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SIMILARITY SOLUTIONS AND EVOLUTION OF WEAK DISCONTINUITIES IN A NO–SLIP DRIFT–FLUX MODEL

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Abstract The basic equations, governing the one–dimensional motion in an isothermal *no–slip* drift–flux model, are considered. Using the invariance group properties of the original system, the new autonomous system is found. The propagation of weak discontinuities is considered in the known particular solution of the autonomous system. The critical time, when these discontinuities culminate into a shock, is also determined.

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1. INTRODUCTION

Multiphase flows are important in a large range of industrial applications, such as in the oil and gas industry, in the chemical and process industry, as well as in the safety analysis of nuclear power plants [1, 5].

As a result, an intense research has been done on such multiphase flows in the recent past. Different models for multiphase flows have been proposed [1, 7].

The mathematical study of the flows of gases in networks is a young field of research and only recently it has been under investigation. The reader is referred to [8, 13], in the context of gas networks.

These kinds of studies are mathematically represented by multiphase drift-flux flow models. These flow models consist in a system of partial differential equations (PDEs) containing several independent and dependent variables.

However, special exact solutions of a system of non-linear PDEs are of great interest. One of the most powerful methods in order to determine particular solutions to PDEs is based upon the study of their invariance with respect to one parameter Lie group of point transformation [18, 22]. Besides similarity methods, another use of Lie symmetries admitted by given PDEs consists in introducing some invertible point-transformations that map the original system to an equivalent one, admitting special solutions.

Classical Lie symmetry with non trivial infinitesimal generators admitted by the governing system are obtained using the straight forward procedure outlined in [24].

Important and significant results have been achieved in recent years, in several fields of mathematical physics, using such techniques [25, 37]. In particular, Donato and Oliveri [23] have shown that if any two of the infinitesimal operators of the invariance group commute, then the system of PDEs can be transformed into an autonomous one.

In this paper, an isothermal *no-slip* drift-flux model for two-phase flows is considered and it is described by the following equations

$$\begin{cases} \partial_t\rho_1 + \partial_x(\rho_1 u) = 0, \\ \partial_t\rho_2 + \partial_x(\rho_2 u) = 0, \\ \partial_t(\rho u) + \partial_x\left[\rho u^2 + p(\rho_1, \rho_2)\right] = 0, \end{cases} \tag{1}$$

where ρ_1 and ρ_2 are the densities of phase 1 and phase 2, respectively, u is the common velocity of the two phases, p is the pressure of the mixture, and we denote $\rho = \rho_1 + \rho_2$.

This model is derived by Evje and Flåtten [1, 2] from the drift-flux model by making the simplifying assumption that the *slip condition*, which is an algebraic relation relating the two velocities of the two phases, has a vanishing *slip function*.

We apply the technique in [23] to obtain the particular exact solution to our system (1). The evolution of weak discontinuities propagating into the above known solution is also discussed and the critical time, where a weak discontinuity culminate into a shock, is determined.

2. MODELING OF A SINGLE PIPE FLOW

In this section we will briefly introduce the *no-slip* drift-flux model [2].

We consider a two-component fluid in a pipe modelled by the so-called drift-flux model. We denote the volume fraction, the density, the pressure, and the velocity of the phase i $(i = 1, 2)$ at position x and time t, by $\alpha_i = \alpha_i(x,t)$, $c_i = c_i(x,t)$, $p_i = p_i(x,t)$, $u_i = u_i(x,t)$, respectively. By definition, it is

$$\alpha_1 + \alpha_2 = 1 \tag{2}$$

and, supposing that

$$p = \alpha_1 p_1 + \alpha_2 p_2, \tag{3}$$

the drift-flux model reads:

$$\begin{cases} \partial_t(\alpha_1 c_1) + \partial_x(\alpha_1 c_1 u_1) = 0, \\ \partial_t(\alpha_2 c_2) + \partial_x(\alpha_2 c_2 u_2) = 0, \\ \partial_t(\alpha_1 c_1 u_1 + \alpha_2 c_2 u_2) + \\ \quad +\partial_x\left(\alpha_1 c_1 u_1^2 + \alpha_2 c_2 u_2^2 + \alpha_1 p_1 + \alpha_2 p_2\right) = Q, \end{cases} \tag{4}$$

where Q denotes the momentum sources that act on both phases and the phasic momentum satisfies a slip relation of the form

$$u_1 - u_2 = \Phi\left(p, u_1, u_2\right).$$

As a further simplification, we discuss the case of a *no-slip* condition $\Phi = 0$, that is we restrict to a flow regime in which the velocities of the two phases are equal.

Moreover, we assume that $Q = 0$ and that each phase is isothermal, i.e. its pressure is given by

$$p_i = a_i^2 c_i, \tag{5}$$

where $a_i = const$ represent the compressibility factor of each phase. Taking into account equation (3), the following expression of pressure is obtained

$$p = \alpha_1 a_1^2 c_1 + \alpha_2 a_2^2 c_2. \tag{6}$$

We also assume that the two fluids are immiscible and, therefore, we denote the total density of each phase as

$$\rho_1 = \alpha_1 c_1, \qquad \rho_2 = \alpha_2 c_2. \tag{7}$$

Then, the following equation of state holds

$$p = a_1^2 \rho_1 + a_2^2 \rho_2. \tag{8}$$

Under this assumptions, the model in (4) simplifies into the form (1) with

$$\rho = \rho_1 + \rho_2 = \alpha_1 c_1 + \alpha_2 c_2. \tag{9}$$

3. LIE SYMMETRIC GROUPS

Here now, we consider Lie groups of transformations with independent variables x, t, and dependent variables ρ_1, ρ_2, u for the problem

$$\begin{aligned} &\tilde{x} = \tilde{x}\left(x, t, u, \rho_1, \rho_2; \varepsilon\right), \\ &\tilde{t} = \tilde{t}\left(x, t, u, \rho_1, \rho_2; \varepsilon\right), \\ &\tilde{u} = \tilde{u}\left(x, t, u, \rho_1, \rho_2; \varepsilon\right), \\ &\tilde{\rho}_k = \tilde{\rho}_k\left(x, t, u, \rho_1, \rho_2; \varepsilon\right) \qquad (k = 1, 2), \end{aligned} \tag{10}$$

where ε is the group parameter. The infinitesimal generator of the group (10) can be expressed in the following vector form [24]

$$X = \xi^x \partial_x + \xi^t \partial_t + \eta^u \partial_u + \eta^{\rho_1} \partial_{\rho_1} + \eta^{\rho_2} \partial_{\rho_2}, \tag{11}$$

in which ξ^x, ξ^t, η^u, η^{ρ_1}, η^{ρ_2} are the infinitesimal functions of the group variables. Then, the corresponding one-parameter Lie group of transformations is given by

$$\begin{aligned}
&\tilde{x} = x + \varepsilon\xi^{x}\left(x,t,u,\rho_1,\rho_2\right) + o\left(\varepsilon^2\right),\\
&\tilde{t} = t + \varepsilon\xi^{t}\left(x,t,u,\rho_1,\rho_2\right) + o\left(\varepsilon^2\right),\\
&\tilde{u} = u + \varepsilon\eta^{u}\left(x,t,u,\rho_1,\rho_2\right) + o\left(\varepsilon^2\right),\\
&\tilde{\rho}_k = \rho_k + \varepsilon\eta^{\rho_k}\left(x,t,u,\rho_1,\rho_2\right) + o\left(\varepsilon^2\right) \qquad (k=1,2).
\end{aligned} \tag{12}$$

Since the system of PDEs (1) has at most first order derivatives, the first prolongation of the generators should be considered in the form

$$\begin{aligned}
pr^1X = X &+ \tau_x^u\partial_{u_x} + \tau_t^u\partial_{u_t} +\\
&+\tau_x^{\rho_1}\partial_{\rho_{1_x}} + \tau_t^{\rho_1}\partial_{\rho_{1_t}} +\\
&+\tau_x^{\rho_2}\partial_{\rho_{2_x}} + \tau_t^{\rho_2}\partial_{\rho_{2_t}},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\tau_t^u &= \eta_t^u + \eta_u^u u_t + \eta_{\rho_1}^u \rho_{1_t} + \eta_{\rho_2}^u \rho_{2_t}\\
&\quad -u_x\left(\xi_t^x + \xi_u^x u_t + \xi_{\rho_1}^x\rho_{1_t} + \xi_{\rho_2}^x\rho_{2_t}\right)\\
&\quad -u_t\left(\xi_t^t + \xi_u^t u_t + \xi_{\rho_1}^t\rho_{1_t} + \xi_{\rho_2}^t\rho_{2_t}\right),\\
\tau_x^u &= \eta_x^u + \eta_u^u u_x + \eta_{\rho_1}^u \rho_{1_x} + \eta_{\rho_2}^u \rho_{2_x}\\
&\quad -u_x\left(\xi_x^x + \xi_u^x u_x + \xi_{\rho_1}^x\rho_{1_x} + \xi_{\rho_2}^x\rho_{2_x}\right)\\
&\quad -u_t\left(\xi_x^t + \xi_u^t u_x + \xi_{\rho_1}^t\rho_{1_x} + \xi_{\rho_2}^t\rho_{2_x}\right),
\end{aligned} \tag{14}$$

the expressions of $\tau_t^{\rho_1}$, $\tau_x^{\rho_1}$, $\tau_t^{\rho_2}$, $\tau_x^{\rho_2}$ are completely analogous.

Now, we apply the first prolongation of the infinitesimal generator (13) to the PDEs system (1). Then we obtain the following PDEs

$$\begin{aligned}
&pr^1X\left(\rho_{k_t} + u\rho_{k_x} + \rho_k u_x\right)_{\rho_{k_t}=-u\rho_{k_x}-\rho_k u_x} = 0,\\
&pr^1X\left[u_t - H\right]_{u_t=H} = 0,
\end{aligned} \tag{15}$$

where $k = 1,2$ and $H = -uu_x - \frac{1}{\rho}\left(a_1^2\rho_{1_x} + a_2^2\rho_{2_x}\right)$.

In addition, we can arrange (15) by the explicit form described in (14), then each of these equations can get a polynomial form in terms of the dependent variables and some of their derivatives with respect to the independent variables, which can be considered as independent function and the coefficients of these functions can be equated to zero. Therefore, we will obtain the explicit solutions of the infinitesimal functions ξ^x, ξ^t, η^u, η^{ρ_1}, and η^{ρ_2}

$$\begin{aligned}
&\xi^x = \alpha_1 x + \alpha_2 t + \alpha_3, \qquad \xi^t = \alpha_1 t + \alpha_4,\\
&\eta^u = \alpha_2, \qquad \eta^{\rho_k} = \alpha_5\rho_k \qquad (k=1,2),
\end{aligned} \tag{16}$$

where α_i, $i = 1,\ldots,5$, are arbitrary constants.

These transformations provide the following 5 Lie point generators

$$\begin{aligned}
&X_1 = x\partial_x + t\partial_t, \qquad X_2 = t\partial_x + \partial_u,\\
&X_3 = \partial_x, \qquad X_4 = \partial_t, \qquad X_5 = \rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2}.
\end{aligned} \tag{17}$$

4. SIMILARITY ANALYSIS AND AUTONOMOUS FORM

It can be verified that the infinitesimal operators Y_1 and Y_2 defined by

$$\begin{aligned} Y_1 &\equiv X_1 = t\partial_t + x\partial_x, \\ Y_2 &\equiv X_2 + X_5 = t\partial_x + \partial_u + \rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2}, \end{aligned} \tag{18}$$

commute, that is $[Y_1, Y_2] = Y_1Y_2 - Y_2Y_1 = 0$.

Hence, using the method given by Donato and Oliveri [23], we introduce a set of canonical variables, $\hat{\tau}$, $\hat{\xi}$, $\hat{U}$, $\hat{R}_1$, and $\hat{R}_2$, such that

$$Y_1\hat{\tau} = 1, \quad Y_1\hat{\xi} = Y_1\hat{U} = Y_1\hat{R}_1 = Y_1\hat{R}_2 = 0. \tag{19}$$

The characteristic equations corresponding to eq. (19) are:

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{\vartheta} = \frac{d\rho_1}{\vartheta} = \frac{d\rho_2}{\vartheta} = \frac{d\hat{\tau}}{1}, \tag{20}$$

which yield the following transformation of variables

$$\hat{\tau} = \ln t, \quad \hat{\xi} = t^{-1}x, \quad u = \hat{U}, \quad \rho_k = \hat{R}_k \quad (k = 1, 2). \tag{21}$$

Then, expressing Y_2 in terms of the new variables, we obtain

$$\hat{Y}_2 = \partial_\xi + \partial_{\hat{U}} + \hat{R}_1\partial_{\hat{R}_1} + \hat{R}_2\partial_{\hat{R}_2}. \tag{22}$$

Similarly, we now choose a second set of canonical variables, τ, ξ, U, R_1 and R_2, such that

$$\hat{Y}_2\xi = 1, \qquad \hat{Y}_2\tau = \hat{Y}_2U = \hat{Y}_2R_1 = \hat{Y}_2R_2 = 0. \tag{23}$$

The characteristic equations corresponding to eq. (23) are:

$$\frac{d\hat{\xi}}{1} = \frac{d\hat{\tau}}{\vartheta} = \frac{d\hat{U}}{1} = \frac{d\hat{R}_1}{\hat{R}_1} = \frac{d\hat{R}_2}{\hat{R}_2} = \frac{d\xi}{1}, \tag{24}$$

which, upon integration, yield

$$\begin{aligned} \tau &= \hat{\tau}, \quad \xi = \hat{\xi}, \quad \widetilde{U} = \hat{U} + \hat{\xi}, \\ \hat{R}_k &= R_k e^{\hat{\xi}} \quad (k = 1, 2). \end{aligned} \tag{25}$$

A combination of equations (21) and (25) yields the following transformation of variables

$$\begin{aligned} \tau &= \ln t, \quad \xi = \tfrac{x}{t}, \quad u = U + \tfrac{x}{t}, \\ \rho_k &= R_k e^{x/t} \quad (k = 1, 2). \end{aligned} \tag{26}$$

Hence, using

$$\partial_x = \frac{1}{t}\partial_\xi, \qquad \partial_t = \frac{1}{t}\partial_\tau - \frac{x}{t^2}\partial_\xi, \tag{27}$$

system (1) takes the following autonomous form:

$$\begin{cases} R\left(\partial_\tau U + U\partial_\xi U\right) + k_1\partial_\xi R_1 + k_2\partial_\xi R_2 = \\ \qquad\qquad = -RU - k_1R_1 - k_2R_2, \\ \partial_\tau R_1 + U\partial_\xi R_1 + R_1\partial_\xi U = -R_1\left(1+U\right), \\ \partial_\tau R_2 + U\partial_\xi R_2 + R_2\partial_\xi U = -R_2\left(1+U\right), \end{cases} \tag{28}$$

with $R = R_1 + R_2$.

Such system admits particular solutions given by

$$U = 1, \quad R_i = R_{i0}\exp\left(\frac{1-k_i}{k_i}\tau - \frac{1+k_i}{k_i}\xi\right), \tag{29}$$

$$U = 0, \quad R_i = R_{i0}\exp\left(-\tau - \xi\right), \tag{30}$$

where R_{i0} are arbitrary constants. Hence, by using (26), (29) and (30), particular exact solutions of system (1) are given by

$$u = 1 + \frac{x}{t}, \quad \rho_i = R_{i0}t^{(1-k_i)/k_i}\exp\left(-\frac{x}{k_i t}\right), \tag{31}$$

$$u = \frac{x}{t}, \quad \rho_i = \frac{R_{i0}}{t}, \tag{32}$$

where $i = 1, 2$.

5. EVOLUTION OF WEAK DISCONTINUITIES

The governing system (1) can be written in the form

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0}, \tag{33}$$

where $\mathbf{u} = (\rho_1, \rho_2, u)^T$, and

$$\mathbf{A} = \begin{bmatrix} u & 0 & \rho_1 \\ 0 & u & \rho_2 \\ \frac{k_1}{\rho} & \frac{k_2}{\rho} & u \end{bmatrix}, \tag{34}$$

whose eigenvalues λ_i, and the corresponding right and left eigenvectors may be written as follows

$$\begin{aligned} \lambda_{1,3} &= u \pm c, & \lambda_2 &= u, \\ \mathbf{r}_{1,3} &= \left(\rho_1, \rho_2, \pm c\right)^T, & \mathbf{r}_2 &= (k_2, -k_1, 0)^T, \\ \mathbf{l}_{1,3} &= \left(\tfrac{k_1}{\rho}, \tfrac{k_2}{\rho}, \pm c\right), & \mathbf{l}_2 &= (\rho_2, -\rho_1, 0), \end{aligned} \tag{35}$$

where

$$c^2 = \frac{k_1\rho_1 + k_2\rho_2}{\rho}, \qquad k_i = a_i^2$$

The evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoulli's law has been studied quite extensively in the literature [14, 17].

Let us consider that a C^1 discontinuity is propagating along the characteristic curve determined by

$$\frac{dx}{dt} = \lambda_1, \tag{36}$$

originating at the point (x_0, t_0).

Then the transport equations for weak discontinuities across the characteristic curve C_1 is given by

$$\begin{aligned} &\mathbf{l}_1\left(\tfrac{d\boldsymbol{\pi}}{dt} + (\mathbf{u}_x + \boldsymbol{\pi})(\nabla\lambda_1 \cdot \boldsymbol{\pi}) + ((\nabla\mathbf{l}_1)\,\boldsymbol{\pi})^T \tfrac{d\mathbf{u}}{dt}\right) + \\ &+ (\mathbf{l}_1 \cdot \boldsymbol{\pi})(\nabla\lambda_1 \cdot \mathbf{u}_x) = 0, \end{aligned} \tag{37}$$

denoting by $\boldsymbol{\pi}$ the jump in $\mathbf{u}_x$ across the C^1 discontinuity, collinear to the right eigenvector $\mathbf{r}_1$, that is

$$\boldsymbol{\pi} = \psi\mathbf{r}_1, \tag{38}$$

where ψ denote the amplitude of the C^1 wave, and $\nabla = (\partial_u, \partial_{\rho_1}, \partial_{\rho_2})$.

Then, on using (33), (36) and (38) in (37), we obtain the following Bernoulli type equation for the wave amplitude ψ:

$$2c\frac{d\psi}{dt} + \mathbf{H}\psi - 2c^2\psi^2 = 0, \tag{39}$$

where

$$\mathbf{H} = 3cu_x - \frac{1}{2\rho}\left[\left(5k_1 - 3c^2\right)\rho_{1x} + \left(5k_2 - 3c^2\right)\rho_{2x}\right]. \tag{40}$$

In view of the particular solution (32), the above equation becomes

$$\frac{d\psi}{dt} + \frac{3}{2t}\psi - c\psi^2 = 0, \qquad c > 0, \tag{41}$$

which yields on integration

$$\psi = \frac{\psi_0}{\eta\left[(1 - 2ct_0\psi_0)\,\eta^{1/2} + 2ct_0\psi_0\right]}, \tag{42}$$

where ψ_0 is the value of ψ at $t = t_0$ and $\eta = t/t_0$.

Equation (42) shows that if $\psi_0 < 0$, then $|\psi| \to 0$ as $\eta \to \infty$, implying thereby that the wave decays and dies out eventually; the corresponding situation is illustrated by the curve in fig. 1. However, if $\psi_0 > 0$, it follows from (42) that there are two possibilities:

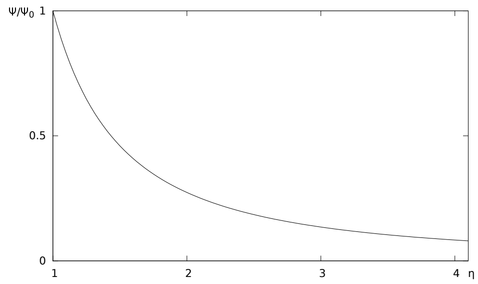

Fig. 1. The behaviour of ψ with η for $\psi_0 < 0$.

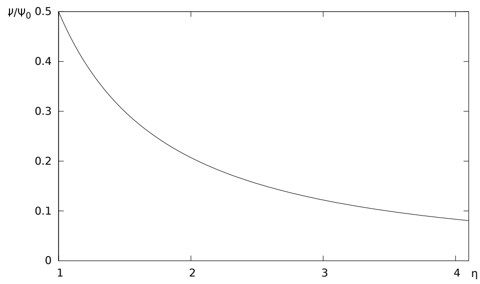

Fig. 2. The behaviour of ψ with η for $0 < \psi_0 < \frac{1}{2ct_0}$.

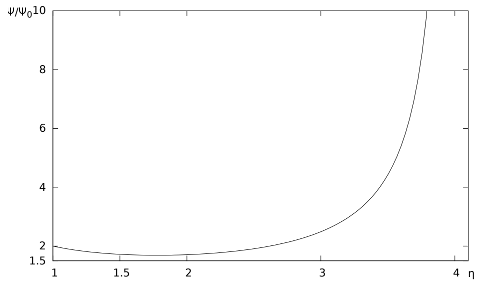

Fig. 3. The behaviour of ψ with η for $\psi_0 > \frac{1}{2ct_0}$.

(a) Let $\psi_0 < \frac{1}{2ct_0}$. Then ψ is finite and non-zero for $\eta < \infty$ and $\psi \to 0$ as $\eta \to \infty$ implying thereby that the wave decays, the corresponding situation is illustrated by the curve in fig. 2.

(b) Let $\psi_0 > \frac{1}{2ct_0}$. Then there exists a finite time $\eta_c > 1$, given by $\eta_c = \frac{2ct_0\psi_0}{2ct_0\psi_0 - 1}$, such that ψ is finite, non-zero and continuous on $[1, \eta_c)$ and $\psi \to \infty$ as $\eta \to \eta_c$. This signifies the weak discontinuity wave culminates into a shock in a finite time only when the initial discontinuity associated with the wave exceeds a critical value. The corresponding situation is illustrated by the curve in fig. 3.

6. CONCLUSIONS

Lie group analysis is used to obtain an exact solution of partial differential equations that describe one dimensional isothermal drift-flux model equations. The evolution of weak discontinuities in a state characterized by exact solution is studied. It is shown that a weak discontinuity wave culminates into a bore after a finite time, only if the initial discontinuity associated with it exceeds a critical value (see fig. 3). However, when $\psi_0 > 0$ and $\psi_0 < 1/2ct_0$ or $\psi_0 < 0$, in both the cases the wave decays eventually (see fig. 1, 2).

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