SETS GOVERNING THE PHASE PORTRAIT (APPROXIMATION OF THE ASYMPTOTIC DYNAMICS)

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Abstract The most important sets governing the phase-portrait (ω-limit sets, unstable manifolds, centre manifolds, inertial manifolds, inertial sets and approximate inertial manifolds) are described.

Examples are worked out and the relationships (of strict inclusion or coincidence) between different governing sets are shown.

1. ω -LIMIT SETS. ATTRACTORS

Let M be a set and let (M, ϕ) be a continuous dynamical system. We say that $p \in M$ is an *ω-limit point of* $u \in M$ if there exists $0 < t_1 < t_2 < \ldots < t_n \to \infty$ such that $\lim_{n\to\infty} \phi_{t_n}(u) = p$ [4]. Similarly, $q \in M$ is called an *α-limit point of* $u \in M$ if there exists $0 > t_1 > t_2 > \ldots > t_n \to -\infty$ such that $\lim_{n\to\infty} \phi_{t_n}(u) = q$ [4].

The set of all ω-limit points of u is called *ω-limit set of* u and we write $\omega(u)$ [4]. The set of all α-limit points of u is called *α-limit set of* u and we write $\alpha(u)$ [4].

Further, we show that the ω-limit sets can be defined not only for a point $u \in M$, but also for a set $\mathcal{A} \subset M$.

These sets are invariant through the dynamics ϕ.

Let $M = H$ be a metric space and assume that the time t runs over $\mathbb{R}^+$. Thus, the forward evolution of the points of phase space is described by a family of operators $S(t)$, $t \geq 0$, $S(t) : H \to H$ enjoying the properties

$$\begin{aligned} S(t+s) &= S(t) \cdot S(s), \\ S(0) &= I \,(\text{identity in H}). \end{aligned} \tag{1}$$

and being associated with a semidynamical system (H, S).

Assume that

$S(t)$ is a continuous (nonlinear) operator from H into itself, for all $t \geq 0$. (2)

For $u \in H$ (or for $\mathcal{A} \subset H$) the *ω-limit set of* u (or $\mathcal{A}$) is defined [11] as

$$\omega(u) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u}\,, \quad \omega(\mathcal{A}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{A}}\,.$$

Notice that $\varphi \in \omega(\mathcal{A})$ if and only if there exists a sequence of elements $\varphi_n \in \mathcal{A}$ and a sequence $t_n \to \infty$ such that

$$S(t_n)\varphi_n \to \varphi \text{ as } n \to \infty. \tag{3}$$

Let $\mathcal{B}$ be a subset of H and let $\mathcal{U}$ be an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is *absorbing* in $\mathcal{U}$ if the orbit of any bounded subset of $\mathcal{U}$ enters into $\mathcal{B}$ after a certain time.

Let A, B be two sets from the phase space M. The invariant set A is an *attractive set* for B if the distance between A and $\phi_t(B)(= \bigcup\limits_{u \in B} \phi_t(u))$ tends to zero for $t \to \infty$ [6].

An *attractor* is an invariant, closed, attractive set for an entire neighborhood [6].

A *global attractor* is the union of all the attractors of the system.

The attractor is included in the absorbing domain and it is also included in the ω -limit set.

For a long time it was understood that the attractors are the most important sets of the phase space. Lately, in the absence of attractors, it was found that some other invariant sets may be of primary importance, for instance, the unstable manifolds, central manifolds etc.

Hypothesis:

$$\text{For } t \text{ large, the operators } S(t) \text{ are uniformly compact.} \tag{4}$$

Alternatively, if H is a Banach space, we may assume that

$$\begin{aligned} &S(t) \text{ is the perturbation of an operator satisfying (4) by} \\ &\qquad\qquad \text{an operator which converges to 0 as } t \to \infty. \end{aligned} \tag{5}$$

The following theorem shows that, with a few hypotheses, the ω-limit set is the attractor.

Theorem 1.1. *[11] Assume that H is a metric space and that the operators $S(t)$ are given and satisfy (1), (2) and either (4) or (5). We also suppose that there exists an open set $\mathcal{U}$ and a bounded set $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the ω-limit set of $\mathcal{B}$ is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$.*

Example 1.1. Consider the two-dimensional dynamical system generated by the Cauchy problem for

$$\begin{cases} \dot{x} = \lambda_1 x, \\ \dot{y} = \lambda_2 y, \end{cases} \tag{6}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}^*$, $|\lambda_2| > |\lambda_1|$ parameters of the same sign.

The solution of this problem reads $x(t) = x_0 e^{\lambda_1 t}$, $y(t) = y_0 e^{\lambda_2 t}$, for all $(x_0, y_0) \in \mathbb{R}^2$. If $\lambda_1, \lambda_2 < 0$, we find that $(0,0)$ is an ω-limit point for all points of $\mathbb{R}^2$ (fig. 1a)). The sequence $t_n = n$ is strictly ascending and $t_n \to \infty$. $\phi_{t_n}(x_0, y_0) = \phi_n(x_0, y_0) = (x_0 e^{\lambda_1 n}, y_0 e^{\lambda_2 n})$ has the limit $(0,0)$ for $n \to \infty$, for all (x_0, y_0). If $\lambda_1, \lambda_2 > 0$, $(0,0)$ is an α-limit point for all points of the phase plane, $\mathbb{R}^2$ (fig. 1b)).

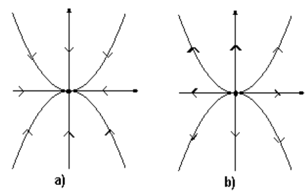

Fig. 1.

The two-dimensional dynamical systems have as ω-limit sets or α-limit sets only the fixed points, homoclinic orbits or limit cycles [4]. In $\mathbb{R}^3$ there exist ω-limit sets which are invariant tori, so the dynamics is not necessary tending to a steady state or a periodic dynamics, but to a quasiperiodic dynamics [4].

2. UNSTABLE MANIFOLDS

Let $(\phi_t)_{t\in\mathbb{R}}$ be the dynamical system generated by the Cauchy problem for the equation $\dot{x} = f(x)$, $x \in M$. Let $x_0 \in M$ be an equilibrium point and let $U \subset M$ be a neighborhood of x_0. By definition the *local unstable manifold* of x_0 [4] is

$$W^u_{loc}(x_0) = \{x \in U \mid \phi_t(x) \in U, \forall t \leq 0 \text{ and } \lim_{t\to-\infty} \phi_t(x) = x_0\} \tag{7}$$

while *the unstable manifold* [4] is

$$W^u(x_0) = \bigcup_{t\geq 0} \phi_t(W^u_{loc}(x_0)). \tag{8}$$

$W^u_{loc}(x_0)$ is a differentiable manifold, while, generally, $W^u(x_0)$ is not. However, $W^u(x_0)$ is an invariant set.

Let $\gamma \subset M$ be a limit cycle described by $x(t+T) = x(t)$ and let $U \subset M$ be a neighborhood of γ. The local unstable manifold of the limit cycle is the set

$$W^u_{loc}(\gamma) = \{x \in U \mid \phi_t(x) \in U, \forall t \leq 0 \text{ and } \lim_{t\to-\infty} |\phi_t(x) - \gamma| = 0\}. \tag{9}$$

Assume that $M = E$ is a Banach space. Consider a semigroup $\{S(t)\}_{t\geq 0}$ which satisfies the properties (1) and suppose that the mapping

$$(t, u_0) \to S(t)u_0 \text{ from } \mathbb{R}_+ \times E \text{ into } E \text{ is continuous.} \tag{10}$$

Let $X \subset E$ be a subset of E (not necessarily a limit cycle).

The *unstable set* of X is the (possibly empty) set of points u which belong to a complete orbit $\{u(t),\, t \in \mathbb{R}\}$ and such that $d(u(t), X) \to 0$ as $t \to -\infty$ [11].

If X is invariant then $W^u(X)$ is invariant too.

Theorem 2.1. *[11] Let E be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a semigroup of operators satisfying (1) and (10) which possessing a global attractor $\mathcal{A}$. Let $X \subset E$ be a compact set invariant through $S(t)$. Then $W^u(X) \subset \mathcal{A}$. For $X = \mathcal{A}$, $W^u(\mathcal{A}) = \mathcal{A}$.*

Again we remark the importance, from the point of view of asymptotic property of attractivity, of the unstable manifold, and not of the stable manifold as expected.

Example 2.1. Consider the system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y. \end{cases} \tag{11}$$

The unique equilibrium point is $(0,0)$ and it is a saddle point. For every (x_0, y_0) initial condition, the solution of the Cauchy problem is $x(t) = x_0 e^t, \quad y(t) = y_0 e^{-t}$, i.e. $\phi_t(x_0, y_0) = (x_0 e^t, y_0 e^{-t})$.

Let $U \subset \mathbb{R}^2$ be a neighborhood of $(0,0)$. We choose U to be the disk of radius 1. Assume first that the initial points $(x_0, y_0) \in U \cap 0x$. Since they satisfy $|x_0| < 1, y_0 = 0$, we have $|x_0 e^t| < 1$, $y_0 e^{-t} = 0$, $\forall\, t \leq 0$, therefore $\phi_t(x_0, y_0) \in U \cap Ox$ too. In addition, $\phi_t(x_0, 0) = (x_0 e^t, 0) \to (0,0)$ as $t \to -\infty$. Hence, the points of $U \cap Ox$ belong to the unstable manifold of $(0,0)$.

Let now $(x_0, y_0) \in U$, $y_0 \neq 0$, $|y_0 e^{-t}| \to +\infty$ for $t \to -\infty$, i.e. $\phi_t(x_0, y_0)$ is not in U, for $t < 0$ sufficiently small. Therefore this (x_0, y_0) does not belong to the unstable manifold of $(0,0)$.

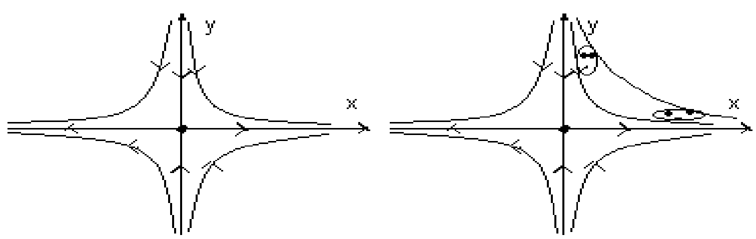

Fig. 2a. *Fig. 2b.*

Hence $W^u_{loc}((0,0)) = [-1,1] \times \{0\}$ and $W^u((0,0)) = \bigcup_{t \geq 0} \phi_t(W^u_{loc}((0,0))) = \mathbb{R} \times \{0\}$. Remark that all trajectories of the dynamical system generated by (11) approach exponentially to the trajectory $x(t) = x_0 e^t$, $y(t) = 0$ from W^u.

$$d((x_0 e^t, y_0 e^{-t}), (x_0 e^t, 0)) = \sqrt{(y_0 e^{-t})^2} = y_0 e^{-t} \to 0.$$

Therefore, the distance between the current points of these two trajectories is equal to the distance between their ordinates in the stable manifold direction and, of course, this tends to zero. It is interesting that this decay to zero is exponential.

As a consequence, as $t \to \infty$, a figure in the phase plane will be contracted in the Oy direction (i.e. of the stable manifold) and magnified in the W^u-direction (fig.2b).

The approach to the unstable manifold is more important than the approach to the stable manifold (because the last one is contracting) and it leads to the notion of axiom A attractor.

The action of the dynamics generated by (11) on the phase space is contractant in one direction and expanding in another one. In the case of finite dynamical systems, if the associated linearized operator around the hyperbolic equilibrium has eigenvalues with n_1 positive real parts and n_2 negative real parts, the effect of the dynamics is the "flattening " in the n_2 directions and "magnification" in the n_1 directions. In the infinite dimensional case the presence of inertial manifolds is associated with "contraction" in an infinity of directions and "magnification" in a finite number of directions. In addition, similarly to the case of the two-dimensional saddle, on the centre manifold there are trajectories towards which the phase space trajectories approach exponentially. In this sense, the asymptotic dynamics (i.e. for $t \to +\infty$) is more similar to the dynamics on the centre manifold. In other words, the study of the dynamics generated by, say, a partial differential equation is better and better approximated by the dynamics on the centre manifold.

3. CENTRE MANIFOLD

In order to study the bifurcation for a dynamical system, one preliminary step is to simplify the problem as much as possible without changing the topological properties of the dynamics of the original system. The linearization principles of Hartman-Grobman type provide conditions under which the stability of an equilibrium point, as well as the behaviour of the solutions around the equilibrium of a nonlinear system are described (up to a homeomorphism) by those of the zero equilibrium of the associated linearized system $\dot{x} = \mathbf{A}x$.

Unfortunately, these priciples are stated only for hyperbolic equilibrium points. When the equilibrium point is not hyperbolic, and, therefore, the Hartman-Grobman theorem does not apply, the reduction theory of the centre manifold is applicable: the dimension of the problem can be reduced, by using a convenient decomposition of the phase space into a direct sum of the stable, unstable and center manifold. Further, we discuss only the finite-dimensional theory of centre manifolds for equilibrium points. The infinite-dimensional case is dealt with in [6].

Consider the particular system

$$\begin{cases} \dot{x} = Ax + f(x,y), \\ \dot{y} = By + g(x,y), \end{cases} \tag{12}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A and B are constant matrices such that the eigenvalues of A have zero real parts, the eigenvalues of B have negative real parts and f and g are functions of class C^2, with $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$ (here f' is the Jacobian matrix of f). It is understood that $(0,0)$ is a nonhyperbolic equilibrium of (12).

If $y = h(x)$ is an invariant manifold for (12) and h is smooth, then it is called a *centre manifold* if $h(0) = 0$, $h'(0) = 0$ [1].

Theorem 3.1. *[1](existence of centre manifolds) Equation (12) has a local centre manifold $y = h(x)$, $|x| < \delta$, where h is of class C^2.*

The dynamics on the centre manifold is governed by the (reduced) n-dimensional system

$$\dot{u} = Au + f(u, h(u)) \tag{13}$$

The following theorem relates the asymptotic behaviour of small solutions of (12), i.e. near the equilibrium $(0,0)$, to solutions of (13).

Theorem 3.2. *[1](reduction principle) Suppose that the zero solution of (13) is stable (asymptotically stable)(unstable). Then the zero solution of (12) is stable (asymptotically stable)(unstable).*

Suppose that the zero solution of (12) is stable. Let $(x(t), y(t))$ be a solution of (12) with $(x(0), y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (13) such that, as $t \to \infty$, $x(t) = u(t) + O(e^{-\gamma t})$, $y(t) = h(u(t)) + O(e^{-\gamma t})$, where $\gamma > 0$ is a constant depending only on B.

In other words, just as remarked in Example 2.1, with a trajectory of the phase space we associate a trajectory on the centre manifold exponentially approaching one to each other as $t \to \infty$.

For functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin let us define

$$(M\phi)(x) = \phi'(x)[Ax + f(x, \phi(x))] - B\phi(x) - g(x, \phi(x)).$$

Theorem 3.3. *[1](approximation of the centre manifold) Suppose that $\phi(0) = 0$, $\phi'(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$ where $q > 1$. Then as $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.*

4. INERTIAL MANIFOLDS

So far we already saw two invariant manifolds on which the dynamics approximates exponentially, asymptotically for $t \to \infty$, the given dynamics. These manifolds were: some unstable manifolds and all centre manifolds.

Now, for an infinite dynamical system, we introduce another invariant manifold, namely the inertial manifold, which is of finite dimension but it describes the large-time behaviour of dynamical systems. The concept of inertial manifold was introduced by Peter Constantin and Ciprian-Ilie Foiaş [2], [3]. It permits the reduction of the infinite-dimensional case to the finite-dimensional one. In addition, the phase space trajectories tend exponentially to the corresponding trajectories situated on the inertial manifold. Moreover, this manifold is Lipschitz.

Consider a Cauchy problem $u(0) = u_0$ for the evolution equation

$$u' = F(u), \tag{14}$$

in a finite or infinite-dimensional Hilbert space H, with which we can associate the semigroup $\{S(t)\}_{t\geq 0}$, where $S(t): u_0 \to u(t)$.

An *inertial manifold* of this system is a finite-dimensional Lipschitz manifold $\mathcal{M}$, positively invariant and which attracts exponentially all the orbits of (14) [11].

Let A be a given linear closed unbounded positive self-adjoint operator in H, with the domain $D(A) \subset H$. Assume that A^{-1} is compact in H. Let us see in which conditions the global attractor of a Cauchy problem for the p.d.e.

$$\frac{du}{dt} + Au + R(u) = 0, \tag{15}$$

is included in a smooth, finite-dimensional, manifold of solutions, $\mathcal{M}$. If $\mathcal{M}$ would be positively invariant (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$), the equation could be restricted to this manifold if the asymptotic behaviour is intended to be obtained. There can be constructed examples with the ratio of attraction arbitrary large. An exponentially attractive manifold would lead to a more complicated asymptotic behaviuor and would make the description of the dynamics on the manifold more relevant for the dynamics in the phase space. In order to satisfy these, there has been introduced the concept of inertial manifold. All inertial manifolds obtained so far are the graphs of some function in a finite-dimensional subspace of H.

Let A^s, $s \in \mathbb{R}$, be the powers of A, defined on $D(A^s)$, and assume that the eigenvalues of A are λ_j and the corresponding eigenvectors w_j of A form an orthonormal basis of H. Define a projector $P_n u = \sum_{i=1}^{n} (u, w_i) w_i$ and let Q_n be its orthogonal complement, i.e. $Q_n = I - P_n$, $Q_n u = \sum_{i=n+1}^{\infty} (u, w_i) w_i$. There exists a function $\phi : P_nH \to Q_nH$ such that on the inertial manifold $\mathcal{M}$ to have $q \equiv Q_n u = \phi(p)$, where $p \equiv P_n u$.

$\mathcal{M}$ can be defined as

$$\mathcal{M} = \{p + \phi(p),\, p \in P_nH\}.$$

Since on the inertial manifold $q = \phi(p)$, restricting (15) to $\mathcal{M}$ we obtain a finite-dimensional e.d.o.

$$\frac{dp}{dt} + Ap + P_nR(p + \phi(p)) = 0, \tag{16}$$

which is called *the inertial form*.

We can state that the dynamics on the inertial manifold is finite-dimensional because every trajectory from $\mathcal{M}$ is given by $u(t) = p(t) + \phi(p(t))$ with $p(t)$ solution of (16).

The theory of inertial manifolds and the theory of centre manifold are particular cases of the reduction principle, which, for a system of equations

$$\dot{x} = F(x, y), \quad \dot{y} = G(x, y),$$

where $F : P \times Q \to P$, $G : P \times Q \to Q$, $F, G \in C^1$ and P, Q are Banach spaces becomes as follows. Suppose that the surface M, which is the graph of φ, is an invariant and exponentially stable set for this system and $\varphi : P \to Q$ is a Lipschitz continuous function. Then, for large times, the dynamics of the system is completely described by the solutions of the reduced system

$$\dot{x} = F(x, \varphi(x)),\, x \in P.$$

The theory which approximates the dynamics for large time is called *the approximate dynamics* [6].

As noticed in Example 2.1, $W^u((0,0))$ is a finite-dimensional invariant manifold which attracts exponentially all the orbits of (11), thus $W^u((0,0))$ is also an inertial manifold. Of course, this was a particular case of unstable manifold. In the general case we do not know if it attracts exponentially the phase trajectories, while the centre manifold *is* an inertial manifold.

5. APPROXIMATE INERTIAL MANIFOLDS

The theorem of existence of an inertial manifold [11] ensures the existence of a function ϕ having as a graph just the inertial manifold. Since the proof of this theorem is not constructive, in applications we do not know the form of this function. Moreover, there exists important equations, e.g. the two-dimensional Navier-Stokes equations, for which the existence of the inertial manifold can not be proved by standard methods. Thus, for numerical reasons, it is of interest and can be found approximate inertial manifolds, which can be described explicitly. The class of constructed approximate inertial manifolds is larger than that of exact inertial manifold.

As the inertial manifold is given by the exact asymptotic relation $q = \phi(p)$, an approximate inertial manifold is given by an approximate relation $q \approx \psi(p)$ and it is related to the nonlinear Galerkin-Faedo-Hopf method; basic applications in the theory of Navier-Stokes equations were carried out by Foiaş and Prodi, whence the fundamental contribution of Foiaş to inertial manifolds and approximate inertial manifolds.

Analyse the equation

$$\frac{du}{dt} + Au + R(u) = 0, \tag{17}$$

and consider the finite-dimensional functions u_m, $u_m \in P_m H$, solutions of the truncated (finite-dimensional) equation.

$$\frac{du_m}{dt} + Au_m + P_m R(u_m) = 0. \tag{18}$$

The solutions u_m of (18) converge to the solution u a lui (17), namely they are uniformly convergent on bounded intervals of time and on compact sets from H, as $m \to \infty$. In the Navier-Stokes case the nonstationary problem (18) is reduced to a stationary one and u_m are the eigenvectors of the corresponding linearized operator. This reasoning was frequently considered by Foiaş and Prodi.

The nonlinear Galerkin method extends (18) by including the neglected terms in the approximation of the inertial manifold, such that (18) becomes

$$\frac{du_m}{dt} + Au_m + P_m R(u_m + P_{2m}\psi(u_m)) = 0,$$

where ψ is the function defining the approximate inertial manifold. The term P_{2m} had to be included because otherwise ψ would have produced an infinite-dimensional term. The standard Galerkin approximation stands now for the approximate inertial manifold $\psi = 0$ [9], [11].

6. INERTIAL SETS

Inertial sets, also called *exponential attractors*, are sets somehow intermediate between the attractors and inertial manifolds. All three objects describe the behaviour for $t \to \infty$ of semidynamical systems. The global attractor is the smallest set from the phase space governing the large time dynamics. Usually, the attractor is sensitive to perturbations and attracts the orbits with a small speed. In most applications the global attractor does not exist (e.g. in Example 2.1 no local or global attractor exist). Inertial manifolds, when they exist, are smooth finite-dimensional manifolds which attract all orbits at an exponential rate; they are stable with respect to perturbations.

Exponential attractors attract all orbits at exponential rate, they are stable with respect to perturbations and they exist for a broad class of evolutionary equations.

In general, the attractors are not manifolds and can have a very complex geometric structure. This is why their characterization is complicated too. Thus, let X be a compact connected subset of a Hilbert space H, let S be a Lipschitz continuous map from X into itself and let us denote the Lipschitz constant of S on X by $Lip_X(S) = L$. If S is restricted to X, then it possesses an universal attractor $\mathcal{A}$ which is a compact connected set given by $\mathcal{A} = \bigcap\limits_{n=1}^{\infty} S^n X$.

A compact set M is called an *exponential fractal attractor* for (S, X) if $\mathcal{A} \subseteq M \subseteq X$ and

i) $SM \subset M$;

ii) M has finite fractal dimension, d_F;

iii) there exist positive constants c_0 and c_1 such that

$$h(S^n X, M) \leq c_0 \exp(-c_1 n), \quad \forall\, n \geq 1,$$

where A, B are compact sets and $h(A, B) = \max\limits_a \min\limits_b |a - b|_H$, is the standard asymmetric Hausdorff pseudodistance.

Exponential attractors share properties of "good" attractors with those of the inertial manifolds.

If a continuous dynamical system has an inertial manifold $\mathcal{M}$, then $\mathcal{M} \bigcap X$ is an exponential fractal attractor.

7. EXAMPLES

7.1. AN INERTIAL MANIFOLD IN THE DYNAMICS OF GAS BUBBLES

In [7] the existence of the inertial manifold for the dynamical system generated by the Cauchy problem for the equation of small oscillations of the radius of a spherical gas bubble is proved.

The small variations of the radius of a spherical gas bubble, surrounded by an incompressible fluid are governed by the following Cauchy problem

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\alpha\varepsilon cos(\omega t), \tag{19}$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{20}$$

where β, ω_0, α, ε and ω are constant real parameters.

The phase space trajectories are

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C\cos(\omega t) + D\sin(\omega t), \tag{21}$$

where

$$A = \frac{\lambda_2 x_0 - \dot{x}_0 + \frac{\alpha\lambda_1}{\lambda_1^2+\omega^2}}{2\sqrt{\beta^2-\omega_0^2}}, \qquad B = \frac{\dot{x}_0 - \lambda_1 x_0 - \frac{\alpha\lambda_2}{\lambda_2^2+\omega^2}}{2\sqrt{\beta^2-\omega_0^2}},$$

$$C = -\alpha\frac{\frac{\alpha\lambda_1}{\lambda_1^2+\omega_0^2} - \frac{\lambda_2}{\lambda_2^2+\omega_0^2}}{2\sqrt{\beta^2-\omega_0^2}}, \quad D = \frac{-\alpha\omega(\lambda_1^2-\lambda_2^2)}{2(\lambda_1^2+\omega_0^2)(\lambda_2^2+\omega_0^2)\sqrt{\beta^2-\omega_0^2}},$$

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2-\omega_0^2}. \tag{22}$$

The attractor of the dynamical system associated with (19), (20) is the ellipse

$$\frac{\dot{x}^2}{\omega^2(C^2+D^2)} + \frac{x^2}{C^2+D^2} = 1, \tag{23}$$

from the plane

$$\ddot{x} + \omega^2 x = 0. \tag{24}$$

The related trajectories on this invariant manifold are $x(t) = C\cos(\omega t) + D\sin(\omega t)$. The distance between the phase space trajectories and the corresponding ones on the invariant manifold (24) is

$$\begin{aligned}\rho(C,I) &= \sqrt{(x_C-x_I)^2+(\dot{x}_C-\dot{x}_I)^2+(\ddot{x}_C-\ddot{x}_I)^2} = \\ &= \sqrt{(Ae^{\lambda_1 t}+Be^{\lambda_2 t})^2+(A\lambda_1 e^{\lambda_1 t}+B\lambda_2 e^{\lambda_2 t})^2+(A\lambda_1^2 e^{\lambda_1 t}+B\lambda_2^2 e^{\lambda_2 t})^2} = \\ &= \sqrt{A^2e^{2\lambda_1 t}(1+\lambda_1^2+\lambda_1^4)+2AB(1+\lambda_1\lambda_2+\lambda_1^2\lambda_2^2)+B^2e^{2\lambda_2 t}(1+\lambda_2^2+\lambda_2^4)} \\ &= \sqrt{A^2e^{2\lambda_1 t}(1+\lambda_1^2+\lambda_1^4)+o(e^{2\lambda_1 t})} \approx Ae^{\lambda_1 t}\sqrt{1+\lambda_1^2+\lambda_1^4} = \mathrm{Ord}(e^{\lambda_1 t}),\end{aligned}$$

and it tends exponentially to zero.

The plane ellipse (23), (24) is proved to be the inertial manifold [7].

7.2. A CENTRE MANIFOLD

In [10] a local analysis of the Cauchy problem for the Gierer-Meinhardt activator-inhibitor normalized system is provided. For a particular case, where only one parameter is variable, a centre manifold is found.

The Gierer-Meinhardt activator-inhibitor model is a Cauchy problem for the system

$$\begin{cases} \dot{a} = \frac{c\rho a^2}{h} - \mu a + \rho\rho_0, \\ \dot{h} = c'\rho a^2 - \gamma h. \end{cases} \tag{25}$$

The normalized form of (25) reads

$$\begin{cases} \frac{dA}{d\tau} = \frac{\rho A^2}{H} - A + \rho\rho_0', \\ \frac{dH}{d\tau} = F(\rho A^2 - H). \end{cases} \tag{26}$$

In order to apply the center manifold theory, in [10] is performed a translation and (25) is linearized around the origin. The obtained system is

$$\begin{cases} A_1' = [\frac{2}{1+\rho\rho_0'} - 1]A_1 - [\frac{1}{\rho}(1+\rho\rho_0')^2]H_1, \\ H_1' = 2\rho F(1+\rho\rho_0')A_1 - FH_1. \end{cases} \tag{27}$$

For an appropriate choice of the parameters, this system is a particular case of the system (12). The system (27) corresponds to the case $m = n = 1$, $A = \frac{2}{1+\rho\rho_0'} - 1$, $B = -F$, $f(A_1, H_1) = -\frac{H_1}{\rho(1+\rho\rho_0')^2}$, $g(A_1, H_1) = 2\rho F(1+\rho\rho_0')A_1$. The condition $f(0,0) = f'(0,0) = g(0,0) = g'(0,0) = 0$ is fulfilled if $\frac{2}{1+\rho\rho_0'} - 1 = 0$, involving $\rho\rho_0' = 1$.

In this case, introducing the notation $A_1 = x, H_1 = y$, the system (27) becomes

$$\begin{cases} x' = [\frac{2}{1+\rho\rho_0'} - 1]x - \frac{y}{\rho}(1+\rho\rho_0')^2, \\ y' = -Fy + 2\rho F(1+\rho\rho_0')x. \end{cases} \tag{28}$$

The equation defining the centre manifold reads [10]

$$h(x) = -2\rho(x^3 - x^2(1+F) + 2Fx).$$

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