INFINITESIMAL GROUPS ASSOCIATED WITH QUADRATIC DYNAMICAL SYSTEMS

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Abstract Any homogeneous quadratic differential system is characterized by a binary algebra determined up to an isomorphism. The homogeneous system, in Yamaguti's sense, associated with this algebra allows us to find out a set of generators for the Lie algebra generated by the left multiplications of the algebra. A realization of this Lie algebra as a Lie algebra of vector fields gives the opportunity to associate an infinitesimal Lie group with any quadratic differential system. Some particular cases are analyzed.

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1. INTRODUCTION

The importance of the nonlinear dynamical systems was very soon remarked [13]. The most simple type of nonlinearity appeared to the quadratic dynamical systems. Many properties of such dynamical systems can be obtained by using an algebraic approach. L. Markus [6] was the first who steps on this way; he made evident the existence of a bijective correspondence between the classes of affinely equivalent quadratic homogeneous systems in plane and the classes of isomorphic 2-dimensional commutative algebras and gives the first classification of such quadratic homogeneous systems up to an central affinity. C. S. Sibirskij [11],[12] deals with the algebraic study of affine invariants and semiinvariants of quadratic differential equations. M. K. Kinyon and A. A. Sagle [5] deal with qualitative aspects of the general theory of quadratic dynamical systems. M. Popa [9] and his coworkers yielded, on the line traced by S. Lie, classification results by means of the orbits of the extended action of affinely transformation groups to the space of parameters of quadratic differential systems; this way follows the direction shown by C. S. Sibirskij [11]. I. Burdujan [2] extended some results of Markus to the case of quadratic differential equations on Banach spaces and tried to give a meaning for the deviations from associativity of the binary algebra associated with a quadratic differential system in connection with the set of solutions of the analyzed system.

In this paper we prove the existence of an infinitesimal group acting on the space where is defined the analyzed quadratic dynamical system. If the algebra associated with the quadratic dynamical system satisfies a certain identity of degree four or five, its homogeneuos system is just a Lie triple system or a general Lie triple system, respectively. Some invariants under the action of an important subgroup of the infinitesimal group are emphasized.

2. HOMOGENEOUS QUADRATIC DIFFERENTIAL SYSTEMS

Although the most part of results concerns the homogeneous quadratic differential systems (briefly, HQDSs) on finite dimensional spaces, the problem of correspondence between the classes of affinely equivalent HQDSs and classes of isomorphic binary algebras is more easily to solve in the case of HQDSs defined on Banach spaces. That is why in this section we prove some results on the Banach space, only.

Let E be a Banach space. An HQDS on E is every autonomous equation of the form $\dot{X} = F(X)$ where F is a continuous quadratic vector form on E. The polar form of F is the symmetric bilinear vector form $G : E \times E \to E$ defined by $G(x, y) = \frac{1}{2} \cdot [F(x+y) - F(x) - F(y)], \quad \forall x, y \in E$. Let us denote by $E(\cdot)$ the commutative algebra defined by means of the multiplication $x \cdot y = G(x, y), \quad \forall x, y \in E$. Two HQDSs $\dot{X} = F(X)$ and $\dot{Y} = F_1(Y)$ are called *affinely equivalent* each other if there exists a continously invertible linear transformation $T : E \to E$ such that X is a solution of the former system if and only if $Y = T(X)$ is a solution for the second system. It can be proved that two HQDSs are affinely equivalent if and only if their associated algebras are isomorphic [2], [6].

On the other hand, with any binary $\mathcal{A}$ algebra a homogeneous (algebraic) system (shortly, h.s.) in the sense of Yamaguti [14] is associated [1]; this h.s. gives the opportunity to find out a family of generators for the Lie algebra $\mathcal{L}$ generated by the left multiplications of the algebra $\mathcal{A}$ [3]. Actually, this association gives the construction of a covariant functor $\mathbf{HF} : \mathcal{ALG}_K \to \mathcal{HS}_K$ whose restriction to the subcategory $\mathcal{A}ss_K$ is just the usual functor associating with any associative algebra the Lie algebra defined on the algebra ground space by means of the commutator-operation. Unfortunately, the h.s. associated with a binary algebra does not characterize it up to an isomorphism.

In particular, the h.s. associated with a commutative algebra $\mathcal{A}(\cdot)$ is defined by the following multilinear operations, given recurrently by

$$[x_1, x_2, \cdots, x_{n+1}] = D_{(x_1, x_2, \cdots, x_n)}(x_{n+1}), \quad \forall (x_1, x_2, \cdots, x_{n+1}) \in \mathcal{A}^{n+1}, \forall n \geq 2$$

where $D_{(x_1,x_2,\cdots,x_n)}$ are the endomorphisms of E defined inductively by

$$D_{(x_1,x_2)} = [L_{x_1}, L_{x_2}],$$

$$D_{(x_1,x_2,\cdots,x_{n+1})} = \left[D_{(x_1,x_2,\cdots,x_n)}, L_{x_{n+1}}\right] - L_{[x_1,x_2,\cdots,x_{n+1}]},\ \forall n \geq 2.$$

Consequently, the h.s. associated with a commutative algebra $\mathcal{A}(\cdot)$ satisfies the following axioms:

(h.s.1) $[x, x, x_1, \cdots, x_k] = 0, \quad \forall k \in \mathbb{N}^\star,\ \forall x, x_1, \cdots, x_k \in \mathcal{A},$

(h.s.2) $[x, y, z] + [y, z, x] + [z, x, y] = 0, \quad \forall x, y, z \in \mathcal{A},$

(h.s.3) $[x, y, z, w] + [y, z, x, w] + [z, x, y, w] = 0, \quad \forall x, y, z, w \in \mathcal{A},$

(h.s.4) $[x_1, \cdots, x_k, y, z] - [x_1, \cdots, x_k, z, y] = 0, \ \forall x_1, \cdots, x_k, y, z \in \mathcal{A},$

(h.s.5) $\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,y_2)}\right] = D_{([x_1,\cdots,x_k,y_1],y_2)} - D_{([x_1,\cdots,x_k,y_2],y_1)} +$
. $+ D_{(x_1,\cdots,x_k,y_1,y_2)} - D_{(x_1,\cdots,x_k,y_2,y_1)},$
. $\forall k = 2, 3, \cdots,\ \forall x_1, \cdots, x_k, y_1, y_2 \in \mathcal{A}$

(h.s.6) $\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell,y_{\ell+1})}\right] = \widetilde{\mathfrak{D}}\big(\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell)}\right], y_{\ell+1}\big) -$
. $-\left[D_{(x_1,\cdots,x_k,y_{\ell+1})}, D_{(y_1,\cdots,y_\ell)}\right] - D_{(x_1,\cdots,x_k,[y_1,\cdots,y_\ell,y_{\ell+1}])} + D_{(y_1,\cdots,y_\ell,[x_1,\cdots,x_k,y_{\ell+1}])},$
. $\forall \ell, k = 2, 3, \cdots,\ \forall x_1, \cdots, x_k, y_1, \cdots, y_{\ell+1} \in \mathcal{A},$

where $\widetilde{\mathfrak{D}}$ is the endomorphism defined by $\widetilde{\mathfrak{D}}\big(D_{(x_1,\cdots,x_n)}, y\big) = D_{(x_1,\cdots 2,x_n,y)}$. The axioms (h.s.5) and (h.s.6) assure us that the linear enveloping of the family of endomorphisms $\widetilde{\mathfrak{L}} = \left\{D_{(x_1,\cdots,x_n)} | \forall x_1, \cdots, x_n \in \mathcal{A}, n \geq 2\right\}$ is just the Lie algebra $\mathfrak{L}$ endowed with the usual Lie bracket from an associative algebra of endomorphisms.

3. INFINITESIMAL GROUP ASSOCIATED WITH A HQDS

In what follows we consider the finite dimensional case, only.

If $E = \mathbb{R}^n$, then any HQDS has the form $\dot{x}^i = a^i_{jk} x^j x^k \quad i, j, k = 1, 2, \cdots, n$, $a^i_{jk} \in \mathbb{R}$ and the binary operation defined on $\mathbb{R}^n$ is $x \cdot y = \big(a^1_{jk} x^j y^k, a^2_{jk} x^j y^k, \cdots, a^n_{jk} x^j y^k\big),\ \ \forall x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. It is clear that the structure constants of the associated algebra are just the coefficients that define the system. In this case, only a finite number of elements in $\widetilde{\mathfrak{L}}$ is a system of generators for $\mathfrak{L}$.

In what follows, we shall prove that $\widetilde{\mathfrak{L}}$ can be realized as a system of local vector fields. Indeed, if $\mathbf{B} = \{e_1, \cdots, e_m\}$ is a basis in E, then the structure constants of the h.s. are defined in the usual way by $[e_{i_1}, e_{i_2}, \cdots, e_{i_k}] = t^j_{i_1 i_2 \cdots i_k} e_j, \quad n \geq 2$. They satisfy the following conditions

(h.s.1)' $t^s_{iij_1\cdots j_k} = 0, \quad k \geq 1$

(h.s.2)' $t^s_{ijk} + t^s_{jki} + t^s_{kij} = 0,$

(h.s.3)' $t^s_{ijkp} + t^s_{jkip} + t^s_{kijp} = 0,$

(h.s.4)' $t^s_{i_1\cdots i_k pq} - t^s_{i_1\cdots i_k qp} = 0, \quad \forall k \geq 2$

(h.s.5)' $t^s_{i_1\cdots i_k p}t^p_{qrm} - t^s_{qrp}t^p_{i_1\cdots i_k m} = t^p_{i_1\cdots i_k q}t^s_{prm} - t^p_{i_1\cdots i_k r}t^s_{pqm} +$
. $+ t^s_{i_1\cdots i_k qrm} - t^s_{i_1\cdots i_k rqm}, \qquad k \geq 2$
(h.s.6)' $t^s_{i_1\cdots i_k p}t^p_{j_1\cdots j_q j_{q+1} m} - t^s_{j_1\cdots j_q j_{q+1} p}t^p_{i_1\cdots i_k m} =$
. $\widetilde{\mathfrak{D}}_{j_{q+1}}\big(t^s_{i_1\cdots i_k p}t^p_{j_1\cdots j_q m} - t^s_{j_1\cdots j_q p}t^p_{i_1\cdots i_k m}\big) -$
. $- t^s_{i_1\cdots i_k j_{q+1} p}t^p_{j_1\cdots j_q m} + t^s_{j_1\cdots j_q p}t^p_{i_1\cdots i_k j_{q+1} m} -$
. $- t^s_{j_1\cdots j_q j_{q+1}}t^s_{i_1\cdots i_k pm} + t^p_{i_1\cdots i_k j_{q+1}}t^s_{j_1\cdots j_q pm}, \quad \forall k, q \geq 2$

where $\widetilde{\mathfrak{D}}_{j_{q+1}}\big(t^s_{i_1\cdots i_k m}\big) = t^s_{i_1\cdots i_k j_{q+1} m}$.

We shall use the notation $D_{i_1 i_2\cdots i_n} = D_{e_{i_1}, e_{i_2}\cdots e_{i_n}}, \quad \forall n \geq 2$. The matrices of $D_{i_1 i_2\cdots i_n}$ in basis $\mathbf{B}$ are defined by $D_{i_1 i_2\cdots i_n}(e_k) = t^j_{i_1\cdots i_n k}e_j, \quad \forall n \geq 2 \ (i_p, k, j = 1, 2, \cdots, m)$. They generates a Lie subalgebra of $g\ell(n, \mathbb{R})$.

The main result of our paper is the following.

Theorem 3.1. *Let $A(\cdot)$ be the binary algebra associated with a HQDS defined on $\mathbb{R}^n$, let a^k_{ij} be the structure constants of $A(\cdot)$ and let $t^j_{i_1 i_2\cdots i_s}$ be the structure constants of the associated h.s.. Then, the following vector fields*

$$X_i = \sum_{j,k=1}^{m} a^k_{ij}x^k\frac{\partial}{\partial x^j}, \quad X_{i_1 i_2\cdots i_n} = \sum_{j,k=1}^{m} t_{i_1 i_2\cdots i_n k}x^j\frac{\partial}{\partial x^j}, \qquad \forall n \geq 2.$$

generates an infinitesimal group.

Proof. The axioms (h.s.3)'-(h.s.6)' lead to the equations

$$\begin{array}{c} X_{ijk} + X_{jki} + X_{kij} = 0, \\ X_{i_1 i_2\cdots i_k js} = X_{i_1 i_2\cdots i_k sj}, \\ \big[X_{i_1 i_2\cdots i_n}, X_{jk}\big] = t^p_{i_1\cdots i_n j}X_{pk} - t^p_{i_1\cdots i_n k}X_{jp}, \qquad k \geq 2, \\ \big[X_{i_1 i_2\cdots i_k}, X_{j_1 j_2\cdots j_s j_{s+1}}\big] = \widetilde{\mathfrak{D}}_{j_{s+1}}\big(\big[X_{i_1 i_2\cdots i_k}, X_{j_1 j_2\cdots j_s}\big]\big) - \\ -\big[X_{i_1 i_2\cdots i_k j_{s+1}}, X_{j_1 j_2\cdots j_s}\big] - t^p_{i_1\cdots i_k j_{s+1}}X_{j_1 j_2\cdots j_s p}, \qquad k, s \geq 2. \end{array}$$

Moreover, the following identities hold

$$\big[X_i, X_j\big] = X_{ij}, \qquad \big[X_{i_1\cdots i_k}, X_j\big] = X_{i_1\cdots i_k j} + t^s_{i_1\cdots i_k j}X_s.$$

These identities assure us that the linear enveloping of the family $\big\{X_i, X_{i_1\cdots i_n} | \forall n \geq 2\big\}$ is a Lie algebra $\mathfrak{L}_{vf}$; further $\mathfrak{L}_{vf1} = \big\{X_{i_1\cdots i_n} | \forall n \geq 2\big\}$ is a Lie subalgebra of $\mathfrak{L}_{vf}$. The mapping

$$L_i \to X_i, \qquad D_{i_1 i_2 \cdot i_n} \to X_{i_1 i_2 \cdot i_n}, \qquad \forall n \geq 2$$

is just a Lie algebra isomorphism. Since $dim\mathfrak{L} = dim\mathfrak{L}_{vf} \leq m^2$, it follows that we have a finite dimensional infinitesimal (local) Lie group G_A whose composition can be locally given by means of Campbell-Hausdorff formula. On E it induces the local multiplication

$$x \star y = x + y + \frac{1}{12}[x, y, y] + \frac{1}{12}[y, x, x] + \frac{1}{24}[y, x, y, x] + \frac{1}{24}[x, y, x, y] + ...$$

4. PARTICULAR CASES

It is well known that the structure of the associated binary algebra is responsible for certain structural properties of the corresponding HQDS. Indeed, if $E(\cdot)$ is an algebra with associative powers, then the Cauchy problem $\dot{x} = x^2, x(t_0) = x_0$ has the solution $x(t) = (I-(t-t_0)L_{x_0})^{-1}(x_0)$, where L_{x_0} denote the left multiplication by x_0. For example, this is the case when the algebra $E(\cdot)$ is a Jordan algebra.

If $E(\cdot)$ is a commutative algebra satisfying the identity [8]

$$2(yx \cdot x)x + y(x^2 \cdot x) = 3(y \cdot x^2) \cdot x$$

then, it implies $[y, x, x, x] = 0$ and, consequently, the h.s. associated with it is just a Lie triple system. It was proved [14] that $\Delta = det[t^s_{ijk}x^j x^k - \rho\delta^s_i]$ is an invariant of the local group generated by X_{ij} (i.e. $X_{ij}(\Delta) = 0$). This result can be extended to the case when the associated algebra satisfies the identity $[y, x, x, x, x] = 0$. This time, the corresponding h.s. is a general Lie triple system [14] and the linear enveloping of $\{X_{ijk}\}$ is a Lie subalgebra of $\mathfrak{L}_{vf}$. Then it can proved the following proposition.

Proposition 4.1. *The functions*

$$\Delta_1 = det[t^s_{ijk}x^j x^k - \rho\delta^s_i],$$
$$\Delta_2 = det[t^s_{ijkp}x^j x^k x^p - \rho\delta^s_i]$$

are invariant under the action of the local group generated by X_{ijk}.

Proof. It is suffices to prove $X_{ijk}(\Delta_1) = 0, X_{ijk}(\Delta_2) = 0$. This is get by straightforward computations and using the axioms of general triple systems.

Remark 4.1. Δ_1 *and* Δ_2 *are the characteristic polynomials of the endomorphisms* $Y \to [Y, X, X]$, $Y \to [Y, X, X, X]$, *respectively.*

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