ROMAI J., v.16, no.2(2020), 37–49

A SEQUENTIAL RANDOM PROBLEM OF AIRY TYPE SOLVED BY THE LOWER AND UPPER METHOD

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Abstract In this work, we investigate a new problem of random differential equations by means of mean square calculus. The introduced problem allows us to obtain the classical Airy random differential equation as a special case. The investigation of the existence and uniqueness of solution via the upper and lowed method is considered. Some illustrative examples are discussed.

Keywords: Airy equation, mean square Caputo derivative, Existence and uniqueness, Ulam stabilty.

2010 MSC: 30C45, 39B72, 39B85.

1. INTRODUCTION

Fractional differential equations theory are of great importance for researchers since it can allow them to describe the mathematical modeling in many fields. For some important details and applications, we invite the reader to see [2, 7, 12, 13] and the references therein. Moreover, the random and stochastic differential equations of fractional order have emerged as a very significant subject, and very quickly, this random theory has became of great attraction to many scientists; it has been developed in both theoretical directions and in applications, see for more details the research works [8, 9, 11, 16]. In this sense, Hafiz et al. [4, 5] have constructed the concept of stochastic fractional calculus in the mean square sense. In [10], the authors have proved the existence of solution of random fractional differential equation with nonlocal condition. Also, in [15] it has been proved the existence and uniqueness of solution for random fractional differential equations with impulses via Banach fixed point technique and Schauder fixed point technique. Recently, Jarad et al. [6] have studied a class random implicit fractional differential equations involving a generalized Hilfer derivative. By using Krasnoselskii and Schaefer

theorems also using Banach contraction, the authors have proved the existence, uniqueness and Ulam stability of solutions of their random class.
In the present paper, we shall be concerned with a new problem of Airy type. Before presenting our results, we recall that the Airy differential equation, and its related functions, have shown many applications in several fields of fluid mechanics, elasticity, and also in quantum physics . The equation has the following form [14]:

$$Y'' - tY = 0, \quad t \in \mathbb{R}$$

To present to the reader other research papers that have motivated the present work, we recall the paper [3] where the authors have been concerned with the random problem that generalizes the classical Airy-type differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0+} Y(t) - Bt^{\beta} Y(t) = 0, \quad t > 0 \\ \quad n-1 < \alpha \leq n, \quad \beta > 0, \\ Y^{(j)}(0) = A_j, \quad j = 0, 1, \cdots, n-1 \end{cases},$$

under the conditions that ${}^{c}\mathcal{D}^{\alpha}_{0+} Y(t)$ is the mean square random Caputo fractional derivative of order α of the stochastic process $Y(t)$.

In the present research work, we study the following random fractional differential problem:

$$\mathcal{D}^{\alpha}_{0+}\left(\mathcal{D}^{\alpha}_{0+} Y(t)\right) = aAf(t, Y(t)) + bBg(t, \mathcal{D}^{\gamma}_{0+} Y(t), \mathcal{D}^{\gamma}_{0+}\left(\mathcal{D}^{\gamma}_{0+} Y(t)\right)) + h(t, \mathcal{I}^{\rho}_{0+} Y(t)), \tag{1}$$

$$t \in J = [0, T]$$

with the initial conditions

$$Y(0) = Y_0, \text{ and } Y(T) = \sum_{i=\overline{1,n}} \lambda_i Y(\zeta_i), \; 0 < \zeta_i < T \tag{2}$$

where $\mathcal{D}^{\alpha}_{0+}$, and $\mathcal{D}^{\gamma}_{0+}$ are the Caputo fractional derivative of orders α, γ, and $\mathcal{I}^{\rho}_{0+}$ is the stochastic mean square fractional integral of order ρ and $0 < 2\gamma < \alpha \leq 1$, $\rho > 0$, $A, B \in L^2(\Omega), a, b \in \mathbb{R}$ and $f, h : J \times L^2(\Omega) \to L^2(\Omega)$, and $g : J \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega)$ are some given functions that will be defined later.

2. PRELIMINARIES

In this section, we introduce some notations and definitions of mean square fractional calculus. Some other auxiliary results on fixed points on ordered metric spaces are also presented. For more details, see [1, 4, 5].

Definition 2.1. *Let $\alpha \in (0,1]$ and $Y \in C(J, L^2(\Omega))$. The stochastic mean square fractional integral $\mathcal{J}_{0^+}^{\alpha} Y(t)$ is defined by*

$$\mathcal{J}_{0^+}^{\alpha} Y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-e)^{\alpha-1} Y(e) de.$$

Definition 2.2. *The Caputo fractional derivative of order $\alpha \in (0,1]$ of the stochastic process Y, denoted by $\mathcal{D}_{0^+}^{\alpha} Y(t)$ is defined by*

$$\mathcal{D}_{0^+}^{\alpha} Y(t) = \mathcal{J}_{0^+}^{1-\alpha} \frac{d}{dt} Y(t),$$

with $\frac{d}{dt} Y(t)$ denotes the mean square derivative of order one for $Y(t)$.

Theorem 2.1. *Let $\alpha \in (0,1]$. If Y is mean square differentiable with mean square integrable second-order derivative, then*

- •) $\lim\limits_{\alpha \to 1} \mathcal{D}_{0^+}^{\alpha} Y(t) = \frac{d}{dt} Y(t)$,
- •) $\lim\limits_{\alpha \to 0} \mathcal{D}_{0^+}^{\alpha} Y(t) = Y(t) - Y(0)$,
- •) $\mathcal{J}_{0^+}^{\alpha} \mathcal{D}_{0^+}^{\alpha} Y(t) = Y(t) - Y(0)$,
- •) $\mathcal{D}_{0^+}^{\alpha} \mathcal{J}_{0^+}^{\alpha} Y(t) = Y(t)$.

Definition 2.3. *Suppose that $(\mathbb{X}, \leq)$ is a partially ordered set and $h : \mathbb{X} \to \mathbb{X}$ is a self mapping. Then h is called increasing (respect. decreasing) if $h(Y) \leq h(U)$ (respect. $h(U) \leq h(Y)$), whenever $Y \leq U$.*

Theorem 2.2. *[1] Let us take $(\mathbb{X}, \leq)$ as a partially ordered set and suppose that there is a metric d in X, such that $(\mathbb{X}, d)$ is a complete metric space. Let also $h : \mathbb{X} \to \mathbb{X}$ be an increasing mapping, such that there exists $Y_0 \in \mathbb{X}; Y_0 \leq h(Y_0)$. Suppose also that there exists $0 \leq N < 1$;*

$$d(h(Y), h(U)) \leq N d(Y, U),$$

for all comparable $Y, U \in \mathbb{X}$. Assume that either h is continuous or $\mathbb{X}$ is such that if an increasing sequence $\{Y_n\} \to Y$ in $\mathbb{X}$, then $Y_n \leq Y$ for all $n \in \mathbb{N}$. If for each $Y, U \in \mathbb{X}$, there exists $Z \in \mathbb{X}$ which is comparable to Y and U, then h has a unique fixed point

Now, let us prove to the show to the reader the following auxiliary result.

Lemma 2.1. *For a given R defining over J, the random integral problem associated to the differential problem*

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} \left(\mathcal{D}_{0^+}^{\alpha} Y(t) \right) = R(t), t \in J \\ Y(0) = Y_0 \\ Y(T) = \sum\limits_{i=\overline{1,n}} \lambda_i Y(\zeta_i), \ 0 < \zeta_i < T \end{cases} \tag{3}$$

has the following expression:

$$\begin{aligned} Y(t) &= \frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1} R(s)ds + \frac{(Y(T)-Y_0)}{T^\alpha}t^\alpha \\ &-\frac{t^\alpha}{T^\alpha\Gamma(2\alpha)}\int_0^T (T-s)^{2\alpha-1} R(s)ds + Y_0. \end{aligned}$$

Proof. Let $0<\alpha\leq 1$. Then, Theorem 2.1 allows us obtain the expression

$$\begin{aligned} Y(t) &= \mathcal{I}^{2\alpha}_{0^+}F(t) + \mathcal{I}^{\alpha}_{0^+}C_1 + C_2, \\ &= \frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1} R(s)ds + \frac{t^\alpha}{\Gamma(\alpha+1)}C_1 + C_2, \end{aligned}$$

where C_1, C_2 are two random variables to be determined.
Using (10), we can write

$$C_2 = Y_0, \quad \text{and} \quad C_1 = \frac{\Gamma(\alpha+1)}{T^\alpha}\left(Y(T) - Y_0 - \mathcal{I}^{2\alpha}_{0^+}R(T)\right).$$

∎

Taking into account the fact that $D^{2\gamma} := D^\gamma(D^\gamma)$, we get the following two expressions

$$\begin{aligned} \mathcal{D}^{\gamma}_{0^+}Y(t) &= \frac{1}{\Gamma(2\alpha-\gamma)}\int_0^t (t-s)^{2\alpha-\gamma-1} R(s)ds + \frac{\Gamma(\alpha+1)(Y(T)-Y_0)}{T^\alpha\Gamma(\alpha-\gamma+1)}t^{\alpha-\gamma} \\ &-\frac{\Gamma(\alpha+1)t^{\alpha-\gamma}}{T^\alpha\Gamma(2\alpha)\Gamma(\alpha-\gamma+1)}\int_0^T (T-s)^{2\alpha-1} R(s)ds \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}^{2\gamma}_{0^+}Y(t) &= \frac{1}{\Gamma(2\alpha-2\gamma)}\int_0^t (t-s)^{2\alpha-1} R(s)ds + \frac{\Gamma(\alpha+1)(Y(T)-Y_0)}{T^\alpha\Gamma(\alpha-2\gamma+1)}t^{\alpha-2\gamma} \\ &-\frac{\Gamma(\alpha+1)t^{\alpha-2\gamma}}{T^\alpha\Gamma(2\alpha)\Gamma(\alpha-2\gamma+1)}\int_0^T (T-s)^{2\alpha-1} R(s)ds. \end{aligned}$$

The lemma is thus proved.

3. MAIN RESULTS

Let $C^{2\gamma}(J, L_2(\Omega)) = \left\{Y : Y, \mathcal{D}^{\gamma}_{0^+}Y, \mathcal{D}^{2\gamma}_{0^+}Y \in C(J, L_2(\Omega))\right\}$ be the Banach space of the mean square continuous second order stochastic processes. This space can be endowed with the norm:

$$\|Y\|_{C^\gamma} = \max\left\{\|Y\|_C, \left\|\mathcal{D}^{\gamma}_{0^+}Y\right\|_C, \left\|\mathcal{D}^{2\gamma}_{0^+}Y\right\|_C\right\},$$

such that

$$\|Y\|_C = \sup_{t\in J} \|Y(t)\|_2 \, , \left\|\mathcal{D}^{\gamma}_{0^+}Y\right\|_C = \sup_{t\in J} \left\|\mathcal{D}^{\gamma}_{0^+}Y(t)\right\|_2 \, , \left\|\mathcal{D}^{2\gamma}_{0^+}Y\right\|_C = \sup_{t\in J} \left\|\mathcal{D}^{2\gamma}_{0^+}Y(t)\right\|_2 .$$

Let $Y_1, Z_1 \in C^{2\gamma}(J, L_2(\Omega))$ be, respectively, the lower and upper solution of (1-2). That is:

$$\begin{aligned} Y_1(t) \quad \leq \quad & \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} R_{Y_1}(s)ds + \frac{(Y_1(T) - Y_0)}{T^{\alpha}} t^{\alpha} \\ & -\frac{t^{\alpha}}{T^{\alpha}\Gamma(2\alpha)} \int_0^T (T-s)^{2\alpha-1} R_{Z_1}(s)ds + Y_0 \end{aligned}$$

and

$$\begin{aligned} Z_1(t) \quad \geq \quad & \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} R_{Z_1}(s)ds + \frac{(Z_1(T) - Y_0)}{T^{\alpha}} t^{\alpha} \\ & -\frac{t^{\alpha}}{T^{\alpha}\Gamma(2\alpha)} \int_0^T (T-s)^{2\alpha-1} R_{Y_1}(s)ds + Y_0, \end{aligned}$$

such that

$$R_Y(s) = aAf(t, Y(t)) + bBg(t, \mathcal{D}^{\gamma}_{0^+}Y(t), \mathcal{D}^{\gamma}_{0^+}\left(\mathcal{D}^{\gamma}_{0^+}Y(t)\right)) + h(t, \mathcal{I}^{\rho}_{0^+}Y(t)).$$

We pass to introduce the assumptions that are needed to prove our results.
• $\mathbb{H}_1$) The given functions f, g and h are continuous.
• $\mathbb{H}_2$) The functions f, g and h are mean square Lipschitz in the following sense:
$\exists\,(k_1, k_2, k_3) \in \left(\mathbb{R}_+^*\right)^3$, such that

$$\sup_{t\in J} \|f(t, Y(t)) - f(t, U(t))\|_2 \leq k_1 \left(\|Y - U\|_{C^{\gamma}}\right),$$

$$\sup_{t\in J} \|g(t, Y(t), Z(t)) - g(t, U(t), V(t))\|_2 \leq k_2 \left(\|Y - U\|_{C^{\gamma}} + \|Z - V\|_{C^{\gamma}}\right)$$

and

$$\sup_{t\in J} \|h(t, Y(t)) - h(t, U(t))\|_2 \leq k_3 \|Y - U\|_{C^{\gamma}},$$

for any $t \in J, Y, Z, U, V \in L^2(\Omega)$.
Then, we consider the quantities:

$$\begin{aligned}
N_{11} &= \frac{T^{2\alpha}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)}+n\sup_{i=\overline{1,n}}\left|\lambda_i\right|,\\
N_{12} &= \frac{T^{2\alpha-\gamma}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha-\gamma+1\right)}+\frac{nT^{-\gamma}\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha-\gamma+1\right)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|,\\
N_{13} &= \frac{T^{2\alpha-2\gamma}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha-2\gamma+1\right)}+\frac{nT^{-2\gamma}\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha-2\gamma+1\right)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|,\\
N_{21} &= \frac{T^{2\alpha}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)},\\
N_{22} &= \frac{T^{2\alpha-\gamma}\Gamma\left(\alpha+1\right)\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|a\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-\gamma+1\right)},\\
N_{23} &= \frac{T^{2\alpha-2\gamma}\Gamma\left(\alpha+1\right)\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|a\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-2\gamma+1\right)}.
\end{aligned}$$

At this step, we are able to present the following main result that deals with the existence and uniqueness of solution for the studied problem.

Theorem 3.1. *Suppose that* $(\mathbb{H}_1)-(\mathbb{H}_2)$ *are valid and (1-2) has a lower and an upper solution. Suppose also that* $N:=\max\{N_{11},N_{12},N_{13},N_{21},N_{22},N_{23}\}$ *is such that* $0<N<1$. *Then (1-2) has a unique solution in* $C^{2\gamma}(J,L_2(\Omega))$.

Proof. It is clear to say that $C^{2\gamma}:=C^{2\gamma}(J,L_2(\Omega))$ is a partially ordered set with the following order relation:

$$Y\leq Z\quad Y,Z\in C^{2\gamma}\Longleftrightarrow Y(t)\leq Z(t)\quad\forall t\in J.$$

Also, $(C^{2\gamma},d)$ is a complete metric space, with $d(Y,Z):=\left\|Y-Z\right\|_{C^{2\gamma}}$. So, if $\{Y_n\}$ is an increasing sequence in $C^{2\gamma}$ which converges to $Y\in C^{2\gamma}$ and $\{Z_n\}$ is a decreasing sequence in $C^{2\gamma}$ that converges to $Z\in C^{2\gamma}$, therefore, $Y_n\leq Y$ and $Z\leq Z_n$ for all n. Also, for any $Y,Z\in C^{2\gamma}$, the functions $\max\{Y,Z\}$ and $\min\{Y,Z\}$ are the upper and the lower bound of Y,Z, respectively. Also, $C^{2\gamma}\times C^{2\gamma}$ is a partially ordered set if we define the following order relation in $C^{2\gamma}\times C^{2\gamma}$:

$$(Y,Z)\,\widehat{\leq}\,(U,V)\Longleftrightarrow Y\leq U,Z\leq V.$$

Furthermore, for every $(Y,Z),(U,V)\in C^{2\gamma}\times C^{2\gamma}$, there is a $(\max\{Y,U\},\min\{Z,V\})\in C^{2\gamma}\times C^{2\gamma}$ that is comparable to (Y,Z) and (U,V). Moreover, $(C^{2\gamma}\times C^{2\gamma},\delta)$ is a complete metric space, where $\delta((Y,Z),(U,V))=:d(Y,U)+d(Z,V)$.

Also, if $\{(Y_n, Z_n)\}$ is an increasing sequence in $C^{2\gamma} \times C^{2\gamma}$ that converges to (Y, U), hence, $(Y_n, Z_n)\widehat{\leq}(Y, Z)$, for each $n \in \mathbb{N}$.
Now let us define $\mathbb{T}_1, \mathbb{T}_2 : C^{2\gamma} \times C^{2\gamma}$ and $\mathbb{T} : C^{2\gamma} \times C^{2\gamma} \to C^{2\gamma} \times C^{2\gamma}$ as follows:

$$\begin{aligned}
\mathbb{T}_1 (Y) (t) &= \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} R_Y(s)ds + \frac{(Y(T) - Y_0)}{T^\alpha} t^\alpha, \\
\mathbb{T}_2 (Z) (t) &= -\frac{t^\alpha}{T^\alpha \Gamma(2\alpha)} \int_0^T (T-s)^{2\alpha-1} R_Z(s)ds + Y_0
\end{aligned}$$

and

$$\mathbb{T}(Y, Z) = (\mathbb{T}_1 (Y) + \mathbb{T}_2(Z), \mathbb{T}_1 (Z) + \mathbb{T}_2(Y)).$$

Based on this, we can write

$$\begin{aligned}
\mathcal{D}_{0+}^{\gamma} (\mathbb{T}_1 Y)(t) &= \frac{1}{\Gamma(2\alpha-\gamma)} \int_0^t (t-s)^{2\alpha-\gamma-1} R_Y(s)ds + \frac{\Gamma(\alpha+1)(Y(T) - Y_0)}{T^\alpha \Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma} \\
\mathcal{D}_{0+}^{\gamma} (\mathbb{T}_2 Z)(t) &= -\frac{\Gamma(\alpha+1) t^{\alpha-\gamma}}{T^\alpha \Gamma(2\alpha) \Gamma(\alpha-\gamma+1)} \int_0^T (T-s)^{2\alpha-1} R_Z(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_{0+}^{2\gamma} (\mathbb{T}_1 Y)(t) &= \frac{1}{\Gamma(2\alpha-2\gamma)} \int_0^t (t-s)^{2\alpha-2\gamma-1} R_Y(s)ds + \frac{\Gamma(\alpha+1)(Y(T) - Y_0)}{T^\alpha \Gamma(\alpha-2\gamma+1)} t^{\alpha-2\gamma} \\
\mathcal{D}_{0+}^{2\gamma} (\mathbb{T}_2 Z)(t) &= -\frac{\Gamma(\alpha+1) t^{\alpha-2\gamma}}{T^\alpha \Gamma(2\alpha) \Gamma(\alpha-2\gamma+1)} \int_0^T (T-s)^{2\alpha-1} R_Z(s)ds.
\end{aligned}$$

The reader can see that $(Y_1, Z_1)\widehat{\leq}\mathbb{T}(Y_1, Z_1)$, $\mathbb{T}_1$ is an increasing mapping and $\mathbb{T}_2$ is a decreasing mapping.
Hence, $\mathbb{T}$ is an increasing mapping in $C^{2\gamma} \times C^{2\gamma}$.
Now, for $(Y, Z), (U, V) \in C^{2\gamma} \times C^{2\gamma}$ with $(Y, Z) \widehat{\leq} (U, V)$ and $t \in J$, we have

$$\begin{aligned}
&\|\mathbb{T}(Y, Z) - \mathbb{T}(U, V)\|_{C^{2\gamma}} \\
&= \|(\mathbb{T}_1 Y + \mathbb{T}_2 Z, \mathbb{T}_1 Z + \mathbb{T}_2 Y) - (\mathbb{T}_1 U + \mathbb{T}_2 V, \mathbb{T}_1 V + \mathbb{T}_2 U)\|_{C^{2\gamma}} \\
&\leq \|\mathbb{T}_1 Y - \mathbb{T}_1 U\|_{C^{2\gamma}} + \|\mathbb{T}_1 Z - \mathbb{T}_1 V\|_{C^{2\gamma}} + \|\mathbb{T}_2 Y - \mathbb{T}_2 U\|_{C^{2\gamma}} + \|\mathbb{T}_2 Z - \mathbb{T}_2 V\|_{C^{2\gamma}}.
\end{aligned}$$

Thanks to $(\mathbb{H}_2)$, we observe that

$$\begin{aligned}
& \|R_Y(s) - R_U(s)\|_2 \\
\leq\; & |a| \|A\| \|f(s, Y(s)) - f(s, U(s))\|_2 \\
& + |b| \|B\|_2 \left\|g(s, \mathcal{D}_{0+}^{\gamma} Y(s), \mathcal{D}_{0+}^{\gamma}\left(\mathcal{D}_{0+}^{\gamma} Y\right)(s)) - g(s, \mathcal{D}_{0+}^{\gamma} U(s), \mathcal{D}_{0+}^{\gamma}\left(\mathcal{D}_{0+}^{\gamma} U\right)(s))\right\|_2 \\
& + \left\|h(s, \mathcal{I}_{0+}^{\rho} Y(s)) - h(s, \mathcal{I}_{0+}^{\rho} U(s))\right\|_2 \\
\leq\; & k_1 |a| \|A\|_2 \|Y - U\|_{C^{2\gamma}} + 2k_2 |b| \|B\|_2 \|Y - U\|_{C^{2\gamma}} \\
& + \tfrac{k_3 T^\rho}{\Gamma(\rho+1)} \|Y - U\|_{C^{2\gamma}} \\
\leq\; & \left(k_1 |a| \|A\|_2 + 2k_2 |b| \|B\|_2 + \tfrac{k_3 T^\rho}{\Gamma(\rho+1)}\right) \|Y - U\|_{C^{2\gamma}}.
\end{aligned} \tag{4}$$

Now, using (4), we obtain the following three inequalities:

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathbb{T}_1Y(t)-\mathbb{T}_1U(t)\right\|_2\\
&\leq \frac{T^{2\alpha}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma(2\alpha+1)}+n\sup_{i=\overline{1,n}}|\lambda_i|\left\|(Y-U)(t)\right\|_2\\
&\leq\left(\frac{T^{2\alpha}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha+1)}+n\sup_{i=\overline{1,n}}|\lambda_i|\right)\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{11}d(Y,U),
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathcal{D}_{0^+}^{\gamma}(\mathbb{T}_1Y)(t)-\mathcal{D}_{0^+}^{\gamma}(\mathbb{T}_1U)(t)\right\|_2\\
&\leq \frac{T^{2\alpha-\gamma}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma(2\alpha-\gamma+1)}+\frac{T^{-\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}\left(n\sup_{i=\overline{1,n}}|\lambda_i|\left\|(Y-U)(t)\right\|_2\right)\\
&\leq\left\{\frac{T^{2\alpha-\gamma}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha-\gamma+1)}\right.\\
&\left.+\frac{nT^{-\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}\sup_{i=\overline{1,n}}|\lambda_i|\right\}\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{12}d(Y,U)
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathcal{D}_{0^+}^{\gamma}\left(\mathcal{D}_{0^+}^{\gamma}(\mathbb{T}_1Y)\right)(t)-\mathcal{D}_{0^+}^{\gamma}\left(\mathcal{D}_{0^+}^{\gamma}(\mathbb{T}_1U)\right)(t)\right\|_2\\
&\leq \frac{T^{2\alpha-2\gamma}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma(2\alpha-2\gamma+1)}+\frac{nT^{-2\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-2\gamma+1)}\sup_{i=\overline{1,n}}|\lambda_i|\left\|(Y-U)(t)\right\|_2\\
&\leq\left\{\frac{T^{2\alpha-2\gamma}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha-2\gamma+1)}\right.\\
&\left.+\frac{nT^{-2\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-2\gamma+1)}\sup_{i=\overline{1,n}}|\lambda_i|\right\}\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{13}d(Y,U).
\end{aligned}$$

Consequently, yield the following relations

$$\left\|\mathbb{T}_1Y-\mathbb{T}_1U\right\|_{C^{2\gamma}}\leq\max\{N_{11},N_{12},N_{13}\}\,d(Y,U), \tag{5}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathbb{T}_1 Z(t)-\mathbb{T}_1 V(t)\right\|_2\\
&\leq \frac{T^{2\alpha}\left\|R_Z(s)-R_V(s)\right\|_2}{\Gamma(2\alpha+1)}+n\sup_{i=\overline{1,n}}\left|\lambda_i\right|\left\|(Z-V)(t)\right\|_2\\
&\leq\left(\frac{T^{2\alpha}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^\rho}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha+1)}+n\sup_{i=\overline{1,n}}\left|\lambda_i\right|\right)\left\|Z-V\right\|_{C^{2\gamma}}\\
&\leq N_{11}d(Z,V),
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_1 Z\right)(t)-\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_1 V\right)(t)\right\|_2\\
&\leq \frac{T^{2\alpha-\gamma}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma(2\alpha-\gamma+1)}+\frac{nT^{-\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|\left\|(Y-U)(t)\right\|_2\\
&\leq\left\{\frac{T^{2\alpha-\gamma}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^\rho}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha-\gamma+1)}\right.\\
&\left.+\frac{nT^{-\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|\right\}\left\|Z-V\right\|_{C^{2\gamma}}\\
&\leq N_{12}d(Z,V)
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_1 Z\right)(t)-\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_1 V\right)(t)\right\|_2\\
&\leq \frac{T^{2\alpha-2\gamma}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma(2\alpha-2\gamma+1)}+\frac{nT^{-2\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-2\gamma+1)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|\left\|(Y-U)(t)\right\|_2\\
&\left\{\frac{T^{2\alpha-2\gamma}\left(k_1|a|\left\|A\right\|_2+2k_2|b|\left\|B\right\|_2+\frac{k_3T^\rho}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha-2\gamma+1)}\right.\\
&\left.+\frac{nT^{-2\gamma}\Gamma(\alpha+1)}{\Gamma(\alpha-2\gamma+1)}\sup_{i=\overline{1,n}}\left|\lambda_i\right|\right\}\left\|Z-V\right\|_{C^{2\gamma}}\\
&\leq N_{13}d(Z,V).
\end{aligned}$$

So, we can write

$$\left\|\mathbb{T}_1 Z-\mathbb{T}_1 V\right\|_{C^{2\gamma}}\leq\max\left\{N_{11},N_{12},N_{13}\right\}d(Z,V). \tag{6}$$

Also, the reader will confirm that we the following inequalities are valid:

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathbb{T}_2Y(t)-\mathbb{T}_2U(t)\right\|_2\\
&\leq \frac{T^{2\alpha}\left\|R_Y(s)-R_U(s)\right\|_2}{\Gamma\left(2\alpha+1\right)}\\
&\leq \frac{T^{2\alpha}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)}\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{21}d(Y,U),
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2Y\right)(t)-\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2U\right)(t)\right\|_2\\
&\leq \frac{T^{2\alpha-\gamma}\Gamma\left(\alpha+1\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-\gamma+1\right)}\left\|R_Y(s)-R_U(s)\right\|_2\\
&\leq \frac{T^{2\alpha-\gamma}\Gamma\left(\alpha+1\right)\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|a\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-\gamma+1\right)}\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{22}d(Y,U)
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in J}\left\|\left(\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2Y\right)\right)\mathcal{D}_{0^+}^{\gamma}(t)-\left(\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2U\right)\right)\mathcal{D}_{0^+}^{\gamma}(t)\right\|_2\\
&\leq \frac{T^{2\alpha-2\gamma}\Gamma\left(\alpha+1\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-2\gamma+1\right)}\left\|R_Y(s)-R_U(s)\right\|_2\\
&\leq \frac{T^{2\alpha-2\gamma}\Gamma\left(\alpha+1\right)\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|a\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)\Gamma\left(\alpha-2\gamma+1\right)}\left\|Y-U\right\|_{C^{2\gamma}}\\
&\leq N_{23}d(Y,U),
\end{aligned}$$

Therefore, we deduce that the following four expressions:

$$\left\|\mathbb{T}_2Y-\mathbb{T}_2U\right\|_{C^{2\gamma}}\leq \max\left\{N_{21},N_{22},N_{23}\right\}d\left(Y,U\right), \tag{7}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\mathbb{T}_2Z(t)-\mathbb{T}_2V(t)\right\|_2\\
&\leq \frac{T^{2\alpha}\left\|R_Z(s)-R_V(s)\right\|_2}{\Gamma\left(2\alpha+1\right)}\\
&\leq \frac{T^{2\alpha}\left(k_1\left|a\right|\left\|A\right\|_2+2k_2\left|b\right|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma\left(2\alpha+1\right)}\left\|Z-V\right\|_{C^{2\gamma}}\\
&\leq N_{21}d(Z,V),
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in J}\left\|\left(\mathcal{D}_{0^+}^{\gamma}\mathbb{T}_2 Z\right)(t)-\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2 V\right)(t)\right\|_2\\
&\le \frac{T^{2\alpha-\gamma}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha-\gamma+1)}\left\|R_Z(s)-R_V(s)\right\|_2\\
&\le \frac{T^{2\alpha-\gamma}\Gamma(\alpha+1)\left(k_1|a|\left\|A\right\|_2+2k_2|a|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha+1)\Gamma(\alpha-\gamma+1)}\left\|Z-V\right\|_{C^{2\gamma}}\\
&\le N_{22}d(Z,V)
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in J}\left\|\left(\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2 Z\right)\right)\mathcal{D}_{0^+}^{\gamma}(t)-\left(\mathcal{D}_{0^+}^{\gamma}\left(\mathbb{T}_2 V\right)\right)\mathcal{D}_{0^+}^{\gamma}(t)\right\|_2\\
&\le \frac{T^{2\alpha-2\gamma}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha-2\gamma+1)}\left\|R_Z(s)-R_V(s)\right\|_2\\
&\le \frac{T^{2\alpha-2\gamma}\Gamma(\alpha+1)\left(k_1|a|\left\|A\right\|_2+2k_2|a|\left\|B\right\|_2+\frac{k_3T^{\rho}}{\Gamma(\rho+1)}\right)}{\Gamma(2\alpha+1)\Gamma(\alpha-2\gamma+1)}\left\|Z-V\right\|_{C^{2\gamma}}\\
&\le N_{23}d(Z,V)
\end{aligned}$$

are valid.
Hence, it yields that

$$\left\|\mathbb{T}_2 Z-\mathbb{T}_2 V\right\|_{C^{2\gamma}}\le \max\left\{N_{21},N_{22},N_{23}\right\}d(Z,V). \tag{8}$$

Consequently, we have

$$\begin{aligned}
&\left\|\mathbb{T}(Y,Z)-\mathbb{T}(U,V)\right\|_{C^{2\gamma}}\\
&\le \max\left\{N_{11},N_{12},N_{13},N_{21},N_{22},N_{23}\right\}\left(d(Y,U)+d(Z,V)\right)\\
&=N\delta\left((Y,Z),(U,V)\right),
\end{aligned}$$

for all $(Y,Z),(U,V)\in C^{2\gamma}\times C^{2\gamma}$, such that $(Y,Z)\,\widehat{\leqq}\,(U,V)$ and $t\in J$.
Hence, for each $(Y,Z),(U,V)\in C^{2\gamma}\times C^{2\gamma}$ with $(Y,Z)\,\widehat{\leqq}\,(U,V)$, we obtain

$$\delta\left(\mathbb{T}(Y,Z),\mathbb{T}(U,V)\right)\le N\delta\left((Y,Z),(U,V)\right).$$

On the other hand let an increasing sequence $\{(Y_n,Z_n)\}$ in $\mathbb{X}$, where

$$(Y_n,Z_n)=\mathbb{T}(Y_{n-1},Z_{n-1})=(\mathbb{T}_1(Y_{n-1})+\mathbb{T}_2(Z_{n-1}),\mathbb{T}_1(Z_{n-1})+\mathbb{T}_2(Y_{n-1}))$$

According to Theorem 2.2 and $\mathbb{X}$ is complete, then

$$(Y^*,Z^*)=\left(\lim_{n\to\infty}Y_n(t),\lim_{n\to\infty}Z_n(t)\right)=(\mathbb{T}_1(Y^*)+\mathbb{T}_2(Z^*),\mathbb{T}_1(Z^*)+\mathbb{T}_2(Y^*))$$

i.e. that $\{(Y_n,Z_n)\}$ converges to $\{(Y^*,Z^*)\}$

At the end, using Theorem 2.2, we conclude that there exists a unique $(Y^*, Z^*) \in C^{2\gamma} \times C^{2\gamma}$ that satisfies $(Y^*, Z^*) = \mathbb{T}(Y^*, Z^*)$.
On the other hand, since (Y^*, Z^*) is another fixed point of $\mathbb{T}$, then $Y^* = Z^*$.
Hence, $Y^* = \mathbb{T}_1(Y^*) + \mathbb{T}_2(Y^*)$, i.e., Y^* is the unique solution of (1-2). ∎

4. EXAMPLES

Example 4.1. *Consider the following random problem*

$$\begin{aligned} \mathcal{D}_{0^+}^{0,7}\left(\mathcal{D}_{0^+}^{0,7}Y(t)\right) &= \tfrac{A}{7^3}f(t,Y(t)) + \tfrac{B}{5^3}g\left(t, \mathcal{D}_{0^+}^{0,25}Y(t), \mathcal{D}_{0^+}^{0,25}\left(\mathcal{D}_{0^+}^{0,25}Y(t)\right)\right) \\ &+ h(t, \mathcal{J}_{0^+}^{0,5}Y(t)), \quad t \in \left[0, \tfrac{1}{7^2},\right] \end{aligned} \tag{9}$$

such that

$$\|A)\|_2 = \frac{1}{5^3}, \|B\|_2 = \frac{1}{7^3}.$$

Taking the conditions

$$\|Y(0)\|_2 = \frac{1}{7^5}, \text{ and } Y(T) = \sum_{i=\overline{1,n}} \frac{1}{n \times 4^i} Y(\zeta_i),$$

so, for any $Y, U \in L^2(\Omega)$ and $t \in J$, we have

$$\begin{aligned} f(t,Y(t)) &= \frac{Y(t)}{7(1+t)}, \qquad h(t,Y(t)) = \frac{\sqrt{\pi}Y(t)}{6(41+t^2)}, \\ g(t,Y(t),U(t)) &= \frac{2Y(t)}{5(1+e^t)} + \frac{2U(t)}{(5+t)}. \end{aligned}$$

Then, we can write

$$\|f(t,Y(t)) - f(t,U(t))\|_2 \le \frac{1}{7}\|Y(t) - U(t)\|_2,$$

$$\|g(t,Y(t),U(t)) - g(t,X(t),V(t))\|_2 \le \frac{1}{5}\left(\|Y(t)-U(t)\|_2 + \|X(t)-V(t)\|_2\right)$$

and

$$\|h(t,Y(t)) - h(t,U(t))\|_2 \le \frac{\sqrt{\pi}}{246}\|Y(t) - U(t)\|_2.$$

Using Theorem 3.1, we get $N \simeq 0,2500212$.
Therefore, thanks to the same theorem, we can state that (9) has a unique solution on $\left[0, \frac{1}{7^2}\right]$.

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