

ON GIRAUX SUBCATEGORIES IN LOCALLY CONVEX SPACES

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Abstract In this paper, we examine the class of Giroux subcategories in the category of locally convex spaces and we show that there exists a bijective correspondence between this lattice of all classes of left and right complete morphisms. Each Giroux subcategories generates some standard bicategorical structure. We describe the classe of injections and the classes of projections of these structures. Every injective object permits to construct a Giroux subcategory. We consider also the reciprocal problem.

Notation. $\mathcal{E}_u$ (resp. $\mathcal{M}_u$) the class of universal epi (resp. mono); $\mathcal{E}_p$ (resp. $\mathcal{M}_p$) the class of precise epi (resp. mono); $\mathcal{E}_p = \mathcal{M}_u^\perp$, $\mathcal{M}_p = \mathcal{E}_u^\perp$; if $\mathcal{K}$ (resp. $\mathcal{R}$) is coreflective (resp. reflective) subcategory, then $k: \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{K}$ ($r: \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{R}$) is the coreflector (resp. reflector) functor; $\mathcal{M}$ (resp. $\mathcal{S}$, Σ) is the subcategory of spaces endowed with Mackey's topology (resp. weak topology; the finest locally convex topology).

1. CLASSES OF LEFT-COMPLETE AND RIGHT-COMPLETE MORPHISMS

In the category of locally convex spaces $\mathcal{C}_2\mathcal{V}$ there are many classes of mophisms which are left-complete and right-complete.

Definition 1.1. *Let $\mathcal{A}$ be a class of morphisms of a category $\mathcal{C}$. The class $\mathcal{A}$ is called left – stable if for any pull-back square $uv' = vu'$ with $u \in \mathcal{A}$, it follows that $u' \in \mathcal{A}$ too.*

The class $\mathcal{A}$ possesses the following properties.

1. $\mathcal{I}so \subset \mathcal{A}$.
2. The class $\mathcal{A}$ is closed with respect to the composition.
3. The class $\mathcal{A}$ is left-stable.
4. The class $\mathcal{A}$ is closed with respect to the products.
5. For any object X of the category $\mathcal{C}$ any family of objects of the category $\mathcal{C}/X$ which belong to $\mathcal{A}$ possess the product.

Similarly we can define the dual notions: a right-stable class and a right-complete class.

Definition 1.2. *Let $\mathcal{L}$ be a class of morphisms of category $\mathcal{C}$. The object $A \in |\mathcal{C}|$ is called $\mathcal{L}$ -injective if for any two morphisms $l : X \rightarrow Y$, $l \in \mathcal{L}$ and $f : X \rightarrow A$ there exists a morphism $g : Y \rightarrow A$ such that $gl = f$.*

It is said that the category $\mathcal{C}$ has sufficiently many $\mathcal{L}$ -injective objects if for any object X of the category $\mathcal{C}$ there exists an object A and it is $\mathcal{L}$ -injective and a morphism $l : X \rightarrow A$, such that $l \in \mathcal{L}$.

Similarly, we can state the dual notions: an $\mathcal{L}$ -projective object and a category with sufficiently many $\mathcal{L}$ -projective objects.

The most frequently the injective objects are examined with respect to the class of injections of a bicategorical structure $(\mathcal{P}, \mathcal{I})$, and the projective ones - with respect to the class of projections.

The following statement is easy to verify.

Theorem 1.1. *Let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure in the category $\mathcal{C}$ which possesses sufficiently many $\mathcal{I}$ -injective objects. Then the class $\mathcal{I}$ is right-complete.*

Remark 1.1. *In the category $\mathcal{C}_2\mathcal{V}$, the class $\mathcal{E}_f$ of the strong epimorphisms is left-complete. However, in this category do not exist sufficiently many $\mathcal{E}_f$ -projective objects as it was proved by V. Geyler [G].*

For any cardinal number α let $m(\alpha)$ be the Banach space of the bounded functions defined on a set X of power α : $|X| = \alpha$. The norm in this space is given by

$$\|f\| = \sup_{x \in X} |f(x)|$$

for any bounded function $f : X \rightarrow \mathbb{R}$. It is known that these objects are $\mathcal{M}_p$ -injective in the category $\mathcal{C}_2\mathcal{V}$ and these objects and their products form a sufficient class of $\mathcal{M}_p$ -injective objects [P].

The class $\mathcal{M}_f$ is right-stable since $\mathcal{M}_f = \text{Ker}(\mathcal{C}_2\mathcal{V})$ and the category $\mathcal{C}_2\mathcal{V}$ is semiabelian.

Theorem 1.2. *In the category $\mathcal{C}_2\mathcal{V}$ the bicategorical structures $(\mathcal{E}_u, \mathcal{M}_p)$ and $(\mathcal{E}_{pi}, \mathcal{M}_f)$ have the right-complete classes of injections.*

In the category $\mathcal{C}_2\mathcal{V}$ the objects of the subcategory Σ is $\mathcal{E}_u$ -projective and obviously they form a sufficient class of $\mathcal{E}_u$ -projective objects.

Theorem 1.3. [B4] *1. $(\mathcal{E}_u, \mathcal{M}_p)$ is the only bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ with sufficient projective objects and with sufficient injective objects.*

2. $(\mathcal{E}_u, \mathcal{M}_p)$ is the only bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ with both left-complete and right-complete classes.

2. ON GIRAUX SUBCATEGORIES IN THE CATEGORY $\mathcal{C}_2\mathcal{V}$

In the category $\mathcal{C}_2\mathcal{V}$ the strict monomorphisms class $\mathcal{M}_f$ is the same as the kernels class. In this way from Condition 2 it follows that the reflector functor preserves kernels. In algebra such subcategories are called the Giraux subcategories. In the category of locally convex spaces $\mathcal{C}_2\mathcal{V}$ there exist many subcategories such as Giraux.

Theorem 2.1. *Let $\mathcal{R}$ be a nonzero reflective subcategory possessing a reflector functor $r : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{R}$. Then the following statements are equivalent:*

1. *The subcategory $\mathcal{R}$ is $\mathcal{E}_u$ -reflective and $r(\mathcal{M}_p) \subset \mathcal{M}_p$.*
2. *The subcategory $\mathcal{R}$ is $\mathcal{E}_u$ -reflective and $r(\mathcal{M}_f) \subset \mathcal{M}_f$.*
3. *The subcategory $\mathcal{R}$ is $\mathcal{E}_u$ -reflective and the functor r commutes and its limits are projective.*
4. *There exists a coreflective subcategory $\mathcal{K}$ with the coreflector functor $k : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{K}$ such that:*
 - a) $kr \sim k$;
 - b) $rk \sim r$.
5. *The functor r has a left-adjoint functor.*
6. *There exists a coreflective subcategory $\mathcal{K}$ of a category $\mathcal{C}_2\mathcal{V}$ with a coreflector functor $k : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{K}$ such that k is the left-adjoint of the functor r .*
7. *The class $\varepsilon\mathcal{R} = \{f \in \mathcal{E}pi \mathcal{C} \mid r(f) \in \mathcal{I}so\}$ is left-complete.*

Remark 2.1. 1. *The Condition 2 follows from the Condition 4, because any reflector functor in the category $\mathcal{C}_2\mathcal{V}$ commutes with the products $[GG]$.*

2. *The morphisms of the class $(\varepsilon\mathcal{R})^\perp$ are called $\mathcal{R}$ -perfect .*
3. *A subcategory that satisfies the equivalent conditions of the above theorem is called a c -reflective subcategory $[B3]$.*
4. *A pair of subcategories $(\mathcal{K}, \mathcal{R})$ of the category $\mathcal{C}_2\mathcal{V}$ that satisfies the Condition 4 of the above theorem is called an adjoint pair of subcategories.*
5. *Any c -reflective subcategory $\mathcal{R}$ determines an adjoint pair of subcategories: $(\mathcal{K}, \mathcal{R})$. The subcategory $\mathcal{K}$ is denoted by $m\mathcal{R} : \mathcal{K} = m\mathcal{R}$.*

Examples. 1. $(\mathcal{M}, \mathcal{S})$ is an adjoint pair of subcategories of the category $\mathcal{C}_2\mathcal{V}$, where $\mathcal{M}$ is the subcategory of the Mackey spaces and $\mathcal{S}$ is the subcategory of the spaces endowed with weak topology. For any adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ in the category $\mathcal{C}_2\mathcal{V}$ one has the inclusions $\mathcal{M} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ $[B3]$.

2. *The reflector functor from the category $\mathcal{C}_2\mathcal{V}$ in the strong nuclear spaces category $s\mathcal{N}$ admits a left-adjoint. Thus, $s\mathcal{N}$ is a c -reflective subcategory.*
3. *The subcategory Sch of the *Schwartz* spaces is also c -reflective $[GG]$.*
4. *The subcategory $\mathcal{N}$ of the nuclear spaces is not c -reflective $[GG]$.*
5. *The subcategories $m(s\mathcal{N})$, $m(Sch)$ were described in the paper $[GG]$.*

Other examples of c -reflective subcategories and of pairs of adjoint subcategories (for which the reflector and co reflector functor satisfies the relations a) and b) of the Condition 4), the Theorem 2.1) suggest us the following theorem.

Theorem 2.2. *Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$, let $\mathcal{L}$ be a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and let $\mathcal{R} \subset \mathcal{L}$. Then $(l(\mathcal{K}), \mathcal{R})$ is an adjoint pair of subcategories of the category $\mathcal{L}$.*

Proof We have that $l(\mathcal{K})$ is a coreflective subcategory in the category $\mathcal{L}$. Indeed, let $X \in |\mathcal{L}|$, $k^X : kX \rightarrow X$ and let $l^{kX} : kX \rightarrow lkX$, where $\mathcal{K}$ is the coreplique and $\mathcal{L}$ is the replique of the respective objects. Since $X \in |\mathcal{L}|$ we have

$$k^X = l_1^X l^{kX}, \quad (1)$$

for one morphism $l_1^X : lkX \rightarrow X$. Let us prove that l_1^X is $l(\mathcal{K})$ -coreplique of the object X .

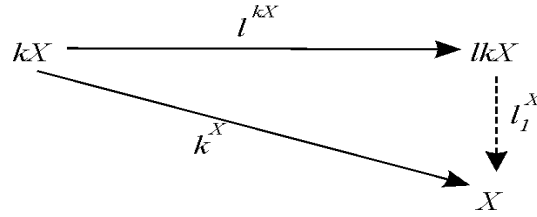


Figure 2.1.

Indeed, let us have a similar diagram built for the object $Y \in |\mathcal{L}|$, and let $f : lkY \rightarrow X$ be an arbitrary morphism. Then

$$f l^{kY} = k^X g, \quad (2)$$

for one morphism $g : kY \rightarrow kX$.

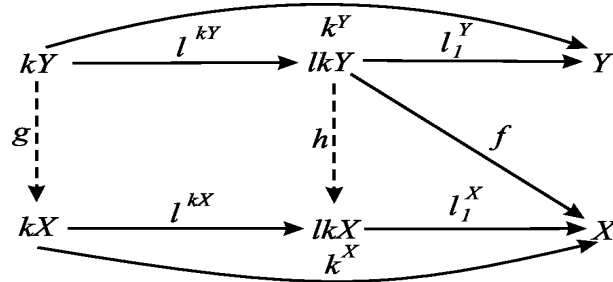


Figure 2.2.

Since l^{kY} is a $\mathcal{L}$ -replique of the object kY , then

$$l^{kX}g = hl^{kY}, \quad (3)$$

for some morphism $h : lkY \rightarrow lkX$. We have

$$l_1^X hl^{kY} \stackrel{(2.3)}{=} l_1^X l^{kX} g \stackrel{(2.1)}{=} k^X g \stackrel{(2.2)}{=} fl^{kY}$$

i.e.

$$l_1^X hl^{kY} = fl^{kY}$$

and, since l^{kY} is an epi, we deduce that

$$l_1^X h = f. \quad (4)$$

As it was mentioned in the above, $\mathcal{M} \subset \mathcal{K}$, such that $k^X \in \mathcal{M}_u$. Taking into account that l^{kX} is an epi it is easy to show that the square

$$l_1^X l^{kX} = k^X \quad (5)$$

is puschout for each object X of the category $\mathcal{C}_2\mathcal{V}$. Thus $l_1^X \in \mathcal{M}_u$, whence the unity of the morphism h with the Property 4.

Now let us verify the relations a) and b) from the Condition 4 of Theorem 2.1. For any object X of the category $\mathcal{L}$ we have the following diagram

$$\begin{array}{ccc} kX & \xrightarrow{l^{kX}} & lkX \\ k^X \downarrow & \nearrow l_l^X & \\ X & \xrightarrow{r^X} & rX \end{array}$$

Figure 2.3.

Obviously $lrX \sim lkX$, since $krX \sim kX$. Thus the $l(\mathcal{K})$ -corepliques of the objects X and rX , are isomorphs. Conversely, since $rkX \sim rX$ we deduce that $rX \sim rlkX$. ■

In the paper [B3] certain constructions permit us to obtain from all adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ of the category $\mathcal{C}_2\mathcal{V}$ another pair of adjoint subcategories $(\mathcal{K}', \mathcal{R})$ in the category $\mathcal{C}_2Ab$ of the locally convex groups this category that contains the category $\mathcal{C}_2\mathcal{V}$.

Let A be a nonzero $\mathcal{M}_p$ -injective object of the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{R} = \mathcal{M}_p P(A)$. The subcategory $\mathcal{R}$ consists of the subspaces of the powers of object A , i.e. from the subspaces of the objects of the form A^τ .

Theorem 2.3. *For any nonzero and $\mathcal{M}_p$ -injective object A of the category $\mathcal{C}_2\mathcal{V}$ the subcategory $\mathcal{R} = \mathcal{M}_p P(A)$ is c-reflective.*

Proof We shall show that the reflector functor $r : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{R}$ satisfies the Condition 1 of the Theorem. 2.1. Indeed, since the subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}_p$ -subobjects, it is $\mathcal{E}_u$ -reflective.

Now let $f : X \rightarrow Y \in \mathcal{M}_p$, and let $r^X : X \rightarrow rX$ and $r^Y : Y \rightarrow rY$ be $\mathcal{R}$ -repliques of the respective objects. Then the object rX is a $\mathcal{M}_p$ -subobject of one object of form A^τ , i.e. there exists a morphism $g : rX \rightarrow A^\tau \in \mathcal{M}_p$.

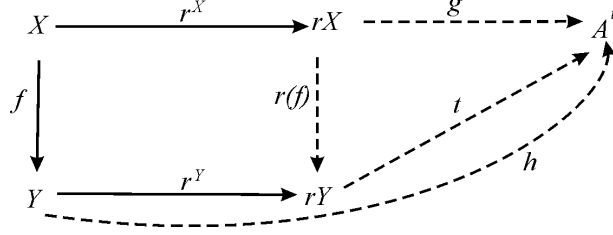


Figure 2.4.

We have

$$r^Y f = r(f) r^X \quad (6)$$

Since A^τ is $\mathcal{M}_p$ -injective, then

$$gr^X = hf \quad (7)$$

for one morphism $h : Y \rightarrow A^\tau$.

Further, $A^\tau \in |\mathcal{R}|$, and r^Y is the $\mathcal{R}$ -replique of the object Y . Therefore, for a morphism $t : rY \rightarrow A^\tau$, we have

$$h = tr^Y. \quad (8)$$

We also have

$$tr(f) r^X \stackrel{(2.6)}{=} tr^Y f \stackrel{(2.8)}{=} hf \stackrel{(2.7)}{=} gr^X,$$

i.e.

$$tr(f) r^X = gr^X.$$

In addition, since r^X is an epi, it follows that

$$tr(f) = g. \quad (9)$$

From the last equality and taking into account that $g \in \mathcal{M}_p$, we deduce that $r(f) \in \mathcal{M}_p$. ■

Theorem 2.4. [B3]. *Let $\omega \leq \alpha < \beta$. Then:*

$$\mathcal{M}_p P(m(\alpha)) \subset \mathcal{M}_p P(m(\beta)) \text{ and } \mathcal{M}_p P(m(\alpha)) \neq \mathcal{M}_p P(m(\beta))$$

In particular, in the category $\mathcal{C}_2\mathcal{V}$ there exists one proper class of c -reflective subcategories.

Proof If a space Banach B is isomorphic with a subspace of the product X^τ by the X locally convex space, then B is isomorphic with a subspace of the space X^n of the n finite cardinal. ■

Corollary 2.1. *In the category $\mathcal{C}_2\mathcal{V}$ there exists a proper class of c -reflective subcategories, and, hence, a proper class of adjoint pairs of subcategories.*

Remark 2.2. *The spaces of the subcategories $m(\mathcal{M}_p P(m(\tau)))$ were described in the paper [GG].*

3. THE CONSERVATION OF THE CLASSES OF INJECTIONS AND PROJECTIONS

Let $\mathcal{R}$ be a reflective subcategory in the category $\mathcal{C}_2\mathcal{V}$. We consider the class $\varepsilon\mathcal{R} = \{f \in \mathcal{Epi} \mathcal{C}_2\mathcal{V} \mid r(f) \in \mathcal{Iso}\}$.

This class together with the class of $\mathcal{R}$ -perfect morphisms $(\varepsilon\mathcal{R})^\perp$ forms a right-bicategorical structure $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$. Thus $\varepsilon\mathcal{R}$ is right-complete.

Lemma 3.1. *Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$ with the respective functors $k : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \rightarrow \mathcal{R}$. Then*

1. $\varepsilon\mathcal{R} = \{f \in \mathcal{C}_2\mathcal{V} \mid r(f) \in \mathcal{Iso}\}$.
2. $\varepsilon\mathcal{R} = \{f \in \mathcal{Mono} \mathcal{C}_2\mathcal{V} \mid k(f) \in \mathcal{Iso}\}$.
3. $\varepsilon\mathcal{R} = \{f \in \mathcal{C}_2\mathcal{V} \mid k(f) \in \mathcal{Iso}\}$.
4. $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$ is a bicategory structure on the right.
5. $((\varepsilon\mathcal{R})^\top, \varepsilon\mathcal{R})$ is a bicategory structure on the left.
6. The class $\varepsilon\mathcal{R}$ is also complete on the right and on the left.
7. $\varepsilon\mathcal{R} \subset \mathcal{E}_u \cap \mathcal{M}_u$.
8. The objects of the subcategory $\mathcal{R}$ form a sufficient class of $\varepsilon\mathcal{R}$ -injective objects.
9. The objects of the subcategory $\mathcal{K}$ form a sufficient class of $\varepsilon\mathcal{R}$ -projective objects.

Proof Let $r(f) \in \mathcal{Iso}$, then in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ r^X \downarrow & & \downarrow r^Y \\ rX & \xrightarrow{r(f)} & rY \end{array}$$

Figure 3.1.

we have

$$r(f) r^X = r^Y f. \quad (10)$$

In this equality the maps $r^X, r(f)$ and r^Y are bijective. That is why f is bijective too. In particular, $f \in \mathcal{E}_u$. Further, since $r(f) r^X \in \mathcal{M}_u$, from (3.1) we deduce that $f \in \mathcal{M}_u$. Hence, we proved the statements 1 and 7.

The statement 2 follows from the definition of the adjoint pair of subcategories and the statement 3 is the dual to the statement 1. The assertions 4 and 5 were mentioned above and the statement 6 is their simple corollary. ■

Theorem 3.1. [B3] . *The correspondence $\mathcal{R} \mapsto \varepsilon\mathcal{R}$ establishes an antiisomorphism of the lattice of c -reflective subcategories and the lattice of the bimorphisms classes of the category $\mathcal{C}_2\mathcal{V}$ which are left-complete and right-complete.*

Proof The left and right-complete class $\varepsilon\mathcal{R}$ is put in correspondence to each c -reflective subcategory $\mathcal{R}$

Conversely. A complete subcategory $\mathcal{R}$ of the A -injective objects is put in correspondence to any complete to the right and to the left class A of bimorphism. Then $m(\mathcal{R})$ is a full subcategory of the A -projective objects. ■

Let $\mathcal{R}$ be a reflective subcategory and let $(\mathcal{P}, \mathcal{J})$ be a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$. In the above we examined some examples when $r(\mathcal{J}) \subset \mathcal{J}$. Let us show a few cases.

Lemma 3.2. *Let $\mathcal{R}$ be a $\mathcal{E}_u$ -reflective subcategory. Then*

1. $r(\mathcal{M}_u) \subset \mathcal{M}_u$.
2. $r(\text{Mono}) \subset \text{Mono}$.

Proof 1. Let $f : X \rightarrow Y \in \mathcal{M}_u$. Then from the equality

$$r^Y f = r(f) r^X \quad (11)$$

it follows that $r(f) r^X \in \mathcal{M}_u$. Since r^X is an epi we deduce that the square

$$1 \cdot (r(f) r^X) = r(f) \cdot r^X \quad (12)$$

is puscout. Therefore, $r(f) \in \mathcal{M}_u$.

$$\begin{array}{ccc} X & \xrightarrow{r(f)r^X} & rY \\ r^X \downarrow & & \downarrow 1 \\ rX & \xrightarrow{r(f)} & rY \end{array}$$

Figure 3.2.

2. See Lemma 1.3 [BT] ■

Lemma 3.3. 1. Let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure and let $\mathcal{R}$ be a $\mathcal{P}$ -reflective subcategory in the category $\mathcal{C}$. Then $r(\mathcal{P}) \subset \mathcal{P}$.

2. Let $(\mathcal{K}, \mathcal{R})$ be a adjoint pair of subcategories in the category $\mathcal{C}$.

a) If $\mathcal{K}$ is $\mathcal{P}$ -coreflective, then $\mathcal{R}$ is $\mathcal{P}$ -reflective, too.

b) If $\mathcal{R}$ is $\mathcal{I}$ -reflective, then $\mathcal{K}$ is $\mathcal{I}$ -reflective, too.

Theorem 3.2. Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$ and let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure. Then the following conditions are equivalent:

a) $r(\mathcal{I}) \subset \mathcal{I}$;

b) $k(\mathcal{P}) \subset \mathcal{P}$.

Proof a) $\Rightarrow$ b). Let $p : X \rightarrow Y \in \mathcal{P}$, and let

$$k(p) = i_1 p_1 \quad (13)$$

be the $(\mathcal{P}, \mathcal{I})$ -factorization of the morphism $k(p)$, where $p_1 : kX \rightarrow Z$.

We examine the $\mathcal{R}$ -replicques of the objects X, Y and Z , by taking into account that $rX \sim rkX$ and $rY \sim rkY$.

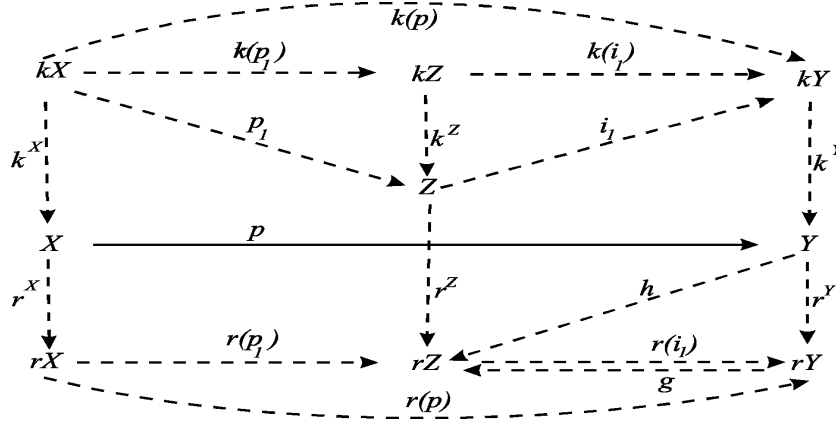


Figure 3.3.

By construction we have the following equalities:

$$pk^X = k^Y k(p); \quad (14)$$

$$p_1 = k^Z k(p_1); \quad (15)$$

$$i_1 k^Z = k(i_1); \quad (16)$$

$$r^Z p_1 = r(p_1) r^X k^X; \quad (17)$$

$$r(i_1) r^Z = r^Y k^Y i_1; \quad (18)$$

$$k(p) = i_1 p_1; \quad (19)$$

$$r(p) r^X = r^Y p. \quad (20)$$

From the equality (3.11) we have

$$rk(p) = r(i_1 p_1) = r(i_1) r(p_1).$$

On the other hand we have

$$rk(p) = r(p).$$

Thus

$$r(p) = r(i_1) r(p_1). \quad (21)$$

We have

$$r(i_1) r(p_1) r^X k^X \stackrel{(3.12)}{=} r(p) r^X k^X \stackrel{(3.11)}{=} r^Y p k^X,$$

i.e.

$$r(i_1) r(p_1) r^X k^X = r^Y p k^X$$

and, since k^X is an epi, it follows that

$$r(i_1) r(p_1) r^X = r^Y p. \quad (22)$$

By the hypothesis a) $r(i_1) \in \mathcal{J}$. Thus, in the equality (3.13) $p \in \mathcal{P}$ and $r(i_1) \in I$, i.e. $p \perp r(i_1)$. Then there exists a morphism $h : Y \rightarrow rZ$ such that

$$r(i_1)h = r^Y \quad (23)$$

$$r(p_1)r^X = hp \quad (24)$$

Then

$$h = gr^Y \quad (25)$$

for some morphism $g : rY \rightarrow rZ$. We have

$$r(i_1)gr^Y \stackrel{(3.16)}{=} r(i_1)h \stackrel{(3.14)}{=} r^Y,$$

i.e.

$$r(i_1)gr^Y = r^Y$$

and, since r^Y is an epi, we deduce that

$$r(i_1)g = 1. \quad (26)$$

From the equality (3.17) and the fact that $r(i_1)$ is a mono it follows that $r(i_1)$ is an izo. We have

$$k(i_1) = kr(i_1),$$

therefore, $k(i_1)$ is an izo, too. From the equality (3.7) and from the fact that $k(i_1)$ is an izo, it follows that the morphisms i_1 and k^Z are izo, too. Thus, $k(p) \sim k(p_1) \sim p_1$. Hence, $k(p) \in \mathcal{P}$.

$b) \Rightarrow a)$. The proof is dual. ■

Examples. 1. For every adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ by definition it follows that

$$k(\mathcal{E}_f) \subset \mathcal{E}_f \text{ and } r(\mathcal{M}_f) \subset (\mathcal{M}_f).$$

By the previous theorem we deduce that

$$r(\text{Mono}) \subset \text{Mono} \text{ and } k(\mathcal{E}pi) \subset \mathcal{E}pi,$$

i.e. r is a monofunctor and k is an epifunctor.

For the assertion that $r(\text{Mono}) \subset \text{Mono}$, see Lemma 3.2. The assertion $k(\mathcal{E}pi) \subset \mathcal{E}pi$ follows from the following topology properties (see [RR]).

2. Since $r(\mathcal{M}_p) \subset \mathcal{M}_p$, for every adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$, then we have that $k(\mathcal{E}_u) \subset \mathcal{E}_u$.

3. By Lemma 3.2 we have $r(\mathcal{M}_u) \subset \mathcal{M}_u$. Thus, $k(\mathcal{E}_p) \subset \mathcal{E}_p$. This assertion is less trivial. It may also be proved directly, by the description of the class $\mathcal{E}_p$ [BG].

4. THE BICATEGORICAL STRUCTURES $\left((\mathcal{J} \circ \varepsilon\mathcal{R})^\top, \mathcal{J} \circ \varepsilon\mathcal{R} \right)$ AND $\left(\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp \right)$

Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories of the category $\mathcal{C}_2\mathcal{V}$ and let $(\mathcal{P}, \mathcal{J})$ be a bicategorical structure. Denote

$$\begin{aligned} \mathcal{J} \circ \varepsilon\mathcal{R} &= \{uv \mid u \in \mathcal{J}, v \in \varepsilon\mathcal{R} \text{ and there exists the composition } uv\} \\ \varepsilon\mathcal{R} \circ \mathcal{P} &= \{uv \mid u \in \varepsilon\mathcal{R}, v \in \mathcal{P} \text{ and there exists the composition } uv\} \end{aligned}$$

Theorem 4.1. *Assume that the adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ possesses the property $r(\mathcal{I}) \subset \mathcal{I}$. Then:*

1. $\left((\mathcal{I} \circ \varepsilon\mathcal{R})^\perp, \mathcal{I} \circ \varepsilon\mathcal{R} \right)$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.
2. $r(\mathcal{I} \circ \varepsilon\mathcal{R}) \subset \mathcal{I}$.
3. If the class of injections $\mathcal{I}$ is right-complete, then so is the class of injections $\mathcal{I} \circ \varepsilon\mathcal{R}$.
4. The pair $\left(\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp \right)$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.
5. $k(\varepsilon\mathcal{R} \circ \mathcal{P}) \subset \mathcal{P}$.
6. If the class of projections $\mathcal{P}$ is left-complete, then the class of the projections $\varepsilon\mathcal{R} \circ \mathcal{P}$ has the same property.

Proof 1. Since each class $\varepsilon\mathcal{R}$ and $\mathcal{I}$ are left-complete, then so the class $\mathcal{I} \circ \varepsilon\mathcal{R}$ is left-stable with respect to the products. It is to prove that it is closed with respect to the composition. Each class $\varepsilon\mathcal{R}$ and $\mathcal{I}$ contain $\mathcal{I} \circ \varepsilon\mathcal{R}$ and are closed according to the composition. Thus, we have to prove the following assertion:

Let $u \in \varepsilon\mathcal{R}$ and $v \in \mathcal{I}$ and assume that there exists the composition uv . Then $uv \in \mathcal{I} \circ \varepsilon\mathcal{R}$. Examine the $\mathcal{R}$ -repliques of the respective objects.

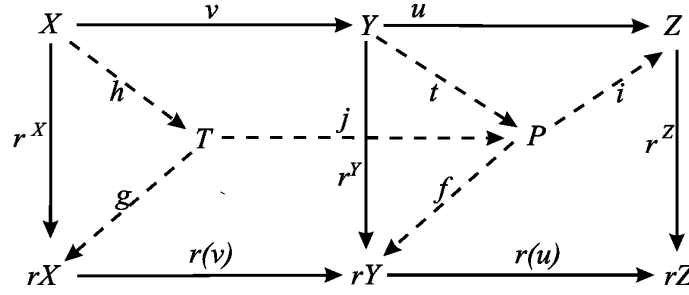


Figure 4.1.

We have

$$r(v)r^X = r^Y v, \quad (27)$$

$$r(u)r^Y = r^Z u. \quad (28)$$

We construct the pull-back squares on the morphisms $r(u)$ and r^Z

$$r(u)f = r^Z i. \quad (29)$$

on the morphisms $r(v)$ and f

$$r(v)g = fj \quad (30)$$

Since $u \in \varepsilon\mathcal{R}$ we deduce that $r(u) \in \mathcal{I}so$ and also that $i \in \mathcal{I}so$; $v \in \mathcal{J}$ therefore $r(v) \in \mathcal{J}$ and $j \in \mathcal{J}$. The squares (4.3) and (4.4) are pull-back, $r^Z \in \varepsilon\mathcal{R}$ and this class is left-stable, hence, $f, g \in \varepsilon\mathcal{R}$. From the equality (4.2) and (4.3) it follows that

$$r^Y = ft; \quad (31)$$

$$u = it, \quad (32)$$

for some morphism t . Further, we have

$$r(v)r^X \stackrel{(4.1)}{=} r^Y v \stackrel{(4.5)}{=} ftv,$$

i.e.

$$r(v)r^X = f(tv). \quad (33)$$

From the equalities (4.7) and (4.4) we deduce that

$$r^X = gh, \quad (34)$$

$$tv = jh, \quad (35)$$

for some morphism h . From the equality (4.8), since r^X and g belong to the class $\varepsilon\mathcal{R}$, i.e. they are bijective maps, we deduce that this holds for the map h . From the fact that h is an epi it follows that g is the $\mathcal{R}$ -replique of the object T . Thus we proved that $r(h) \in \mathcal{I}so$, i.e. $h \in \varepsilon\mathcal{R}$. We have

$$uv \stackrel{(4.6)}{=} itv \stackrel{(4.9)}{=} ijh,$$

i.e.

$$uv = (ij)h, \quad (36)$$

with $ij \in \mathcal{J}$ and $h \in \varepsilon\mathcal{R}$.

Thus, we proved that the class $\mathcal{J} \circ \varepsilon\mathcal{R}$ is closed with respect to the composition.

2. We take into account that $r(\varepsilon\mathcal{R}) \subset \mathcal{I}so$.
3. The right-completeness of the classes $\mathcal{J}$ and $\varepsilon\mathcal{R}$ implies the fact that the same holds for the class $\mathcal{J} \circ \varepsilon\mathcal{R}$.
4. The proof dual to that of the assertion 1 follows. Indeed, the Theorem 3.2 asserts that

$$r(\mathcal{J}) \subset \mathcal{J} \Leftrightarrow k(\mathcal{P}) \subset \mathcal{P}$$

5. The assertion is true because $k(\varepsilon\mathcal{R}) \subset \mathcal{I}so$ and $k(\mathcal{P}) \subset \mathcal{P}$

6. Dual to the proof of assertion 2. ■

Let $\mathbb{R}_c$ be the class of all c -reflective subcategories in the category $\mathcal{C}_2\mathcal{V}$. By Theorem 2.5, $\mathbb{R}_c$ is a proper class.

Corollary 4.1. . 1. *The classes of the bicategorical structures*

$$\begin{aligned} & \{((\mathcal{M}_f \circ \varepsilon\mathcal{R})^\perp, \mathcal{M}_f \circ \varepsilon\mathcal{R}) \mid \mathcal{R} \in \mathbb{R}_c\} \\ & \{((\mathcal{M}_p \circ \varepsilon\mathcal{R})^\perp, \mathcal{M}_p \circ \varepsilon\mathcal{R}) \mid \mathcal{R} \in \mathbb{R}_c\} \end{aligned}$$

are proper classes of bicategorical structures in the category $\mathcal{C}_2\mathcal{V}$ with the right-complete class of injections.

2. *The class of bicategorical structures*

$$\left\{ \left(\varepsilon\mathcal{R} \circ \mathcal{E}_p, (\varepsilon\mathcal{R} \circ \mathcal{E}_p)^\perp \right) \mid \mathcal{R} \in \mathbb{R}_c \right\}$$

is a proper class of bicategorical structures in the category $\mathcal{C}_2\mathcal{V}$ having a left-complete class of projections.

Corollary 4.2. [BG] . *Let $\mathcal{S}$ be the subcategory of the spaces endowed with the weak topology. Then*

1. $((\mathcal{M}_f \circ \varepsilon\mathcal{S})^\perp, \mathcal{M}_f \circ \varepsilon\mathcal{S})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.
2. $((\mathcal{M}_p \circ \varepsilon\mathcal{S})^\perp, \mathcal{M}_p \circ \varepsilon\mathcal{S})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.

Corollary 4.3. *Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$, let $(\mathcal{P}, \mathcal{J})$ be a bicategorical structure and let $r(\mathcal{J}) \subset \mathcal{J}$. Then:*

1. $\mathcal{K}$ is a $(\varepsilon\mathcal{R} \circ \mathcal{P})$ - and $(\mathcal{J} \circ \varepsilon\mathcal{R})$ - coreflective subcategory.
2. $\mathcal{R}$ is a $(\varepsilon\mathcal{R} \circ \mathcal{P})$ - and $(\mathcal{J} \circ \varepsilon\mathcal{R})$ - reflective subcategory.
3. If $\mathcal{M}_p \subset \mathcal{J}$, then the class $(\mathcal{J} \circ \varepsilon\mathcal{R})$ is $\mathcal{E}_p$ -extremal.

5. CLASSES OF PROJECTIONS $(\mathcal{J} \circ \varepsilon\mathcal{R})^\perp$

As in the previous section, we suppose that $(\mathcal{K}, \mathcal{R})$ is an adjoint pair of subcategories, $(\mathcal{P}, \mathcal{J})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ and these pairs possess the property $r(\mathcal{J}) \subset \mathcal{J}$.

By the previous theorem, the bicategorical structure $(\mathcal{P}, \mathcal{J})$ generates two new bicategorical structures

$$((\mathcal{J} \circ \varepsilon\mathcal{R})^\perp, \mathcal{J} \circ \varepsilon\mathcal{R}) \text{ and } (\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp).$$

We mention the following inclusions

$$(\mathcal{J} \circ \varepsilon\mathcal{R})^\perp \subset \mathcal{P} \subset \varepsilon\mathcal{R} \circ \mathcal{P},$$

$$(\varepsilon\mathcal{R} \circ \mathcal{P})^\perp \subset \mathcal{I} \subset \mathcal{I} \circ \varepsilon\mathcal{R}.$$

The class $(\mathcal{I} \circ \varepsilon\mathcal{R})^\perp$. Since this class is contained in the class $\mathcal{P}$, we shall show which elements of the class $\mathcal{P}$ are contained in this class. Let $f : (X, u) \rightarrow (Y, t) \in \mathcal{P}$, and let

$$f = me \quad (37)$$

be the $(\mathcal{E}_f, \text{Mono})$ -factorization of the morphism f . Then u'' is the factor topology of the topology u and m is a bimorphism, i.e. it is an injective map with dense image in the space (Y, t) . Let $k^Y : (Y, k(t)) \rightarrow (Y, t)$ be the $\mathcal{K}$ -coreplique of the respective object. Obviously, we have

$$f \in (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp \Leftrightarrow m \in (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp$$

Figure 5.1.

Theorem 5.1. *The morphism $f \in (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp$ iff the topology t is the strongest locally convex topology for which the maps m and k^Y are continuous.*

Proof Let $m \in (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp$ and let be t_1 the strongest locally convex topology on the space Y for which the maps m and k^Y are continuous. Then $t_1 \geq t$, and the corresponding maps m_1 (defined on m) and k_1^Y (defined on k^Y) and the identity map i are continuous and we have the following commutative diagram

Figure 5.2.

From the equality

$$ik_1^Y = k^Y \quad (38)$$

it follows that $i \in \varepsilon\mathcal{R}$, and k_1^Y is a $\mathcal{K}$ -coreplique of the resp. object. From the equality

$$im_1 = m \quad (39)$$

it follows that $i \in (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp$. Thus, $i \in \varepsilon\mathcal{R} \cap (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp \subset (\mathcal{I} \circ \varepsilon\mathcal{R}) \cap (\mathcal{I} \circ \varepsilon\mathcal{R})^\perp = \text{Iso}$ and $t = t_1$.

Conversely. Let t be the strongest locally convex topology for which the maps m and k^Y remain continuous. We shall prove that $m \perp (\mathcal{J} \circ \varepsilon \mathcal{R})$. Since $m \in P$ it is sufficient prove that $m \perp (\varepsilon \mathcal{R})$. Indeed, let

$$lg = hm \quad (40)$$

hold with $l \in \varepsilon \mathcal{R}$, and let $k^V : kV \rightarrow V$ be the $\mathcal{K}$ -coreplique of the resp. object. Then lk^V is the $\mathcal{K}$ -coreplique of the object W and

$$hk^Y = lk^V l_1 \quad (41)$$

holds for some morphism l_1 .

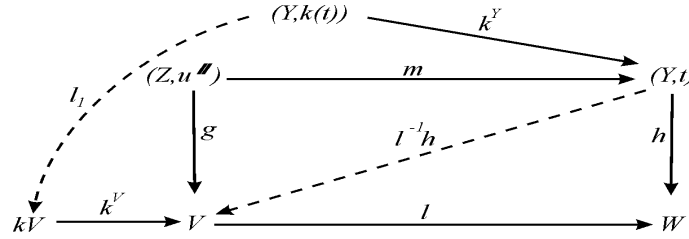


Figure 5.3.

Let us examine the map $l^{-1}h$. We have

$$l^{-1}hk^Y \stackrel{(5.5)}{=} l^{-1}lk^V l_1 = k^V l_1,$$

i.e.

$$(l^{-1}h)k^Y = k^V l_1. \quad (42)$$

We also have

$$l^{-1}hm \stackrel{(5.4)}{=} l^{-1}l g = g,$$

i.e.

$$(l^{-1}h)m = g \quad (43)$$

From the equalities (5.6) and (5.7) it follows that the composition of the map $l^{-1}h$ with the maps m and k^Y are continuous. Then by the definition of the topology t we deduce that the map $l^{-1}h$ is continuous, too. ■

Let us assume that $\mathcal{M}_p \subset \mathcal{J}$. Then $\mathcal{P} \subset \mathcal{E}_u$.

Thus, the morphisms of the class $(\mathcal{J} \circ \varepsilon \mathcal{R})^\perp$ are surjective maps. If we come back to the first diagram we see that the morphism f and, together with it, the morphism m belong to the class $\mathcal{E}_u$. Thus, the vector spaces Z and Y coincide. In this case, the description of the class $(\mathcal{J} \circ \varepsilon \mathcal{R})^\perp$ is simpler.

Theorem 5.2. *The continuous and surjective linear map $f : (X, u) \rightarrow (Y, t) \in \mathcal{P}$ belongs to the class $(\mathcal{J} \circ \varepsilon \mathcal{R})^\perp$ iff*

$$t = \min(k(t), u'')$$

where $k(t)$ is the $\mathcal{K}$ -coreplique that corresponds to the topology t , and u'' is the factor topology on the vector space Y of the topology u .

Remark 5.1. *In the paper [BG] we have the same description of the classes $(\mathcal{M}_f \circ \varepsilon \mathcal{S})^\perp$ and $(\mathcal{M}_p \circ \varepsilon \mathcal{S})^\perp$.*

Theorem 5.3. *We have that $p : X \rightarrow Y \in (\mathcal{J} \circ \varepsilon \mathcal{R})^\perp$ iff $p \cdot k^X = k^Y \cdot k(p)$ is a puscout square.*

Proof Let $p : (X, u) \rightarrow (Y, t) \in (\mathcal{J} \circ \varepsilon \mathcal{R})^\perp$.

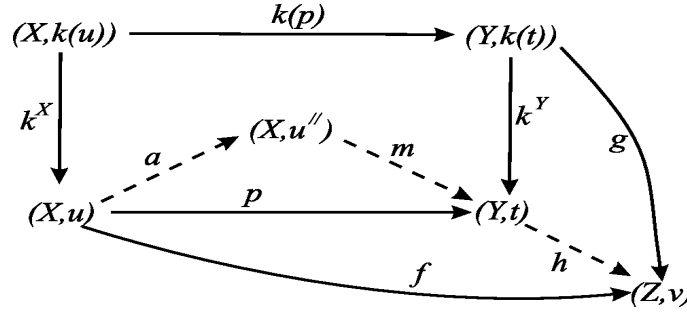


Figure 5.4.

Examine the commutative square

$$pk^X = k^Y k(p). \quad (44)$$

Let us prove that it is a puscout. Let

$$f \cdot k^X = g \cdot k(p). \quad (45)$$

Define the linear map

$$h = g \cdot (k^Y)^{-1}. \quad (46)$$

Let us prove that it is continuous. Let

$$f = me \quad (47)$$

be the $(\mathcal{E}_f, \text{Mono})$ -factorization of the corresponding morphism. By Theorem 5.1 it is sufficient to prove that the maps hm and hk^Y are continuous. Since e is a factorial map it follows that the map hm will be continuous iff the map hme , i.e. hp , will be continuous. We mention that by the equality (5.8), due

to the fact that k^X is a bijective map, we have the following equality of linear maps

$$p = k^Y k(p) \cdot (k^X)^{-1}. \quad (48)$$

Thus

$$\begin{aligned} hp &\stackrel{(5.10)}{=} g(k^Y)^{-1} p \stackrel{(5.12)}{=} g(k^Y)^{-1} k^Y k(p) (k^X)^{-1} = gk(p) (k^X)^{-1} \stackrel{(5.9)}{=} \\ &fk^X (k^X)^{-1} = f, \end{aligned}$$

i.e

$$hp = f. \quad (49)$$

Further we have

$$hk^Y \stackrel{(5.10)}{=} g(k^Y)^{-1} k^Y = g$$

or

$$hk^Y = g. \quad (50)$$

The equalities (5.13) and (5.14) show that the map h is continuous, and the square (5.8) is puschout.

Conversely. Let $p : X \rightarrow Y \in \mathcal{P}$, and assume that the square (5.8) is puschout. Let us prove that $p \in (\mathcal{J} \circ \varepsilon\mathcal{R})^\perp$. Since $p \in \mathcal{P}$, it follows that $p \perp \mathcal{J}$. Let us prove that $p \perp (\varepsilon\mathcal{R})$. Indeed, let $b \in \varepsilon\mathcal{R}$, and assume that

$$bf = gp. \quad (51)$$

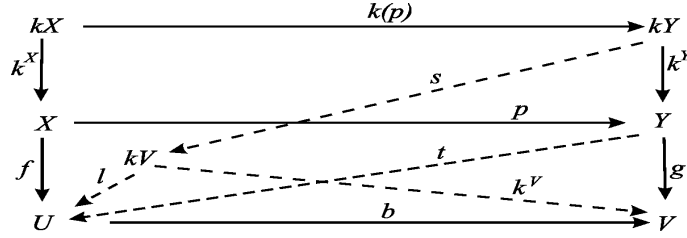


Figure 5.5.

Since $b \in \varepsilon\mathcal{R}$ it follows that

$$k^V = bl \quad (52)$$

for some morphism l . Then

$$gk^Y = k^V s \quad (53)$$

for some morphism s . We have

$$b l s k(p) \stackrel{(5.16)}{=} k^V s k(p) \stackrel{(5.17)}{=} g k^Y k(p) \stackrel{(5.8)}{=} g p k^X \stackrel{(5.15)}{=} b f k^X$$

i.e.

$$b l s k(p) = b f k^X \quad (54)$$

and, since b is a mono, it follows that

$$l s k(p) = f k^X. \quad (55)$$

By the hypothesis that (5.8) is a puscout square we deduce that

$$f = t p, \quad (56)$$

$$l s = t k^Y. \quad (57)$$

Since p is an epi then from the equalities (5.15) and (5.20) it follows that

$$b t = g. \quad (58)$$

The equalities (5.20) and (5.22) show that $p \perp (\varepsilon \mathcal{R})$. ■

6. CLASSES OF PROJECTIONS $\varepsilon \mathcal{R} \circ \mathcal{P}$

The description of the morphisms of this class follows from its definition. Let $f : X \rightarrow Y \in \mathcal{C}_2 \mathcal{V}$, and let

$$f = i p \quad (59)$$

be the $(\mathcal{P}, \mathcal{I})$ -factorization. Since $\mathcal{P} \subseteq \varepsilon \mathcal{R} \circ \mathcal{P}$ it follows that $f \in \varepsilon \mathcal{R} \circ \mathcal{P}$ iff $p \in \varepsilon \mathcal{R} \circ \mathcal{P}$.

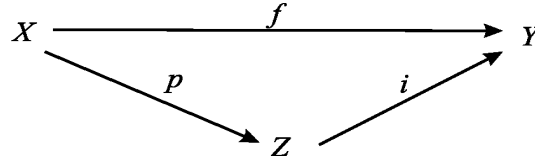


Figure 6.1.

Theorem 6.1. *The morphism $f \in (\varepsilon \mathcal{R} \circ \mathcal{P})$ iff $i \in \varepsilon \mathcal{R}$.*

Proof If $i \in \varepsilon \mathcal{R}$ then it is evident that $f \in \varepsilon \mathcal{R} \circ \mathcal{P}$.

Conversely. Let $f \in \varepsilon \mathcal{R} \circ \mathcal{P}$. Let us prove that $i \in \varepsilon \mathcal{R}$. By hypothesis, we have

$$f = e_1 p_1, \quad (60)$$

where $e_1 \in \varepsilon\mathcal{R}$ and $p_1 \in \mathcal{P}$. We have the commutative square

$$ip = e_1 p_1, \quad (61)$$

with $i \in \mathcal{I}$ and $p_1 \in \mathcal{P}$. Thus,

$$p = h p_1, \quad (62)$$

$$ih = e_1, \quad (63)$$

for some morphism h . Since $e_1 \in \varepsilon\mathcal{R} \subset \mathcal{E}_u \cap \mathcal{M}_u$ and m is a mono we deduce that the square

$$i \cdot h = e_1 \cdot 1 \quad (64)$$

is a pull-back square. Thus, $h \in \mathcal{E}_u \cap \mathcal{M}_u$. Since $e_1 \in \varepsilon\mathcal{R}$ it follows that the morphism $r^Y e_1$ is the $\mathcal{R}$ -replique of the object P and from the equality (6.5) and from the fact that h is an epi, we deduce that the morphism $r^Y i$ is the $\mathcal{R}$ -replique of the object Z . Thus, $r(i) \in \mathcal{I}so$ and $i \in \varepsilon\mathcal{R}$.

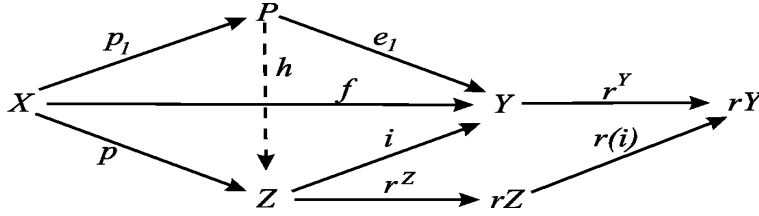


Figure 6.2.

■

7. CLASSES OF INJECTIONS $(\varepsilon\mathcal{R} \circ \mathcal{P})^\perp$

Let $f : (X, u) \rightarrow (Y, t) \in \mathcal{I}$, and let $f = me$ be the $(\mathcal{E}pi, \mathcal{M}_f)$ -factorization of the morphisms. Since $m \in \mathcal{M}_f \subset (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp$ we deduce that $f \in (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp$ iff the morphism e belongs to this class. Let $r^X : (X, u) \rightarrow (X, r(u))$ be the $\mathcal{R}$ -replique of (X, u) .

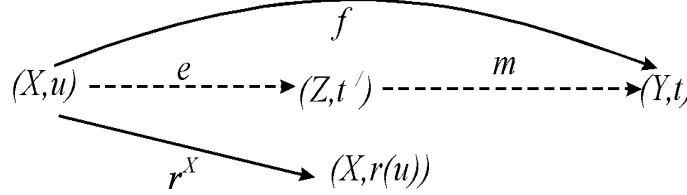


Figure 7.1.

The topology t' is the topology induced from the space (Y, t) on the linear space $Z = cl(f(X))$.

Theorem 7.1. *The morphism f belongs to the class $(\varepsilon\mathcal{R} \circ \mathcal{P})^\perp$ iff u is the weakest locally convex topology for which the maps e and r^X remain continuous.*

Proof Dual to the proof of Theorem 5.1. ■

Theorem 7.2. *Let $i : X \rightarrow Y \in \mathcal{J}$. Then $i \in (\varepsilon\mathcal{R} \circ \mathcal{P})^\perp$ iff the square*

$$r(i)r^X = r^Y i$$

is pull-back.

Proof Dual to the proof of Theorem 5.3. ■

8. CLASSES OF INJECTIONS $\mathcal{J} \circ \varepsilon\mathcal{R}$

The class $\mathcal{J} \circ \varepsilon\mathcal{R}$. Let $f : X \rightarrow Y \in \mathcal{C}_2\mathcal{V}$, and let $f = ip$ be the $(\mathcal{P}, \mathcal{J})$ -factorization of the morphisms. Since $\mathcal{J} \subset \mathcal{J} \circ \varepsilon\mathcal{R}$ it follows that $f \in \mathcal{J} \circ \varepsilon\mathcal{R}$ iff $p \in \mathcal{J} \circ \varepsilon\mathcal{R}$.

Theorem 8.1. *The morphism $f \in \mathcal{J} \circ \varepsilon\mathcal{R}$ iff $p \in \varepsilon\mathcal{R}$.*

Proof Dual to the proof of Theorem 6.1. ■

Corollary 8.1. [BG] . *Let $f : (X, u) \rightarrow (Y, t)$ be a mono and let t' be the topology induced by the topology t on the subspace X . The monomorphism f belongs to the class $\mathcal{M}_p \circ \varepsilon\mathcal{S}$ iff the spaces (X, u) and (X, t') have the same dual space.*

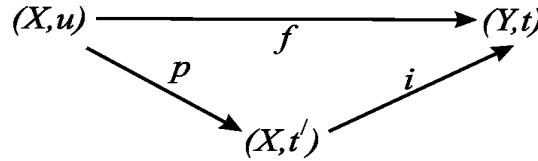


Figure 8.1.

9. THE CASE OF ONE INJECTIVE OBJECT

Let A be a nonzero $\mathcal{M}_p$ -injective object in the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{R} = \mathcal{M}_p P(A)$ and let $\mathcal{K} = m\mathcal{R}$. Then by Theorem 2.3 $(\mathcal{K}, \mathcal{R})$ is a pair of adjoint subcategories in the category $\mathcal{C}_2\mathcal{V}$. In this case $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ and $\mathcal{M}_f \circ \varepsilon\mathcal{R}$ admit another description, too.

Theorem 9.1. *1. $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ is the class of all monomorphisms with respect to which the object A remains injective.*

2. $\mathcal{M}_f \circ \varepsilon\mathcal{R}$ is the class of all morphisms of the class $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ that have a closed image.

Proof 1. Let $f : X \rightarrow Y \in \mathcal{M}_p \circ \varepsilon\mathcal{R}$. Then, by definition of the class $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ we have

$$f = ip \quad (65)$$

with $i \in \mathcal{M}_p$ and $p \in \varepsilon\mathcal{R}$. Let us show that the object A is injective with respect to the morphism f . Indeed, let $g : X \rightarrow A$.

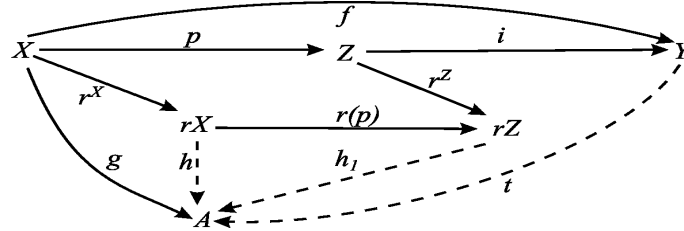


Figure 9.1.

Since $A \in |\mathcal{R}|$, we have

$$g = hr^X \quad (66)$$

for some morphism h . Further, $p \in \varepsilon\mathcal{R}$, therefore, $r(p)$ is an isomorphism. Thus,

$$g = h_1 r^Z p \quad (67)$$

for some morphism h_1 . Definitely A is $\mathcal{M}_p$ -injective and $i \in \mathcal{M}_p$. So,

$$h_1 r^Z = ti \quad (68)$$

for a morphism t . In this case

$$g = tf \quad (69)$$

and the morphism g is an extension of f .

Conversely. Let $f : X \rightarrow Y$ be a monomorphism such that all morphism from the object X in the object A extend the morphism f . Let us show that $f \in \mathcal{M}_p \circ \varepsilon\mathcal{R}$. Let

$$f = ip \quad (70)$$

be the $(\mathcal{E}_u, \mathcal{M}_p)$ -factorization of the morphism f . Then each morphisms from the object X in the object A extends in a unique way the morphism p . This permits us to consider the sets $\text{Hom}(X, A)$ and $\text{Hom}(rX, A)$ as biing equal. In order to obtain the $\mathcal{R}$ -replices of these objects, we must examine the morphisms $g^X : X \rightarrow A^{\text{Hom}(X, A)}$ and $g^Z : Z \rightarrow A^{\text{Hom}(Z, A)}$.

All mentioned in the above show that there exists an isomorphism $j = p^{\text{Hom}(X, A)}$ such that

$$jg^X = g^Z p \quad (71)$$

the $(\mathcal{E}_u, \mathcal{M}_p)$ -factorization of the morphisms g^X and g^Z gives us the $\mathcal{R}$ -repliques of the objects X and Z respectively. Let

$$g^X = i^X r^X, \quad (72)$$

$$g^Z = i^Z r^Z \quad (73)$$

be these extensions. Then

$$g^X = i^X r^X = (j^{-1} i^Z) (r^Z p)$$

are two $(\mathcal{E}_u, \mathcal{M}_p)$ -factorizations of the morphism g^X . This implies that there exists an isomorphism h such that

$$hr^X = r^Z p, \quad (74)$$

$$j^{-1} i^Z h = i^X. \quad (75)$$

From the equality (9.10) we deduce that $h = r(p)$, i.e. $r(p)$ is an isomorphism and $p \in \varepsilon\mathcal{R}$.

$$\begin{array}{ccccc}
 X & \xrightarrow{p} & Z & \xrightarrow{i} & Y \\
 \downarrow r^X & & \downarrow r^Z & & \\
 rX & \xrightarrow{h=r(p)} & rZ & & \\
 \downarrow i^X & & \downarrow i^Z & & \\
 A^{\text{Hom}(X,A)} & \xrightarrow{j=p} & A^{\text{Hom}(Z,A)} & &
 \end{array}$$

The diagram illustrates the commutativity of the following relations: $g^X = i^X r^X$, $g^Z = i^Z r^Z$, $h = r(p)$, and $j = p$. A curved arrow labeled g^X points from X to $A^{\text{Hom}(X,A)}$, and another curved arrow labeled g^Z points from Z to $A^{\text{Hom}(Z,A)}$.

Figure 9.2.

2. Follows from 9.1. ■

Corollary 9.1. [BG].1. $\mathcal{M}_u$ is the class of all morphisms with respect to which the field $\mathbb{K}$ is injective: $\mathcal{M}_u = \mathcal{M}_p \circ \varepsilon\mathcal{S}$.

2. $\mathcal{M}_f \circ \varepsilon\mathcal{S}$ is the class of all universal monomorphisms with a closed image.

In the conditions of Theorem 9.1, the following theorem holds.

Theorem 9.2. In the category $\mathcal{C}_2\mathcal{V}$ the objects of form A^τ form a sufficient class of $(\mathcal{M}_p \circ \varepsilon\mathcal{R})$ -injective objects.

Proof These objects are $\mathcal{M}_p$ -injective and $\varepsilon\mathcal{R}$ -injective. Therefore, they also are $(\mathcal{M}_p \circ \varepsilon\mathcal{R})$ -injective. Further, for each object X of the category $\mathcal{C}_2\mathcal{V}$ we have $r^X \in \varepsilon\mathcal{R}$ and $g^X \in \mathcal{M}_p$. Thus, the morphism $g^X r^X$ belongs to the

class $\mathcal{M}_p \circ \varepsilon \mathcal{R}$. This fact means that in the category $\mathcal{C}_2\mathcal{V}$ there exist sufficiently many objects which are $(\mathcal{M}_p \circ \varepsilon \mathcal{R})$ - injective.

$$X \xrightarrow{r^X} rX \xrightarrow{g^{rX}} A^{Hom(rX, A)}$$

Figure 9.3.

■

Corollary 9.2. *In the category $\mathcal{C}_2\mathcal{V}$ there exists a proper class of the bicategorical structures with sufficiently many injective objects.*

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