ECONOMETRIC AND GEOMETRIC ANALYSIS OF COBB-DOUGLAS AND CES PRODUCTION FUNCTIONS

M. Zakhirov
"M. Ulugbek" National University of Uzbekistan

Abstract Due to the fact that the sign of mixed second order partial derivatives of the functions depends on the scale, in the theory of production functions the scale plays an important role. The laws of changes of scale can be analyzed mathematically. Known analytic holomorphic functions, so far they were studied for the elliptic type of space. However, since the production functions can be different, it is necessary to study the theory of analytic functions of hyperbolic and parabolic types too. This needed theory is developed herein.

I. Cobb- Douglas and CES production functions are
(Cobb-Douglas)$F(x,y) = Ax^s y^t$, $x \geq 0, y \geq 0$, $A > 0$, $0 < s,t < 1$,
(CES)$P(x,y) = a_0(a_1x^n + a_2y^n)^{\frac{1}{\rho}}$,$x \geq 0, y \geq 0$,$a_2 = 1 - a_1$,$-1 < n, \rho < 0$, $0 < a_1, a_2 < 1$, $a_0 > 0$, where x is the capital invested and y is the labor force involved in the production. A, s, t, a_0, a_1, a_2, n, ρ are constants, evaluated according to the real production data and statistical information. First the Cobb-Douglas production function was studied for $s+t = 1$, but later, Douglas allowed the cases $s + t > 1$ and $s + t < 1$ too. The CES production function first was studied for $n - \rho = 0$ and later for $n - \rho > 0$ and $n - \rho < 0$ too. Assume that the volume of capital and labor force increase by 10%. Then
$F(1.10x, 1.10y) = A(1.10x)^s(1.10y)^t = (1.10)^{s+t}Ax^sy^t = (1.10)^{s+t}F(x,y)$and

$$P(1.10x, 1.10y) = a_0[a_1(1.10)^nx^n + a_2(1.10)^ny^n]^{\frac{1}{\rho}} = (1.10)^{n/\rho}P(x,y).$$

In this case, the volume of production increases $(1.10)^{s+t}$(Cobb-Douglas) and $(1.10)^{n/\rho}$(CES) times. The answer to the question "will this growth be greater or less than 10%?" depends on $s + t$ and n/ρ. Summarizing, we have:
$s + t = 1$and $n - \rho = 0$ – constant result of scale;
$s + t > 1$and $n - \rho$¿0 – positive result of scale;
$s + t < 1$and $n - \rho$¡0 – negative result of scale.

Cobb-Douglas and CES production functions can correspond to any value of the result of scale. This is the cause of their popularity among econometricians. However, the feature, providing their popularity is that their elasticity of substitution in their domain is equal to one, feature invariant relatively to the allowance of the sum of $s$and t and the quotient of n and ρ.

II. Given the surface representing the graph of the function $z = f(x, y)$, $(x, y) \in D \to \Delta z, D \in R^2$, $z \in R$, then the theorema ergerium of the surface at the point (x, y) is evaluated by formula $K = \frac{f_{xx}f_{yy}-f_{xy}^2}{1+f_x^2+f_y^2}$, where $f_{xx}, f_{yy}, f_{xy}, f_x, f_y$are partial derivatives of the function $f$with respect to the corresponding arguments. Since the denominator is never negative and zero, it follows that the sign of theorema ergerium depends on the nominator. We have

$$F_{xx}F_{yy} + F_{xy}^2 = A^2stx^{2s-2}y^{2t-2}[1-(s+t)] = X[1-(s+t)],$$

$$P_{xx}P_{yy}-P_{xy}^2 = \frac{a_0^2}{\rho^2}a_1a_2n^2(xy)^{n-2}(1-n)(a_1x^n+a_2y^n)^{\frac{2}{\rho}-2}(1-n/\rho) = Y(1-n/\rho),$$

hence the sign of theorema ergerium depends on $[1-(s+t)]$ and $(1-n/\rho)$ respectively (because other factors (that is $X$and$Y$) are always positive). We conclude that:

$s+t = 1$ and $n-\rho = 0$-constant result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of parabolic type;

$s+t > 1$and $n-\rho > 0$–positive result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of hyperbolic type;

$s+t < 1$and $n-\rho < 0$–negative result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of elliptic type.

The three types of surfaces are determined by three geometries: Euclidean, of Bolyai-Lobachevsky, and Riemann spherical geometry. Thus:

1) if the production gives constant result of scale, then the computations must be performed on the basis of the law of the dual numbers of Euclidean geometry with an imaginary unity $i_p$where $i_p^2 = 0$ but $i_p \neq 0$;

2) if the production gives positive result of scale, then the computations must be performed on the basis of the law of the double numbers of the Bolyai-Labochevsky geometry with an imaginary unity $i_h$where $i_h^2 = 1$ but $i_h \neq \pm 1$ and there is a great probability of avoiding every possible crises;

3) if the production gives negative result of scale, then the computations must be performed on the basis of the law of the complex numbers of the

Riemann geometry with an imaginary unity i_e where $i_e^2 = -1$ and there is possibility of transferring the production to a higher level.

In the three dimensional space, three geometries correspond to three types of production.

Basic goal of economic theory is to find the ways of transferring the production with negative result of scale to the production with constant and henceforth to the production with positive result of scale.

III. Denote by D the economic space limited to the capital x and labor y. Suppose that a geodetic line passes through$A(x_0, y_0) \in D$, i.e. with the given (x_0, y_0)(capital and labor) we associate an ideal production. In other words, the ideal production corresponds to the least expense. Let the point$M(x, y) \in D$ which does not correspond to the ideal production. Then:

1) if the production is in the routine of constant result of scale, then with the coordinates x, y of the point M, a single ideal production can correspond;

2) if the production is in the routine of positive result of scale, then with the coordinates x, y of the point M, at lest two ideal productions can correspond;

3) if the production is in the routine of negative result of scale, then with the coordinates x, y of the point M, cannot correspond any ideal productions.

Perhaps, these results are obvious from the economic point of view, but we figured them out as a mathematic investigation and this is the power of our investigation.

Since the systems of coordinates are based on the parallelism of geodetic lines, this means that in the case of positive and negative result of scales, the ideal production is impossible to correspond and so the effectiveness of other productions cannot be compared with. However, this problem can be solved by leading the local system of coordinates or by defining the surface of production functions in the three dimensional Euclidean space.

Leading the system of curvilinear coordinates, with respect to which (consequently with respect to the scale) all types of surfaces are invariant is the road (calibrating) invariance and the corresponding geometry is the Wale geometry W_2. The importance of this invariance is stated at the beginning of the article.

Thus, leading the isothermal system of coordinates is directly connected with MES (Mixed Equation System). Stereographic transformation of the sphere S on the tangent plane $T_pS$of the sphere on the southern pole P determines MES and the vector fields on the sphere, if this transformation satisfies the commutativity of automorphism $T_pS \to S \to T_pS$, $u = u[x(u, v), y(u, v)]$, $v = [x(u, v), y(u, v)]$, where $(u, v) \in S - T_pS$ are the local coordinates of the point MES. If the above-stated invariance (invariance of elasticity of substitution) is satisfied relatively to the system of coordinates(u, v), then we can obtain the system of equations MES relatively to (x, y)

$$\frac{\partial u}{\partial u} = \frac{\partial v}{\partial v}; \quad \frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} i_m^2,$$

where $i_m^2 = 0, \pm 1$.

Relatively to the isothermal system of coordinates (x, y) we conclude that $u_x = \beta v_x + \gamma v_y, , \; i_m^2 v_x = \beta u_x + \gamma u_y$ where $\beta = \beta(x, y), \gamma = \gamma(x, y)$ describe the metrical features of the surface.

For $i_m^2 = 0$, we can get parabolic, for $i_m^2 = 1$, hyperbolic and for $i_v^2 = -1$, elliptic system of equations MES. The coefficients β, γ are related to the functions $u = u(x, y)$ and $v = v(x, y)$ as follows

$$\beta = -\frac{u_x u_y - v_x v_y i_m^2}{u_x v_y + u_y v_x i_m^2}; \quad \gamma = \frac{u_x^2 - i_m^2 v_x^2}{u_x v_y + i_m^2 u_y v_x};$$

Holomorphic solutions of the above stated system of equations appear are

$$w = u + i_m v = f(\xi(x, y) + i_m \eta(x, y)) = u[\xi(x, y), \eta(x, y)] + i_m v[\xi(x, y) + i_m \eta(x, y)],$$

where f is a holomorphic function with respect to its argument that does not depend on the conjugate function $\xi(x, y) - i_m \eta(x, y)$. Relatively to ξ, η the following system of equations

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \quad ; \quad \frac{\partial u}{\partial \eta} = i_m^2 \frac{\partial v}{\partial \xi}$$

is satisfied.

This is called the Cauchy-Riemann system and it was studied only for the elliptic types of surfaces. It is easy to show its power for the hyperbolic and parabolic types too.

To this aim, we must demonstrate the connection of the holomorphic function with Pauli matrices, namely

$\begin{pmatrix} 0 & -i_e \\ i_e & 0 \end{pmatrix}$, which is the σ_x matrix of Pauli, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is the σ_y matrix of Pauli and the σ_z matrix of Pauli, obtained by multiplying the first two matrices. We can conclude that holomorphic functions must satisfy the equality

$\begin{pmatrix} 0 & 1 \\ i_m^2 & 0 \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix} = i_m \begin{pmatrix} w_x \\ w_y \end{pmatrix}$,where $w = u + i_m v$

Rather

1)$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_\xi + i_e v_\xi \\ u_\eta + i_e v_\eta \end{pmatrix} = i_e \begin{pmatrix} u_\xi + i_e v_\xi \\ u_\eta + i_e v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for elliptic types of system of equations MES;

2) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_\xi + i_h v_\xi \\ u_\eta + i_h v_\eta \end{pmatrix} = i_h \begin{pmatrix} u_\xi + i_h v_\xi \\ u_\eta + i_h v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for hyperbolic types of system of equations MES;

3) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_\xi + i_p v_\xi \\ u_\eta + i_p v_\eta \end{pmatrix} = i_p \begin{pmatrix} u_\xi + i_p v_\xi \\ u_\eta + i_p v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for parabolic types of system of equations MES.

In fact, the Pauli matrices are the particular cases of a certain Qmatrix, the connection of which with the holomorphic functions gives us the above stated system of equations MES. More precisely

$$Q = \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2 - i_m^2}{\gamma} & -\beta \end{pmatrix},$$

and when $\beta = 0$and $\gamma = 1$, we obtain the Pauli matrices. The system of equations MES (quasi-conformal transformation) will be found by means of the following relation

$$\begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2 - i_m^2}{\gamma} & -\beta \end{pmatrix} \begin{pmatrix} w_x \\ \\ w_y \end{pmatrix} = i_m \begin{pmatrix} w_x \\ \\ w_y \end{pmatrix}.$$

Specifically, if we denote by A the Q matrix when the imaginary unity is $i_e (i_m^2 = i_e^2 = -1)$, then we can conclude that$A^4 = E$, which means that$i_e^4 = 1$. Thus, there are two solutions of this equation, $i_e^2 = -1$ and $i_e^2 = 1$. Most mathematicians have omitted the second solution (i.e., $i_e^2 = 1$) supposing it to have real solutions, but in fact, it is also the imaginary unity.

If we denote by B the Q matrix when its imaginary unity is $i_h (i_m^2 = i_h^2 = 1)$, then we can conclude that$B^2 = E$, which demonstrates that$i_h^2 = 1$.

If we denote by C the Q matrix when its imaginary unity is $i_p (i_m^2 = i_p^2 = 0)$, then we can conclude that$C^2 = 0$, which demonstrates that$i_p^2 = 0$.

An arbitrary solution $w = u(x, y) + i_m v(x, y)$of the system of equations MES reads

$$u(x, y) + i_m v(x, y) = f \left[\int_c M\gamma dx - M\beta dy + i_m M dy \right],$$

where c is an arbitrary Jordan curve, which connects the fixed point $(x_0, y_0) \in D$ with the arbitrary point $(x, y) \in D$, and compactly belonging to D. Here f is an arbitrary function holomorphic with respect to its argument, and $M$is the integrating multiplier of the form $\gamma dx - \beta dy$, i.e. $M = \sqrt{\frac{1}{\gamma} \cdot \frac{\partial(\xi,\eta)}{\partial(x,y)}}$ and $\frac{\partial(\xi,\eta)}{\partial(x,y)}$ is the Jacobian of the transformation, $(x,y) \to (\xi,\eta)$.

In case of production functions (rather Cobb-Douglas and CES) we have $\xi = F(x,y), \quad \eta = y$, and $\beta = -F_y(x,y), \quad \gamma = F_x(x,y)$. Then $dF = \gamma dx - \beta dy = F_x dx + F_y dy$, therefore $\int\limits_c dF = \int\limits_c F_x dx + F_y dy = 0$, if c is a simple closed curve. Moreover, $\frac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x \eta_y - \xi_y \eta_x = F_x$ and $M = \sqrt{\frac{1}{F_x} \cdot F_x} = 1$.

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