).$$

This gives immediately one part of our theorem. Indeed, we have now obviously

$$l_1 \leq C_2 \|f - f_2\|_{A_t^\infty} = C_3 \|f_1\|_{A_t^\infty} \leq C_4 \epsilon.$$

It remains to prove that $l_2 \leq l_1$. Let us assume $l_1 < l_2$. Then there are two numbers ϵ and ϵ_1,both positive such that there exists f_{ϵ_1} ,so that this function is in A_ν^2 and $\epsilon > \epsilon_1$ and also the condition

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

holds and $G(f) = \infty$, where G is a double integral in formulation of theorem in l_2 (see (6)).

Next from

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

we have the following two estimates,the second one is a direct corollary of first one. First we have

$$(\epsilon - \epsilon_1)\tau_{N_{\epsilon,t}}(z)\gamma_t^{-1}(z) \leq C|f_{\epsilon_1}(z)|$$

,where $\tau_{N_{\epsilon,t}}(z)$ is a characteristic function of $N = N_{\epsilon,t}(f)$ set we defined above.

And from last estimate we have directly multiplying both sides by Bergman kernel $B_\beta(z, w)$ and integrating by tube Ω both sides with measure dm_β

$$G(f) \leq C \int_\Omega (L(f_{\epsilon_1}))^2 \gamma_\nu(z) dm(z),$$

where

$$L = L(f_{\epsilon_1}, z)$$

and

$$L(f_{\epsilon_1}, z) = \int_\Omega |f_{\epsilon_1}(w)||B_\beta(z, w)| dm_\beta(w).$$

Denote this expression by I.

Put $\beta + n + 1 = k_1 + k_2$, where $k_1 = \beta + 1 - n - \mu$, $k_2 = \mu + 2n(\frac{1}{2} + \frac{1}{2})$ where the additional parameter will be chosen by us later.

By classical Hölder inequality we obviously have

$$I^2 \leq C I_1 I_2,$$

where

$$I_1(f) = \int_\Omega |f_1(z)|^2 |(z - \overline{w})^s| \gamma_{2\beta}(z) dm(z),$$

$$I_2 = \int_\Omega |(z - \overline{w})^\nu| dm(z)$$

and where $f_1 = f_{\epsilon_1}$ and

$$s = 2\mu - 2 - 2\beta,$$

$$\nu = -2n - 2\mu.$$

Choosing finally the μ parameter, so that the estimate (A) namely

$$\int_\Omega \gamma_{\tilde{t}}(\tau)|B_{\tilde{t}+\tilde{\beta}}(\tau, w)|dm(\tau) \le C\gamma_{\tilde{\beta}}^{-1}(w),$$

$$w \in \Omega$$

can be used twice above with some restrictions on parameters and finally making some additional easy calculations we will get what we need.

Indeed we have now obviously,

$$\int_\Omega (\int_\Omega |f_{\epsilon_1}(z)|B_\beta(z, w)|dm_\beta(z))^2 \gamma_\nu(w) dm(w) \le C\|f_{\epsilon_1}\|^2_{A^2_\nu}$$

and

$$G(f) \le C\|f_{\epsilon_1}\|_{A^2_\nu},$$

but we also have

$$f_{\epsilon_1} \in A^2_\nu.$$

This is in contradiction with our previous assumption above that $G(f) = \infty$. So we proved the estimate which we wanted to prove. The proof of our first theorem in Siegel domains of first type is now complete. ∎

3. NEW ESTIMATES FOR DISTANCES IN BERGMAN TYPE SPACES IN SIEGEL DOMAINS OF THE SECOND TYPE

We first recall some basic facts on Siegel domains of second type and then establish notations for our second main theorem. Recall first the explicit formula for the Bergman kernel function is known for very few domains.The explicit forms and zeros of the Bergman kernel function for Hartogs domains and Hartogs type domains (Cartan-Hartogs domains) were found only recently [2]. On the other hand in strictly pseudoconvex domains the principle part of the Bergman kernel can be expressed explicitly by kernels closely related to so-called Henkin -Ramirez kernel see for example [11] and references there. In [10] the Bergman kernel

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right)$$

for the Siegel domain of the second type was computed explicitly. It is an integral via V^* a convex homogeneous open irreducible cone of rank l in R^n, a conjugate cone of V cone and which also contains no straight line and in that integral the fixed Hermitian form from definition of D Siegel domain(see below for definition) participates (see details for this [6]). This fact was heavily used in [6] in solutions of several classical problems in Siegel domains of the second type and we will also use one estimate from [6] for this kernel, but we define it otherwise ,representing it otherwise in this paper (see also [6]). We will need now some short, but more concrete review of certain results from [6] to make this exposition more complete. To be more precise the authors in [6] showed that on homogeneous Siegel domain of type 2 under certain conditions on parameters the subspace of a weighted L^p space for all positive p consisting of holomorphic functions is reproduced by a concrete weighted Bergman kernel which we just mentioned. They also obtain some L^p estimates for weighted Bergman projections in this case. The proof relies on direct generalization of the Plancherel-Gindikin formula for the Bergman space A^2 (see[10]). We remind the reader that the Siegel domain of type 2 associated with the open convex homogeneous irreducible cone V of rank l which contains no straight line,$V \in R^n$,and a V-Hermitian homogeneous form F which act from product of two C^m into C^n is a set of points (w, τ) from C^{m+n} so that the difference D of $\Im w$ and the value of F on (τ, τ) is in V cone. This domain is affine homogeneous and we now should recall the following expression for the Bergman kernel of $D = D(V, F)$. Let D be an affine-homogeneous Siegel domain of type 2. Let $dv(z)$ denote the Lebesgue measure on D and let $H(D)$ denote the space of all holomorphic functions on D. The Bergman kernel is given by the following formula (see[6]) for $(\tau_1, \tau_2) \in D$ and $(\tau_3, \tau_4) \in D$

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right) = (\frac{\tau_1 - \overline{\tau}_3}{2i} - (F(\tau_2, \tau_4))^{2d-q},$$

where two vectors $q = (q_i)$ and $d = (d_i)$ and in addition $n = (n_i)$ here the i index is running from 1 to l are specified via $n_{i,k}$, where these $n_{i,k}$ numbers are dimensions of certain $(R_{i,k})$ and $(C_{i,j})$ subspaces of the certain canonical decomposition of C^{m+n} and R^n via the V cone from definition of our D domain (see for some additional details about this [10] and [6]). We will call this family of triples parameters of a Siegel domain D of second type. They will appear in our main theorem and it is short proof. The standard Bergman projection here on D as usual is denoted by P, it is the orthogonal projection of $L^2(D, dv)$ onto it is analytic subspace $A^2(D, dv)$ consisting of all holomorphic functions. The authors in [6] showed that some well-known facts of much simpler domains holds also here, for example there is an integral operator on L^2 space defined by the certain $b(\tau, z)$ Bergman kernel. And for this types of Siegel domains as it was mentioned above this Bergman kernel was computed explicitly previously in [10]. Further, let ϵ be a real number. Now for all positive finite p we define a space of integrable functions (weighted L^p spaces with $b^{-\epsilon}(z, z)$ weights) for all $\epsilon > \epsilon_0$

$$L^{p,\epsilon}(D) = L^p(D, b^{-\epsilon}(z, z)dv(z))$$

and we denote as usual by $A^{p,\epsilon}$ the analytic subspace of this space with usual modification when $p = \infty$. Note the restriction is meaningful since there is an ϵ_0 so that for all those ϵ which are smaller than this fixed ϵ_0 the $A^{2,\epsilon}$ is an empty class (see [6]). We denote by P_ϵ the corresponding Bergman projection which is the orthogonal projection of $L^{2,\epsilon}$ to it is analytic subspace $A^{2,\epsilon}$. In [6] the authors give a condition on real numbers and vectors r, p, ϵ, so that the weighted Bergman projection reproduces all functions in $A^{p,r}(D)$.This vital property for our theorem was partly deduced by them from Plancherel-Gindikin formula and the fact that

$$P_\epsilon(f)(z) = c_\epsilon \int_D f(u) b^{1+\epsilon}(z, u) b^{-\epsilon}(u) du,$$

so it defines as in simpler cases an integral operator on $L^{2,\epsilon}(D)$ by the kernel $b^{1+\epsilon}(\tau, z)$ (see for this [6]), it is a weighted Bergman projection from $L^{2,\epsilon}$ onto $A^{2,\epsilon}$ (see, for example, [6] and references therein.).

The following several assertions concerning Bergman projection acting in analytic spaces in Siegel domain of the second type and estimates of Bergman kernel which we mentioned above and in addition to this some facts on spaces of integrable functions and their analytic subspaces we defined above on these Siegel domains were proved in [6] and some are crucial for this paper. We will formulate immediately after them our main result on distances in Siegel domains of the second type. Then providing a comment on a proof of that assertion which contains no new ideas when we compare it with the proof of previous theorem we will finish this paper. We use the following notation. The i index below is running from 1 to l everywhere and to make the reading easier we accept this from advance. We also use below everywhere standard rules of calculations between two vectors as they were seen by us for example in [6], also sometimes we write

$$d\tilde{V}(\tau_1, \tau_2)$$

not $dv(\tau)$ meaning

$$\tau = (\tau_1, \tau_2) \in D.$$

In the following assertions

$$(n_i), (q_i), (d_i)$$

will always act as parameters of the Siegel domain D we introduced above and they are playing a crucial role. We write always D below meaning

$$D(n, q, d)$$

,where $n = (n_i)$, $d = (d_i)$, $q = (q_i)$. We write $c_i \le b_i$ for two vectors from R^l below meaning as usual that this is true for all values of i from 1 to l. If $c \le bi$ (or $c < b_i$) then all b_i are bigger or equal (or bigger)than c.

Proposition 3.1. *Let* $\epsilon \in R^l$, $r \in R^l$, $p \in R_+$, $0 \leq r_j$. *Then there are two sets of numbers* (k_i), (m_i) *depending on parameters of D Siegel domain so that if* $1 \leq p < k_i$ *and* $\epsilon_i > m_i$, *then*

$$P_\epsilon f = f$$

for all

$$f \in A^{p,r}(D).$$

Let $\epsilon \in R^l$, $r \in R^l$, $p \in (0, \infty)$, $v_i < r_i$, *for some* v_i *numbers depending on parameters of D domain. Then there are two sets of numbers* (k_i^1),(m_i^1), *depending from parameters of D Siegel domain, so that if*

$$0 < p < k_i^1$$

and if $\epsilon_i > m_i^1$, *then*

$$P_\epsilon f = f$$

for all

$$f \in A^{p,r}(D).$$

Proposition 3.2. *If*

$$\epsilon_i > \frac{n+2}{2(2d-q)_i}$$

where $\epsilon \in R^l$, *then* P_ϵ *is an integral operator with*

$$b^{1+\epsilon}(t_1, z_1)(t_2, z_2)$$

kernel on $L^{2,\epsilon}$ *and*

$$P_\epsilon f = f$$

for all $f \in A^{p,0}(D)$, *when* $p \in (0, p_0)$, *where*

$$p_0 \leq \frac{n_i - 2(2d-q)_i}{n_i}.$$

If there is an index i so that

$$2\epsilon_i \leq \frac{n_i + 2}{(2d-q)_i}$$

then we have $A^{2,\epsilon} = 0$, *moreover if the reverse estimate holds for all i and* ϵ_i *instead of* $2\epsilon_i$ *then the intersection of* $A^{2,\epsilon}$ *and* $A^{p,r}$ *is dense in* $A^{p,r}$, *if* $1 \leq p < \infty$, $0 \leq r_i$, $\epsilon \in R^l$.

The following embedding which is taken also from [6] is important for us. It allows us as in previous simpler case to pose a distance problem in this domain showing that

Bergman spaces $A^{p,r}$ are subspaces of $A^{\infty}_{\frac{1+r}{p}}$ Bergman-type spaces. Let $r \in R^l$ and $p \in (0, \infty)$, then

$$|f(z)|^p \leq Cb^{1+r}(z,z)\|f\|^p_{p,r}, \quad z \in D.$$

Further let ϵ and r are from R^l. If

$$\epsilon_i > \frac{n_i}{-2(2d-q)_i}$$

and

$$r_i > \frac{n_i + 2}{2(2d-q)_i} + \epsilon_i,$$

then we have

$$P_r(f) = f$$

as soon as f belongs to A^{∞}_{ϵ}(see[6],[13]). This will also be needed in the proofs of main result of this section (see for this also the parallel proof of our previous theorem from previous section).

Proposition 3.3. *Let $\beta \in R^l$ and all β_i are nonnegative then the following estimate holds*

$$b^{\beta}(z+\tau, z+\tau) \leq b^{\beta}(z,z)$$

and also

$$|b^{\beta}(\tau, z)| \leq b^{\beta}(z,z)$$

for all τ and z from D.

The following estimate to be more precise it is direct analogue can be found in the proof of previous theorem where it was used three times.

Proposition 3.4. *Let α and ϵ be two vectors from R^l and (τ, z) be a point of D. Then if*

$$\frac{n_i + 2}{2(2d-q)_i} < \epsilon_i$$

and

$$\epsilon_i - \frac{n_i}{2(2d-q)_i} < \alpha_i,$$

then the integral

$$\int_D |b^{\alpha+1}((\tau, v), (z, u))| b^{-\epsilon}((z,u)(z,u)) d\tilde{V}(z, v)$$

is equal with

$$c_{\alpha,\epsilon} b^{\alpha-\epsilon}((\tau, v), (\tau, v)).$$

We are able now based only on last proposition and two comments concerning integral representations before previous proposition to formulate a theorem on distances in Siegel domains of the second type which is a direct analogue of our previous results (see, for example, [22], [23], [21]) and our previous theorem on distances in this situation. All facts and preliminaries which are needed here for our proof can be found above in assertions from [6] which we just formulated, all lines of arguments for our proof of this theorem can be also found above in the proof of our previous theorem though some not very long technical calculations with indexes should be added. Note that one implication in this theorem below is easier and we just repeat here arguments of our previous theorem.

Theorem 3.1. *Let*

$$N_{\tilde{\epsilon},r}(f) = \left\{z \in D, |f(z)|b^{1+r}(z,z) > \tilde{\epsilon}\right\},$$

where $\tilde{\epsilon}$ is a positive number. Then the following two quantities are equivalent

$$\mathrm{dist}_{A^{\infty}_{1+r}}(f, A^{1,r})$$

and

$$\inf\left\{\tilde{\epsilon} > 0, \int_D \left(\int_{N_{r,\tilde{\epsilon}}(f)} b^{-k+1+r}(\tau,\tau)|b(\tau,z)|^{k+1} dv(\tau)\right) b^{-r}(z,z) dv(z) < \infty\right\},$$

for all r and k so that $r \in (r_0, \infty)$ and $k \in (k_0, \infty)$ and for certain fixed vectors r_0 and k_0 depending on parameters of the Siegel D domain (d_i) and (q_i) and (n_i).

We finally remark that the theorem above is probably valid for all $p > 1$ (not only $p = 1$ when calculations are simpler) and the reader can formulate easily that theorem in general case following the formulation of our previous theorem. The proof probably is parallel to the proof of previous theorem and it is based on estimates from propositions above. Note also our assertion is true for all homogeneous Siegel domains not only symmetric Siegel domains of the second type (see [6], [8], [12]). We remark as r_0 we can take $\max(r_1, r_2, 0)$ where r_1 and r_2 are depending on parameters of domain $r_1 = \frac{n_i+2}{2(2d-q)_i}$ and $r_2 = \frac{-n_i}{2(2d-q)_i} - 1$.

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SOME CONDITIONS FOR UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract The main object of the present paper is to discuss some univalence conditions for a general integral operator $G_{n,m}$ defined by means of Al-Oboudi differential operator. Many known univalence conditions are written to prove our results.

Keywords: analytic functions, general Schwarz lemma; differential operator.
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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$. For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator

$$D^0 f(z) = f(z), \tag{2}$$

$$D^1_\delta f(z) = (1-\delta) f(z) + \delta z f'(z), \quad \delta \geq 0, \tag{3}$$

$$\ldots$$

$$D^n_\delta f(z) = D_\delta \left(D^{n-1} f(z)\right), \quad (n \in \mathbb{N} = \{1, 2, ...\}). \tag{4}$$

If f is given by (1), then from (3) and (4) we see that

$$D^n_\delta f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup 0). \tag{5}$$

It results $D^n_\delta f(0) = 0$.

Remark 2. *When $\delta = 1$, we get the Sălăgean differential operator [5].*

In the proof of our main result (Theorem 2.1) we need a univalence criterion. The univalence criterion, asserted by Theorem 1.1, is a generalization of Ahlfor's and Becker's univalence criterion; it was proved by Pescar [4].

Theorem 1.1. *[4] Let β be a complex number, with $\mathrm{Re}\beta > 0$, and c a complex number such that $|c| \leq 1$, $c \neq -1$ and $f(z) = z + ...$ a regular function in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1,$$

for all $z \in U$, then the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}} = z + ...$$

is regular and univalent in $\mathbb{U}$.

For the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [3] proved the following univalence condition.

Theorem 1.2. *[3] Let $f \in \mathcal{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq 1 \qquad (z \in \mathbb{U}). \tag{6}$$

Then the function f is univalent in $\mathbb{U}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [2]).

Lemma 1. *[2] Let be a regular function f in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has a single, unique zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (\forall z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Here, in our investigation, we introduce a general integral operator by means of the Al-Oboudi differential operator as follows:

$$G_{n,m}(f_1, ..., f_n, g_1, ..., g_n)(z) =$$

$$\left(\left(1+\sum_{i=1}^{n}\alpha_i\right)\int_0^z\prod_{i=1}^{n}\left(D^m f_i(t)\right)^{\alpha_i}\left(g_i'(t)\right)^{\gamma_i}dt\right)^{\frac{1}{\left(1+\sum_{i=1}^{n}\alpha_i\right)}}, \tag{7}$$

where $\alpha_i, \gamma_i \in \mathbb{C}$, $f_i, g_i \in \mathcal{A}$ for all $i \in \{1,2,...,n\}$ and D^m is the Al-Oboudi differential operator for $m \in \mathbb{N}_0$.
In this paper, we study the univalence conditions involving the general integral operator defined by (7).

2. MAIN RESULTS

Theorem 2.1. *Let the functions $f_i, g_i \in \mathcal{A}$, where f_i satisfy the condition*

$$\left|\frac{z^2\left(D^m f_i(z)\right)'}{\left[D^m f_i(z)\right]^2}-1\right| \le 1 \quad \left(z \in \mathbb{U};\quad m \in \mathbb{N}_0\right), \tag{8}$$

let $M_i \ge 1$, $N_i \ge 1$ and $\alpha_i, \gamma_i, \beta$ be complex numbers such that $\beta = 1+\sum_{i=1}^{n}\alpha_i$ and

$$\mathrm{Re}\beta \ge \sum_{i=1}^{n}\left[\left|\alpha_i\right|\left(2M_i+1\right)+\left|\gamma_i\right|N_i\right] > 0$$

for all $i \in \{1,2,...,n\}$. If

$$\left|D^m f_i(z)\right| \le M_i \quad (z \in \mathbb{U}), \qquad \left|\frac{z g_i''(z)}{g_i'(z)}\right| \le N_i \quad (z \in \mathbb{U})$$

and

$$|c| \le 1-\frac{1}{\mathrm{Re}\beta}\sum_{i=1}^{n}\left[\left|\alpha_i\right|\left(2M_i+1\right)+\left|\gamma_i\right|N_i\right] \tag{9}$$

for all $i \in \{1,2,...,n\}$, then the integral operator $G_{n,m}(f_1,...,f_n,g_1,...,g_n)(z)$ defined by (7) is in the class $\mathcal{S}$.

Proof. For (5) we have

$$\frac{D^m f_i(z)}{z} = 1+\sum_{k=2}^{\infty}\left[1+(k-1)\delta\right]^m a_{k,i}z^{k-1} \quad (m \in \mathrm{N}_0)$$

and

$$\frac{D^m f_i(z)}{z} \ne 0$$

for all $i \in \{1,2,...,n\}$, $z \in \mathbb{U}$.
The integral operator $G_{n,m}(f_1,...,f_n,g_1,...,g_n)(z)$ defined by (7) can be rewritten as follows:

$$G_{n,m}(f_1,...,f_n,g_1,...,g_n)(z) =$$

$$\left(\left(1+\sum_{i=1}^{n}\alpha_i\right)\int_0^z t^{\sum_{i=1}^{n}\alpha_i}\prod_{i=1}^{n}\left(\frac{D^m f_i(t)}{t}\right)^{\alpha_i}\left(g_i'(t)\right)^{\gamma_i}dt\right).$$

Let us define the function $h(z)$ by

$$h(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{D^m f_i(t)}{t}\right)^{\alpha_i}\left(g_i'(t)\right)^{\gamma_i}dt$$

where $f_i, g_i \in \mathcal{A}$ for all $i \in \{1,2,...,n\}$.
The function $h(z)$ is indeed regular in $\mathbb{U}$ and satisfy the following usual normalization condition:

$$h(0)=h'(0)-1=0.$$

Now, calculating the derivates of the first and second orders, we readily obtain

$$h'(z)=\prod_{i=1}^{n}\left(\frac{D^m f_i(z)}{z}\right)^{\alpha_i}\left(g_i'(z)\right)^{\gamma_i}$$

and

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\left[\alpha_i\left(\frac{z\left(D^m f_i(z)\right)'}{D^m f_i(z)}-1\right)+\gamma_i\frac{zg_i''(z)}{g_i'(z)}\right]. \tag{10}$$

From the equation (10), we have

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|$$

$$\begin{aligned}
&=\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{1}{\beta}\sum_{i=1}^{n}\left[\alpha_i\left(\frac{z\left(D^m f_i(z)\right)'}{D^m f_i(z)}-1\right)+\gamma_i\frac{zg_i''(z)}{g_i'(z)}\right]\right|\\
&\le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\alpha_i|\left(\left|\frac{z\left(D^m f_i(z)\right)'}{D^m f_i(z)}\right|+1\right)+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right]\\
&\le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left[|\alpha_i|\left(\left|\frac{z^2\left(D^m f_i(z)\right)'}{\left[D^m f_i(z)\right]^2}\right|\left|\frac{D^m f_i(z)}{z}\right|+1\right)+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right].
\end{aligned} \tag{11}$$

From the hypothesis of Theorem 2.1, we have

$$\left|D^m f_i(z)\right|\le M_i \quad (z\in\mathbb{U}), \qquad \left|\frac{zg_i''(z)}{g_i'(z)}\right|\le N_i \quad (z\in\mathbb{U}),$$

then by the General Schwarz Lemma for the functions f_i, we obtain that

$$\left|D^m f_i(z)\right|\le M_i\,|z| \qquad (z\in\mathbb{U}, m\in\mathbb{N}_0)$$

for all $i \in \{1, 2, ..., n\}$.
We apply this result in the inequality (11) and from (8) we obtain

$$\begin{aligned}
&\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| \\
&\qquad \le |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left[|\alpha_i| \left(\left(\left| \frac{z^2\,(D^m f_i(z))'}{[D^m f_i(z)]^2} - 1 \right| + 1 \right) M_i + 1 \right) + \left|\gamma_i\right| N_i \right] \\
&\qquad \le |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} \left[\, |\alpha_i|\,(2M_i + 1) + \left|\gamma_i\right| N_i \right] \\
&\qquad \le |c| + \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[\, |\alpha_i|\,(2M_i + 1) + \left|\gamma_i\right| N_i \right].
\end{aligned}$$

So, from (9), we have

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)} \right| \le 1.$$

Applying Theorem 1.1, we obtain that the integral operator $G_{n,m}(z)$ defined by (7) is in the class $\mathcal{S}$. ∎

If we set $m = 0$ in Theorem 2.1, we can obtain the following interesting consequence of this theorem.

Corollary 2.1. *Let the functions $f_i, g_i \in \mathcal{A}$, where f_i satisfy the condition*

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| \le 1 \qquad (z \in \mathbb{U}),$$

let $M_i, N_i \ge 1$ and $\alpha_i, \gamma_i, \beta$ be complex numbers such that $\beta = 1 + \sum_{i=1}^{n} \alpha_i$ and

$$\mathrm{Re}\beta \ge \sum_{i=1}^{n} \left[\, |\alpha_i|\,(2M_i + 1) + \left|\gamma_i\right| N_i \right] > 0$$

for all $i \in \{1, 2, ..., n\}$. If

$$|f_i(z)| \le M_i \qquad (z \in \mathbb{U}), \qquad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \le N_i \qquad (z \in \mathbb{U})$$

and

$$|c| \le 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left[\, |\alpha_i|\,(2M_i + 1) + \left|\gamma_i\right| N_i \right]$$

for all $i \in \{1, 2, ..., n\}$, *then the integral operator*

$$G_n(f_1, ..., f_n, g_1, ...g_n)(z) =$$

$$\left(\left(1 + \sum_{i=1}^{n} \alpha_i\right) \int_0^z \prod_{i=1}^{n} (f_i(t))^{\alpha_i} \left(g_i'(t)\right)^{\gamma_i} dt\right)^{\frac{1}{1+\sum_{i=1}^{n} \alpha_i}}$$

is in the class $\mathcal{S}$.

Setting $n = 1$ in Theorem 2.1 we have

Corollary 2.2. *Let the functions* $f, g \in \mathcal{A}$, *where* f *satisfies the condition*

$$\left|\frac{z^2(D^m f(z))'}{[D^m f(z)]^2} - 1\right| \le 1 \quad (z \in \mathbb{U}, m \in \mathbb{N}_0)\,,$$

let $M \ge 1$, $N \ge 1$ *and* α, γ, β *be complex numbers such that* $\beta = 1 + \alpha$ *and*

$$\mathrm{Re}\beta \ge \left[\, |\alpha|\,(2M + 1) + |\gamma|\,N\right] > 0.$$

If

$$\left|D^m f(z)\right| \le M \quad (z \in \mathbb{U})\,, \qquad \left|\frac{zg''(z)}{g'(z)}\right| \le N \quad (z \in \mathbb{U})$$

and

$$|c| \le 1 - \frac{1}{\mathrm{Re}\beta}\left[\, |\alpha|\,(2M + 1) + |\gamma|\,N\right]$$

then the integral operator

$$G_m(f, g)(z) = \left((1 + \alpha) \int_0^z (D^m f(t))^{\alpha} (g'(t))^{\gamma} dt\right)^{\frac{1}{1+\alpha}}$$

is in the class $\mathcal{S}$.

Setting $m = 0$ in Corollary 2.2 we have

Corollary 2.3. *Let the functions* $f, g \in \mathcal{A}$, *where* f *satisfies the condition (6), let* $M \ge 1$, $N \ge 1$ *and* α, γ, β *be complex numbers such that* $\beta = 1 + \alpha$ *and*

$$\mathrm{Re}\beta \ge \left[\, |\alpha|\,(2M + 1) + |\gamma|\,N\right] > 0.$$

If

$$|f(z)| \le M \quad (z \in \mathbb{U})\,, \qquad \left|\frac{zg''(z)}{g'(z)}\right| \le N \quad (z \in \mathbb{U})$$

and

$$|c| \le 1 - \frac{1}{\mathrm{Re}\beta}\left[\, |\alpha|\,(2M + 1) + |\gamma|\,N\right]$$

then the integral operator

$$G(z) = \left((1+\alpha)\int_0^z (f(t))^\alpha (g'(t))^\gamma \, dt\right)^{\frac{1}{1+\alpha}}$$

is in the class $\mathcal{S}$.

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FUNCTIONAL CONTRACTIONS IN LOCAL BRANCIARI METRIC SPACES

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Abstract A fixed point result is given for a class of functional contractions over local Branciari metric spaces. It extends some contributions in the area due to Fora et al [Mat. Vesnik, 61 (2009), 203-208].

Keywords: Symmetric, polyhedral inequality, local Branciari metric, convergent/Cauchy sequence, Matkowski function, contraction, periodic and fixed point.

2010 MSC: 47H10 (Primary), 54H25 (Secondary).

1. INTRODUCTION

Let X be a nonempty set. By a *symmetric* over X we shall mean any map $d : X \times X \to R_+$ with (cf. Hicks [11])

(a01) $d(x, y) = d(y, x), \ \forall x, y \in X$ (d is *symmetric*);

the couple (X, d) will be referred to as a *symmetric space*. Further, let $T : X \to X$ be a selfmap of X, and put Fix(T) = $\{z \in X; z = Tz\}$; any such point will be called *fixed* under T. According to Rus [24, Ch 2, Sect 2.2], we say that $x \in X$ is a *Picard point* (modulo (d, T)) if **1a)** $(T^n x; n \geq 0)$ is d-convergent, **1b)** each point of $\lim_n(T^n x)$ is in Fix(T). If this happens for each $x \in X$, then T is referred to as a *Picard operator* (modulo d); if (in addition) Fix(T) is a *singleton* ($x, y \in$ Fix$(T) \Longrightarrow x = y$), then T is called a *global Picard operator* (modulo d). [We refer to Section 2 for all unexplained notions]. Sufficient conditions for these properties to be valid require some additional conditions upon d; the usual ones are

(a02) $d(x, y) = 0$ iff $x = y$ (d is *reflexive sufficient*)

(a03) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ (d is *triangular*);

when both these hold, d is called a *(standard) metric* on X. In this (classical) setting, a basic result to the question we deal with is the 1922 one due to Banach [3]; it says that, whenever (X, d) is complete and (for some λ in $[0, 1[$)

(a04) $d(Tx, Ty) \leq \lambda d(x, y), \ \forall x, y \in X,$

then, T is a global Picard operator (modulo d). This result found various applications in operator equations theory; so, it was the subject of many extensions. A natural way of doing this is by considering "functional" contractive conditions like (a05) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \quad \forall x, y \in X;$

where $F : R_+^5 \to R_+$ is an appropriate function. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [23]; see also Turinici [28]. Another way of extension is that of conditions imposed upon d being modified. For example, in the class of symmetric spaces, a relevant paper concerning the contractive question is the 2005 one due to Zhu et al [29]. Here, we shall be interested in fixed point results established over *generalized* metric spaces, introduced as in Branciari [5]; where, the triangular property (a03) is to be substituted by the *tetrahedral* one:

(a06) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$,
whenever $x, y, u, v \in X$ are distinct to each other.

Some pioneering results in the area were given by Das [7], Miheţ [21], and Samet [25]; see also Azam and Arshad [2]. In parallel to such developments, certain technical problems involving these structures were considered. For example, Sarma et al [27] observed that Branciari's result may not hold, in view of the Hausdorff property for (X, d) being not deductible from (a06). This remark was followed by a series of results founded on this property being *ab initio* imposed; see, in this direction, Chen and Sun [6] or Lakzian and Samet [18]. However, in 2011, Kikina and Kikina [16] noticed that such a regularity condition is ultimately superfluous for the ambient space; so, the initial setting will suffice for these results being retainable. It is our aim in the present exposition to confirm this, within a class of "local" Branciari metric spaces. Further aspects will be delineated elsewhere.

2. PRELIMINARIES

Let $N = \{0, 1, ...\}$ denote the set of all natural numbers. For each $n \geq 1$ in N, let $N(n, >) := \{0, ..., n-1\}$ stand for the *initial interval* (in N) induced by n. Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*; also denoted as: $\text{card}(P) = \aleph_0$. In addition, given some $n \geq 1$, any set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; and we write this: $\text{card}(Q) = n$; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

(A) Let (X, d) be a symmetric space. Given $k \geq 1$, any ordered system $C = (x_1, ..., x_k)$ in X^k will be called a *k-chain* of X; the class of all these will be re-denoted as $\text{chain}(X; k)$. Given such an object, put $[C] = \{x_1, ..., x_k\}$. If $\text{card}([C]) = k$, then C will be referred to as a *regular k-chain* (in X); denote the class of all these as $\text{rchain}(X; k)$. In particular, any point $a \in X$ may be identified with a regular 1-chain of X. For any $C \in \text{chain}(X; k)$, where $k \geq 2$, denote

$$\Lambda(C) = d(x_1, x_2) + ... + d(x_{k-1}, x_k), \text{ whenever } C = (x_1, ..., x_k)$$

(the "length" of C). Given $h \geq 1$ and the h-chain $D = (y_1, ..., y_h)$ in X, let $(C; D)$ stand for the $(k+h)$-chain $E = (z_1, ..., z_{k+h})$ in X introduced as

$$z_i = x_i,\ 1 \le i \le k;\ \ z_{k+j} = y_j,\ 1 \le j \le h;$$

it will be referred to as the "product" between C and D. This operation may be extended to a finite family of such objects.

Having these precise, let us say that the symmetric d is a *local Branciari metric* when it is reflexive sufficient and has the property: for each effectively denumerable $M \subseteq X$, there exists $k = k(M) \ge 1$ such that

(b01) $d(x, y) \le \Lambda(x; C; y)$, for all $x, y \in M$, $x \ne y$, and all $C \in \mathrm{rchain}(M; k)$, with $(x; C; y) \in \mathrm{rchain}(M; 2 + k)$

(referred to as: the $(2 + k)$-*polyhedral inequality*). Note that, the triangular inequality (a03) and the tetrahedral inequality (a06) are particular cases of this one, corresponding to $k = 1$ and $k = 2$, respectively. On the other hand, (b01) is not reducible to (a03) or (a06); because, aside from $k > 2$ being allowed, the index in question depends on each effectively denumerable subset M of X.

Suppose that we introduced such an object. Define a d-convergence structure over X as follows. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), d-converges to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$; i.e.,

(b02) $\forall \varepsilon > 0, \exists i = i(\varepsilon)$: $n \ge i \Longrightarrow d(x_n, x) < \varepsilon$.

(This concept meets the standard requirements in Kasahara [14]; we do not give details). The set of all such points x will be denoted $\lim_n(x_n)$; when it is nonempty, (x_n) is called *d-convergent*. Note that, in this last case, $\lim_n(x_n)$ may be not a singleton, even if (a06) holds; cf. Samet [26]. Further, call the sequence (x_n), *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; i.e.,

(b03) $\forall \varepsilon > 0, \exists j = j(\varepsilon)$: $j \le m < n \Longrightarrow d(x_m, x_n) < \varepsilon$.

Clearly, a necessary condition for this is

$$d(x_m, x_{m+i}) \to 0 \text{ as } m \to \infty, \text{ for each } i > 0;$$

referred to as: (x_n) is *d-semi-Cauchy*; but the converse is not in general true. Note that, by the adopted setting, a d-convergent sequence need not be d-Cauchy, even if d is tetrahedral; see the quoted paper for details. Despite of this, (X, d) is called *complete*, if each d-Cauchy sequence is d-convergent.

(B) As already precise, the (nonempty) set of limit points for a convergent sequence is not in general a singleton. However, in the usual (metric) fixed point arguments, the convergence property of this sequence comes from the d-Cauchy property of the same. So, we may ask whether this supplementary condition upon (x_n) will suffice for such a property. Call (X, d), *Cauchy-separated* if, for each d-convergent d-Cauchy sequence (x_n) in X, $\lim_n(x_n)$ is a singleton.

Proposition 2.1. *Assume that d is a local Branciari metric (see above). Then, (X, d) is Cauchy-separated.*

Proof. Let (x_n) be a d-convergent d-Cauchy sequence. Assume by contradiction that $\lim_n(x_n)$ has at least two distinct points:

(b04) $\exists u, v \in X$ with $u \neq v$, such that $x_n \xrightarrow{d} u$, $x_n \xrightarrow{d} v$.

i) Denote $A = \{n \in N; x_n = u\}$, $B = \{n \in N; x_n = v\}$. We claim that both A and B are finite. In fact, if A is effectively denumerable, then $A = \{n(j); j \geq 0\}$, where $(n(j); j \geq 0)$ is strictly ascending (hence $n(j) \to \infty$ as $j \to \infty$) and $x_{n(j)} = u$, $\forall j \geq 0$. Since, on the other hand, $x_{n(j)} \to v$ as $j \to \infty$, we must have $d(u, v) = 0$; so that, $u = v$, contradiction. An identical reasoning is applicable when B is effectively denumerable; hence the claim. As a consequence, there exists $p \in N$, such that: $x_n \neq u$, $x_n \neq v$, for all $n \geq p$. Without loss, one may assume that $p = 0$; i.e.,

$$\{x_n; n \geq 0\} \cap \{u, v\} = \emptyset \ \ [x_n \neq u \text{ and } x_n \neq v, \text{ for all } n \geq 0]. \tag{1}$$

ii) Put $h(0) = 0$. We claim that the set $S_0 = \{n \in N; x_n = x_{h(0)}\}$ is finite. For, otherwise, it has the representation $S_0 = \{m(j); j \geq 0\}$, where $(m(j); j \geq 0)$ is strictly ascending (hence $m(j) \to \infty$ as $j \to \infty$) and $x_{m(j)} = x_0$, $\forall j \geq 0$. Combining with (b04) gives $x_0 = u$, $x_0 = v$; hence, $u = v$, contradiction. As a consequence of this, there exists $h(1) > h(0)$ with $x_{h(1)} \neq x_{h(0)}$. Further, by a very similar reasoning, $S_{0,1} = \{n \in N; x_n \in \{x_{h(0)}, x_{h(1)}\}\}$ is finite too; hence, there exists $h(2) > h(1)$ with $x_{h(2)} \notin \{x_{h(0)}, x_{h(1)}\}$; and so on. By induction, we get a subsequence $(y_n := x_{h(n)}; n \geq 0)$ of (x_n) with

$$y_i \neq y_j, \text{ for } i \neq j;\ \ y_n \xrightarrow{d} u, y_n \xrightarrow{d} v \text{ as } n \to \infty. \tag{2}$$

The subset $M = \{y_n; n \geq 0\} \cup \{u, v\}$ is effectively denumerable; let $k = k(M) \geq 1$ stand for the natural number assured by the local Branciari metric property of d. From the $(2 + k)$-polyhedral inequality (b01) we have, for each $n \geq 0$,

$$d(u, v) \leq d(u, y_{n+1}) + ... + d(y_{n+k}, v).$$

(The possibility of writing this is assured by (1) and (2) above). On the other hand, (y_n) is a d-Cauchy sequence; because, so is (x_n); hence $d(y_m, y_{m+1}) \to 0$ as $m \to \infty$. Passing to limit in the above relation gives $d(u, v) = 0$; whence, $u = v$, contradiction. So, (b04) is not acceptable; and this concludes the argument. ∎

(B) Let $\mathcal{F}(R_+)$ stand for the class of all functions $\varphi : R_+ \to R_+$. Denote

(b05) $\mathcal{F}_r(R_+) = \{\varphi \in \mathcal{F}(R_+); \varphi(0) = 0;\ \varphi(t) < t,\ \forall t > 0\}$;

each $\varphi \in \mathcal{F}_r(R_+)$ will be referred to as *regressive*. Note that, for any such function,

$$\forall u, v \in R_+ :\ v \leq \varphi(\max\{u, v\}) \Longrightarrow v \leq \varphi(u). \tag{3}$$

Call $\varphi \in \mathcal{F}_r(R_+)$, *strongly regressive*, provided

(b06) $\forall \gamma > 0, \exists \beta \in]0, \gamma[, (\forall t)\colon \gamma \le t < \gamma + \beta \Longrightarrow \varphi(t) \le \gamma$;
or, equivalently: $0 \le t < \gamma + \beta \Longrightarrow \varphi(t) \le \gamma$.

Some basic properties of such functions are given below.

Proposition 2.2. *Let $\varphi \in \mathcal{F}_r(R_+)$ be strongly regressive. Then,*

i) *for each sequence $(r_n; n \ge 0)$ in R_+ with $r_{n+1} \le \varphi(r_n)$, $\forall n$, we have $r_n \to 0$ [we then say that φ is iteratively asymptotic]*

ii) *in addition, for each sequence $(s_n; n \ge 0)$ in R_+ with $s_{n+1} \le \varphi(\max\{s_n, r_n\})$, $\forall n$ we have $s_n \to 0$.*

Proof. i) Let $(r_n; n \ge 0)$ be as in the premise of this assertion. As φ is regressive, we have $r_{n+1} \le r_n$, $\forall n$. The sequence $(r_n; n \ge 0)$ is therefore descending; hence $\gamma := \lim_n(r_n)$ exists in R_+. Assume by contradiction that $\gamma > 0$; and let $\beta \in]0, \gamma[$ be the number indicated by the strong regressiveness of φ. As $r_n \ge \gamma > 0$, $\forall n$ (and φ=regressive), one gets $r_{n+1} < r_n$, $\forall n$; hence, $r_n > \gamma$, $\forall n$. Further, as $r_n \to \gamma$, there exists some rank $n(\beta)$ in such a way that (combining with the above) $n \ge n(\beta) \Longrightarrow \gamma < r_n < \gamma + \beta$. The strong regressiveness of φ then gives (for the same ranks, n) $\gamma < r_{n+1} \le \varphi(r_n) \le \gamma$; contradiction. Consequently, $\gamma = 0$; and we are done.

ii) Let $(r_n; n \ge 0)$ and $(s_n; n \ge 0)$ be as in the premise of these assertions. Denote for simplicity $(t_n := \max\{s_n, r_n\}; n \ge 0)$. For each n, we have $r_{n+1} \le r_n \le t_n$ and (as φ is regressive) $s_{n+1} \le \varphi(t_n) \le t_n$; hence $[t_{n+1} \le t_n, \forall n]$. The sequence $(t_n; n \ge 0)$ is therefore descending; wherefrom, $t := \lim_n(t_n)$ exists in R_+ and $t_n \ge t$, $\forall n$. Assume by contradiction that $t > 0$. As $r_n \to 0$, there must be some rank $n(t)$ such that $n \ge n(t) \Longrightarrow r_n < t$. Combining with the above, one gets $t_n \ge t > r_n$, for all $n \ge n(t)$; whence $t_n = s_n$, for all $n \ge n(t)$. But then, the choice of $(s_n; n \ge 0)$ gives $s_{n+1} \le \varphi(s_n)$, for all $n \ge n(t)$. This, along with the first part of the proof, gives $s_n \to 0$; hence $t_n \to 0$; contradiction. Consequently, $t = 0$; and, from this, the conclusion follows. ∎

Now, let us give two basic examples of such functions.

B1) Suppose that $\varphi \in \mathcal{F}_r(R_+)$ is a *Boyd-Wong* function [4]; i.e.

(b07) $\limsup_{t \to s+} \varphi(t) < s$, for all $s > 0$.

Then, φ is strongly regressive. The verification is immediate, by definition; so, we do not give details.

B2) Suppose that $\varphi \in \mathcal{F}_r(R_+)$ is a *Matkowski function* [20]; i.e.

(b08) φ is increasing and $[\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0]$.

(Here, for each $n \ge 0$, φ^n stands for the n-th iterate of φ). Then, φ is strongly regressive. The verification of this assertion is to be found to Jachymski [12]; however, for completeness reasons, we shall provide it, with some modifications. Assume by contradiction that φ is not strongly regressive; that is (for some $\gamma > 0$)

$\forall \beta \in]0, \gamma[$, $\exists t \in [\gamma, \gamma + \beta[$: $\varphi(t) > \gamma$ (hence, $\gamma < t < \gamma + \beta$).

As φ=increasing, this yields $\varphi(t) > \gamma$, $\forall t > \gamma$. By induction, we get $\varphi^n(t) > \gamma$, for all n, and all $t > \gamma$. Fixing some $t > \gamma$, we have (passing to limit as $n \to \infty$) $0 \geq \gamma$, contradiction; hence the claim.

3. MAIN RESULT

Let X be a nonempty set; and $d(.,.)$ be a local Branciari metric over it, with

(c01) (X, d) is complete (each d-Cauchy sequence is d-convergent).

Note that, by Proposition 2.1, for each d-Cauchy sequence (x_n) in X, $\lim_n(x_n)$ is a (nonempty) singleton, $\{z\}$; as usually, we write $\lim_n(x_n) = \{z\}$ as $\lim_n(x_n) = z$.

Let $T : X \to X$ be a selfmap of X. We say that $x \in X$ is a *Picard point* (modulo (d, T)) if **3a)** $(T^n x; n \geq 0)$ is d-Cauchy (hence d-convergent), **ii)** $\lim_n(T^n x)$ is in $\text{Fix}(T)$. If this happens for each $x \in X$, then T is referred to as a *Picard operator* (modulo d); if (in addition) $\text{Fix}(T)$ is a singleton, then T is called a *globally Picard operator* (modulo d).

Now, concrete circumstances guaranteeing such properties involve functional contractive (modulo d) conditions upon T. Precisely, denote for $x, y \in X$:

(c02) $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

It is easy to see that

$$M(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x)\}, \quad \forall x, y \in X. \tag{4}$$

Given $\varphi \in \mathcal{F}_r(R_+)$, we say that T is $(d, M; \varphi)$*-contractive* if

(c03) $d(Tx, Ty) \leq \varphi(M(x, y))$, $\forall x, y \in X$.

The main result of this note is

Theorem 3.1. *Suppose that T is $(d, M; \varphi)$-contractive, where $\varphi \in \mathcal{F}_r(R_+)$ is strongly regressive. Then, T is a globally Picard operator (modulo d).*

Proof. First, we check the singleton property. Let $z_1, z_2 \in \text{Fix}(T)$ be arbitrary fixed. By this very choice,

$$M(z_1, z_2) = \max\{d(z_1, z_2), 0, 0\} = d(z_1, z_2).$$

Combining with the contractive condition yields

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \varphi(d(z_1, z_2));$$

wherefrom $d(z_1, z_2) = 0$; hence $z_1 = z_2$; so that, $\text{Fix}(T)$ is (at most) a singleton. It remains now to establish the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \geq 0$. There are several steps to be passed.

I) If $x_n = x_{n+1}$ for some $n \geq 0$, we are done. So, it remains to discuss the remaining situation; i.e. (by the reflexive sufficiency of d)

(c04) $\rho_n := d(x_n, x_{n+1}) > 0$, for all $n \geq 0$.

By the contractive property and (4), $\rho_{n+1} \leq \varphi(\max\{\rho_n, \rho_{n+1}\})$, for all $n \geq 0$; so that (taking (3) into account)

$$\rho_{n+1} \leq \varphi(\rho_n), \quad \forall n \geq 0. \tag{5}$$

Combining with (c04) one gets that $(\rho_n; n \geq 0)$ is strictly descending; moreover, by Proposition 2.2, $\rho_n \to 0$ as $n \to \infty$.

II) Fix $i \geq 1$, and put $(\sigma_n^i := d(x_n, x_{n+i}); n \geq 0)$. Again by the contractive condition, we get the evaluation

$$\sigma_{n+1}^i = d(Tx_n, Tx_{n+i}) \leq \varphi(M(x_n, x_{n+i})) = \varphi(\max\{\sigma_n^i, \rho_n, \rho_{n+i}\}), \ \forall n \geq 0;$$

wherefrom, by (5)

$$\sigma_{n+1}^i \leq \varphi(\max\{\sigma_n^i, \rho_n\}), \quad \forall n \geq 0. \tag{6}$$

This yields (again via Proposition 2.2) $\sigma_n^i \to 0$, for each $i \geq 1$; that is,

$$d(x_n, x_{n+i}) \to 0 \text{ as } n \to \infty, \text{ for each } i \geq 1; \tag{7}$$

or, in other words: (x_n) is d-semi-Cauchy.

III) Suppose that

(c05) there exists $i, j \in N$ such that $i < j$, $x_i = x_j$.

Denoting $p = j - i$, we thus have $p > 0$ and $x_i = x_{i+p}$; so that (by the very definition of our iterative sequence)

$$x_i = x_{i+np}, \ x_{i+1} = x_{i+np+1}, \ \text{ for all } n \geq 0.$$

By the introduced notations this yields (via (c04) and (7))

$$0 < \rho_i = \rho_{i+np} \to 0 \text{ as } n \to \infty;$$

contradiction. Hence, (c05) cannot hold; wherefrom, we must have

$$\text{for all } i, j \in N\colon \ i \neq j \text{ implies } x_i \neq x_j. \tag{8}$$

IV) As a consequence of this fact, the map $n \mapsto x_n$ is injective; so that, $Y := \{x_n; n \geq 0\}$ is effectively denumerable. Let $k = k(Y) \geq 1$ be the natural number attached to it, by the local Branciari property of d. Also, let $\gamma > 0$ be arbitrary fixed; and $\beta \in]0, \gamma[$ be given by the strong regressivity of φ. By the d-semi-Cauchy property (7), there exists $j(\beta) \in N$ such that

$$d(x_n, x_{n+i}) < \beta/2k \ (\leq \beta/2 < \gamma + \beta/2), \quad \forall n \geq j(\beta), \ \forall i \in \{1, ..., k+1\}. \tag{9}$$

We now claim that

$$(\forall q \geq 1): \ d(x_n, x_{n+q}) < \gamma + \beta/2, \ \ \forall n \geq j(\beta); \tag{10}$$

and, from this, the d-Cauchy property for $(x_n; n \geq 0)$ follows. The case of $q \in \{1, ..., k+1\}$ is clear, via (9). Assume that (10) holds, for all $q \leq p$ (where $p \geq k+1$); we show that it holds as well for $q = p+1$. So, let $n \geq j(\beta)$ be arbitrary fixed. By the inductive hypothesis and (9),

$$d(x_{n+k}, x_{n+p}) < \gamma + \beta/2 < \gamma + \beta$$
$$d(x_{n+k}, x_{n+k+1}) < \beta/2k < \beta < \gamma + \beta$$
$$d(x_{n+p}, x_{n+p+1}) < \beta/2k < \beta < \gamma + \beta;$$

whence, by definition,

$$M(x_{n+k}, x_{n+p}) < \gamma + \beta.$$

This, by the contractive condition and (b06), gives

$$d(x_{n+k+1}, x_{n+p+1}) \leq \varphi(M(x_{n+k}, x_{n+p})) \leq \gamma.$$

Combining with the $(2+k)$-polyhedral inequality (for $C = (x_{n+2}, ..., x_{n+k+1})$),

$$d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+p+1})$$
$$< k\beta/2k + \gamma \leq \beta/2 + \gamma;$$

and the assertion follows. As (X, d) is complete, we have

$$x_n \xrightarrow{d} x \text{ as } n \to \infty, \text{ for some } z \in X; \tag{11}$$

moreover, by Proposition 2.1, z is uniquely determined by this relation. We claim that this is our desired point. Assume by contradiction that $z \neq Tz$; or, equivalently, $\rho := d(z, Tz) > 0$.

V) Denote $A = \{n \in N; x_n = z\}$, $B = \{n \in N; x_n = Tz\}$. If A is effectively denumerable, we have $A = \{m(j); j \geq 0\}$, where $(m(j); j \geq 0)$ is strictly ascending (hence $m(j) \to \infty$). As $x_{m(j)} = z$, $\forall j \geq 0$, we have $x_{m(j)+1} = Tz$, $\forall j \geq 0$. Combining with $x_{m(j)+1} \xrightarrow{d} z$ as $j \to \infty$, we must have $d(z, Tz) = 0$; hence $z = Tz$, contradiction. On the other hand, if B is effectively denumerable, we have $B = \{n(j); j \geq 0\}$, where $(n(j); j \geq 0)$ is strictly ascending (hence $n(j) \to \infty$). As $x_{n(j)} = Tz$, $\forall j \geq 0$, one gets (via $x_{n(j)} \xrightarrow{d} z$ as $j \to \infty$) $d(z, Tz) = 0$; whence $z = Tz$, again a contradiction. It remains to discuss the case of both A and B being finite; i.e.,

(c06) there exists $h \geq 0$ such that: $\{x_n; n \geq h\} \cap \{z, Tz\} = \emptyset$.

The subset $Y := \{x_n; n \geq h\} \cup \{z, Tz\}$ is therefore effectively denumerable. Let $k = k(Y) \geq 1$ be the natural number attached to it, by the local Branciari property of

d. We have, for each $n \geq k$ (by the $(2+k)$-polyhedral inequality applied to $C := (x_{n+2}, ..., x_{n+k+1})$)

$$\rho \leq d(z, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, Tz) \tag{12}$$

By (7) and (11), there exists $j(\rho) \geq h$ in such a way that

$$n \geq j(\rho) \Longrightarrow d(x_n, z), d(x_n, x_{n+1}) < \rho/2.$$

As a consequence, we must have

$$M(x_{n+k}, z) = \rho, \ \ \forall n \geq j(\rho).$$

so that, by the contractive condition,

$$d(x_{n+k+1}, Tz) \leq \varphi(\rho), \forall n \geq j(\rho).$$

Replacing in (12), we get an evaluation like

$$\rho \leq d(z, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + \varphi(\rho), \ \ \forall n \geq j(\rho).$$

Passing to limit as n tends to infinity gives $\rho \leq \varphi(\rho)$; wherefrom (as φ is regressive) $\rho = 0$; contradiction. Hence, $z = Tz$; and the proof is complete. ∎

In particular, when the regressive function φ is a Boyd-Wong one, our main result covers the one due to Das and Dey [8]; note that, by the developments in Jachymski [13], it includes as well the related statements in Di Bari and Vetro [9]. On the other hand, when $d(.,.)$ is a standard metric, Theorem 3.1 reduces to the statement in Leader [19]. Further aspects may be found in Kikina et al [17]; see also Khojasteh et al [15].

4. FURTHER ASPECTS

A direct inspection of the proof above shows that conclusion of Theorem 3.1 is retainable even if one works with orbital completeness of the ambient space. Some conventions are in order. Let X be a nonempty set; and $d(.,.)$ be a reflexive sufficient symmetric over it; supposed to be a local Branciari metric. Further, take a selfmap T of X. Call the sequence $(y_n; n \geq 0)$ in X, *T-orbital* when $y_n = T^n x$, $n \geq 0$, for some $x \in X$. In this case, let us say that (X, d) is *T-orbital complete* when each T-orbital d-Cauchy sequence is d-convergent.

The following extension of Theorem 3.1 is available. Let the general conditions above be fulfilled; as well as (in place of (c01))

(d01) (X, d) is T-orbital complete.

Theorem 4.1. *Suppose that T is $(d, M; \varphi)$-contractive, where $\varphi \in \mathcal{F}_r(R_+)$ is strongly regressive. Then, T is a global Picard operator (modulo d).*

The proof mimics the one of Theorem 3.1; so, we omit it.
Call the regressive function $\varphi \in \mathcal{F}_r(R_+)$, *admissible* provided

(d02) φ is increasing, usc and $\sum_n \varphi^n(t) < \infty$, $\forall t > 0$.

Clearly, φ is a Matkowski function; hence, in particular, a strongly regressive one. This, in the particular case of d fulfilling the tetrahedral inequality, tells us that the main result in Fora et al [10] is a particular case of Theorem 4.1 above. In addition, we note that the usc condition posed by the authors may be removed. Note that, the introduced framework may be also used to get an extension of the contributions due to Akram and Siddiqui [1]; see also Moradi and Alimohammadi [22]. These will be discussed elsewhere.

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CALCULATION OF ADSORPTION ISOTHERMS OF NAF FROM AQUEOUS SOLUTIONS BY THE SAMPLES OF ALUMINUM OXIDE

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Abstract In the paper the constants of adsorption equilibrium have been applied for calculation of adsorption isotherms of fluoride ions from model aqueous solutions with initial concentrations from $5 \cdot 10^{-4}$ to 0,50mmol/l. As fluorine adsorbents the aluminum oxihydroxides obtained by calcination at 200 and 800°C (A200 and A800) the electrochemical dimensional machining products of aluminum alloy have been used.

Comparison is given of theoretically calculated and experimentally obtained adsorption isotherms in the system $Al_2O_3 - H_2O - NaF$. It has been shown a satisfactory correlation of experimental and theoretically calculated values of isotherms of fluorine adsorption on studied samples.

Keywords: adsorption, adsorption isotherms, constants of equilibrium, fluorine, aluminum oxihydroxides, activity coefficient.

2010 MSC: 80A50.

1. INTRODUCTION

The isotherm of adsorption equilibrium calculation is very important from theoretical viewpoint and for practical application of an adsorption method in water purification technology. Knowing the isotherm form and equilibrium constant, one can estimate the dose of adsorbent and the method of its application – in a static or dynamic mode, as well as to conclude, what is the mechanism of capture of adsorbate on the adsorbent surface. To calculate the isotherm, we first should determine the constant of adsorption equilibrium.

The constant of adsorption equilibrium (K_a) is widely used for calculation of many thermodynamic functions of an adsorption process – standard diminution of mole adsorption energy ($-\Delta F^0$), standard adsorption enthalpy (ΔH^0) and standard adsorption entropy (ΔS^0) [1-4].

There are a lot of references for these calculations presented in the literature with well-known models of adsorption application: Freundlich, Langmuir, Dubinin-Radushkevich, Riedlich-Peterson adsorption models and others [5-10].

However, not always the experimental data can be linearized with these equations. Therefore, the authors have based the calculation of the constant on the ratio of the concentration of adsorbed substance on the solid surface to its concentration in the equilibrium solution [11-13].

2. RESULTS AND DISCUSSION

The method for calculation of the apparent adsorption equilibrium constant, which has been used at adsorption of non-polar low-molecular organic molecules from aqueous solutions on activated carbon [14], can be also applied for description of adsorption equilibrium of other systems, in particular at fluoride ions adsorption on pore aluminum oxide sorbents [15].

We deal in our investigation with a binary solution where water is a solvent (component 1) and NaF molecules – are a solute (component 2). For the constant calculation the following formula has been used [14,15]:

$$K_{a2} = \frac{\Theta_2}{\left[1 - \Theta_2\left(\frac{v_2^0 - v_1^0}{v_2^0}\right)\right] \cdot C_2 \cdot v_2^0} \cdot \frac{\gamma_{a2}}{\gamma_2} \tag{1}$$

In equation (1) Θ_2 – the NaF molecules surface covering degree which is equal to $\frac{a_2 \cdot v_2^\circ}{v_m}$ where a_2 – the adsorption value, mmol/g, v_1° and v_2° – mole volumes of water and adsorbed NaF equal 0.0180 and 0.0217cm^3/mmol, respectively, v_m – the sorbent maximum adsorption pore volume cm^3/g, C_2 – NaF solution equilibrium concentration, mmol/l, γ_2 and γ_{a2} – the NaF molecules activity coefficients in the equilibrium bulk solution and in the adsorption layer, respectively. Substituting numerical values of mole volumes of the water and sodium fluoride into equation (1) and expressing NaF bulk concentration in mmol/cm^3 we obtain:

$$K_{a2} = \frac{4.6 \cdot 10^4 \cdot \Theta_2 \cdot \gamma_{a2}}{(1 - 0.172 \cdot \Theta_2) \cdot C_2 \cdot \gamma_2} \tag{2}$$

At the constant K_{a2} calculation it has been accepted that the relation of activity coefficients γ_{a2}/γ_2 in rather dilute solutions is equal to 1. In that case if the experimental adsorption data are presented in coordinates lg $\frac{\cdot \Theta_2}{(1-0.172 \cdot \Theta_2) \cdot C_2}$ versus Θ_2 one can get linear extrapolation dependence and the intercept is numerically equal to adsorption constant logarithm (lgK_{a2}).

The adsorption equilibrium constant for the oxyhydroxide aluminum sample A200 has been calculated in such a way [14].

However for more concentrated NaF solutions (more than 0,01mol/l) it is necessary to take into account the activity coefficients both in the adsorption layer $\gamma_{\acute{f}2}$ and in the equilibrium bulk solution γ_2 and to use activities instead of concentrations.

Knowing the adsorption constant K_{a2} value we can make estimation of adsorption isotherms, i.e. to find the adsorption sodium fluoride dependence on its equilibrium concentration in solution (C_2) it is enough to get a functional dependence of $\gamma_{\acute{f}2}$ on surface covering degree Θ_2.

The adsorption isotherms calculation are carried out as follows: fixed values of Θ_2 (or that is the same – adsorption values, a_2) are defined and for these values the equilibrium NaF concentrations in bulk solution (C_2) (or its activities) are computed from equation (2). Calculated adsorption values are determined for any Θ_2 from the expression $\acute{r}_{2calc.} = \Theta_2\ a_m$, where a_m – the maximum adsorption, mmol/g which is calculated from the relation:

$$v_a = a_m \cdot v_2^0 \tag{3}$$

Then the isotherm is constructed in coordinates a_2 vs C_2 (or $a_2$vs $C_2 \cdot \gamma_2$) and compared with experimental determined adsorption values a_{2exp} for corresponding equilibrium NaF concentrations or activities in solution.

It is known [14] that the coefficient γ_{a2} includes 3 items: γ_{c2} , γ_{a2-2}, and γ_{a2-H2O}, which characterize different types of interactions in the adsorption layer and can be presented as their sum:

$$RT ln\gamma_{a2} = RT ln\gamma_{C2} + RT ln\gamma_{a2-2} + RT ln\gamma_{a2-H2O} \tag{4}$$

The term γ_{c2} reflects the influence of NaF concentration growth in equilibrium solution on its content in the adsorption layer of the sorbent. The second term γ_{a2-2} is responsible for interaction between the adsorbed substance molecules and the term γ_{a2-H2O} notes the degree of interaction of adsorbed substance molecules with water in the adsorption layer.

Taking into account that in the system NaF-H_2O there is no association of NaF molecules (i.e there is no interaction between NaF molecules in adsorption layer) we can accept that $\gamma_{a2-2} = 1$ and $\lg\gamma_{a2-2} = 0$. On the other hand it is known [16] that sodium fluoride do not hydrolyze in water and therefore we can suppose the coefficient $\gamma_{a2-H20} = 1$ and hence, $\lg\gamma_{a2-H2O} = 0$ as well. Thus, the activity coefficient γ_{a2} will mainly be defined by γ_{C2} that allows the term γ_{a2} to be substituted for γ_{C2} in equation (2) and makes concentration (activity) computation from the following expression:

$$C_2 \cdot \gamma_2 = \frac{4.6 \cdot 10^4 \cdot \Theta_2 \cdot \gamma_{C2}}{(1 - 0.172 \cdot \Theta_2) \cdot K_{a2}} \tag{5}$$

The maximum γ_{c2} value takes place at $\Theta_2 = 1$ when the total sorbent surface is occupied with NaF molecules (H_2O molecules are absent). Since in that case the adsorption layer is at equilibrium with saturated NaF solution, $\lg\gamma_{c2}$ will be determined by adsorption equilibrium constant K_{a2} (at $\Theta_2 = 1$) and NaF solubility in water at corresponding temperature C_{sNaF} (C_{sNaF} – saturated NaF solution concentration i.e. maximum solubility of NaF salt in water, mol/kg of water).

$$\lg\gamma_{C2_{\Theta_2=1}} = \lg\frac{K_{a2} \cdot C_{sNaF}}{C_{sNaf} + C_{H_2O}} = \lg\frac{K_{a2}}{56,5}, \tag{6}$$

since C_{sNaF} = 1 mol/kg [17] and C_{H2O} = 55.5mol/kg.

As $\lg\gamma_{C2}$ changes from 0 at $\Theta_2 = 0$ to $\lg\gamma_{c2} = \lg\frac{K_{a2}}{56,5}$ at $\Theta_2 = 1$, γ_{C2} for any Θ_2 value can be estimated from interpolation dependence of γ_{C2} on Θ_2:

$$\gamma_{C2} = \left(\frac{K_{a2}}{56,5}\right)^{\Theta_2}, \tag{7}$$

or

$$lg\gamma_{c2} = \Theta_2 \cdot \lg\gamma_{C2_{\Theta_2=1}}. \tag{8}$$

As to NaF activity coefficient in equilibrium bulk solution γ_2, it can be calculated from a well-known equation [18, 19]:

$$\ln\gamma_{2\pm} = -0,5066 \cdot z^2 \cdot \left(\frac{\sqrt{I}}{1+\sqrt{I}} - 0,2I\right), \tag{9}$$

where I – ionic strength of the solution which for 1-1 electrolyte NaF is equal to molar solution concentration, mol/kg, z – atomic charge. The calculated data are shown in Table 1.

In Table 2 there are shown the calculated NaF adsorption isotherms on A200 sample with surface area 358m^2/g and sorption pore volume 0,457cm^3/g [20]. The equilibrium NaF solution concentration was calculated from equation (5) under the assumption that $\gamma_{a2}/\gamma_2 \neq 1$.

K_{a2} = 3311, V_2^0 = 0.0217cm^3/mmol, S_{sp} = 358m^2/g, a_m calc. = 36.0mmol NaF/g, a_2 calc. = $\theta_2 \cdot 36.0$ mmol NaF/g, K_{a2} = 358 at $\theta_2 = 1$.

The NaF adsorption isotherms on the sample of Al oxide A800 (surface area is 145,6m^2/g and sorption pore volume – 0,221cm^3/g) calculated from equation (5) taking into account the NaF activity coefficients in adsorption layer and in equilibrium bulk solution (Table 1) are shown in Table 3 and Fig. 1.

K_{a2} = 5937, V_2^0 = 0.0217cm^3/mmol, $S_{sp.}$ = 145.6m^2/g, v_m = 0.221cm^3/g, a_m calc. = v_m/v_2^0 = 10.18mmol NaF/g, K_{a2} = 367 at $\theta_2 = 1$, a_2 calc. = $\theta_2 \cdot 10.18$ mmol/g.

As it is seen from the listed data in both cases there is a good agreement of calculated from equation (5) values of NaF adsorption – a_2 calc. with experimentally determined ones – a_2 exp. Discrepancies of calculated and experimental adsorption data do not exceed 12% for A200 and 8.5% for A800 oxides.

3. CONCLUSION

Experimental and theoretically calculated values of isotherms of fluorine adsorption on studied samples are in good agreement.

For adsorption equilibrium isotherm calculation in the system aluminum oxide – aqueous NaF solution is quite enough data of adsorption equilibrium constants and NaF solubility data.

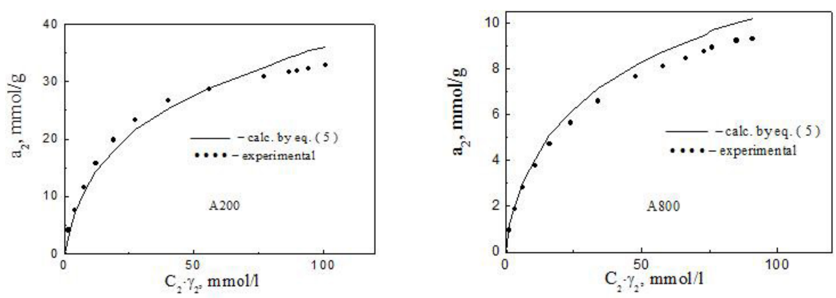
Table 1 Activity coefficients of NaF molecules in the equilibrium bulk solution in the system NaF – H_2O – Al_2O_3

A200			A800		
C_2, mmol/l	γ_2	NaF activity, mmol/l	C_2, mmol/l	γ_2	NaF activity, mmol/l
1.7	0.980	1.7	1.4	0.982	1.4
4.2	0.970	4.1	3.5	0.973	3.4
7.8	0.961	7.5	6.6	0.964	6.4
12.8	0.952	12.2	11.5	0.954	11.0
20.3	0.942	19.1	17.1	0.946	16.2
29.4	0.933	27.4	25.6	0.936	24.0
43.4	0.923	40.1	36.8	0.927	34.1
61.4	0.913	56.1	52.2	0.917	47.9
85.5	0.902	77.2	63.7	0.911	58.0
96.7	0.899	86.9	73.2	0.907	66.7
100.1	0.897	89.8	80.7	0.904	73.0
105.2	0.896	94.3	84.2	0.903	76.0
113.0	0.894	101.1	94.5	0.899	85.0
-	-	-	101.4	0.897	91.0

For NaF solutions concentrations more than 0.01mol/L at adsorption equilibrium constants calculation one should take into account the activity coefficients of NaF molecules in the adsorption layer and in the bulk equilibrium solution.

Table 2 Adsorption isotherms of NaF from aqueous solutions by the sample A200 at 20^0C

θ_2	$\frac{\Theta_2}{(1-0.172\cdot\Theta_2)}$	$\lg\gamma_{a2}$	γ_{a2}	$C_2 \cdot \gamma_2$, calc. by eq. (5), mmol/l	a_2, calc., mmol/g	a_2, experim., mmol/g	Relative deviation, Δ, %
0.1	0.101	0.08	1.17	1.7	3.6	4.1	12.2
0.2	0.207	0.15	1.41	4.1	7.2	7.6	5.3
0.3	0.316	0.23	1.70	7.5	10.8	11.6	6.7
0.4	0.429	0.31	2.04	12.2	14.4	15.7	8.3
0.5	0.547	0.40	2.51	19.1	18.0	19.8	9.1
0.6	0.669	0.47	2.95	27.4	21.6	23.2	6.7
0.7	0.796	0.56	3.63	40.1	25.2	26.6	5.2
0.8	0.926	0.64	4.36	56.1	28.8	28.6	0.7
0.9	1.060	0.72	5.24	77.2	32.4	30.8	5.2
0.95	1.140	0.74	5.49	86.9	34.2	31.7	7.9
0.96	1.150	0.75	5.62	89.8	34.7	31.8	9.1
0.98	1.180	0.76	5.75	94.3	35.5	32.2	10.2
1.0	1.208	0.78	6.02	101.0	36.0	32.8	9.8



(a) (b)

Fig. 1. Calculated and experimental adsorption isotherms of NaF on oxyhydroxides aluminum samples: (a) – A200, (b) – A800

Table 3 Adsorption isotherms of NaF from aqueous solutions by the sample A800 at 20^0C

θ_2	$\frac{\Theta_2}{(1-0.172\cdot\Theta_2)}$	$\lg\gamma_{c2}$	γ_{c2}	$C_2 \cdot \gamma_2$, calc. by eq. (5), mmol/l	a_2, calc., mmol/g	a_2, experim., mmol/g	Relative deviation, Δ, %
0.1	0.101	0.081	1.20	1.4	1.2	0.94	8.5
0.2	0.207	0.162	1.45	3.4	2.04	1.88	8.5
0.3	0.316	0.243	1.75	6.4	3.05	2.82	8.2
0.4	0.429	0.324	2.11	11.0	4.07	3.76	8.2
0.5	0.547	0.405	2.54	16.2	5.09	4.70	8.3
0.6	0.669	0.486	3.06	24.0	6.10	5.64	8.2
0.7	0.796	0.567	3.69	34.1	7.13	6.58	8.4
0.8	0.926	0.648	4.45	47.9	8.14	7.66	6.3
0.86	1.00	0.697	4.98	58.0	8.75	8.10	8.0
0.9	1.065	0.729	5.36	66.4	9.16	8.46	8.3
0.93	1.107	0.753	5.66	73.0	9.46	8.74	8.2
0.95	1.136	0.769	5.87	76.0	9.67	8.93	8.3
0.98	1.178	0.794	6.22	85.0	9.97	9.21	8.2
1.00	1.207	0.810	6.46	91.0	10.18	9.3	9.5

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