ROMAI J., v.16, no.2(2020), 117–135

SEQUENTIAL RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS: NEW EXISTENCE AND NEW DATA DEPENDENCE RESULTS

Hafssa Yfrah[1], Zoubir Dahmani[2], Farooq Ahmed Gujar[3], Mehmet Zeki Sarikaya[4]
[1,2]*Laboratory of Pure and Applied Mathematics, Faculty of Exact Sciences and Informatics, University of Mostaganem, Algeria*
[3]*School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore*
[4]*Department of Mathematics, Ducze University, Turkey*
hafssa2014yfrah@gmail.com, zzdahmani@yahoo.fr, farooqgujar@gmail.com, sarikayamz@gmail.com

Abstract In this work, we are concerned with a new sequential nonlinear random fractional differential equation of fractional order with two sequential different orders and nonlocal conditions. An existence and uniqueness of solutions for the problem is obtained by means of an appropriate fixed point theorem. Then, new concepts on the sequential continuous and fractional derivative dependence are introduced. At the end, some results of stability on random as well for deterministic, related to the data dependence, are discussed. To the best of our knowledge, this is the first time where such random problem is considered.

Keywords: Sequential random differential equation, existence and uniqueness, mean square solution.
2010 MSC: 30C45, 39B72, 39B82.

1. INTRODUCTION

Fractional calculus is appearing in the different fields of scientific research such as: applied mathematics, physics, control theory, mechanical structures, thermodynamics, etc. [2, 7, 16]. For some recent studies on fractional calculus and fractional differential equations (FDEs), we refer the reader to a series of papers [3, 4, 6, 14, 17, 19].
Random fractional differential equations, as natural extensions of deterministic ones, arise in numerous fields with anomalous dynamics, such as network traffic, signal transmissions through strong magnetic fields [1, 5, 15, 20, 21]. Recently, for the initial random fractional differential problems have been investigated by several authors, to cite a few, we begin by the paper [11], where the authors have studied the following very interesting nonlinear random FDE

with a nonlocal condition:

$$\begin{cases} \mathbf{D}^{\alpha}X(t) &= c(t)f(X(t)) + b(t),\ t \in [0,T] \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k),\ a_k > 0,\ \tau_k \in]0,T[. \end{cases},$$

where, $\mathbf{D}^{\alpha}$ represent the mean square Caputo fractional derivative of order $\alpha \in]0,1]$.
Then, based on the above paper, the authors of [18] have been concerned with the following random problem

$$\begin{cases} \mathbf{D}^{\alpha}X(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\alpha-1}X(t)\big) \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k) \\ X_1 &= X'(0). \end{cases},$$

where, $\mathbf{D}^{\alpha}$ represent the mean square Caputo fractional derivative of order $\alpha \in]1,2]$.
Very recently, the authors of the paper [22] have studied the following high order nonlinear random FDE.

$$\begin{cases} \mathbf{D}^{\alpha}X(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\alpha-1}X(t), \ldots, \mathbf{D}^{\alpha-n+1}X(t)\big), \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k), \\ X_j &= X^{(j)}(0) \quad j = 1, \ldots, n-1. \end{cases}.$$

Some recent concepts have been introduced and other data dependance of the solutions have been discussed.
In this paper, we are concerned with a new class of nonlinear random differential equation with nonlocal conditions and two sequential fractional derivatives. So, we consider the problem:

$$\begin{cases} \mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\beta}X(t)\big), \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k),\ a_k > 0,\ \tau_k \in]0,T[\\ X_1 &= X^{(\beta)}(0), \end{cases}, \quad (1)$$

where, $\mathbf{D}^{\alpha}$ and $\mathbf{D}^{\beta}$ represents the mean square Caputo fractional derivative, with, α and β are in $]0,1]$, $X(\cdot)$ is a second random function, X_0, X_1 are a second random variable and a_k are positive real numbers, $f : \mathbb{L}_2(\Omega) \to \mathbb{R}$, $g : \mathbb{L}_2(\Omega) \to \mathbb{R}$, c and $b : J \to \mathbb{R}$, with, $J = [0,T]$.
The paper is organized as follows: in the next section, we recall all the necessary definitions and lemmas used in the rest of our work. Then, we present

the integral solution of the problem studied. After that, we obtain an existence and uniqueness of the solution in a Banach space. In the last section, we establish other results of random continuous and differentially dependence.

2. PRELIMINARIES

In this section, we present some definitions and notations of fractional calculus, and some basic mean square results that we need it in this work [9, 10, 12, 13].

Let $(\Omega, E, \mathbb{P})$ be a compete probability space.

Let $X(t,\omega) = \{X(t), t \in J = [0,T], \omega \in \Omega\}$, be a second-order random variable, i.e., $E(X^2(t)) := \int_\Omega X^2 d\mathbb{P} < \infty$. Let $\mathbb{L}_2(\Omega)$ is the Banach space of random variables, such that their $E(X^2)$ are finite.
Let $\mathcal{C} = \mathcal{C}(J, \mathbb{L}_2(\Omega))$ the Banach space of the class of all mean continuous second order random processes with:

$$||X||_{\mathcal{C}} = \sup_{t\in J} ||X||_2 = \sup_{t\in J} \sqrt{E(X(t))^2}.$$

Now, we have the following definitions:

Definition 2.1. *Let $X(t) \in \mathcal{C}$ and $p > 0$. The mean square Riemann-Liouville fractional integral of order p of $X(t)$ is defined as*

$$\mathbf{I}^p X(t) := \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} X(s) ds,$$

where, $\Gamma(\cdot)$ denotes the gamma function.

Definition 2.2. *Let $X(t) \in \mathcal{C}$ and $q > 0$. The mean square Caputo derivative of fractional order q is defined as:*

$$\mathbf{D}^q X(t) := \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} X^{(n)}(s) ds, n-1 < q < n,\ n = [q]+1,$$

where $n \in \mathbb{N}^*$, $X^{(n)}$ denotes the mean square differentiation, and $X(t)$ is assumed to be mean square differentiable.

Lemma 2.1. *Let $X(t) \in \mathcal{C}$. For $q > 0$, the general solution of the differential equation $\mathbf{D}^q X(t) = 0$, is given by*

$$X(t) = C_0 + C_1 t - + \cdots + C_{n-1} t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1, \ldots, n-1, n = [q]+1$.

Lemma 2.2. *Let $X(t) \in \mathcal{C}$. Let $q > 0$, so,*

$$\mathbf{I}^q \mathbf{D}^q X(t) = X(t) + C_0 + C_1 t + \cdots + C_{n-1} t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1, \ldots, n-1, n = [q]+1$.

3. A SEQUENTIAL RANDOM INTEGRAL SOLUTION

Lemma 3.1. *The integral solution of the sequential random FDE* (1) *is given by:*

$$\begin{aligned} X(t) =& a^{-1}\Bigg[X_0 - X_1\sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \\ &- \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\alpha-1}X(s)\big)\Big]ds\Bigg] \\ &+ X_1\frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\alpha-1}X(s)\big)\Big]ds, \end{aligned} \tag{2}$$

where $a = 1 + \sum_{k=1}^{n} a_k$.

Proof. We note

$$Y(t) := c(t)f\big(X(t)\big) + b(t)g\big(\mathbf{D}^{\beta}X(t)\big)$$

and we consider

$$\mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) = Y(t) \tag{3}$$

for which we apply the mean square Riemann-Liouville fractional integral of order α to (3) to obtain

$$\mathbf{D}^{\beta}X(t) = \gamma_0 + \mathbf{I}^{\alpha}Y(t), \tag{4}$$

where, $\gamma_0 \in \mathbb{R}$. Again, we apply the mean square Riemann-Liouville fractional integral of order β to (4). We can write

$$X(t) = \gamma_1 + \gamma_0\frac{t^{\beta}}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta}Y(t), \tag{5}$$

where, $\gamma_1 \in \mathbb{R}$. We take $t = 0$ in (5), we get $X(0) = \gamma_1$, and we take $t = \tau_k$ in (5), we get,

$$X(\tau_k) = \gamma_1 + \gamma_0\frac{\tau_k^{\beta}}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta}Y(t)\Big|_{t=\tau_k},$$

so,

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \gamma_1 + \sum_{k=1}^{n} a_k\left[\gamma_1 + \gamma_0\frac{\tau_k^{\beta}}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta}Y(t)\Big|_{t=\tau_k}\right]. \tag{6}$$

The derivative d'order $\mathbf{D}^\beta$ of (5) is

$$X^{(\beta)}(t) = \gamma_0 + \mathbf{I}^\alpha Y(t),$$

and we take $t = 0$, we have

$$X^{(\beta)}(0) = \gamma_0 = X_1.$$

Substituting the value of γ_0 in (6), we get the value of γ_1

$$\gamma_1 = \frac{1}{1 + \sum\limits_{k=1}^{n} a_k}\left[X_0 - X_1 \sum_{k=1}^{n} a_k \frac{\tau_k^\beta}{\Gamma(\beta+1)} - \sum_{k=1}^{n} a_k\, \mathbf{I}^{\alpha+\beta} Y(t)\Big|_{t=\tau_k}\right].$$

The proof is thus achieved. ∎

4. A UNIQUE SEQUENTIAL SOLUTION IN $\|.\|_F$ SENSE

Now, we introduce the Banach space

$$F := \{X \in \mathcal{C}, \mathbf{D}^\beta X \in \mathcal{C}\},$$

equipped with the norm

$$\|X\|_F = \|X\|_{\mathcal{C}} + \|\mathbf{D}^\beta X\|_{\mathcal{C}}.$$

We define the new operator over F as follows

$$\begin{array}{llll} \Phi & : F & \to & F \\ & X & \to & \Phi X, \end{array} \tag{7}$$

$$\Phi X(t) := a^{-1}\Bigg[X_0 - X_1 \sum_{k=1}^{n} a_k \frac{\tau_k^\beta}{\Gamma(\beta+1)}$$
$$-\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\alpha-1}X(s)\big)\Big]ds\Bigg] + X_1 \frac{t^\beta}{\Gamma(\beta+1)}$$
$$+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\alpha-1}X(s)\big)\Big]ds,$$

where $a = 1 + \sum\limits_{k=1}^{n} a_k$.

We prove the following result.

Lemma 4.1. *Suppose that $f, g : \mathbb{L}_2(\Omega) \to \mathbb{R}$ and, $c, b : J \to \mathbb{R}$ are continuous functions. In addition, we assume that*

(H1): $\exists\, K_1, K_2 > 0$, $\forall x, y \in \mathbb{L}_2(\Omega)$

$$||f(x) - f(y)||_2 \leq K_1 ||x - y||_2,$$

and,

$$||g(x) - g(y)||_2 \leq K_2 ||x - y||_2$$

(H.2): $\sup_{t\in J} |c(t)| = u < \infty,$ *and* $\sup_{t\in J} |b(t)| = v < \infty.$

Then, we have $\Phi : F \to F$.

Proof. Before starting the proof, we suppose $f(0) \leq m_1 < \infty$, and, $g(0) \leq m_2 < \infty$.
Let $X \in F$, and $\forall t_1, t_2 \in J$, where, $|t_2 - t_1| \leq \delta$, we have

$$\begin{aligned}
&\Phi X(t_2) - \Phi X(t_1)\\
&= X_1 \frac{t_2^\beta}{\Gamma(\beta+1)} + \int_0^{t_2} \frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds\\
&- X_1 \frac{t_1^\beta}{\Gamma(\beta+1)} - \int_0^{t_1} \frac{(t_1 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(t)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds,\\
=& \frac{t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} X_1 + \int_0^{t_1} \frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds\\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds\\
&- \int_0^{t_1} \frac{(t_1 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(t)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds,\\
=& \frac{t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} X_1 + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds\\
&+ \int_0^{t_1} \left[\frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_1 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right] \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds.
\end{aligned}$$

By using the above introduced norm

$$\begin{aligned}
&||\Phi X(t_2) - \Phi X(t_1)||_2\\
&\leq \frac{t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} X_1 + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[|c(s)| ||f\big(X(s)\big)||_2 + |b(s)| ||g\big(\mathbf{D}^\beta X(s)\big)||_2\Big] ds +\\
&\int_0^{t_1} \left[\frac{(t_2 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_1 - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right] \Big[|c(s)| ||f\big(X(s)\big)||_2 + |b(s)| ||g\big(\mathbf{D}^\beta X(s)\big)||_2\Big] ds.
\end{aligned} \tag{8}$$

We have

$$||f(X(t))||_2 - |f(0)| \leq ||f(X(t)) - f(0)||_2,$$

and, according to the assumptions $(H.1)$, we obtain,

$$\sup_{t\in[0,T]} ||f(X(t))||_2 \leq K_1||X(t)||_2 + m_1.$$

With the same arguments for g and according to $(H.1)$, we get

$$\sup_{t\in[0,T]} ||g(\mathbf{D}^{\beta}X(t))||_2 \leq K_2||X(t)||_2 + m_2.$$

Therefore, we have

$$\begin{aligned}
&||\Phi X(t_2) - \Phi X(t_1)||_{\mathcal{C}} \\
&\leq \frac{t_2^{\beta} - t_1^{\beta}}{\Gamma(\beta+1)}X_1 + \frac{(t_2-t_1)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_2 + m_{\mathcal{C}}\big)\Big] \\
&+ \left[\frac{-(t_2-t_1)^{\alpha+\beta} + t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\Big[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big)\Big], \\
\leq& \frac{t_2^{\beta} - t_1^{\beta}}{\Gamma(\beta+1)}X_1 + \left[\frac{t_2^{\alpha+\beta} - t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\Big[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big)\Big],
\end{aligned} \tag{9}$$

On the other hand, we have

$$\begin{aligned}
&\mathbf{D}^{\beta}\Phi X(t_2) - \mathbf{D}^{\beta}\Phi X(t_1) \\
&= X_1 + \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds \\
&- X_1 - \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds,
\end{aligned} \tag{10}$$

Hence,

$$\begin{aligned}
&||\mathbf{D}^{\beta}\Phi X(t_2) - \mathbf{D}^{\beta}\Phi X(t_1)||_2 \\
&\leq \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)||_2\Big]ds \\
&- \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)||_2\Big]ds,
\end{aligned} \tag{11}$$

we pass to the sup of $||\cdot||_2$ on J, we obtain

$$\begin{aligned}
&||\mathbf{D}^{\beta}\Phi X(t_2) - \mathbf{D}^{\beta}\Phi X(t_1)||_{\mathcal{C}} \\
&\leq \left[\frac{t_2^{\alpha} - t_1^{\alpha}}{\Gamma(\alpha+1)}\right]\Big[u\big(K_1||X(t)||_{\mathcal{C}} + m_1\big) + v\big(K_2||X(t)||_{\mathcal{C}} + m_2\big)\Big].
\end{aligned} \tag{12}$$

From inequalities (9) and (12), it yields that

$$\begin{aligned}||\Phi X(t_2)-\Phi X(t_1)||_F \leq &||\Phi X(t_2)-\Phi X(t_1)||_{\mathcal{C}}+||\mathbf{D}^\beta \Phi X(t_2)-\mathbf{D}^\beta \Phi X(t_1)||_{\mathcal{C}},\\ \leq &\frac{t_2^\beta - t_1^\beta}{\Gamma(\beta+1)}X_1+\left[\frac{t_2^{\alpha+\beta}-t_1^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}\right]\\ &\times\left[u\big(K_1||X(t)||_{\mathcal{C}}+m_1\big)+v\big(K_2||X(t)||_{\mathcal{C}}+m_2\big)\right]\to 0,\end{aligned} \tag{13}$$

as $t_2 \to t_1$. Hence, the lemma is proved. ∎

Using the above Banach space with its introduced norm, we shall prove the existence and uniqueness of sequential solutions of the random FDE.

Theorem 4.1. *Assume that* $(H.1)$ *and* $(H.2)$ *hold. The random problem* (1) *has a unique solution provided that* $A<1$, *where, we have*

$$A:=\left(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\left(uK_1+vK_2\right).$$

Proof. Let $X,Y\in F$, we have

$$\begin{aligned}&\Phi X(t)-\Phi Y(t)=\\ &-a^{-1}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds\\ &+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^\beta X(s)\big)\right]ds\\ &+a^{-1}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(Y(s)\big)+b(s)g\big(\mathbf{D}^\beta Y(s)\big)\right]ds\\ &-\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left[c(s)f\big(Y(s)\big)+b(s)g\big(\mathbf{D}^\beta Y(s)\big)\right]ds,\end{aligned} \tag{14}$$

and this implies that

$$\begin{aligned}&\Phi X(t)-\Phi Y(t)=\\ &-a^{-1}\sum_{k=1}^n a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)\big(f\big(X(s)\big)-f\big(Y(s)\big)\big)\\ &\qquad\qquad+b(s)\big(g\big(\mathbf{D}^\beta X(s)\big)-g\big(\mathbf{D}^\beta Y(s)\big)\big)\Big]ds\\ &+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)\big(f\big(X(s)\big)-f\big(Y(s)\big)\big)+b(s)\big(g\big(\mathbf{D}^\beta X(s)\big)-g\big(\mathbf{D}^\beta Y(s)\big)\big)\Big]ds,\end{aligned} \tag{15}$$

consequently,

$$
\begin{aligned}
&||\Phi X(t)-\Phi Y(t)||_2 \leq \\
&a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(Y(s)\big)||_2 \\
&\qquad\qquad + |b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big]ds \qquad (16)\\
&+\int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(Y(s)\big)||_2 \\
&\qquad\qquad + |b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big]]ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||\Phi X(t)-\Phi Y(t)||_{\mathcal{C}} \leq & a^{-1}\sum_{k=1}^{n} a_k \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||X-Y||_{\mathcal{C}}\Big] \\
&+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||X-Y||_{\mathcal{C}}\Big],
\end{aligned} \tag{17}
$$

thus,

$$
||\Phi X(t)-\Phi Y(t)||_{\mathcal{C}} \leq 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||X-Y||_{\mathcal{C}}\Big]. \tag{18}
$$

Moreover, we have

$$
\begin{aligned}
\mathbf{D}^{\beta}\Psi X(t)-\mathbf{D}^{\beta}\Psi Y(t) = & \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds \\
& -\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(Y(s)\big)+b(s)g\big(\mathbf{D}^{\beta}Y(s)\big)\Big]ds, \\
= & \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)\big(f\big(X(s)\big)-f\big(Y(s)\big)\big) \\
& + b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)\big)\Big]ds.
\end{aligned} \tag{19}
$$

Hence, it yields that

$$||\mathbf{D}^{\beta}\Psi X(t)-\mathbf{D}^{\beta}\Psi Y(t)||_2 \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)-f\big(Y(s)\big)||_2 + |b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big]ds, \tag{20}$$

so,

$$||\mathbf{D}^{\beta}\Psi X(t)-\mathbf{D}^{\beta}\Psi Y(t)||_{\mathcal{C}} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||X-Y||_{\mathcal{C}}\Big]. \tag{21}$$

By the inequalities (18) and (21), we get

$$||\Phi X(t)-\Phi Y(t)||_F \leq \Big[2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big]\Big[uK_1+vK_2\Big]||X-Y||_{\mathcal{C}}, \tag{22}$$

At the end, we conclude that

$$||\Phi X(t)-\Phi Y(t)||_F \leq A||X-Y||_F. \tag{23}$$

Finally, Φ is contractive as $A<1$. This ends the proof. ∎

5. DEPENDANCE ON RANDOM DATA

In this section, we establish new concepts for the above sequential random FDE with its nonlocal condition, in addition, we prove the results for the continuous and differentially dependence on random/deterministic data.

So let us consider the following sequential random FDE with the nonlocal conditions:

$$\begin{cases} \mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) &= c(t)f(X(t))+b(t)g\big(\mathbf{D}^{\beta}X(t)\big), \\ \tilde{X}_0 &= X(0)+\sum_{k=1}^{n}a_kX(\tau_k), \\ \tilde{X}_1 &= X^{(\beta)}(0), \end{cases}, \tag{24}$$

and, we study the continuous dependance on the random data X_0 and X_1 of the solution of the sequential random problem (1).

Definition 5.1. *The solution $X\in F$ of the random problem* (1) *is continuously and β-differentially dependent on the random data X_0 and X_1 if for all*

$\epsilon > 0$, $\exists \delta_0 > 0, \delta_1 > 0$ *such that* $||X_0 - \tilde{X}_0||_2 \leq \delta_0$, *and,* $||X_1 - \tilde{X}_1||_2 \leq \delta_1$, $\Rightarrow ||X_0 - \tilde{X}_0||_F \leq \epsilon$.

Theorem 5.1. *Assume that* $(H.1)$ *and* $(H.2)$ *hold. Then, the solution of the random FDE is continuously and* β*-differentially dependent on* X_0 *and* X_1*.*

Proof. Let $X(t)$ as defined in (2) be the solution of the problem (1) and

$$\begin{aligned}\tilde{X}(t) =& a^{-1}\Bigg[\tilde{X}_0 - \tilde{X}_1 \sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \\ &- \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(\tilde{X}(s)\big) + b(s) g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds\Bigg] \\ &+ \tilde{X}_1 \frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(\tilde{X}(s)\big) + b(s) g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds,\end{aligned} \tag{25}$$

be the solution of the problem (24). Then,

$$\begin{aligned}&X(t) - \tilde{X}(t) = \\ &a^{-1}\big(X_0 - \tilde{X}_0\big) - a^{-1} \sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \big(X_1 - \tilde{X}_1\big) - a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &\times \Big[c(s)\big(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big) - g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\big)\Big] ds \\ &+ \frac{t^{\beta}}{\Gamma(\beta+1)} \big(X_1 - \tilde{X}_1\big) \\ &+ \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)\big(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big) - g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\big)\Big] ds.\end{aligned} \tag{26}$$

By using $|| \cdot ||_2$ on J, we get

$$\begin{aligned}&||X(t) - \tilde{X}(t)||_2 \leq \\ &a^{-1}||X_0 - \tilde{X}_0||_2 + a^{-1} \sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} ||X_1 - \tilde{X}_1||_2 + a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &\times \Big[|c(s)|||f\big(X(s)\big) - f\big(\tilde{X}(s)\big)||_2 + |b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big) - g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)||_2\Big] ds\end{aligned} \tag{27}$$

$$
\begin{aligned}
&+\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}||X_1-\tilde{X}_1||_2+\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(\tilde{X}(s)\big)||_2 \\
&+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)||_2\Big]ds,
\end{aligned}
\tag{28}
$$

hence

$$
\begin{aligned}
&||X(t)-\tilde{X}(t)||_{\mathfrak{C}}\leq \\
&a^{-1}\delta_0+a^{-1}\sum_{k=1}^{n}a_k\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}\delta_1+\frac{t^{\beta}}{\Gamma(\beta+1)}\delta_1 \\
&+a^{-1}\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X(s)-\tilde{X}(s)||_{\mathfrak{C}}+vK_2||X(s)-\tilde{X}(s)||_{\mathfrak{C}}\Big]ds \\
&+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X(s)-\tilde{X}(s)||_{\mathfrak{C}}+vK_2||X(s)-\tilde{X}(s)||_{\mathfrak{C}}\Big]ds, \\
\leq& a^{-1}\delta_0+a^{-1}\sum_{k=1}^{n}a_k\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1+\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1 \\
&+a^{-1}\sum_{k=1}^{n}a_k\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_{\mathfrak{C}}+vK_2||X(s)-\tilde{X}(s)||_{\mathfrak{C}}\Big] \\
&+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_{\mathfrak{C}}+vK_2||X(s)-\tilde{X}(s)||_{\mathfrak{C}}\Big].
\end{aligned}
\tag{29}
$$

So

$$
\begin{aligned}
||X(t)-\tilde{X}(t)||_{\mathfrak{C}}\leq& a^{-1}\delta_0+\big(a^{-1}\sum_{k=1}^{n}a_k+1\big)\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1 \\
&+\Big(a^{-1}\sum_{k=1}^{n}a_k\tau_k+1\Big)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big(uK_1+vK_2\Big)||X(s)-\tilde{X}(s)||_{\mathfrak{C}},
\end{aligned}
$$

$$
||X(t)-\tilde{X}(t)||_{\mathfrak{C}}\leq \delta_0+2\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1+2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big(uK_1+vK_2\Big)||X(s)-\tilde{X}(s)||_{\mathfrak{C}}.
\tag{30}
$$

Furthermore,

$$\mathbf{D}^{\beta}X(t)-\mathbf{D}^{\beta}\tilde{X}(t)=X_1+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds$$
$$-\tilde{X}_1-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(\tilde{X}(s)\big)+b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big]ds. \tag{31}$$

Then, we have

$$\begin{aligned}||\mathbf{D}^{\beta}X(t)-\mathbf{D}^{\beta}\tilde{X}(t)||_{\mathcal{C}}\leq&||X_1-\tilde{X}_1||_2+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)-f\big(\tilde{X}(s)\big)||_2\\&+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)||_2\Big]ds,\\\leq&\delta_1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(uK_1+vK_2\Big)||X-\tilde{X}||_{\mathcal{C}}.\end{aligned} \tag{32}$$

Combining the inequalities (30) and (32), we obtain

$$\begin{aligned}||X(t)-\tilde{X}(t)||_F\leq&\delta_0+\Big(2\frac{T^{\beta}}{\Gamma(\beta+1)}+1\Big)\delta_1\\&+\Big(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(uK_1+vK_2\Big)||X(s)-\tilde{X}(s)||_F\end{aligned} \tag{33}$$

witch implies that

$$||X(t)-\tilde{X}(t)||_F\leq\frac{\delta_0+\Big(2\frac{T^{\beta}}{\Gamma(\beta+1)}+1\Big)\delta_1}{1-A}=\epsilon. \tag{34}$$

This ends the proof. ∎

We pass to study the dependance on the deterministic data $a_k>0$ of the solution of the sequential random problem (1).
We consider the sequential random FDE with the nonlocal conditions

$$\left\{\begin{array}{rcl}\mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t)&=&c(t)f(X(t))+b(t)g\big(\mathbf{D}^{\beta}X(t)\big),\\X_0&=&X(0)+\displaystyle\sum_{k=1}^{n}\tilde{a}_kX(\tau_k),\\X_1&=&X^{(\beta)}(0),\end{array}\right. , \tag{35}$$

and we introduce the following definition.

Definition 5.2. *The solution $X \in F$ of the sequential random FDEs* (1) *is continuously and β-differentially dependent on the deterministic data a_k if for all $\epsilon > 0, \exists \delta > 0$ such that*
$|a_k - \tilde{a}_k| < \delta \Rightarrow ||X - \tilde{X}||_F \leq \epsilon$.

No, we present to the reader the following result:

Theorem 5.2. *Assume that $(H.1)$ and $(H.2)$ hold. Then, the solution of the sequential random FDE is continuously and β-differentially dependent on a_k.*

Proof. Before starting the proof, we introduce the following notations

$$\begin{aligned}
\mathcal{K}_1 =& a^{-1} - \tilde{a}^{-1},\\
\mathcal{K}_2 =& \tilde{a}^{-1}\sum_{k=1}^{n}\tilde{a}_k - a^{-1}\sum_{k=1}^{n}a_k,\\
\mathcal{K}_3 =& \tilde{a}^{-1}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k}\frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta\tilde{X}(s)\big)\Big]ds\\
&- a^{-1}\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\Big]ds,\\
\mathcal{K}_4 =& \int_0^{t}\frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^\beta X(s)\big)\Big]ds\\
&- \int_0^{t}\frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta\tilde{X}(s)\big)\Big]ds.
\end{aligned}$$

Let $X(t)$ (as defined in (2)) be the solution of (1) and

$$\begin{aligned}
\tilde{X}(t) =& \tilde{a}^{-1}\Big[X_0 - X_1\sum_{k=1}^{n}\tilde{a}_k\frac{\tau_k^\beta}{\Gamma(\beta+1)}\\
&- \sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k}\frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta\tilde{X}(s)\big)\Big]ds\Big]\\
&+ X_1\frac{t^\beta}{\Gamma(\beta+1)} + \int_0^{t}\frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^\beta\tilde{X}(s)\big)\Big]ds,
\end{aligned}\tag{36}$$

be the solution of (35).
Then,

$$X(t) - \tilde{X}(t) = \mathcal{K}_1X_0 + \frac{\tau_k^\beta}{\Gamma(\beta+1)}\mathcal{K}_2X_1 + \mathcal{K}_3 + \mathcal{K}_4. \tag{37}$$

Hence, we get

$$
\begin{aligned}
|\mathcal{K}_1| \leq& |\sum_{k=1}^{n}\tilde{a}_k - \sum_{k=1}^{n} a_k| \\
\leq& n\delta,
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
\mathcal{K}_3 =& \tilde{a}^{-1}\Big(1+\sum_{k=1}^{n}\tilde{a}_k\Big)\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big)+b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big]ds \\
&- a^{-1}\Big(1+\sum_{k=1}^{n}a_k\Big)\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds \\
&- \tilde{a}^{-1}\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(\tilde{X}(s)\big)+b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big]ds \\
&+ a^{-1}\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds,
\end{aligned}
\tag{39}
$$

so

$$
\begin{aligned}
\mathcal{K}_3 =& -\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)(f\big(X(s)\big)-f\big(\tilde{X}(s)\big)) \\
&\qquad + b(s)(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big))\Big]ds \\
&+ (a^{-1}-\tilde{a}^{-1})\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds \\
&+ \tilde{a}^{-1}\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)(f\big(X(s)\big)-f\big(\tilde{X}(s)\big)) \\
&\qquad + b(s)(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big))\Big]ds,
\end{aligned}
\tag{40}
$$

we know that

$$
\sup_{t\in[0,T]} ||f\big(X(t)\big)||_2 \leq K_1||X(t)||_2 + m_1,
$$

and

$$
\sup_{t\in[0,T]} ||g\big(\mathbf{D}^{\beta}X(t)\big)||_2 \leq K_2||X(t)||_2 + m_2.
$$

By using our hypotheses, we get

$$\begin{aligned}||\mathcal{K}_3||_2 \leq &\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\Big]ds \\ &+ n\delta \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[u(K_1||X(s)||_2 + m_1) + v(K_2||X(s)||_2 + m_2)\Big]ds \\ &+ \tilde{a}^{-1}\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\Big]ds,\end{aligned} \tag{41}$$

hence,

$$\begin{aligned}||\mathcal{K}_3||_2 \leq &\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\Big] \\ &+ n\delta\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[u(K_1||X(s)||_2 + m_1) + v(K_2||X(s)||_2 + m_2)\Big] \\ &+ \tilde{a}^{-1}\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\Big].\end{aligned} \tag{42}$$

By $(H.1)$ and $(H.2)$, we get

$$||\mathcal{K}_4||_2 \leq \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2 + vK_2||X(s)-\tilde{X}(s)||_2\Big]. \tag{43}$$

Then,

$$\begin{aligned}&||X(t)-\tilde{X}(t)||_2 \leq \\ &|\mathcal{K}_1|||X_0||_2 + \frac{\tau_k^{\beta}}{\Gamma(\beta+1)}||\mathcal{K}_2||_2||X_1||_2 + ||\mathcal{K}_3||_2 + ||\mathcal{K}_4||_2 \\ \leq &n\delta||X_0||_2 + \frac{\tau_k^{\beta}}{\Gamma(\beta+1)}n\delta||X_1||_2 \\ &+ \big(1+\tilde{a}^{-1}\big)\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(uK_1 + vK_2\big)||X(s)-\tilde{X}(s)||_2 \\ &+ n\delta\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[u(K_1||X(s)||_2 + m_1) + v(K_2||X(s)||_2 + m_2)\Big] \\ &+ \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(uK_1 + vK_2\big)||X(s)-\tilde{X}(s)||_2,\end{aligned} \tag{44}$$

we pass now to the sup over J, it yields that

$$\begin{aligned}||X-\tilde{X}||_{\mathcal{C}} \leq & n\delta\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2 \\ & +\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\big)\Big] \\ & +3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(uK_1+vK_2\big)||X-\tilde{X}||_{\mathcal{C}}.\end{aligned} \tag{45}$$

Also, we have

$$\begin{aligned}\mathbf{D}^{\beta}X(t)-\mathbf{D}^{\beta}\tilde{X}(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}&\Big[c(s)(f\big(X(s)\big)-f\big(\tilde{X}(s)\big)) \\ &+b(s)(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big))\Big]ds\end{aligned} \tag{46}$$

so,

$$||\mathbf{D}^{\beta}X-\mathbf{D}^{\beta}\tilde{X}||_{\mathcal{C}} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)}\big(uK_1+vK_2\big)||X-\tilde{X}||_{\mathcal{C}}. \tag{47}$$

By the inequalities (45) and (47), we observe that

$$\begin{aligned}&||X-\tilde{X}||_F \\ &\leq n\delta\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2 \\ &+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\big)\Big] \\ &+\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\big[uK_1+vK_2\big]||X-\tilde{X}||_{\mathcal{C}} \\ &\leq \frac{n\delta\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||X||_{\mathcal{C}}+m_2)\big)\Big]}{1-L},\end{aligned} \tag{48}$$

where $L=\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\big[uK_1+vK_2\big]$. ∎

References

[1] S. Abbas, N. A. Arifi, M. Bechohra, Y. Zhou, *Random coupled Hilfer and Hadamard fractional differential systems in generalized Banach spaces*, Mathematics. **7** (2019), 285.

[2] S. Abbas, M. Benchohra, J. R. Greaf, J. Henderson, *Implicit fractional differential and inegral equations: Existence ane Stability*, Walter de Gryter Gmbh Co KG. (2018), 26.

[3] A. Anber, Z. Dahmani, *The variational iteration method for solving the fractional coupled lotka-volterra equation*, Journal of Interdisciplinary Mathematics. **14(4)** (2011), 373-388.

[4] A. Benzidane, Z. Dahmani, *A class of nonlinear singular differential equations*, Journal of Interdisciplinary Mathematics. **22(6)** (2019), 991-1007.

[5] C. Burgos, J.-C. Crtés, L. Villafuerte, R.J. Villanueva, *Mean square convergent numerical solutions of random fractional differential equations: Approximations of moments and density*, Journal of Computation and Applied Mathematics. 3 April 2020.

[6] Z. Dahmani, M.A. Abdelaoui, M. Houas, *Polynomial solutions for a class of fractional differential equations and systems*, Journal of Interdisciplinary Mathematics. **21(3)** (2018), 669-680.

[7] L. Debnath , *Recent applications of fractional calculus to science and engineering*, International Journal of Mathematics and Mathematical Sciences. **54** (2003), 3413-3442.

[8] B. Dumitru, Fahd Jarad, Ekin Uğurlu, *Singular conformable sequential differential equations with distributional potentials*, Quaestiones Mathematicae. **42(3)** (2018), 277-287.

[9] A. M. A. El-Sayed, *The mean square Riemann-Liouville stochastic fractional derivative and stochastic fractional order differential equation*, Math. Sei. Res. J. **9** (2005),142-150.

[10] A.M.A. El-Sayed, *On the stochastic fractional calculus operators*, Journal of Fractional Calculus and Applications. **6(1)** (Jan. 2015), 101-109.

[11] A. M. A. El- Sayed, F. Gaafar, M. El-Gendy, *Continuous dependence of the solution of random fractional-order differential equation with nonlocal conditions*, J. Fractional Differential Calculus, **7(1)**(2017), 135-149.

[12] F.M. Hafiz, *The fractional calculus for some stochastic processes*, Stoch. Anal. Appl. **22** (2004), 507-523.

[13] F.M. Hafiz, A.M.A. El-Sayed, M.A. El-Tawil, *On a stochastic fractional calculus*, Frac. Calc; Appl. Anal. **4** (2001), 81-90.

[14] M. Houas, Z. Dahmani, *Coupled systems of inegro-differential equations involving Reimann-Liouville integrals and Caputo derivatives*, Acta Univ. Apulensis Math. Inform. **28** (2014), 133-6150.

[15] K. Kanagarajan, E. M. Alsayed, S. Harikrishnan, *A general study on random integro-differential equations of arbitrary order*, Journal of Applied Analysis and Computation. **9(4)** (August 2019), 1407-1424.

[16] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[17] M.Z. Sarikaya ,M. Bezziou, Z. Dahmani, *New operators for fractional integration theory with some applications*, Journal of Mathematical Extension. **12(1)** (2018), 87-100.

[18] I. Slimane, Z. Dahmani, *A continuous and fractional derivative dependence of random differential equations with nonlocal conditions*, Journal of Interdiscip Math. Accepted, April 2020.

[19] A. Taieb, Z. Dahmani, *Fractional system of nonlinear integro-differential equations*, Journal of Fractional calculus and Applications. **10(1)** (2019), 55-67.

[20] H. Vu, Truong Vinh An, Hoa Van Ngo, *Random fractional differential equations with Riemann-Liouville -type fuzzy differentiability concept*, Journal of Intelligent and Fuzzy Systems. **36(6)** (2019), 6467-6480.

[21] H. Vu, Hoa Van Ngo, *On initial value problem of random fractional differential equation with impulses*, Hacettepe Journal of Mathematics and Statistics, **49(1)** (2020), 282-293.

[22] H. Yfrah, Z. Dahmani, L. Tabharit, A. Abdelenbi, *High order random fractional differential equations: existence, uniqueness and data dependence*, Journal of Interdisciplinary Mathematics. Accepted, November 2020.