AVERAGE VELOCITY OF THE TURBULENT FLOW NEAR A SMOOTH WALL

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Abstract

The universal law concerning the distribution of the turbulent flow average velocity field near a rigid smooth wall is established by considering the field in the transition regime from the laminar motion in the laminar sub-layer, to the turbulent motion in the fully turbulent layer, subsequently passing on the application of the effects of molecular viscosity and turbulent viscosity. Consequently, the expression of the universal function of the correlation between the longitudinal and transversal components of the turbulent flow velocity pulsation near the wall is deduced.

1. Determination of the universal law concerning the average velocity of the turbulent flow near a smooth wall

The general equations of a turbulent flow determined by O. Reynolds are

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} = \bar{X}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\upsilon \frac{\partial \bar{u}_i}{\partial x_k} - \overline{u_i' u_k'}\right),\ i = 1,\, 2,\, 3. \tag{1}$$

We investigate the plane-parallel stationary turbulent motion of a fluid along the Ox axis in the semi-space $z > 0$, assuming there is no gradient of the average pressure. In this case $(1)_1$ has the form

$$\upsilon \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{d}{dz}\left(\overline{u'w'}\right) = 0. \tag{2}$$

This equation is integrated to have

$$\eta \frac{\partial \bar{u}}{\partial z} - \rho\left(\overline{u'w'}\right) = \tau_0, \tag{3}$$

where $\eta = \rho\upsilon$ is the dynamic coefficient of viscosity, and τ_0 is the friction tension at the wall.

The first term in equations (2) and (3), represents the contribution of the molecular viscosity, while the second one the contribution of the velocity pulsation when the friction tension τ_0 is formed. We admit the classical model of the turbulent motion near a wall i.e. the flow field is divided in three domains: the viscous sub-layer, the transition region and the region of fully developed turbulence. The notation of the dynamic coefficient of turbulent viscosity is η'= $\rho\upsilon$'. It was introduced for the first time by J. Boussinesq. We have

$$\tau' = -\rho(\overline{u'w'}) = \eta' \frac{d\bar{u}}{dz}. \tag{4}$$

Therefore the equation of motion near a rigid wall is written as

$$\left(\eta + \eta^{'}\right) \frac{d\bar{u}}{dz} = \tau_0. \tag{5}$$

Assume that this equation is available everywhere near the wall (which means that the principle of superposition of both the molecular viscosity and of the turbulent viscosity is supposed to hold), specifying that in the viscous sub-layer $\eta >> \eta'$, in the transition region η and η' have the same order of magnitude and in the field of full turbulence $\eta \ll \eta^{'}$.

For the turbulent viscosity we propose an expression of the type $\eta = \rho l u_*$ [1], where the mixing length for distances larger than the thickness d of the laminar sub-layer is expressed by the relation $l = \kappa (z - d)$where κ is the constant of Kármán and $u_* = \sqrt{\tau_0/\rho}$ the friction velocity. It follows that

$$\eta' = \kappa\rho\,(z - d)\,u_*, \tag{6}$$

where $\kappa = 0$ for $z < d$ and $k \neq 0$ for $z \geq d$. Introducing the dimensionless values $\zeta = zu_*/\upsilon$ and $\delta = \frac{du*}{\upsilon}$, the equation (5) is written in the dimensionless form

$$[1 + \kappa\,(\zeta - \delta)]\,\frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) = 1, \tag{7}$$

where $\kappa = 0$ for $\zeta < \delta$, $\kappa \neq 0$ for $\zeta \geq \delta$. The equation (7) is integrated, taking into account the value of κ and the evident condition of the adherence to the wall, $\bar{u} = 0$ for $\zeta = 0$. The integration constant is determinated provided that the transition from the laminar profile to the turbulent profile $\bar{u}/u_* = \delta$ occur. The following universal function is obtained for the profile of the turbulent velocities near a rigid smooth plane wall

$$\frac{\bar{u}}{u_k} = \begin{cases} \zeta, & \text{for} \quad 0 \leq \zeta \leq \delta, \\ \frac{1}{\kappa} \ln\left[1 + \kappa\,(\zeta - \delta)\right], & \text{for} \quad \zeta \geq \delta. \end{cases} \tag{8}$$

The constant δ introduces the influence of the molecular viscosity, while the constant κ introduces the influence of the turbulent viscosity when forming the average velocity profile. The numerical values of these constants are $\kappa = 0.4$ and $\delta = 7.5$ [2].

In Table 1 we present comparatively some characteristic elements of the classical theories of the turbulence and of our theory presented herein. In Table 2 we reproduced the expressions of the universal profile of the average velocity $u_+ = \bar{u}/u_*$ of the turbulent flow near a wall.

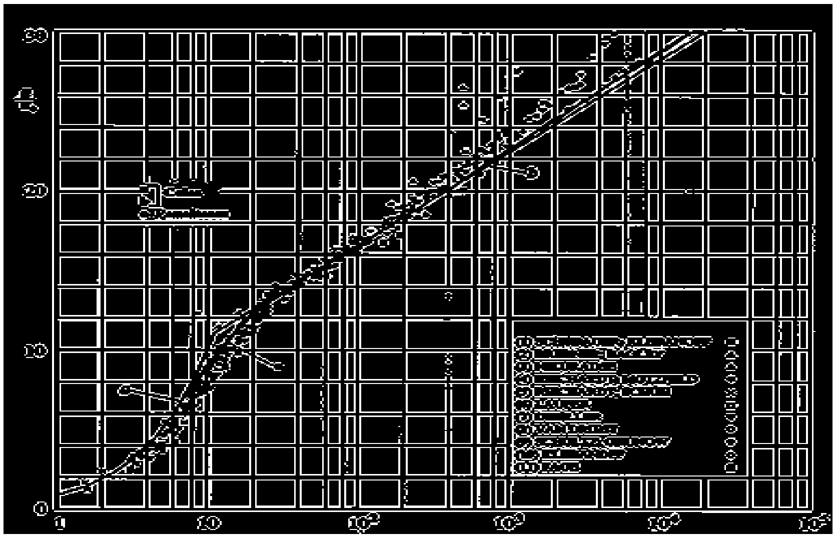
Table 1

Mixing length	Dynamic coefficient of turbulent viscosity	Authors
$l = \kappa z$	$\mu' = \kappa^2 \rho z^2 \frac{d\bar{u}}{dz}$	Prandtl
$l = \kappa \left(\frac{d\bar{u}}{dz}\right) \Big/ \left(\frac{d^2\bar{u}}{dz^2}\right)$	$\mu' = \kappa^2 \rho \left(\frac{d\bar{u}}{dz}\right)^3 \Big/ \left(\frac{d^2\bar{u}}{dz^2}\right)^2$	Kármán
$l = \kappa(z-d)$	$\mu' = \kappa\rho(z-d)u_*$	Panaitescu

Table 2

Formula $\bar{u}_+ = \frac{\bar{u}}{u_*}$	Range	Authors
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 11.5$ $\zeta > 11.5$	Prandtl and Taylor
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 2.5\ln\zeta + 5.1$	$0 \le \zeta \le 11.1$ $\zeta > 11.1$	Coles
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 5.0\ln\zeta - 3.05$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 5$ $5 < \zeta \le 30$ $\zeta > 30$	Kármán
$\bar{u}_+ = 14.53\,\mathrm{th}(\zeta/14.53)$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 27.5$ $\zeta > 27.5$	Rannie
$\frac{d\bar{u}_+}{d\zeta} = \frac{1}{1 - n^2\bar{u}_*\zeta\left[1-\exp\left(-n^2\bar{u}_*\zeta\right)\right]}$ $\bar{u}_+ = 2.78\ln\zeta + 3.8$	$0 \le \zeta \le 26$ $n = 0.124$ $\zeta > 26$	Deissler
$\zeta = \bar{u}_+ + A\left[\exp B\bar{u}_+ - 1 - B\bar{u}_+ - \frac{1}{2}(B\bar{u}_+)^2 - \frac{1}{6}(B\bar{u}_+)^3 - \frac{1}{24}(B\bar{u}_+)^4 - \mathrm{K}\right]$	for any ζ $A = 0.1108$ $B = 0.4$	Spalding
$\frac{d\bar{u}_+}{d\zeta} = \frac{1}{1 + \left\{1 + 4\kappa^2\zeta\left[1-\exp\left(-\frac{\zeta}{A}\right)\right]^2\right\}^{1/2}}$	for any ζ $A = 26$	Van Driest
$u_+ = \begin{cases} \zeta & 0 \le \zeta \le \delta \\ \frac{1}{\kappa}\ln\left[1+\kappa(\zeta-\delta)\right]+\delta & \zeta > \delta \end{cases}$	for any ζ $\kappa = 0.4$ $\delta = 7.5$	Panaitescu

Fig. 1. Universal velocity profiles.



Comparing the profile (8) proposed by us with the experimental data of a large number of research workers [3], [4] a very good agreement is obtained (fig. 1).

2. Deduction of the expression of universal function of correlation between the longitudinal and transversal components of the pulsation of turbulent velocity near a wall

The equation of motion near a wall (3) is transcribed in the dimensionless form as

$$\frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) - \frac{\overline{u'w'}}{u_*^2} = 1. \tag{9}$$

For the universal law of distribution of average velocities (8) it follows

$$-\frac{\overline{u'w'}}{u_*^2} = 1 - \frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) = \frac{\kappa\,(\zeta-\delta)}{1+\kappa\,(\zeta-\delta)}. \tag{10}$$

We specify that the function (10) makes sense for $\zeta \geq \delta$ only therefore except for the viscous sub-layer, which is very well confirmed experimentally [1], [4].

3. Conclusions

The form achieved for the average velocity profile and the comparison with other results of the literature lead to the folowing conclusions:

- the average velocity profile of the turbulent flow near a smooth wall is expressed by a continuous function together with its first order derivative on the whole field of definition, i.e., for any distance from the wall (other formulae, e.g. the ones proposed by Prandtl and Taylor, Coles, Kármán, Rannie, Deissler are represented by functions that have angular points at the common points of the three regions, which is inconsistent with the physical reality);
- the logarithmic form obtained for the average velocity profile (8) is analogous to the classic one of Prandtl – Kármán, but it is more general in the sense that it is valid for the region of transition from the laminar motion in the viscous sub-layer, to the turbulent motion in the fully developed turbulent layer;
- we estimate that our form (8), for the average velocity profile, is simpler (e.g. as compared to the formulae proposed by Diessler, Spalding, Van Driest for the description of the distribution of the average velocity over all distances from the wall, therefore in the region of transition as well);
- the constants κ and δ, occurring in the expression of the universal function of the average velocity profile, have a physical importance well determinated, previously specified;
- the universal function of the correlation between the longitudinal and transversal components of the velocity pulsations (10) is obtained analytically and is confirmed by experimental results.

References

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