ROMAI J., v.16, no.2(2020), 63–74

FRACTIONAL BULLEN TYPE INEQUALITIES FOR DIFFERENTIABLE PREINVEX FUNCTIONS

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Abstract In this paper, we first establish an integral identity, by the use of this identity we derive some new fractional Bullen type inequalities for differentiable preinvex functions.

Keywords: Bullen's inequality, Hölder inequality, preinvex function, fractional integrals.
2010 MSC: 26D10, 26D15, 26A51.

1. INTRODUCTION

The classical inequality for convex functions known as the Hermite-Hadamard inequality which can be stated as follows: For any convex function f over the finite interval $[a, b]$, we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction (see [14]).

The above inequality have not stopped to attract the attention of researchers, various generalizations, improvements, and novel inequalities, see for example [1, 4, 5, 6, 8, 9, 10, 11, 12, 13, 16], and the references therein..

Without being exhaustive, we mention here the work of Bullen where he established the following inequality, see [2, 3]

$$\tfrac{2}{b-a}\int_a^b f(x)dx \leq f\left(\tfrac{a+b}{2}\right) + \tfrac{f(a)+f(b)}{2}.$$

In this work, we establish a new integral identity, and we use it to derive some new Bullen's inequalities for preinvex function via Riemann-Liouville integral operators.

2. PRELIMINARIES

In this section, we recall some definitions and lemma

Definition 2.1. *[14] A set $I \subseteq \mathbb{R}^n$ is said to be convex, if*

$$tx + (1-t)\,y \in I.$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. *[14] A function $f : I \to \mathbb{R}$ is said to be convex on I where I is an interval of $\mathbb{R}$, if*

$$f\,(tx + (1-t)\,y) \leq tf\,(x) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.3. *[17] A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if*

$$x + t\eta\,(y,x) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.4. *[17] A function $f : K \subset (0,+\infty) \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f\,(x + t\eta\,(y,x)) \leq (1-t)\,f\,(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.5. *[5] The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a,$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad b > x,$$

respectively, for $f \in L_1[a,b]$ where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function, and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

Lemma 2.1. *[15] For $0 < u \leq 1$ and $0 \leq x < y$, we have*

$$|x^u - y^u| \leq (y-x)^u\,.$$

3. MAIN RESULTS

In order to prove our results, we need the following lemma

Lemma 3.1. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be an absolutely continuous function on $(a,a+\eta(b,a))$ where $\eta(b,a)>0$ and assume that $f'\in L([a,a+\eta(b,a)])$, then one has the following equality*

$$F(a,b,\eta,f) = \frac{\eta(b,a)}{8}\int_0^1 \left(t^\alpha-(1-t)^\alpha\right)\left(f'\left(a+\tfrac{t}{2}\eta(b,a)\right)+f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right)dt, \tag{2}$$

where

$$F(a,b,\eta,f) = \frac{1}{2}\left[\frac{f(a)+f(a+\eta(b,a))}{2}+f\left(\frac{2a+\eta(b,a)}{2}\right)\right]-\frac{\alpha}{4}\left(\frac{2}{\eta(b,a)}\right)^\alpha\Gamma(\alpha+1) \times\left(J^\alpha_{a^+}f\left(\frac{2a+\eta(b,a)}{2}\right)+J^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f(a) + J^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f(a+\eta(b,a))+J^\alpha_{(a+\eta(b,a))^-}f\left(\frac{2a+\eta(b,a)}{2}\right)\right). \tag{3}$$

Proof. The integration by parts of the right side of (22) gives

$$\begin{aligned}
F(a,b,\eta,f) &= \frac{\eta(b,a)}{8}\int_0^1 \left(t^\alpha-(1-t)^\alpha\right)\left(f'\left(a+\tfrac{t}{2}\eta(b,a)\right)+f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right)dt\\
&= \frac{1}{4}\left(t^\alpha-(1-t)^\alpha\right)\left(f\left(a+\tfrac{t}{2}\eta(b,a)\right)+f\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right)\Big|_0^1\\
&\quad -\frac{\alpha}{4}\int_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)\left(f\left(a+\tfrac{t}{2}\eta(b,a)\right)+f\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right)dt\\
&= \frac{1}{4}\left(f(a)+2f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right)\\
&\quad -\frac{\alpha}{4}\left(\int_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)f\left(a+\tfrac{t}{2}\eta(b,a)\right)dt\right.\\
&\quad \left.+\int_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)f\left(a+\tfrac{1+t}{2}\eta(b,a)\right)dt\right)
\end{aligned}$$

$$=\frac{1}{2}\left[\frac{f(a)+f(a+\eta(b,a))}{2}+f\left(\frac{2a+\eta(b,a)}{2}\right)\right]-\frac{\alpha}{4}\left(\frac{2}{\eta(b,a)}\right)^{\alpha}$$

$$\times\left(\int_{a}^{\frac{2a+\eta(b,a)}{2}}(t-a)^{\alpha-1}f(t)dt+\int_{a}^{\frac{2a+\eta(b,a)}{2}}\left(\frac{2a+\eta(b,a)}{2}-t\right)^{\alpha-1}f(t)\,dt\right.$$

$$\left.+\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)}\left(t-\frac{2a+\eta(b,a)}{2}\right)^{\alpha-1}f(t)\,dt+\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)}(a+\eta(b,a)-t)^{\alpha-1}f(t)\,dt\right)$$

$$=\frac{1}{2}\left[\frac{f(a)+f(a+\eta(b,a))}{2}+f\left(\frac{2a+\eta(b,a)}{2}\right)\right]$$

$$-\frac{1}{4}\left(\frac{2}{\eta(b,a)}\right)^{\alpha}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^{-}}f(a)+J^{\alpha}_{a^{+}}f\left(\frac{2a+\eta(b,a)}{2}\right)\right.$$

$$\left.+J^{\alpha}_{(a+\eta(b,a))^{-}}f\left(\frac{2a+\eta(b,a)}{2}\right)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^{+}}f(a+\eta(b,a))\right),$$

which is the desired result. ∎

Theorem 3.1. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable mapping on $(a,a+\eta(b,a))$ where $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$. If $|f'|$ is preinvex with respect to $\eta(b,a)$, then the following fractional inequality holds*

$$|F(a,b,\eta,f)|\leq\frac{\eta(b,a)}{4(\alpha+1)}\left(1-\left(\tfrac{1}{2}\right)^{\alpha}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Proof. From Lemma 3.1, properties of modulus, and preinvexity of $|f'|$ on $[a,a+\eta(b,a)]$, we have

$$|F(a,b,\eta,f)|$$

$$\leq\frac{\eta(b,a)}{8}\left(\int_0^1|t^{\alpha}-(1-t)^{\alpha}|\left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|dt\right.$$

$$\left.+\int_0^1|t^{\alpha}-(1-t)^{\alpha}|\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|dt\right)$$

$$\leq \frac{\eta(b,a)}{8}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|dt\right.$$

$$+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|dt$$

$$+\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|dt$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|dt\right)$$

$$\leq \frac{\eta(b,a)}{8}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\left(1-\tfrac{t}{2}\right)\left|f'(a)\right|+\left(\tfrac{t}{2}\right)^{s}\left|f'(b)\right|\right)dt\right.$$

$$+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(\left(1-\tfrac{t}{2}\right)\left|f'(a)\right|+\left(\tfrac{t}{2}\right)\left|f'(b)\right|\right)dt$$

$$+\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\left(1-\tfrac{1+t}{2}\right)\left|f'(a)\right|+\left(\tfrac{1+t}{2}\right)\left|f'(b)\right|\right)dt$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(\left(1-\tfrac{1+t}{2}\right)\left|f'(a)\right|+\left(\tfrac{1+t}{2}\right)\left|f'(b)\right|\right)dt\right)$$

$$=\frac{\eta(b,a)}{8}\left(\left|f'(a)\right|\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{3}{2}-t\right)dt+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(\tfrac{3}{2}-t\right)dt\right)\right.$$

$$\left.+\left|f'(b)\right|\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{1}{2}+t\right)dt+\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(\tfrac{1}{2}+t\right)dt\right)\right)$$

$$=\frac{\eta(b,a)}{8}\left(\left|f'(a)\right|\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{3}{2}-t\right)dt+\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{1}{2}+t\right)dt\right)\right.$$

$$+\left|f'(b)\right|\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{1}{2}+t\right)dt+\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(\tfrac{3}{2}-t\right)dt\right)\Bigg)$$

$$=\tfrac{\eta(b,a)}{4}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha}-t^{\alpha}\right)dt\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

$$=\tfrac{\eta(b,a)}{4(\alpha+1)}\left(1-\left(\tfrac{1}{2}\right)^{\alpha}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

The proof is completed. ∎

Corollary 3.1. *In Theorem 3.1, if we choose $\eta(b,a)=b-a$ we get*

$$\Bigg|\tfrac{1}{2}\left[\tfrac{f(a)+f(b)}{2}+f\left(\tfrac{a+b}{2}\right)\right]-\tfrac{\alpha}{4}\left(\tfrac{2}{b-a}\right)^{\alpha}\Gamma(\alpha+1)$$
$$\times\left(J_{a^+}^{\alpha}f\left(\tfrac{a+b}{2}\right)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{(b)^-}^{\alpha}f\left(\tfrac{a+b}{2}\right)\right)\Bigg|$$
$$\leq\tfrac{b-a}{4(\alpha+1)}\left(1-\left(\tfrac{1}{2}\right)^{\alpha}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Corollary 3.2. *In Theorem 3.1, if we take $\alpha=1$, we obtain*

$$\left|\tfrac{1}{2}\left[\tfrac{f(a)+f(a+\eta(b,a))}{2}+f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right]-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\right|$$
$$\leq\tfrac{\eta(b,a)}{16}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Moreover, if we choose $\eta(b,a)=b-a$, we get

$$\left|\tfrac{1}{2}\left[\tfrac{f(a)+f(b)}{2}+f\left(\tfrac{a+b}{2}\right)\right]-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|\leq\tfrac{b-a}{16}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Theorem 3.2. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable mapping on $(a,a+\eta(b,a))$ where $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$. If $|f'|^q$ where $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$ is preinvex with respect to $\eta(b,a)$, then the following fractional inequality holds*

$$|F(a,b,\eta,f)|$$
$$\leq\tfrac{\eta(b,a)}{8}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Proof. Using Lemma 3.1, properties of modulus, Hölder's inequality, preinvexity of $|f'|$ on $[a, a+\eta(b,a)]$, and Lemma1, we get

$$
\begin{aligned}
|F(a,b,\eta,f)| \leq & \frac{\eta(b,a)}{8}\left(\int_0^1 |t^\alpha-(1-t)^\alpha|\left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|dt\right.\\
&\left.+\int_0^1 |t^\alpha-(1-t)^\alpha|\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|dt\right)\\
\leq & \frac{\eta(b,a)}{8}\left(\int_0^1 |t^\alpha-(1-t)^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\left(\int_0^1 \left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right)\\
\leq & \frac{\eta(b,a)}{8}\left(\int_0^{\frac{1}{2}} |t^\alpha-(1-t)^\alpha|^p\,dt+\int_{\frac{1}{2}}^1 |t^\alpha-(1-t)^\alpha|^p\,dt\right)^{\frac{1}{p}}\\
&\times\left(\left(\int_0^1 \left(\left(1-\tfrac{t}{2}\right)|f'(a)|^q+\left(\tfrac{t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \left(\left(1-\tfrac{1+t}{2}\right)|f'(a)|^q+\left(\tfrac{1+t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right)\\
\leq & \frac{\eta(b,a)}{8}\left(\int_0^{\frac{1}{2}} (1-2t)^{\alpha p}\,dt+\int_{\frac{1}{2}}^1 (2t-1)^{\alpha p}\,dt\right)^{\frac{1}{p}}\\
&\times\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right)\\
\leq & \frac{\eta(b,a)}{8}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

The proof is completed. ∎

Corollary 3.3. *In Theorem 3.2, if we choose $\eta(b,a)=b-a$ we get*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{\alpha}{4}\left(\frac{2}{b-a}\right)^{\alpha}\Gamma(\alpha+1)\right.$$
$$\left.\times\left(J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)+J^{\alpha}_{(b)^-}f\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq\frac{b-a}{8}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.4. *In Theorem 3.2, if we take $\alpha=1$, we obtain*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(a+\eta(b,a))}{2}+f\left(\frac{2a+\eta(b,a)}{2}\right)\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(u)\,du\right|$$
$$\leq\frac{\eta(b,a)}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right)$$

Moreover, if we choose $\eta(b,a)=b-a$, we get

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\frac{b-a}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Theorem 3.3. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable mapping on $(a,a+\eta(b,a))$ where $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$. If $|f'|^q$ where $q\geq 1$ is preinvex with respect to $\eta(b,a)$, then the following fractional inequality holds*

$$|F(a,b,\eta,f)|$$
$$\leq\frac{\eta(b,a)}{4(\alpha+1)}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 3.1, properties of modulus, and preinvexity of $|f'|$ on $[a,a+\eta(b,a)]$, we have

$$|F(a,b,\eta,f)|\leq\frac{\eta(b,a)}{8}\left(\int_0^1|t^{\alpha}-(1-t)^{\alpha}|\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|dt\right.$$

$$+\int_0^1 |t^\alpha-(1-t)^\alpha|\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|dt\Bigg)$$

$$\leq\tfrac{\eta(b,a)}{8}\left(\int_0^1 |t^\alpha-(1-t)^\alpha|\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\int_0^1 |t^\alpha-(1-t)^\alpha|\left|f'\left(a+\tfrac{t}{2}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 |t^\alpha-(1-t)^\alpha|\left|f'\left(a+\tfrac{1+t}{2}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq\tfrac{\eta(b,a)}{8}\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)\,dt+\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f'(a)\right|^q\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)\left(1-\tfrac{t}{2}\right)+\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)\left(1-\tfrac{t}{2}\right)dt\right)\right.$$

$$\left.+\left|f'(b)\right|^q\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)\tfrac{t}{2}dt+\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)\tfrac{t}{2}dt\right)\right)^{\frac{1}{q}}$$

$$+\left(\left|f'(a)\right|^q\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)\left(1-\tfrac{1+t}{2}\right)dt+\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)\left(1-\tfrac{1+t}{2}\right)dt\right.\right.$$

$$\left.\left.+\left|f'(b)\right|^q\left(\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)\tfrac{1+t}{2}dt+\int_{\frac{1}{2}}^1(t^\alpha-(1-t)^\alpha)\tfrac{1+t}{2}dt\right)\right)\right)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{\eta(b,a)}{4(\alpha+1)}\left(1-\left(\tfrac{1}{2}\right)^\alpha\right)\left(\left(\tfrac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\tfrac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

The proof is completed. ∎

Corollary 3.5. *In Theorem 3.3, if we choose $\eta(b,a) = b - a$ we get*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{\alpha}{4}\left(\frac{2}{b-a}\right)^{\alpha}\Gamma(\alpha+1)\right.$$
$$\left.\times\left(J^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) + J^{\alpha}_{(b)^-} f\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq \frac{b-a}{4(\alpha+1)}\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.6. *In Theorem 3.3, if we take $\alpha = 1$, we obtain*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(a+\eta(b,a))}{2} + f\left(\frac{2a+\eta(b,a)}{2}\right)\right] - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)\,du\right|$$
$$\leq \frac{\eta(b,a)}{16}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

Moreover, if we choose $\eta(b,a) = b - a$, we get

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{16}\left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

4. APPLICATIONS INVOLVING THE ARITHMETIC AND LOGARITHMIC MEANS

We shall consider the means for arbitrary real numbers a, b.

The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{0,-1\}$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^3, (a+A(b,a))^3\right) + A^3(a, a+A(b,a)) - 2L_3^3(a, a+A(b,a))\right|$$
$$\leq \frac{3}{16}(a+b)\left(a^2+b^2\right).$$

Proof. The assertion follows from Theorem 3.1 with $\alpha = 1$ and $\eta(b,a) = A(a,b)$ applied to the function $f(x) = x^3$ which $f'(x) = 3x^2$ is preinvex function. ∎

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\begin{aligned}&\left|A\left(a^2, b^2\right) + A^2(a,b) - 2L_2^2(a,b)\right| \\ &\leq \tfrac{b-a}{2}\left(\tfrac{q-1}{2q-1}\right)^{\frac{1}{p}}\left(\left(\tfrac{3a^q+b^q}{4}\right)^{\frac{1}{q}} + \left(\tfrac{a^q+3b^q}{4}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. The assertion follows from Theorem 3.2 with $\alpha = 1, \eta(b,a) = b-a$ and $q \geq 2$ applied to the function $f(x) = \frac{1}{2}x^2$ which $f'(x) = x$ and $|f'(x)|^q = x^q$ convex function. ∎

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