Contents

SECOND ORDER DIFFERENCE EQUATIONS GOVERNED BY MONOTONE OPERATORS

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Abstract Some second order difference equations governed by maximal monotone operators are studied. Monotone boundary conditions are associated. The problem is the discrete variant of a class of abstract second order differential equations in a Hilbert space.

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1. INTRODUCTION

Suppose that $A : D(A) \subseteq H \to H$ and $\Phi : D(\Phi) \subseteq H \times H \to H \times H$ are maximal monotone operators in a Hilbert space H endowed with the scalar product $(.,.)$ and the corresponding norm $||.||$. In [2] the author studied the existence for the boundary problem

$$\begin{cases} pu'' + ru' \in Au + f, \text{ a.e. on } (0,T) \\ (u'(0), -u'(T)) \in \Phi(u(0) - a, u(T) - b), \end{cases}$$

where $a, b \in D(A)$ are given elements in the domain of A and $p, r : [0,T] \to \mathbb{R}$ are continuous functions. In the beginning, the existence was investigated for $p \equiv 1$, $r \equiv 0$ and a nonlinear semigroup was defined by V. Barbu [5]. Later on, several authors studied the above problem ([1], [2], [6], [9]).

In the sequel, we put $p \equiv 1$ and $r \equiv 0$. We are going to study the existence and the uniqueness of the solution to the discrete variant of the above boundary value problem, namely

$$\begin{cases} u_{i+1} - 2u_i + u_{i-1} \in c_i A u_i + f_i, \ i = \overline{1,N} \\ (u_1 - u_0, -u_{N+1} + u_N) \in \Phi(u_0 - a, u_{N+1} - b). \end{cases} \tag{1.1}$$

Here $a, b \in H$ and $c_i > 0$, $(f_i)_{i=\overline{1,N}} \in H^N$, for $i = \overline{1,N}$ are given sequences. Existence, asymptotic behavior and the equivalence with an optimization problem for some particular cases of (1.1) were analyzed in [3], [4], [7], [8], [9].

The boundary condition generalizes the boundary conditions in [5] : $u_1 - u_0 \in \alpha(u_0 - a)$, $-u_{N+1} + u_N \in \beta(u_{N+1} - b)$, with α, β maximal monotone in H.

We prove that, if A is also strongly monotone in H, then problem (1.1) has a unique solution $u = (u_i)_{i=\overline{1,N}} \in H^N$.

In the next section, we give an auxiliary result and in the last section we prove the existence and uniqueness of the solution of problem (1.1).

2. AN AUXILIARY RESULT

Denote by $B : H^N \to H^N$ the operator defined by

$$B\left((u_i)_{i=\overline{1,N}}\right) = (-u_{i+1} + 2u_i - u_{i-1})_{i=\overline{1,N}}, \tag{2.1}$$

$$D(B) = \{(u_i)_{i=\overline{1,N}} \in H^N,$$
$$(u_1 - u_0, -u_{N+1} + u_N) \in \Phi(u_0 - a, u_{N+1} - b)\}. \tag{2.2}$$

We show that B is maximal monotone in the product space H^N. Recall that the scalar product in H^N is

$$< (u_i)_{i=\overline{1,N}}, (v_i)_{i=\overline{1,N}} >= \sum_{i=1}^{N} (u_i, v_i), \; (\forall)\,(u_i), (v_i) \in H^N$$

and the scalar product in $H \times H$ is

$$< (x_1, y_1), (x_2, y_2) >_{H\times H} = (x_1, x_2) + (y_1, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in H \times H$.

The following lemma is a particular case of some auxiliary result in [3].

Lemma 2.1. *Let $c > 0$ be a given constant. Then, the problem*

$$\begin{cases} \xi_{i+1} - 3\xi_i + \xi_{i-1} = 0, \; i = \overline{1,N} \\ \xi_0 = 0, \quad \xi_1 = c \end{cases} \tag{2.3}$$

has a strictly increasing solution $\xi_i > 0$, for all $i = \overline{1, N+1}$ and the problem

$$\begin{cases} \zeta_{i+1} - 3\zeta_i + \zeta_{i-1} = 0, \; i = \overline{1,N} \\ \zeta_{N+1} = 0, \quad \zeta_N = -c \end{cases} \tag{2.4}$$

has a strictly increasing solution $\zeta_i < 0, \; i = \overline{0,N}$.

Now we establish the main auxiliary result.

Proposition 2.1. *If $a, b \in H$ and Φ is maximal monotone in $H \times H$, then the operator B given by $(2.1) - (2.2)$ is maximal monotone in H^N.*

Proof Let $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}}$ be two given sequences in $D(B)$. It is easy to check that

$$< B\left((u_i)_{i=\overline{1,N}}\right) - B\left((v_i)_{i=\overline{1,N}}\right), (u_i - v_i)_{i=\overline{1,N}} >=$$

$$= \sum_{i=1}^{N} ||u_{i+1} - u_i - v_{i+1} + v_i||^2 + ||u_1 - u_0 - v_1 + v_0||^2 +$$

$$+(u_1 - u_0 - v_1 + v_0, u_0 - v_0) - (u_{N+1} - u_N - v_{N+1} + v_N, u_{N+1} - v_{N+1}) \geq 0.$$

Since $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}} \in D(B)$, by the monotonicity of Φ, we deduce that

$$(u_1 - u_0 - v_1 + v_0, u_0 - v_0) - (u_{N+1} - u_N - v_{N+1} + v_N, u_{N+1} - v_{N+1}) \geq 0,$$

so we obtain the monotonicity of B in H^N.

Now we show that B is maximal monotone, that is $R(I + B) = H^N$, i.e. for every sequence $(g_i)_{i=\overline{1,N}} \in H^N$, there is a sequence $(u_i)_{i=\overline{1,N}} \in H^N$ such that

$$\begin{cases} u_{i+1} - 3u_i + u_{i-1} = g_i, \ i = \overline{1,N} \\ (u_1 - u_0, -u_{N+1} + u_N) \in \Phi(u_0 - a, u_{N+1} - b). \end{cases} \tag{2.5}$$

We are looking for the solution of (2.5) under the form

$$u_i = w_i + x\xi_i + y\zeta_i, \ i = \overline{1,N}, \tag{2.6}$$

where $w_i, \ \xi_i, \ \zeta_i$ are the solutions of the problems

$$\begin{cases} w_{i+1} - 3w_i + w_{i-1} = g_i, \ i = \overline{1,N} \\ w_0 = 0, \quad w_1 = 0 \end{cases} \tag{2.7}$$

and (2.3), (2.4) respectively. But u_i verifies the equation from (2.5) for all $x, y \in H$. We are looking for $x, y \in H$ such that u_i satisfies also the boundary condition in (2.5). One writes it under the form

$$(y\zeta_1 - y\zeta_0 + cx, -w_{N+1} + w_N - x\xi_{N+1} + x\xi_N - cy) \in$$

$$\in \Phi(y\zeta_0 - a, w_{N+1} + x\xi_{N+1} - b).$$

This inclusion becomes

$$(-w_{N+1} + w_N, 0) \in F(x, y) + G(x, y), \tag{2.8}$$

where

$$F(x, y) = ((\xi_{N+1} - \xi_N)x + cy, \ cx + (\zeta_1 - \xi_0)y),$$

$$G(x,y) = (z_2, -z_1),$$

where $(z_1, z_2) \in \Phi(y\zeta_0 - a, w_{N+1} + x\xi_{N+1} - b)$.

Function F is everywhere defined, linear, continuous and strongly monotone in $H \times H$.

Like in Proposition 2.1 from [3], we can prove that function G is maximal monotone in $H \times H$. Hence $F + G$ is maximal monotone and coercive and therefore, B is maximal monotone. The lemma is proved.

3. THE MAIN RESULT

In this section we prove the main result concerning the existence and uniqueness of the solution to problem (1.1), provided that A is strongly monotone in H. Let $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = (I - J_\lambda)/\lambda$ be the resolvent of A and the Yosida approximation of A, respectively.

Theorem 3.1. *Let $A : D(A) \subseteq H \to H$ be a maximal monotone and strongly monotone operator in the real Hilbert space H, $0 \in D(A)$, $0 \in A0$. Suppose that Φ is maximal monotone in $H \times H$, $(0,0) \in D(\Phi)$, $(0,0) \in \Phi(0,0)$. Let $a, b \in H$, $c_i > 0$, $f_i \in H$, $i = \overline{1,N}$ be given sequences. Then, problem* (1.1) *has a unique solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$.*

Proof If A is strongly monotone with the constant $\omega > 0$, then

$$\left(x' - y', x - y\right) \geq \omega ||x - y||^2, \ (\forall) y \in Ax, \ y' \in Ax' \tag{3.1}$$

and

$$(A_\lambda x - A_\lambda y, x - y) \geq \frac{\omega}{1 + \lambda\omega} ||x - y||^2 \geq \frac{\omega}{2} ||x - y||^2, \tag{3.2}$$

for $0 < \lambda < 1/\omega$. Let $\mathcal{A}$ be the operator

$$\mathcal{A}\left((u_i)_{i=\overline{1,N}}\right) = (c_1 A u_1, ..., c_N A u_N) \tag{3.3}$$

and let B be the operator given by $(2.1) - (2.2)$. Then problem (1.1) can be written as

$$0 \in B\left((u_i)_{i=\overline{1,N}}\right) + \mathcal{A}\left((u_i)_{i=\overline{1,N}}\right) + (f_i)_{i=\overline{1,N}}. \tag{3.4}$$

Since B, $\mathcal{A}_\lambda$ are maximal monotone in H^N and $\mathcal{A}_\lambda$ is everywhere defined on H^N, it follows that $B + \mathcal{A}_\lambda$ is also maximal monotone in H^N. By the strong monotonicity of A, it follows that $B + \mathcal{A}_\lambda$ is coercive, so for every sequence $(f_i)_{i=\overline{1,N}} \in H^N$, there is $\left(u_i^\lambda\right)_{i=\overline{1,N}} \in D(B)$ such that $B\left((u_i^\lambda)_{i=\overline{1,N}}\right) + \mathcal{A}_\lambda\left((u_i^\lambda)_{i=\overline{1,N}}\right) = -(f_i)_{i=\overline{1,N}}$. This means that problem

$$\begin{cases} u_{i+1}^\lambda - 2u_i^\lambda + u_{i-1}^\lambda = c_i A_\lambda u_i^\lambda + f_i, \ i = \overline{1,N} \\ (u_1^\lambda - u_0^\lambda, -u_{N+1}^\lambda + u_N^\lambda) \in \Phi\left(u_0^\lambda - a, u_{N+1}^\lambda - b\right) \end{cases} \tag{3.5}$$

has at least one solution $\left(u_i^\lambda\right)_{i=\overline{1,N}} \in D\left(B\right)$.

We prove that $\left(u_i^\lambda\right)_{i=\overline{1,N}}$ is bounded with respect to λ in H^N. To this end, one multiplies (3.5) by u_i^λ, one uses (3.1), $0 \in A0$, $(0,0) \in \Phi(0,0)$, the boundary condition, together with the monotonicity of Φ.

Thus we find that $\sum\limits_{i=1}^{N} ||u_i^\lambda||^2$ and $||u_i^\lambda||$, $i = \overline{0, N+1}$ are bounded with respect to λ. Equation (3.5) implies also the boundedness with respect to λ of $A_\lambda u_i^\lambda$.

Next we are going to pass to the limit as $\lambda \to 0$ in (3.5). To this end, let $\lambda,\ \mu > 0$ be fixed. One subtracts the equation (3.5) for λ and for μ, one multiplies the difference by $u_i^\lambda - u_i^\mu$ and one sums from $i = 1$ to $i = N$:

$$\sum_{i=1}^{N}(u_{i+1}^\lambda - u_{i+1}^\mu - u_i^\lambda + u_i^\mu, u_i^\lambda - u_i^\mu) - \sum_{i=1}^{N}(u_i^\lambda - u_i^\mu - u_{i-1}^\lambda + u_{i-1}^\mu, u_i^\lambda - u_i^\mu) =$$

$$= \sum_{i=1}^{N} c_i(A_\lambda u_i^\lambda - A_\mu u_i^\mu, u_i^\lambda - u_i^\mu). \tag{3.6}$$

Using the same computation as in $[3]$, we find that $u_i^\lambda \to u_i$ as $\lambda \searrow 0$ in H, where $u_i \in D\left(A\right)$ and verifies (1.1). The existence is proved.

In order to verify the uniqueness, let $(u_i)_{i=\overline{1,N}}$, $(v_i)_{i=\overline{1,N}}$ be two solutions of (1.1) and let $x_i = u_i - v_i$. Then we subtract the equations for u_i and for v_i, multiply the difference by x_i and sum up from $i = 1$ to $i = N$. Since u_i and v_i verify the boundary condition of problem (1.1), by (3.1) we get

$$c\omega \sum_{i=1}^{N} ||x_i||^2 \le -||x_{N+1} - x_N||^2 + (x_{N+1} - x_N, x_{N+1}) - (x_1 - x_0, x_0) \le 0,$$

so the uniqueness is proved.

References

[1] A.R. Aftabizadeh, N.H. Pavel, Boundary value problems for second order differential equations and a convex problem of Bolza, *Diff. Integral Eqns.* **2**(1989), 495-509.

[2] N.C. Apreutesei, A boundary value problem for second order differential equations in Hilbert spaces, *Nonlinear Analysis, TMA*, **24**(1995), 1235-1246.

[3] N.C. Apreutesei, A finite difference scheme in Hilbert spaces, *Annals Univ. Craiova*, **30**(2003), 21-29.

[4] N. Apreutesei, Finite difference schemes with monotone operators, *Adv. Difference Eq.* **1**(2004), 11-22.

[5] V. Barbu, A class of boundary problems for second order abstract differential equations, *J. Fac. Sci. Univ. Tokyo,Sect 1,* **19**(1972), 295-319.

[6] 12. H. Brézis, Équations d'évolution du second ordre associées à des opérateurs monotones, *Israel J. Math.* **12**(1972), 51-60.

[7] E. Mitidieri, G. Moroşanu, Asymptotic behaviour of the solutions of second order difference equations associated to monotone operators, *Numerical Funct. Anal. Optim.* **8**(1986-1987), 419-434.

[8] G. Moroşanu, Second order difference equations of monotone type, *Numerical Funct. Anal.Optim.***1**(1979), 441-450.

[9] E. Poffald, S. Reich, An incomplete Cauchy problem, *J. Math. Anal. Appl.*, **113**(1986), 514-543.

ON GIRAUX SUBCATEGORIES IN LOCALLY CONVEX SPACES

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Abstract In this paper, we examine the class of Giraux subcategories in the category of locally convex spaces and we show that there exists a bijective correspondence between this lattice of all classes of left and right complete morphisms. Each Giraux subcategories generates some standard bicategorical structure. We describe the classe of injections and the classes of projections of these structures. Every injective object permits to construct a Giraux subcategory. We consider also the reciprocal problem.

Notation. $\mathcal{E}_u$ (resp. $\mathcal{M}_u$) the class of universal epi (resp. mono); $\mathcal{E}_p$ (resp. $\mathcal{M}_p$) the class of precise epi (resp. mono): $\mathcal{E}_p = \mathcal{M}_u^{\rceil}$, $\mathcal{M}_p = \mathcal{E}_u^{\lfloor}$; if $\mathcal{K}$ (resp. $\mathcal{R}$) is coreflective (resp. reflective) subcategory, then k: $\mathcal{C}_2\mathcal{V} \to \mathcal{R}$ (r: $\mathcal{C}_2\mathcal{V} \to \mathcal{R}$) is the coreflector (resp. reflector) functor; $\mathcal{M}$ (resp. $\mathcal{S}$, Σ) is the subcategory of spaces endowed with Mackey's topology (resp. weak topology; the finest locally convex topology).

1. CLASSES OF LEFT-COMPLETE AND RIGHT-COMPLETE MORPHISMS

In the category of locally convex spaces $\mathcal{C}_2\mathcal{V}$ there are many classes of mophisms which are left-complete and right-complete.

Definition 1.1. *Let $\mathcal{A}$ be a class of morphisms of a category $\mathcal{C}$. The class $\mathcal{A}$ is called left − stable if for any pull-back square $uv' = vu'$ with $u \in \mathcal{A}$, it follows that $u' \in \mathcal{A}$ too.*

The class $\mathcal{A}$ possesses the following properties.

1. $\mathcal{I}so \subset \mathcal{A}$.
2. The class $\mathcal{A}$ is closed with respect to the composition.
3. The class $\mathcal{A}$ is left-stable.
4. The class $\mathcal{A}$ is closed with respect to the products.
5. For any object X of the category $\mathcal{C}$ any family of objects of the category $\mathcal{C}/ X$ which belong to $\mathcal{A}$ possess the product.

Similarly we can define the dual notions: a right-stable class and a right-complete class.

Definition 1.2. *Let $\mathcal{L}$ be a class of morphisms of category $\mathcal{C}$. The object $A \in |\mathcal{C}|$ is called $\mathcal{L}$-injective if for any two morphisms $l : X \to Y$, $l \in \mathcal{L}$ and $f : X \to A$ there exists a morphism $g : Y \to A$ such that $gl = f$.*

It is said that the category $\mathcal{C}$ has sufficiently many $\mathcal{L}$-injective objects if for any object X of the category $\mathcal{C}$ there exists an object A and it is $\mathcal{L}$-injective and a morphism $l : X \to A$, such that $l \in \mathcal{L}$.

Similarly, we can state the dual notions: an $\mathcal{L}$-projective object and a category with sufficiently many $\mathcal{L}$-projective objects.

The most frequently the injective objects are examined with respect to the class of injections of a bicategorical structure $(\mathcal{P}, \mathcal{I})$, and the projective ones - with respect to the class of projections.

The following statement is easy to verify.

Theorem 1.1. *Let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure in the category $\mathcal{C}$ which possesses sufficiently many $\mathcal{I}$-injective objects. Then the class $\mathcal{I}$ is right-complete.*

Remark 1.1. *In the category $\mathcal{C}_2\mathcal{V}$, the class $\mathcal{E}_f$ of the strong epimorphisms is left-complete. However, in this category do not exist sufficiently many $\mathcal{E}_f$ - projective objects as it was proved by V. Geyler [G].*

For any cardinal number α let $m(\alpha)$ be the Banach space of the bounded functions defined on a set X of power α: $|X| = \alpha$. The norm in this space is given by

$$\|f\| = \sup_{x \in X} |f(x)|$$

for any bounded function $f : X \to \mathbb{R}$. It is known that these objects are $\mathcal{M}_p$-injective in the category $\mathcal{C}_2\mathcal{V}$ and these objects and their products form a sufficient class of $\mathcal{M}_p$-injective objects [P] .

The class $\mathcal{M}_f$ is right-stable since $\mathcal{M}_f = \mathcal{K}er(\mathcal{C}_2\mathcal{V})$ and the category $\mathcal{C}_2\mathcal{V}$ is semiabelian.

Theorem 1.2. *In the category $\mathcal{C}_2\mathcal{V}$ the bicategorical structures $(\mathcal{E}_u, \mathcal{M}_p)$ and $(\mathcal{E}pi, \mathcal{M}_f)$ have the right-complete classes of injections.*

In the category $\mathcal{C}_2\mathcal{V}$ the objects of the subcategory $\sum$ is $\mathcal{E}_u$-projective and obviously they form a sufficient class of $\mathcal{E}_u$-projective objects.

Theorem 1.3. [*B*4] *1. $(\mathcal{E}_u, \mathcal{M}_p)$ is the only bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ with sufficient projective objects and with sufficient injective objects.*

2. $(\mathcal{E}_u, \mathcal{M}_p)$ is the only bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ with both left-complete and right-complete classes.

2. ON GIRAUX SUBCATEGORIES IN THE CATEGORY $\mathcal{C}_2\mathcal{V}$

In the category $\mathcal{C}_2\mathcal{V}$ the strict monomorphisms class $\mathcal{M}_f$ is the same as the kernels class. In this way from Condition 2 it follows that the reflector functor preserves kernels. In algebra such subcategories are called the Giraux subcategories. In the category of locally convex spaces $\mathcal{C}_2\mathcal{V}$ there exist many subcategories such as Giraux.

Theorem 2.1. *Let $\mathcal{R}$ be a nonzero reflective subcategory possessing a reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$. Then the following statements are equivalent:*

1. The subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective and $r(\mathcal{M}_p) \subset \mathcal{M}_p$.

2. The subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective and $r(\mathcal{M}_f) \subset \mathcal{M}_f$.

3. The subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective and the functor r commutes and its limits are projective.

4. There exists a coreflective subcategory $\mathcal{K}$ with the coreflector functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ such that:

a) $kr \sim k$;

b) $rk \sim r$.

5. The functor r has a left-adjoint functor.

6. There exists a coreflective subcategory $\mathcal{K}$ of a category $\mathcal{C}_2\mathcal{V}$ with a coreflector functor $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ such that k is the left-adjoint of the functor r.

7. The class $\varepsilon\mathcal{R} = \{f \in \mathcal{E}pi\ \mathcal{C} \mid r(f) \in \mathcal{I}so\}$ is left-complete.

Remark 2.1. *1. The Condition 2 follows from the Condition 4, because any reflector functor in the category $\mathcal{C}_2\mathcal{V}$ commutes with the products* [*GG*].

2. The morphisms of the class $(\varepsilon\mathcal{R})^{\perp}$ are called $\mathcal{R}$-perfect .

3. A subcategory that satisfies the equivalent conditions of the above theorem is called a c-reflective subcategory [*B*3] .

4. A pair of subcategories ($\mathcal{K}$,$\mathcal{R}$) of the category $\mathcal{C}_2\mathcal{V}$ that satisfies the Condition 4 of the above theorem is called an adjoint pair of subcategories.

5. Any c-reflective subcategory $\mathcal{R}$ determines an adjoint pair of subcategories: ($\mathcal{K}$,$\mathcal{R}$) . The subcategory $\mathcal{K}$ is denoted by $m\mathcal{R} : \mathcal{K} = m\mathcal{R}$.

Examples. 1. ($\mathcal{M}$,$\mathcal{S}$) is an adjoint pair of subcategories of the category $\mathcal{C}_2\mathcal{V}$, where $\mathcal{M}$ is the subcategory of the Mackey spaces and $\mathcal{S}$ is the subcategory of the spaces endowed with weak topology. For any adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ in the category $\mathcal{C}_2\mathcal{V}$ one has the inclusions $\mathcal{M} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ [*B*3].

2. The reflector functor from the category $\mathcal{C}_2\mathcal{V}$ in the strong nuclear spaces category $s\mathcal{N}$ admits a left-adjoint. Thus, $s\mathcal{N}$ is a c-reflective subcategory.

3. The subcategory $\mathcal{S}ch$ of the *Schwarts* spaces is also c-reflective [*GG*] .

4. The subcategory $\mathcal{N}$ of the nuclear spaces is not c-reflective [*GG*].

5. The subcategories $m(s\mathcal{N})$, $m\,(Sch)$ were described in the paper [*GG*].

Other examples of c-reflective subcategories and of pairs of adjoint subcategories (for which the reflector and co reflector functor satisfies the relations a) and b) of the Condition 4), the Theorem 2.1) suggest us the following theorem.

Theorem 2.2. *Let* $(\mathcal{K}, \mathcal{R})$ *be an adjoint pair of subcategories in the category* $\mathcal{C}_2\mathcal{V}$*, let* $\mathcal{L}$ *be a reflective subcategory of category* $\mathcal{C}_2\mathcal{V}$ *and let* $\mathcal{R} \subset \mathcal{L}$*. Then* $(l(\mathcal{K}), \mathcal{R})$ *is an adjoint pair of subcategories of the category* $\mathcal{L}$*.*

Proof We have that $l(\mathcal{K})$ is a coreflective subcategory in the category $\mathcal{L}$. Indeed, let $X \in |\mathcal{L}|$, $k^X : kX \to X$ and let $l^{kX} : kX \to lkX$, where $\mathcal{K}$ is the coreplique and $\mathcal{L}$ is the replique of the respective objects. Since $X \in |\mathcal{L}|$ we have

$$k^X = l_1^X l^{kX}, \tag{1}$$

for one morphism $l_1^X : lkX \to X$. Let us prove that l_1^X is $l(\mathcal{K})$-coreplique of the object X.

$$\begin{tikzcd} kX \arrow[r, "l^{kX}"] \arrow[dr, "k^X"'] & lkX \arrow[d, dashed, "l_1^X"] \\ & X \end{tikzcd}$$

Figure 2.1.

Indeed, let us have a similar diagram built for the object $Y \in |\mathcal{L}|$, and let $f : lkY \to X$ be an arbitrary morphism. Then

$$fl^{kY} = k^X g, \tag{2}$$

for one morphism $g : kY \to kX$.

$$\begin{tikzcd} kY \arrow[r, "l^{kY}"] \arrow[d, dashed, "g"'] \arrow[rr, bend left, "k^Y"] & lkY \arrow[r, "l_1^Y"] \arrow[d, dashed, "h"'] \arrow[dr, "f"] & Y \\ kX \arrow[r, "l^{kX}"] \arrow[rr, bend right, "k^X"'] & lkX \arrow[r, "l_1^X"] & X \end{tikzcd}$$

Figure 2.2.

Since l^{kY} is a $\mathcal{L}$-replique of the object kY, then

$$l^{kX}g = hl^{kY}, \tag{3}$$

for some morphism $h : lkY \to lkX$. We have

$$l_1^X hl^{kY} \overset{(2.3)}{=} l_1^X l^{kX} g \overset{(2.1)}{=} k^X g \overset{(2.2)}{=} fl^{kY}$$

i.e.

$$l_1^X hl^{kY} = fl^{kY}$$

and, since l^{kY} is an epi, we deduce that

$$l_1^X h = f. \tag{4}$$

As it was mentioned in the above, $\mathcal{M} \subset \mathcal{K}$, such that $k^X \in \mathcal{M}_u$. Taking into account that l^{kX} is an epi it is easy to show that the square

$$l_1^X l^{kX} = k^X \tag{5}$$

is puschout for each object X of the category $\mathcal{C}_2\mathcal{V}$. Thus $l_1^X \in \mathcal{M}_u$, whence the unity of the morphism h with the Property 4.

Now let us verify the relations a) and b) from the Condition 4 of Theorem 2.1. For any object X of the category $\mathcal{L}$ we have the following diagram

$$\begin{tikzcd}
kX \arrow[r, "l^{kX}"] \arrow[d, "k^X"'] & lkX \arrow[dl, "l_1^X"'] \\
X \arrow[r, "r^X"] & rX
\end{tikzcd}$$

Figure 2.3.

Obvionsly $lrX \sim lkX$, since $krX \sim kX$. Thus the $l(\mathcal{K})$-corepliques of the objects X and rX, are isomorphs. Conversely, since $rkX \sim rX$ we deduce that $rX \sim rlkX$. ■

In the paper $[B3]$ certain constructions permit us to obtain from all adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ of the category $\mathcal{C}_2\mathcal{V}$ another pair of adjoint subcategories $(\mathcal{K}', \mathcal{R})$ in the category $\mathcal{C}_2\mathcal{A}b$ of the locally convex groups this category that contains the category $\mathcal{C}_2\mathcal{V}$.

Let A be a nonzero $\mathcal{M}_p$-injective object of the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{R} = \mathcal{M}_pP(A)$. The subcategory $\mathcal{R}$ consists of the subspaces of the powers of object A, i.e. from the subspaces of the objects of the form A^τ.

Theorem 2.3. *For any nonzero and $\mathcal{M}_p$-injective object A of the category $\mathcal{C}_2\mathcal{V}$ the subcategory $\mathcal{R} = \mathcal{M}_pP(A)$ is c-reflective.*

Proof We shall show that the reflector functor $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ satisfies the Condition 1 of the Theorem. 2.1. Indeed, since the subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}_p$-subobjects, it is $\mathcal{E}_u$-reflective.

Now let $f : X \to Y \in \mathcal{M}_p$, and let $r^X : X \to rX$ and $r^Y : Y \to rY$ be $\mathcal{R}$-repliques of the respective objects. Then the object rX is a $\mathcal{M}_p$-subobject of one object of form A^τ, i.e. there exists a morphism $g : rX \to A^\tau \in \mathcal{M}_p$.

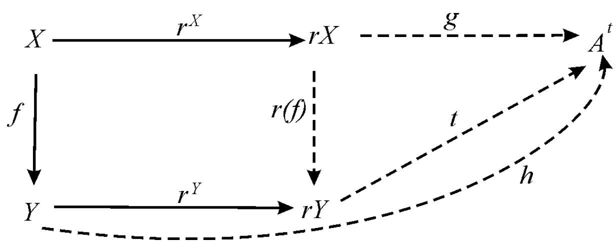

Figure 2.4.

We have

$$r^Y f = r\,(f)\,r^X \tag{6}$$

Since A^τ is $\mathcal{M}_p$-injective, then

$$gr^X = hf \tag{7}$$

for one morphism $h : Y \to A^\tau$.

Further, $A^\tau \in |\mathcal{R}|$, and r^Y is the $\mathcal{R}-$replique of the object Y. Therefore, for a morphism $t : rY \to A^\tau$, we have

$$h = tr^Y. \tag{8}$$

We also have

$$tr\,(f)\,r^X \overset{(2.6)}{=} tr^Y f \overset{(2.8)}{=} hf \overset{(2.7)}{=} gr^X,$$

i.e.

$$tr\,(f)\,r^X = gr^X.$$

In addition, since r^X is an epi, it follows that

$$tr(f) = g. \tag{9}$$

From the last equality and taking into account that $g \in \mathcal{M}_p$, we deduce that $r\,(f) \in \mathcal{M}_p$. ■

Theorem 2.4. $[B3]$. *Let $\omega \leq \alpha < \beta$. Then:*
$\mathcal{M}_pP\,(m\,(\alpha)) \subset \mathcal{M}_pP\,(m\,(\beta))$ *and* $\mathcal{M}_pP\,(m\,(\alpha)) \neq \mathcal{M}_pP\,(m\,(\beta))$

In particular, in the category $\mathcal{C}_2\mathcal{V}$ there exists one proper class of c-reflective subcategories.

Proof If a space Banach B is isomorphic with a subspace of the product X^{τ} by the X locally convex space, then B is isomorphic with a subspace of the space X^n of the n finite cardinal. ■

Corollary 2.1. *In the category $\mathcal{C}_2\mathcal{V}$ there exists a proper class of c-reflective subcategories, and, hence, a proper class of adjoint pairs of subcategories.*

Remark 2.2. *The spaces of the subcategories $m(\mathcal{M}_p P(m(\tau)))$ were described in the paper* $[GG]$.

3. THE CONSERVATION OF THE CLASSES OF INJECTIONS AND PROJECTIONS

Let $\mathcal{R}$ be a reflective subcategory in the category $\mathcal{C}_2\mathcal{V}$. We consider the class $\varepsilon\mathcal{R} = \{f \in \mathcal{E}pi\ \mathcal{C}_2\mathcal{V}\ |r(f) \in \mathcal{I}so\}$.

This class together with the class of $\mathcal{R}$-perfect morphisms $(\varepsilon\mathcal{R}\ )^{\perp}$ forms a right-bicategorical structure $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^{\perp})$. Thus $\varepsilon\mathcal{R}$ is right-complete.

Lemma 3.1. *Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$ with the respective functors $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$. Then*

1. $\varepsilon\mathcal{R} = \{f \in \mathcal{C}_2\mathcal{V} \mid r\,(f) \in \mathcal{I}so\}$.

2. $\varepsilon\mathcal{R} = \{f \in \mathcal{M}ono\ \mathcal{C}_2\mathcal{V} \mid k\,(f) \in \mathcal{I}so\}$.

3. $\varepsilon\mathcal{R} = \{f \in \mathcal{C}_2\mathcal{V} \mid k\,(f) \in \mathcal{I}so\}$.

4. $\left(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^{\perp}\right)$ *is a bicategory structure on the right.*

5. $\left((\varepsilon\mathcal{R})^{\top}, \varepsilon\mathcal{R}\right)$ *is a bicategory structure on the left.*

6. The class $\varepsilon\mathcal{R}$ is also complete on the right and on the left.

7. $\varepsilon\mathcal{R} \subset \mathcal{E}_u \cap \mathcal{M}_u$.

8. The objects of the subcategory $\mathcal{R}$ form a sufficient class of $\varepsilon\mathcal{R}$-injective objects.

9. The objects of the subcategory $\mathcal{K}$ form a sufficient class of $\varepsilon\mathcal{R}$-projective objects.

Proof Let $r\,(f) \in \mathcal{I}so$, then in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r^X"'] & Y \arrow[d, "r^Y"] \\ rX \arrow[r, "r(f)"] & rY \end{tikzcd}$$

Figure 3.1.

we have

$$r\,(f)\,r^X = r^Y f. \tag{10}$$

In this equality the maps $r^X, r\,(f)$ and r^Y are bijective. That is why f is bijective too. In particular, $f \in \mathcal{E}_u$. Further, since $r\,(f)\,r^X \in \mathcal{M}_u$, from (3.1) we deduce that $f \in \mathcal{M}_u$. Hence, we proved the statements 1 and 7.

The statement 2 follows from the definition of the adjoint pair of subcategories and the statement 3 is the dual to the statement 1. The assertions 4 and 5 were mentioned above and the statement 6 is their simple corollary. ■

Theorem 3.1. $[B3]$. *The correspondence $\mathcal{R} \longmapsto \varepsilon\mathcal{R}$ establishes an antiisomorphism of the lattice of c-reflective subcategories and the lattice of the bimorphisms classes of the category $\mathcal{C}_2\mathcal{V}$ which are left-complete and right-complete.*

Proof The left and right-complete class $\varepsilon\mathcal{R}$ is put in correspondence to each c-reflective subcategory $\mathcal{R}$

Conversely. A complete subcategory $\mathcal{R}$ of the A-injective objects is put in correspondence to any complete to the right and to the left class A of bimorphism. Then $m\,(\mathcal{R})$ is a full subcategory of the A-projective objects. ■

Let $\mathcal{R}$ be a reflective subcategory and let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$. In the above we examined some examples when $r(\mathcal{I}) \subset \mathcal{I}$. Let us show a few cases.

Lemma 3.2. *Let $\mathcal{R}$ be a $\mathcal{E}_u$-reflective subcategory. Then*

1. $r\,(\mathcal{M}_u) \subset \mathcal{M}_u$.

2. $r\,(\mathcal{M}ono) \subset \mathcal{M}ono$.

Proof 1. Let $f : X \to Y \in \mathcal{M}_u$. Then from the equality

$$r^Y f = r\,(f)\,r^X \tag{11}$$

it follows that $r\,(f)\,r^X \in \mathcal{M}_{u.}$ Since r^X is an epi we deduce that the square

$$1 \cdot \left(r\,(f)\,r^X\right) = r\,(f) \cdot r^X \tag{12}$$

is puschout. Therefore, $r\,(f) \in \mathcal{M}_u$.

$$\begin{tikzcd}
X \arrow[r, "r(f)r^X"] \arrow[d, "r^X"'] & rY \arrow[d, "1"] \\
rX \arrow[r, "r(f)"] & rY
\end{tikzcd}$$

Figure 3.2.

2. See Lemma 1.3 [*BT*] ■

Lemma 3.3. *1. Let $(\mathcal{P},\mathcal{I})$ be a bicategorical structure and let $\mathcal{R}$ be a $\mathcal{P}$-reflective subcategory in the category $\mathcal{C}$. Then $r(\mathcal{P}) \subset \mathcal{P}$.*

2. Let $(\mathcal{K},\mathcal{R})$ be a adjoint pair of subcategories in the category $\mathcal{C}$.

a) If $\mathcal{K}$ is $\mathcal{P}$-coreflective, then $\mathcal{R}$ is $\mathcal{P}$-reflective, too.

b) If $\mathcal{R}$ is $\mathcal{I}$-reflective, then $\mathcal{K}$ is $\mathcal{I}$-reflective, too.

Theorem 3.2. *Let $(\mathcal{K}\ ,\mathcal{R})$ be an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$ and let $(\mathcal{P},\mathcal{I})$ be a bicategorical structure. Then the following conditions are equivalent:*

a)$r(\mathcal{I}) \subset \mathcal{I}$;

b)$k(\mathcal{P}) \subset \mathcal{P}$.

Proof $a) \Rightarrow b)$. Let $p : X \to Y \in \mathcal{P}$, and let

$$k(p) = i_1 p_1 \tag{13}$$

be the $(\mathcal{P},\mathcal{I})$ $-$factorization of the morphism $k(p)$, where $p_1 : kX \to Z$.

We examine the $\mathcal{R}$-replicques of the objects X, Y and Z, by taking into account that $rX \sim rkX$ and $rY \sim rkY$.

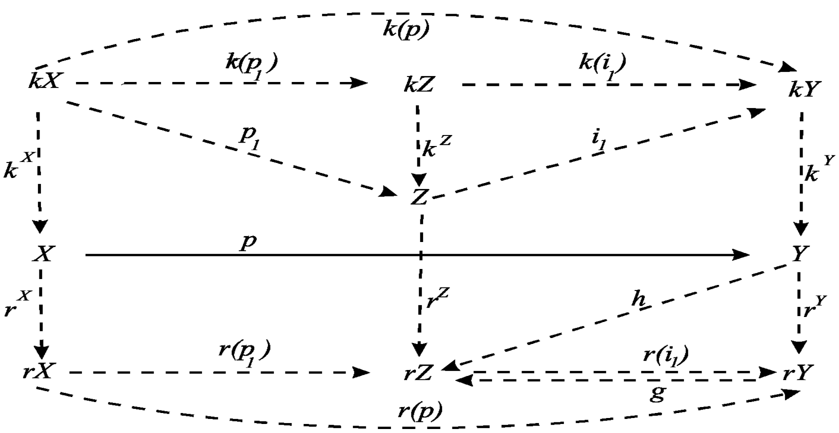

Figure 3.3.

By construction we have the following equalities:

$$pk^X = k^Y k(p); \tag{14}$$

$$p_1 = k^Z k(p_1); \tag{15}$$

$$i_1 k^Z = k(i_1); \tag{16}$$

$$r^Z p_1 = r\,(p_1)\, r^X k^X; \tag{17}$$

$$r\,(i_1)\, r^Z = r^Y k^Y i_1; \tag{18}$$

$$k\,(p) = i_1 p_1; \tag{19}$$

$$r\,(p)\, r^X = r^Y p. \tag{20}$$

From the equality (3.11) we have

$$rk\,(p) = r\,(i_1 p_1) = r\,(i_1)\, r\,(p_1)\,.$$

On the other hand we have

$$rk\,(p) = r\,(p)\,.$$

Thus

$$r\,(p) = r\,(i_1)\, r\,(p_1)\,. \tag{21}$$

We have

$$r\,(i_1)\, r\,(p_1)\, r^X k^X \overset{(3.12)}{=} r\,(p)\, r^X k^X \overset{(3.11)}{=} r^Y p k^X,$$

i.e.

$$r\,(i_1)\, r\,(p_1)\, r^X k^X = r^Y p k^X$$

and, since k^X is an epi, it follows that

$$r\,(i_1)\, r\,(p_1)\, r^X = r^Y p. \tag{22}$$

By the hypothesis a) $r\,(i_1) \in \mathcal{I}$. Thus, in the equality (3.13) $p \in \mathcal{P}$ and $r\,(i_1) \in I$, i.e. $p \perp r\,(i_1)$. Then there exists a morphism $h : Y \to rZ$ such that

$$r(i_1)h = r^Y \tag{23}$$

$$r(p_1)r^X = hp \tag{24}$$

Then

$$h = gr^Y \tag{25}$$

for some morphism $g : rY \to rZ$. We have

$$r(i_1)gr^Y \overset{(3.16)}{=} r(i_1)h \overset{(3.14)}{=} r^Y,$$

i.e.

$$r(i_1)gr^Y = r^Y$$

and, since r^Y is an epi, we deduce that

$$r(i_1)g = 1. \tag{26}$$

From the equality (3.17) and the fact that $r(i_1)$ is a mono it follows that $r(i_1)$ is an izo. We have

$$k(i_1) = kr(i_1),$$

therefore, $k(i_1)$ is an izo, too. From the equality (3.7) and from the fact that $k(i_1)$ is an izo, it follows that the morphisms i_1 and k^Z are izo, too. Thus, $k(p) \sim k(p_1) \sim p_1$. Hence, $k(p) \in \mathcal{P}$.

$b) \Rightarrow a)$. The proof is dual. ■

Examples. 1. For every adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$ by definition it follows that

$$k(\mathcal{E}_f) \subset \mathcal{E}_f \text{ and } r(\mathcal{M}_f) \subset (\mathcal{M}_f).$$

By the previous theorem we deduce that

$$r(\mathcal{M}ono) \subset \mathcal{M}ono \text{ and } k(\mathcal{E}pi) \subset \mathcal{E}pi,$$

i.e. r is a monofunctor and k is an epifunctor.

For the assertion that $r(\mathcal{M}ono) \subset \mathcal{M}ono$, see Lemma 3.2. The assertion $k(\mathcal{E}pi) \subset \mathcal{E}pi$ follows from the following topology properties (see [*RR*]).

2. Since $r(\mathcal{M}_p) \subset \mathcal{M}_p$, for every adjoint pair of subcategories $(\mathcal{K}, \mathcal{R})$, then we have that $k(\mathcal{E}_u) \subset \mathcal{E}_u$.

3. By Lemma 3.2 we have $r(\mathcal{M}_u) \subset \mathcal{M}_u$. Thus, $k(\mathcal{E}_p) \subset \mathcal{E}_p$. This assertion is less trivial. It may also be proved directly, by the description of the class $\mathcal{E}_p$ [*BG*].

4. THE BICATEGORICAL STRUCTURES $\left((\mathcal{J} \circ \varepsilon\mathcal{R})^{\rceil}, \mathcal{J} \circ \varepsilon\mathcal{R}\right)$ AND $\left(\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^{\llcorner}\right)$

Let $(\mathcal{K}, \mathcal{R})$ be an adjoint pair of subcategories of the category $\mathcal{C}_2\mathcal{V}$ and let $(\mathcal{P}, \mathcal{J})$ be a bicategorical structure. Denote

$$\mathcal{J} \circ \varepsilon\mathcal{R} = \{uv \mid u \in \mathcal{J},\ v \in \varepsilon\mathcal{R} \text{ and there exists the composition } uv\}$$
$$\varepsilon\mathcal{R} \circ \mathcal{P} = \{uv \mid u \in \varepsilon\mathcal{R},\ v \in \mathcal{P} \text{ and there exists the composition } uv\}$$

Theorem 4.1. *Assume that the adjoint pair of subcategories* $(\mathcal{K}, \mathcal{R})$ *possesses the property* $r(\mathcal{I}) \subset \mathcal{I}$*. Then:*

1. $\left((\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}, \mathcal{I} \circ \varepsilon\mathcal{R}\right)$ *is a bicategorical structure in the category* $C_2\mathcal{V}$*.*

2. $r(\mathcal{I} \circ \varepsilon\mathcal{R}) \subset \mathcal{I}$*.*

3. If the class of injections $\mathcal{I}$ *is right-complete, then so is the class of injections* $\mathcal{I} \circ \varepsilon\mathcal{R}$*.*

4. The pair $\left(\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^{\lfloor}\right)$ *is a bicategorical structure in the category* $\mathcal{C}_2\mathcal{V}$*.*

5. $k(\varepsilon\mathcal{R} \circ \mathcal{P}) \subset \mathcal{P}$*.*

6. If the class of projections $\mathcal{P}$ *is left-complete , then the class of the projections* $\varepsilon\mathcal{R} \circ \mathcal{P}$ *has the same property.*

Proof 1.Since each class $\varepsilon\mathcal{R}$ and $\mathcal{I}$ are left-complete, then so the class $\mathcal{I} \circ \varepsilon\mathcal{R}$ is left-stable with respect to the products. It is to prove that it is closed with respect to the composition. Each class $\varepsilon\mathcal{R}$ and $\mathcal{I}$ contain $\mathcal{I} \circ \varepsilon\mathcal{R}$ and are closed according to the composition. Thus, we have to prove the following assertion:

Let $u \in \varepsilon\mathcal{R}$ and $v \in \mathcal{I}$ and assume that there exists the composition uv. Then $uv \in \mathcal{I} \circ \varepsilon\mathcal{R}$. Examine the $\mathcal{R}$-repliques of the respective objects.

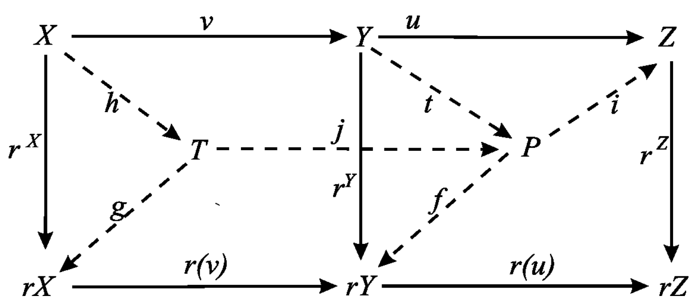

Figure 4.1.

We have

$$r(v)r^X = r^Y v, \tag{27}$$

$$r(u)r^Y = r^Z u. \tag{28}$$

We construct the pull-back squares on the morphisms $r(u)$ and r^Z

$$r(u)f = r^Z i. \tag{29}$$

on the morphisms $r(v)$ and f

$$r(v)\, g = fj \tag{30}$$

Since $u \in \varepsilon\mathcal{R}$ we deduce that $r(u) \in \mathcal{I}so$ and also that $i \in \mathcal{I}so$; $v \in \mathcal{I}$ therefore $r(v) \in \mathcal{I}$ and $j \in \mathcal{I}$. The squares (4.3) and (4.4) are pull-back, $r^Z \in \varepsilon\mathcal{R}$ and this class is left-stable, hence, $f, g \in \varepsilon\mathcal{R}$. From the equality (4.2) and (4.3) it follows that

$$r^Y = ft; \tag{31}$$

$$u = it, \tag{32}$$

for some morphism t. Further, we have

$$r(v)r^X \overset{(4.1)}{=} r^Y v \overset{(4.5)}{=} ftv,$$

i.e.

$$r(v)r^X = f(tv). \tag{33}$$

From the equalities (4.7) and (4.4) we deduce that

$$r^X = gh, \tag{34}$$

$$tv = jh, \tag{35}$$

for some morphism h. From the equality (4.8), since r^X and g belong to the class $\varepsilon\mathcal{R}$, i.e. they are bijective maps, we deduce that this holds for the map h. From the fact that h is an epi it follows that g is the $\mathcal{R}$-replique of the object T. Thus we proved that $r(h) \in \mathcal{I}so$, i.e. $h \in \varepsilon\mathcal{R}$. We have

$$uv \overset{(4.6)}{=} itv \overset{(4.9)}{=} ijh,$$

i.e.

$$uv = (ij)\,h, \tag{36}$$

with $ij \in \mathcal{I}$ and $h \in \varepsilon\mathcal{R}$.

Thus, we proved that the class $\mathcal{I} \circ \varepsilon\mathcal{R}$ is closed with respect to the composition.

2. We take into account that $r(\varepsilon\mathcal{R}) \subset \mathcal{I}so$.

3. The right-completeness of the classes $\mathcal{I}$ and $\varepsilon\mathcal{R}$ implies the fact that the same holds for the class $\mathcal{I} \circ \varepsilon\mathcal{R}$.

4. The proof dual to that of the assertion 1 follows. Indeed, the Theorem 3.2 asserts that

$$r(\mathcal{I}) \subset \mathcal{I} \Leftrightarrow k(\mathcal{P}) \subset \mathcal{P}$$

5.The assertion is true because $k(\varepsilon\mathcal{R}) \subset \mathcal{I}so$ and $k(\mathcal{P}) \subset \mathcal{P}$

6. Dual to the proof of assertion 2. ■

Let $\mathbb{R}_c$ be the class of all c-reflective subcategories in the category $\mathcal{C}_2\mathcal{V}$. By Theorem 2.5, $\mathbb{R}_c$ is a proper class.

Corollary 4.1. . *1. The classes of the bicategorical structures*

$$\left\{((\mathcal{M}_f \circ \varepsilon\mathcal{R})^{\rceil}, \mathcal{M}_f \circ \varepsilon\mathcal{R}) \mid \mathcal{R} \in \mathbb{R}_c\right\}$$
$$\left\{((\mathcal{M}_p \circ \varepsilon\mathcal{R})^{\rceil}, \mathcal{M}_p \circ \varepsilon\mathcal{R}) \mid \mathcal{R} \in \mathbb{R}_c\right\}$$

are proper classes of bicategorical structures in the category $\mathcal{C}_2\mathcal{V}$ with the right-complete class of injections.

2.The class of bicategorical structures

$$\left\{\left(\varepsilon\mathcal{R} \circ \mathcal{E}_p, (\varepsilon\mathcal{R} \circ \mathcal{E}_p)^{\lfloor}\right) \mid \mathcal{R} \in \mathbb{R}_c\right\}$$

is a proper class of bicategorical structures in the category $\mathcal{C}_2\mathcal{V}$ having a left-complete class of projections.

Corollary 4.2. $[BG]$. *Let $\mathcal{S}$ be the subcategory of the spaces endowed with the weak topology. Then*

1.$((\mathcal{M}_f \circ \varepsilon\mathcal{S})^{\rceil}, \mathcal{M}_f \circ \varepsilon\mathcal{S})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.

2.$((\mathcal{M}_p \circ \varepsilon\mathcal{S})^{\rceil}, \mathcal{M}_p \circ \varepsilon\mathcal{S})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$.

Corollary 4.3. *Let $(\mathcal{K}, \mathcal{R})$ te an adjoint pair of subcategories in the category $\mathcal{C}_2\mathcal{V}$, let $(\mathcal{P}, \mathcal{I})$ be a bicategorical structure and let $r(\mathcal{I}) \subset \mathcal{I}$. Then:*

1. $\mathcal{K}$ is a $(\varepsilon\mathcal{R} \circ \mathcal{P})$ -and $(\mathcal{I} \circ \varepsilon\mathcal{R})$ - coreflective subcategory.

2. $\mathcal{R}$ is a $(\varepsilon\mathcal{R} \circ \mathcal{P})$ -and $(\mathcal{I} \circ \varepsilon\mathcal{R})$ - reflective subcategory.

3. *If $\mathcal{M}_p \subset \mathcal{I}$, then the class $(\mathcal{I} \circ \varepsilon\mathcal{R})$* is *$\mathcal{E}_p-$extremal.*

5. CLASSES OF PROJECTIONS $(\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}$

As in the previous section, we suppose that $(\mathcal{K}, \mathcal{R})$ is an adjoint pair of subcategories, $(\mathcal{P}, \mathcal{I})$ is a bicategorical structure in the category $\mathcal{C}_2\mathcal{V}$ and these pairs possess the property $r(\mathcal{I}) \subset \mathcal{I}$.

By the previous theorem, the bicategorical structure $(\mathcal{P}, \mathcal{I})$ generates two new bicategorical structures

$$((\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}, \mathcal{I} \circ \varepsilon\mathcal{R}) \text{ and } (\varepsilon\mathcal{R} \circ \mathcal{P}, (\varepsilon\mathcal{R} \circ \mathcal{P})^{\lfloor}).$$

We mention the following inclusions

$$(\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil} \subset \mathcal{P} \subset \varepsilon\mathcal{R} \circ \mathcal{P},$$

$$(\varepsilon\mathcal{R}\circ\mathcal{P})^{\lfloor}\subset\mathcal{I}\subset\mathcal{I}\circ\varepsilon\mathcal{R}.$$

The class $(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}$. Since this class is contained in the class $\mathcal{P}$, we shall show which elements of the class $\mathcal{P}$ are contained in this class. Let $f:(X,u)\to(Y,t)\in\mathcal{P}$, and let

$$f=me \tag{37}$$

be the $(\mathcal{E}_f,\mathcal{M}ono)$-factorization of the morphism f. Then u'' is the factor topology of the topology u and m is a bimorphism, i.e. it is an injective map with dense image in the space (Y,t). Let $k^Y:(Y,k(t))\to(Y,t)$ be the $\mathcal{K}$-coreplique of the respective object. Obvionsly, we have

$$f\in(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}\Leftrightarrow m\in(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}$$

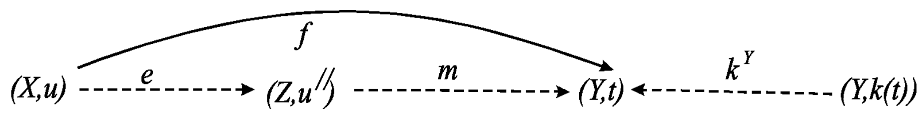

Figure 5.1.

Theorem 5.1. *The morphism* $f\in(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}$ *iff the topology* t *is the strongest locally convex topology for which the maps* m *and* k^Y *are continuous.*

Proof Let $m\in(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}$ and let be t_1 the strongest locally convex topology on the space Y for which the maps m and k^Y are continuous. Then $t_1\geq t$, and the corresponding maps m_1 (defined on m) and k_1^Y (defined on k^Y) and the identity map i are continuous and we have the following commutative diagram

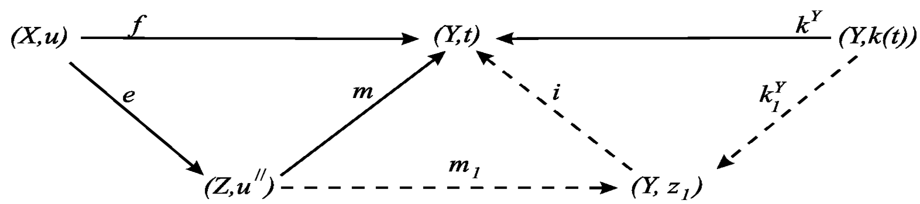

Figure 5.2.

From the equality

$$ik_1^Y=k^Y \tag{38}$$

it follows that $i\in\varepsilon\mathcal{R}$, and k_1^Y is a $\mathcal{K}$-coreplique of the resp. object. From the equality

$$im_1=m \tag{39}$$

it follows that $i\in(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}$. Thus, $i\in\varepsilon\mathcal{R}\cap(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}\subset(\mathcal{I}\circ\varepsilon\mathcal{R})\cap(\mathcal{I}\circ\varepsilon\mathcal{R})^{\rceil}=\mathcal{I}so$ and $t=t_1$.

Conversely. Let t be the strongest locally convex topology for which the maps m and k^Y remain continuous. We shall prove that $m \perp (\mathcal{I} \circ \varepsilon\mathcal{R})$. Since $m \in P$ it is sufficient prove that $m \perp (\varepsilon\mathcal{R})$. Indeed, let

$$lg = hm \tag{40}$$

hold with $l \in \varepsilon\mathcal{R}$, and let $k^V : kV \to V$ be the $\mathcal{K}$-coreplique of the resp. object. Then lk^V is the $\mathcal{K}$-coreplique of the object W and

$$hk^Y = lk^V l_1 \tag{41}$$

holds for some morphism l_1.

$$\begin{tikzcd}
 & & (Y,k(t)) \arrow[dr, "k^Y"] \arrow[ddll, dashed, bend right, "l_1"'] & \\
 & (Z,u'') \arrow[rr, "m"] \arrow[d, "g"] & & (Y,t) \arrow[d, "h"] \arrow[dll, dashed, "l^{-1}h"'] \\
kV \arrow[r, "k^V"] & V \arrow[rr, "l"] & & W
\end{tikzcd}$$

Figure 5.3.

Let us examine the map $l^{-1}h$. We have

$$l^{-1}hk^Y \overset{(5.5)}{=} l^{-1}lk^V l_1 = k^V l_1,$$

i.e.

$$(l^{-1}h)k^Y = k^V l_1. \tag{42}$$

We also have

$$l^{-1}hm \overset{(5.4)}{=} l^{-1}l\ g = g,$$

i.e.

$$(l^{-1}h)m = g \tag{43}$$

From the equalities (5.6) and (5.7) it follows that the composition of the map $l^{-1}h$ with the maps m and k^Y are continuous. Then by the definition of the topology t we deduce that the map $l^{-1}h$ is continuous, too. ■

Let us assume that $\mathcal{M}_p \subset \mathcal{I}$. Then $\mathcal{P} \subset \mathcal{E}_u$.

Thus, the morphisms of the class $(\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}$ are surjective maps. If we come back to the first diagram we see that the morphism f and, together with it, the morphism m belong to the class $\mathcal{E}_u$. Thus, the vector spaces Z and Y coincide. In this case, the description of the class $(\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}$ is simpler.

Theorem 5.2. *The continuous and surjective linear map* $f : (X, u) \to (Y, t) \in \mathcal{P}$ *belongs to the class* $(\mathcal{J} \circ \varepsilon\mathcal{R})^{\rceil}$ *iff*

$$t = min(k(t), u'')$$

where $k(t)$ *is the* $\mathcal{K}$*-coreplique that corresponds to the topology* t*, and* u'' *is the factor topology on the vector space* Y *of the topology* u*.*

Remark 5.1. *In the paper* $[BG]$ *we have the same description of the classes* $(\mathcal{M}_f \circ \varepsilon\mathcal{S})^{\rceil}$ *and* $(\mathcal{M}_p \circ \varepsilon\mathcal{S})^{\rceil}$*.*

Theorem 5.3. *We have that* $p : X \to Y \in (\mathcal{J} \circ \varepsilon\mathcal{R})^{\rceil}$ *iff* $p \cdot k^X = k^Y \cdot k\,(p)$ *is a puschout square.*

Proof Let $p : (X, u) \to (Y, t) \in (\mathcal{J} \circ \varepsilon\mathcal{R})^{\rceil}$.

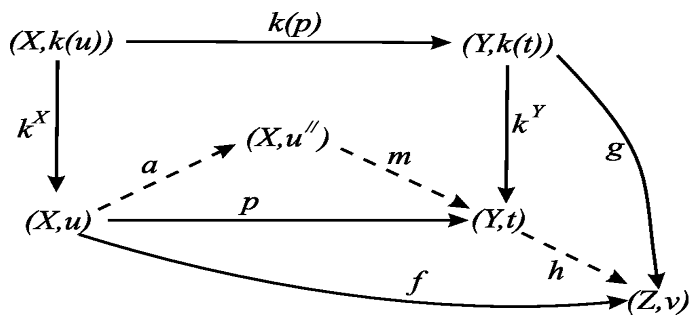

Figure 5.4.

Examine the commutative square

$$pk^X = k^Y k\,(p)\,. \tag{44}$$

Let us prove that it is a puschout. Let

$$f \cdot k^X = g \cdot k\,(p)\,. \tag{45}$$

Define the linear map

$$h = g \cdot \left(k^Y\right)^{-1}. \tag{46}$$

Let us prove that it is continuous. Let

$$f = me \tag{47}$$

be the $(\mathcal{E}_f, \mathcal{M}ono)$-factorization of the corresponding morphism. By Theorem 5.1 it is sufficient to prove that the maps hm and hk^Y are continuous. Since e is a factorial map it follows that the map hm will be continuous iff the map hme, i.e. hp, will be continuous. We mention that by the equality (5.8), due to the fact that k^X is a bijective map, we have the following equality of linear maps

$$p = k^Y k(p) \cdot (k^X)^{-1}. \tag{48}$$

Thus

$$hp \overset{(5.10)}{=} g(k^Y)^{-1}\, p \overset{(5.12)}{=} g(k^Y)^{-1} k^Y k(p)(k^X)^{-1} = gk(p)(k^X)^{-1} \overset{(5.9)}{=} fk^X(k^X)^{-1} = f,$$

i.e

$$hp = f. \tag{49}$$

Further we have

$$hk^Y \overset{(5.10)}{=} g(k^Y)^{-1} k^Y = g$$

or

$$hk^Y = g. \tag{50}$$

The equalities (5.13) and (5.14) show that the map h is continuous, and the square (5.8) is puschout.

Conversely. Let $p : X \to Y \in \mathcal{P}$, and assume that the square (5.8) is puschout. Let us prove that $p \in (\mathcal{I} \circ \varepsilon\mathcal{R})^{\rceil}$. Since $p \in \mathcal{P}$, it follows that $p \perp \mathcal{I}$. Let us prove that $p \perp (\varepsilon\mathcal{R})$. Indeed, let $b \in \varepsilon\mathcal{R}$, and assume that

$$bf = gp. \tag{51}$$

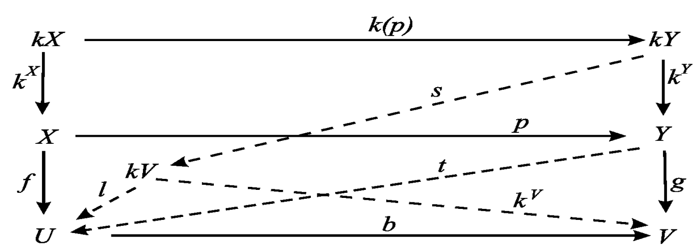

Figure 5.5.

Since $b \in \varepsilon\mathcal{R}$ it follows that

$$k^V = bl \tag{52}$$

for some morphism l. Then

$$gk^Y = k^V s \tag{53}$$

for some morphism s. We have

$$blsk(p) \overset{(5.16)}{=} k^V sk(p) \overset{(5.17)}{=} gk^Y k(p) \overset{(5.8)}{=} gpk^X \overset{(5.15)}{=} bfk^X$$

i.e.

$$blsk(p) = bfk^X \tag{54}$$

and, since b is a mono, it follows that

$$lsk(p) = fk^X. \tag{55}$$

By the hypothesis that (5.8) is a puschout square we deduce that

$$f = tp, \tag{56}$$

$$ls = tk^Y. \tag{57}$$

Since p is an epi then from the equalities (5.15) and (5.20) it follows that

$$bt = g. \tag{58}$$

The equalities (5.20) and (5.22) show that $p \perp (\varepsilon\mathcal{R})$. ■

6. CLASSES OF PROJECTIONS $\varepsilon\mathcal{R} \circ \mathcal{P}$

The description of the morphisms of this class follows from its definition. Let $f : X \to Y \in \mathcal{C}_2\mathcal{V}$, and let

$$f = ip \tag{59}$$

be the $(\mathcal{P}, \mathcal{I})$-factorization. Since $\mathcal{P} \subset \varepsilon\mathcal{R} \circ \mathcal{P}$ it follows that $f \in \varepsilon\mathcal{R} \circ \mathcal{P}$ iff $p \in \varepsilon\mathcal{R} \circ \mathcal{P}$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & Z \arrow[ur, "i"'] & \end{tikzcd}$$

Figure 6.1.

Theorem 6.1. *The morphism* $f \in (\varepsilon\mathcal{R} \circ \mathcal{P})$ *iff* $i \in \varepsilon\mathcal{R}$.

Proof If $i \in \varepsilon\mathcal{R}$ then it is evident that $f \in \varepsilon\mathcal{R} \circ \mathcal{P}$.

Conversely. Let $f \in \varepsilon\mathcal{R} \circ \mathcal{P}$. Let us prove that $i \in \varepsilon\mathcal{R}$. By hypothesis, we have

$$f = e_1 p_1, \tag{60}$$

where $e_1 \in \varepsilon\mathcal{R}$ and $p_1 \in \mathcal{P}$. We have the commutative square

$$ip = e_1 p_1, \tag{61}$$

with $i \in \mathcal{I}$ and $p_1 \in \mathcal{P}$. Thus,

$$p = hp_1, \tag{62}$$

$$ih = e_1, \tag{63}$$

for some morphism h. Since $e_1 \in \varepsilon\mathcal{R} \subset \mathcal{E}_u \cap \mathcal{M}_u$ and m is a mono we deduce that the square

$$i \cdot h = e_1 \cdot 1 \tag{64}$$

is a pull-back square. Thus, $h \in \mathcal{E}_u \cap \mathcal{M}_u$. Since $e_1 \in \varepsilon\mathcal{R}$ it follows that the morphism $r^Y e_1$ is the $\mathcal{R}$-replique of the object P and from the equality (6.5) and from the fact that h is an epi, we deduce that the morphism $r^Y i$ is the $\mathcal{R}$-replique of the object Z. Thus, $r(i) \in \mathcal{I}so$ and $i \in \varepsilon\mathcal{R}$.

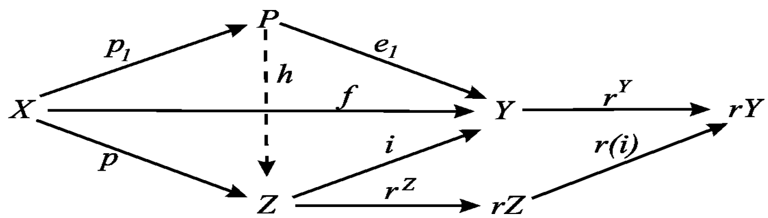

Figure 6.2.

■

7. CLASSES OF INJECTIONS $(\varepsilon\mathcal{R} \circ \mathcal{P})^{\llcorner}$

Let $f : (X, u) \to (Y, t) \in \mathcal{I}$, and let $f = me$ be the $(\mathcal{E}pi, \mathcal{M}_f)$-factorization of the morphisms. Since $m \in \mathcal{M}_f \subset (\varepsilon\mathcal{R} \circ \mathcal{P})^{\llcorner}$ we deduce that $f \in (\varepsilon\mathcal{R} \circ \mathcal{P})^{\llcorner}$ iff the morphism e belongs to this class. Let $r^X : (X, u) \to (X, r(u))$ be the $\mathcal{R}$ - replique of (X, u) .

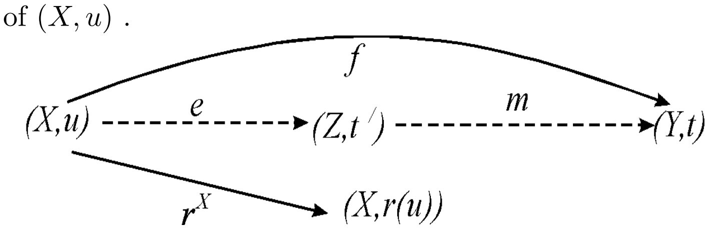

Figure 7.1.

The topology t' is the topology induced from the space (Y, t) on the linear space $Z = cl\,(f(X))$.

Theorem 7.1. *The morphism f belongs to the class $(\varepsilon\mathcal{R}\circ\mathcal{P})^{\llcorner}$ iff u is the weakest locally convex topology for which the maps e and r^X remain continuous.*

Proof Dual to the proof of Theorem 5.1. ∎

Theorem 7.2. *Let $i : X \to Y \in \mathcal{I}$. Then $i \in (\varepsilon\mathcal{R}\circ\mathcal{P})^{\llcorner}$ iff the square*

$$r\,(i)\,r^X = r^Y i$$

is pull-back.

Proof Dual to the proof of Theorem 5.3. ∎

8. CLASSES OF INJECTIONS $\mathcal{I}\circ\varepsilon\mathcal{R}$

The class $\mathcal{I}\circ\varepsilon\mathcal{R}$. Let $f : X \to Y \in \mathcal{C}_2\mathcal{V}$, and let $f = ip$ be the $(\mathcal{P},\mathcal{I})$-factorization of the morphisms. Since $\mathcal{I} \subset \mathcal{I}\circ\varepsilon\mathcal{R}$ it follows that $f \in \mathcal{I}\circ\varepsilon\mathcal{R}$ iff $p \in \mathcal{I}\circ\varepsilon\mathcal{R}$.

Theorem 8.1. *The morphism $f \in \mathcal{I}\circ\varepsilon\mathcal{R}$ iff $p \in \varepsilon\mathcal{R}$.*

Proof Dual to the proof of Theorem 6.1. ∎

Corollary 8.1. [*BG*] . *Let $f : (X,u) \to (Y,t)$ be a mono and let t' be the topology induced by the topology t on the subspace X. The monomorphism f belongs to the class $\mathcal{M}_p\circ\varepsilon\mathcal{S}$ iff the spaces (X,u) and (X,t') have the same dual space.*

$$\begin{tikzcd}
(X,u) \arrow[rr, "f"] \arrow[dr, "p"'] & & (Y,t) \\
& (X,t') \arrow[ur, "i"'] &
\end{tikzcd}$$

Figure 8.1.

9. THE CASE OF ONE INJECTIVE OBJECT

Let A be a nonzero $\mathcal{M}_p$-injective object in the category $\mathcal{C}_2\mathcal{V}$, $\mathcal{R} = \mathcal{M}_pP(A)$ and let $\mathcal{K} = m\mathcal{R}$. Then by Theorem 2.3 $(\mathcal{K},\mathcal{R})$ is a pair of adjoint subcategories in the category $\mathcal{C}_2\mathcal{V}$. In this case $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ and $\mathcal{M}_f \circ \varepsilon\mathcal{R}$ admit another description, too.

Theorem 9.1. *1. $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ is the class of all monomorphisms with respect to which the object A remains injective.*

2. $\mathcal{M}_f \circ \varepsilon\mathcal{R}$ *is the class of all morphisms of the class* $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ *that have a closed image.*

Proof 1. Let $f : X \to Y \in \mathcal{M}_p \circ \varepsilon\mathcal{R}$. Then, by definition of the class $\mathcal{M}_p \circ \varepsilon\mathcal{R}$ we have

$$f = ip \tag{65}$$

with $i \in \mathcal{M}_p$ and $p \in \varepsilon\mathcal{R}$. Let us show that the object A is injective with respect to the morphism f. Indeed, let $g : X \to A$.

$$\begin{tikzcd}
X \arrow[rr, "p"] \arrow[rrrr, bend left, "f"] \arrow[dr, "r^X"] \arrow[ddr, "g"'] & & Z \arrow[rr, "i"] \arrow[dr, "r^Z"] & & Y \arrow[ddlll, dashed, bend left, "t"] \\
& rX \arrow[rr, "r(p)"] \arrow[d, dashed, "h"] & & rZ \arrow[dll, dashed, "h_1"] & \\
& A & & &
\end{tikzcd}$$

Figure 9.1.

Since $A \in |\mathcal{R}|$, we have

$$g = hr^X \tag{66}$$

for some morphism h. Further, $p \in \varepsilon\mathcal{R}$, therefore, $r(p)$ is an isomorphism. Thus,

$$g = h_1 r^Z p \tag{67}$$

for some morphism h_1. Definitely A is $\mathcal{M}_p$-injective and $i \in \mathcal{M}_p$. So,

$$h_1 r^Z = ti \tag{68}$$

for a morphism t. In this case

$$g = tf \tag{69}$$

and the morphism g is an extension of f.

Conversely. Let $f : X \to Y$ be a monomorphism such that all morphism from the object X in the object A extend the morphism f. Let us show that $f \in \mathcal{M}_p \circ \varepsilon\mathcal{R}$. Let

$$f = ip \tag{70}$$

be the $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of the morphism f. Then each morphisms from the object X in the object A extends in a unique way the morphism p. This permits us to consider the sets $Hom(X, A)$ and $Hom(rX, A)$ as biing equal. In order to obtain the $\mathcal{R}$-repliques of these objects, we must examine the morphisms $g^X : X \to A^{Hom(X,A)}$ and $g^Z : Z \to A^{Hom(Z,A)}$.

All mentioned in the above show that there exists an isomorphism $j = p^{Hom(X,A)}$ such that

$$jg^X = g^Z p \tag{71}$$

the $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of the morphisms g^X and g^Z gives us the $\mathcal{R}$ -repliques of the objects X and Z respectively. Let

$$g^X = i^X r^X, \tag{72}$$

$$g^Z = i^Z r^Z \tag{73}$$

be these extensions. Then

$$g^X = i^X r^X = \left(j^{-1} i^Z\right)\left(r^Z p\right)$$

are two $(\mathcal{E}_u, \mathcal{M}_p)$-factorizations of the morphism g^X. This implies that there exists an isomorphism h such that

$$hr^X = r^Z p, \tag{74}$$

$$j^{-1} i^Z h = i^X. \tag{75}$$

From the equality (9.10) we deduce that $h = r\,(p)$, i.e. $r\,(p)$ is an isomorphism and $p \in \varepsilon\mathcal{R}$.

$$\begin{tikzcd}
X \arrow[rr, "p"] \arrow[d, dashed, "r^X"] \arrow[dd, bend right, "g^X"'] & & Z \arrow[r, "i"] \arrow[d, dashed, "r^Z"] & Y \\
rX \arrow[rr, dashed, "h=r(p)"] \arrow[d, dashed, "i^X"] & & rZ \arrow[d, dashed, "i^Z"] & \\
A^{Hom(X,A)} \arrow[rr, dashed, "j=p^{Hom(X,A)}"'] & & A^{Hom(Z,A)} &
\end{tikzcd}$$

Figure 9.2.

2. Follows from 9.1. ■

Corollary 9.1. $[BG]$.*1.* $\mathcal{M}_u$ *is the class of all morphisms with respect to which the field* $\mathbb{K}$ *is injective:* $\mathcal{M}_u = \mathcal{M}_p \circ \varepsilon\mathcal{S}$.

2.$\mathcal{M}_f \circ \varepsilon\mathcal{S}$ *is the class of all universal monomorphisms with a closed image.*

In the conditions of Theorem 9.1, the following theorem holds.

Theorem 9.2. *In the category* $\mathcal{C}_2\mathcal{V}$ *the objects of form* A^τ *form a sufficient class of* $(\mathcal{M}_p \circ \varepsilon\mathcal{R})$*-injective objects.*

Proof These objects are $\mathcal{M}_p$-injective and $\varepsilon\mathcal{R}$-injective. Therefore, they also are $(\mathcal{M}_p \circ \varepsilon\mathcal{R})$-injective. Further, for each object X of the category $\mathcal{C}_2\mathcal{V}$ we have $r^X \in \varepsilon\mathcal{R}$ and $g^{rX} \in \mathcal{M}_p$. Thus, the morphism $g^{rX} r^X$ belongs to the class $\mathcal{M}_p \circ \varepsilon\mathcal{R}$. This fact means that in the category $\mathcal{C}_2\mathcal{V}$ there exist sufficiently many objects which are $(\mathcal{M}_p \circ \varepsilon\mathcal{R}\ )$- injective.

$$X \xrightarrow{r^X} rX \xrightarrow{g^{rX}} A^{Hom(rX,A)}$$

Figure 9.3.

■

Corollary 9.2. *In the category $\mathcal{C}_2\mathcal{V}$ there exists a proper class of the bicategorical structures with sufficiently many injective objects.*

References

[*B*1] Botnaru D., On a bicategorical structure of the locally convex spaces categories, Functs. Analiz, Ulianovsk, 13(1979), 79-86 (in Russian).

[*B*2] Botnaru D., Some categorical aspects of vectorial locally convex spaces, Scientific Annals. of MSU, The series "Physical-Mathematical sciences", Chisinau, 2000, 77-86.

[*B*3] Botnaru D., Right-bicategorical structures and pairs of adjoint subcategories., PhD Thesis, Moscow, 1979 (in Russian).

[*B*4] Botnaru D., On a bicategorical structure of the locally convex spaces categories, Functs. Analiz, Ulianovsk, **13** (1979), 79-86(in Russian).

[*BG*] Botnaru D., Gysin V. B., Stable monomorphisms in the category of separating locally convex spaces., Izv. A.N. R.S.S.M., 1 (1973), 3-7 (in Russian).

[*BT*] Botnaru D., Turcanu A., Les produits de gauche et de droite de deux souscatégories, Acta e Commentations, **3**, Ştiinţe Fizice şi Matematice, Chişinău, 2003, Analele Univ. de Stat din Tiraspol, 57-73.

[*G*] Geyler V.A., About projective objects in the category of locally convex spaces., Functs. Analiz i evo Prilojenia, **6**, 2 (1972), 79-80 (in Russian).

[*GG*] Geyler V.A., Gysin V. B., Generalized duality for locally convex spaces., Functs. Analiz, Ulianovsk, **11** (1978), 41-50 (in Russian).

[*P*] Palomadov D. P., Homology methods in the theory of locally convex spaces., UMN, **26** 1 (157) (1971) 3-65 (in Russian).

[*RR*] Robertson A.P., Robertson W., Topological vector spaces, Campridge, 1964.

NUMBER OF $GL(2,\mathbb{R})$-ORBITS FOR EACH DIMENSION OF DIFFERENTIAL SYSTEM WITH QUADRATIC HOMOGENEITIES

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We consider the system of ordinary differential equations

$$\begin{cases} \dot{x} = gx^2 + 2hxy + ky^2, \\ \dot{y} = lx^2 + 2mxy + ny^2, \end{cases} \tag{1}$$

and the group $GL(2,\mathbb{R})$ of center-afine transformations

$$\begin{cases} x' = \alpha x + \beta y \\ y' = \gamma x + \delta y \end{cases} \qquad \left(\Delta = det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0\right), \tag{2}$$

where x', y' are new variables. The variables and all coefficients are real.

Denote by $E(a)$ the Euclidean space of the coefficients from the right-hand sides of system (1) , where $a = (g, h, k, l, m, n) \in E(a)$, and the point from $E(a)$, which coresponds to the system obtained from the system (1) with coefficients a under a transformation $q \in GL(2,\mathbb{R})$, by $a(q)$.

Definition 1. *The polynomial $K(x, y, a)$ of the system (1) coefficients and the phase-plane coordonates is called a center-affine comitant of this system, if the identity*

$$K(x', y', b) = \Delta^{-g} K(x, y, a),$$

where $g \in \mathbb{Z}$, and b is a vector of new coefficients, is fullfiled for any $q \in GL(2,\mathbb{R})$, $a \in E(a)$, and x, y.

If K does not depend on variables x, y, then it is usually called the center-affine invariant of system (1).

Hereinafter we need the following center-affine invariants and comitants from [2] for system (1):

$$I_7 = a^{\alpha}_{pr} a^{\beta}_{\alpha q} a^{\gamma}_{\beta s} a^{\delta}_{\gamma\delta} \varepsilon^{pq} \varepsilon^{rs}, \quad I_8 = a^{\alpha}_{pr} a^{\beta}_{\alpha q} a^{\gamma}_{\delta s} a^{\delta}_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs},$$

$$I_9 = a^{\alpha}_{pr} a^{\beta}_{\beta q} a^{\gamma}_{\gamma s} a^{\delta}_{\alpha\delta} \varepsilon^{pq} \varepsilon^{rs}, \quad K_1 = a^{\alpha}_{\alpha\beta} x^{\beta},$$

$$K_5 = a^p_{\alpha\beta} x^\alpha x^\beta x^q \varepsilon^{pq}, \quad K_6 = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} x^\gamma x^\delta,$$

$$K_7 = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} x^\gamma x^\delta, \quad K_9 = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\alpha\beta} x^\delta, \tag{3}$$

where $a^1_{11} = g$, $a^1_{12} = a^1_{21} = h$, $a^1_{22} = k$, $a^2_{11} = l$, $a^2_{12} = a^2_{21} = m$, $a^2_{22} = n$.

Definition 2. *The set $O(a) = \{a(q) | q \in GL(2,\mathbb{R})\}$ is called the $GL(2,\mathbb{R})$-orbit of the point a for system (1).*

It is considered that $\dim_{\mathbb{R}} O(a) = \mathrm{rank}(M_1)$, where M_1 is the matrix, constructed on coordinate vectors of the Lie algebra operators for system (1).

The set $S \subseteq E(a)$ is $GL(2,\mathbb{R})$-invariant, if for any point $a \in M$ the orbit $O(a) \subseteq M$.

In [1, pag.208] the following theorem is proved.

Theorem 1. *$GL(2,\mathbb{R})$-orbit of a system (9) has dimension:*

$$4, for K_5(K_9 + \beta) \not\equiv 0; \tag{4}$$

$$3, for K_5(K_1 + K_7) \not\equiv 0,\ K_9 + \beta \equiv 0; \tag{5}$$

$$2, for K_5(K_1 + K_7) \equiv 0,\ K_1 + K_7 \not\equiv 0; \tag{6}$$

$$0, for K_1 \equiv K_5 \equiv 0. \tag{7}$$

where K_1, K_5, K_7, K_9 are form (3), and β is the discriminant of K_5 and is writen as

$$\beta = 27 I_8 - I_9 + 18 I_7. \tag{8}$$

Remark 1. *The sets S_1, S_2, S_3, S_4, defined by conditions (4), (5), (6), (7) form $GL(2,\mathbb{R})$ decomposition of the space $E(a)$ of system (1) coefficients, i.e*

$$\bigcup_{i=1}^{4} S_i = E(a), \quad S_i \bigcap_{i \neq j} S_j = \oslash,$$

and each S_i is $GL(2,\mathbb{R})$-invariant.

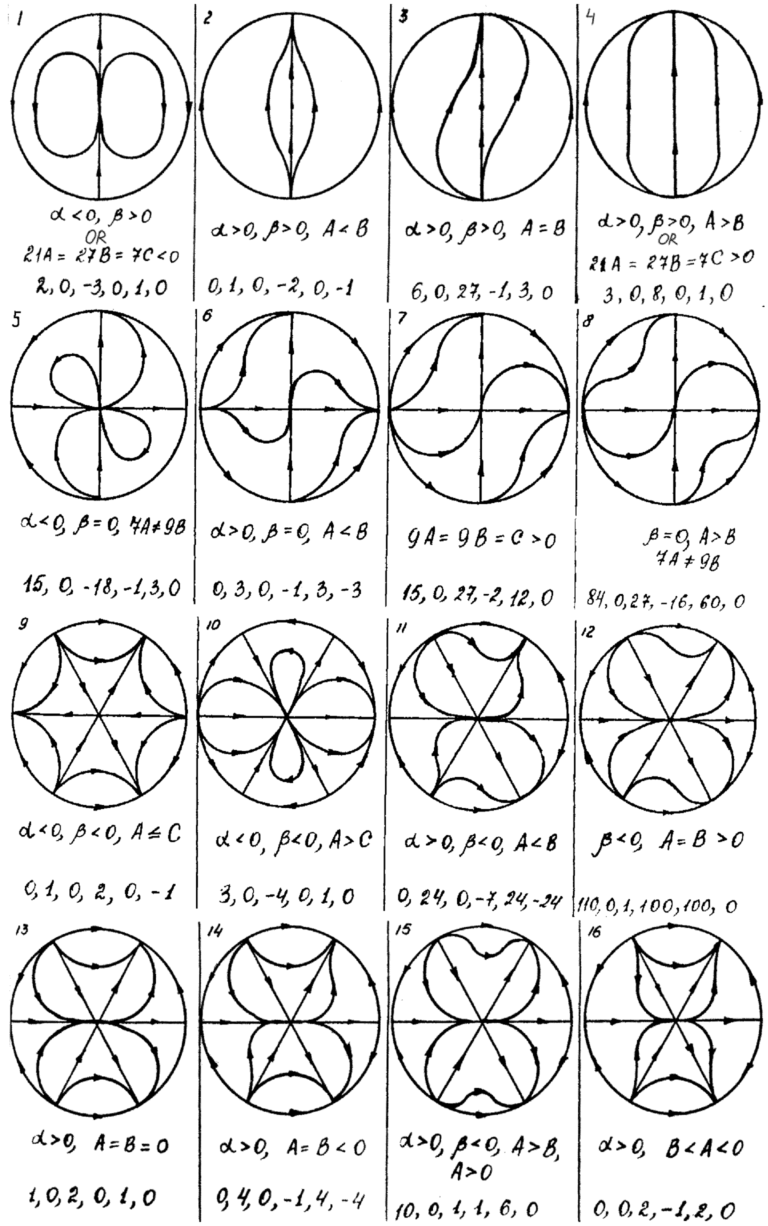
In [2, pag. 75, 78] 32 geometrical pictures for the system (1) are drawn up in the Poincaré circle. For every picture the necessary and sufficient conditions for their realization, and the coeficients g, h, k, l, m, n of one exemple of corresponding classe are written.

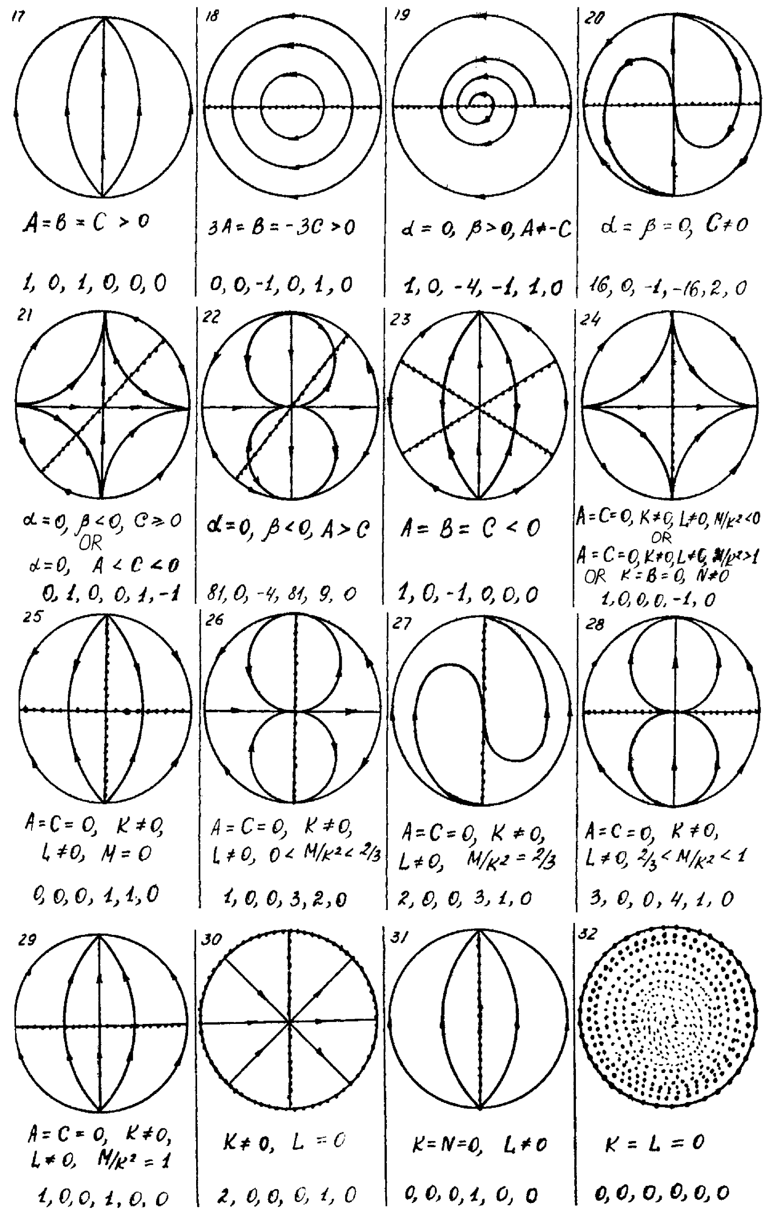
The following notation was used

$$A = I_7, \quad B = I_8, \quad C = I_9, \quad K = K_1, \quad L = K_5,$$

$$M = K_6, \quad N = K_7, \quad \alpha = B + C - 2A. \tag{9}$$

Hereinafter the 32 geometrical pictures will be presented.


1
α < 0, β > 0
OR
21A = 27B = 7C < 0
2, 0, -3, 0, 1, 0
2
α > 0, β > 0, A < B
0, 1, 0, -2, 0, -1
3
α > 0, β > 0, A = B
6, 0, 27, -1, 3, 0
4
α > 0, β > 0, A > B
OR
21A = 27B = 7C > 0
3, 0, 8, 0, 1, 0
5
α < 0, β = 0, 7A ≠ 9B
15, 0, -18, -1, 3, 0
6
α > 0, β = 0, A < B
0, 3, 0, -1, 3, -3
7
9A = 9B = C > 0
15, 0, 27, -2, 12, 0
8
β = 0, A > B
7A ≠ 9B
84, 0, 27, -16, 60, 0
9
α < 0, β < 0, A ≤ C
0, 1, 0, 2, 0, -1
10
α < 0, β < 0, A > C
3, 0, -4, 0, 1, 0
11
α > 0, β < 0, A < B
0, 24, 0, -7, 24, -24
12
β < 0, A = B > 0
110, 0, 1, 100, 100, 0
13
α > 0, A = B = 0
1, 0, 2, 0, 1, 0
14
α > 0, A = B < 0
0, 4, 0, -1, 4, -4
15
α > 0, β < 0, A > B,
A > 0
10, 0, 1, 1, 6, 0
16
α > 0, B < A < 0
0, 0, 2, -1, 2, 0



(10)

Remark 2.[2] *a) The pictures 2, 3, 4, 17 are topologically equivalent;*
b) The pictures 6-8 are topologically equivalent;
c) The pictures 11-16 are topologically equivalent.

From the proof of the Theorem 1 we get the following two lemas.

Lemma 1. *If $K_5 \equiv 0$, then $K_9 + \beta \equiv 0$.*

Lemma 2. *If $\beta \not\equiv 0$, then $K_5 \not\equiv 0$ and $K_5(K_9 + \beta) \not\equiv 0$.*

It is possible to show, that the following takes place:

Lemma 3. *If $K_5 \equiv 0$, then $I_7 = I_8 = I_9 = \alpha = \beta = 0$.*

Lemma 4. *If $K_9 + \beta \equiv 0$, then $I_7 = I_8 = I_9 = 0$.*

Lemma 5. *If $I_7 \neq 0$ or $I_8 \neq 0$ or $I_9 \neq 0$, then $K_9 + \beta \not\equiv 0$.*

Lemma 6. *If $I_7 = I_9 = 0$, then $I_8 = 0$.*

Taking into account Lemma 2, Lemma 5 and pictures (10), we can prove the following theorem.

Theorem 2. *The dimension of $GL(2,\mathbb{R})$-orbits of systems, to which the pictures 1-23 correspond, is equal to 4.*

Let us analyse the following:

Case 24. Assume that

$$A = C = 0,\ KL \neq 0,\ \frac{M}{K^2} \in \mathbb{R} \setminus [0,1]\, or\, K = B = 0,\ N \neq 0. \tag{11}$$

In this case the system (1) have the form

$$\begin{cases} \dot{x} = gx^2, \\ \dot{y} = lx^2 + 2xy, \end{cases} \tag{12}$$

for which

$$K = K_1 = (g+m)x,\ M = K_6 = (g^2 + gm)x^2,$$
$$N = K_7 = (g^2 + m^2)x^2,\ K_9 = gm^2 y. \tag{13}$$

Assume that $K_9 \equiv 0$.

If $m = 0$, then from (13) it follows $K = gx$, $M = g^2x^2$, and $\dfrac{M}{K^2} = 1$ (which contradicts (11)). Therefore, $m \neq 0$.

If $g = 0$, then $M = 0$ (which contradicts (11)).

We obtain that $K_9 = gm^2y \not\equiv 0$ and $K_9 + \beta \not\equiv 0$. Taking into account Lemma 1 and Theorem 1 we obtain that in this case $\dim_{\mathbb{R}} O(a) = 4$.

Remark 3. *Similarly, (taking into account the Lemmas 1-6, it is possible to show, that:*

in the cases 26-28 $\dim_{\mathbb{R}} O(a) = 4$*;*

in the cases 25, 29 $\dim_{\mathbb{R}} O(a) = 3$*;*

in the cases 30, 31 $\dim_{\mathbb{R}} O(a) = 2$*.*

The case 32 is evident: $\dim_{\mathbb{R}} O(a) = 0$*.*

Theorem 3. *Dimension of $GL(2,\mathbb{R})$-orbits, to which the pictures:*

1-24, 26-28 correspond, is equal to 4;

25, 29 correspond, is equal to 3;

30, 31 correspond, is equal to 2;

32 correspond, is equal to 0.

Taking into account the Remark 2, we can prove the following

Theorem 4. *For the system (1) there are*
17 $GL(2,\mathbb{R})$*-orbits with dimension 4;*
2 $GL(2,\mathbb{R})$*-orbits with dimension 3;*
2 $GL(2,\mathbb{R})$*-orbits with dimension 2;*
1 $GL(2,\mathbb{R})$*-orbit with dimension 0.*

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References

[1] Popa, M. N., *Application of algebraic methods to differential systems*, Series in Applied and Industrial Mathematics **15**, University of Pitesti, 2004 (in Romanian)

[2] Sibirschi, K. S., *Introduction to algebraic theory of invariants of differential equations*, Chishinau, Shtiintsa, 1982. (in Russian, published in English in 1988)

INFINITESIMAL GROUPS ASSOCIATED WITH QUADRATIC DYNAMICAL SYSTEMS

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Abstract Any homogeneous quadratic differential system is characterized by a binary algebra determined up to an isomorphism. The homogeneous system, in Yamaguti's sense, associated with this algebra allows us to find out a set of generators for the Lie algebra generated by the left multiplications of the algebra. A realization of this Lie algebra as a Lie algebra of vector fields gives the opportunity to associate an infinitesimal Lie group with any quadratic differential system. Some particular cases are analyzed.

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1. INTRODUCTION

The importance of the nonlinear dynamical systems was very soon remarked [13]. The most simple type of nonlinearity appeared to the quadratic dynamical systems. Many properties of such dynamical systems can be obtained by using an algebraic approach. L. Markus [6] was the first who steps on this way; he made evident the existence of a bijective correspondence between the classes of affinely equivalent quadratic homogeneous systems in plane and the classes of isomorphic 2-dimensional commutative algebras and gives the first classification of such quadratic homogeneous systems up to an central affinity. C. S. Sibirskij [11],[12] deals with the algebraic study of affine invariants and semiinvariants of quadratic differential equations. M. K. Kinyon and A. A. Sagle [5] deal with qualitative aspects of the general theory of quadratic dynamical systems. M. Popa [9] and his coworkers yielded, on the line traced by S. Lie, classification results by means of the orbits of the extended action of affinely transformation groups to the space of parameters of quadratic differential systems; this way follows the direction shown by C. S. Sibirskij [11]. I. Burdujan [2] extended some results of Markus to the case of quadratic differential equations on Banach spaces and tried to give a meaning for the deviations from associativity of the binary algebra associated with a quadratic differential system in connection with the set of solutions of the analyzed system.

In this paper we prove the existence of an infinitesimal group acting on the space where is defined the analyzed quadratic dynamical system. If the algebra associated with the quadratic dynamical system satisfies a certain identity of degree four or five, its homogeneuos system is just a Lie triple system or a general Lie triple system, respectively. Some invariants under the action of an important subgroup of the infinitesimal group are emphasized.

2. HOMOGENEOUS QUADRATIC DIFFERENTIAL SYSTEMS

Although the most part of results concerns the homogeneous quadratic differential systems (briefly, HQDSs) on finite dimensional spaces, the problem of correspondence between the classes of affinely equivalent HQDSs and classes of isomorphic binary algebras is more easily to solve in the case of HQDSs defined on Banach spaces. That is why in this section we prove some results on the Banach space, only.

Let E be a Banach space. An HQDS on E is every autonomous equation of the form $\dot{X} = F(X)$ where F is a continuous quadratic vector form on E. The polar form of F is the symmetric bilinear vector form $G : E \times E \to E$ defined by $G(x, y) = \frac{1}{2} \cdot [F(x+y) - F(x) - F(y)], \;\; \forall x, y \in E$. Let us denote by $E(\cdot)$ the commutative algebra defined by means of the multiplication $x \cdot y = G(x, y), \;\; \forall x, y \in E$. Two HQDSs $\dot{X} = F(X)$ and $\dot{Y} = F_1(Y)$ are called *affinely equivalent* each other if there exists a continously invertible linear transformation $T : E \to E$ such that X is a solution of the former system if and only if $Y = T(X)$ is a solution for the second system. It can be proved that two HQDSs are affinely equivalent if and only if their associated algebras are isomorphic [2], [6].

On the other hand, with any binary $\mathcal{A}$ algebra a homogeneous (algebraic) system (shortly, h.s.) in the sense of Yamaguti [14] is associated [1]; this h.s. gives the opportunity to find out a family of generators for the Lie algebra $\mathcal{L}$ generated by the left multiplications of the algebra $\mathcal{A}$ [3]. Actually, this association gives the construction of a covariant functor $\mathbf{HF} : \mathcal{ALG}_K \to \mathcal{HS}_K$ whose restriction to the subcategory $\mathcal{A}ss_K$ is just the usual functor associating with any associative algebra the Lie algebra defined on the algebra ground space by means of the commutator-operation. Unfortunately, the h.s. associated with a binary algebra does not characterize it up to an isomorphism.

In particular, the h.s. associated with a commutative algebra $\mathcal{A}(\cdot)$ is defined by the following multilinear operations, given recurrently by

$$[x_1, x_2, \cdots, x_{n+1}] = D_{(x_1, x_2, \cdots, x_n)}(x_{n+1}), \quad \forall (x_1, x_2, \cdots, x_{n+1}) \in \mathcal{A}^{n+1}, \forall n \geq 2$$

where $D_{(x_1,x_2,\cdots,x_n)}$ are the endomorphisms of E defined inductively by

$$D_{(x_1,x_2)} = [L_{x_1}, L_{x_2}],$$

$$D_{(x_1,x_2,\cdots,x_{n+1})} = \left[D_{(x_1,x_2,\cdots,x_n)}, L_{x_{n+1}}\right] - L_{[x_1,x_2,\cdots,x_{n+1}]},\ \forall n \geq 2.$$

Consequently, the h.s. associated with a commutative algebra $\mathcal{A}(\cdot)$ satisfies the following axioms:

(h.s.1) $[x, x, x_1, \cdots, x_k] = 0, \quad \forall k \in \mathbb{N}^\star,\ \forall x, x_1, \cdots, x_k \in \mathcal{A},$

(h.s.2) $[x, y, z] + [y, z, x] + [z, x, y] = 0, \quad \forall x, y, z \in \mathcal{A},$

(h.s.3) $[x, y, z, w] + [y, z, x, w] + [z, x, y, w] = 0, \quad \forall x, y, z, w \in \mathcal{A},$

(h.s.4) $[x_1, \cdots, x_k, y, z] - [x_1, \cdots, x_k, z, y] = 0, \quad \forall x_1, \cdots, x_k, y, z \in \mathcal{A},$

(h.s.5) $\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,y_2)}\right] = D_{([x_1,\cdots,x_k,y_1],y_2)} - D_{([x_1,\cdots,x_k,y_2],y_1)} +$
. $+ D_{(x_1,\cdots,x_k,y_1,y_2)} - D_{(x_1,\cdots,x_k,y_2,y_1)},$
. $\forall k = 2, 3, \cdots, \quad \forall x_1, \cdots, x_k, y_1, y_2 \in \mathcal{A}$

(h.s.6) $\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell,y_{\ell+1})}\right] = \widetilde{\mathfrak{D}}\left(\left[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell)}\right], y_{\ell+1}\right) -$
. $-\left[D_{(x_1,\cdots,x_k,y_{\ell+1})}, D_{(y_1,\cdots,y_\ell)}\right] - D_{(x_1,\cdots,x_k,[y_1,\cdots,y_\ell,y_{\ell+1}])} + D_{(y_1,\cdots,y_\ell,[x_1,\cdots,x_k,y_{\ell+1}])},$
. $\forall \ell, k = 2, 3, \cdots, \quad \forall x_1, \cdots, x_k, y_1, \cdots, y_{\ell+1} \in \mathcal{A},$

where $\widetilde{\mathfrak{D}}$ is the endomorphism defined by $\widetilde{\mathfrak{D}}\left(D_{(x_1,\cdots,x_n)}, y\right) = D_{(x_1,\cdots 2,x_n,y)}$. The axioms (h.s.5) and (h.s.6) assure us that the linear enveloping of the family of endomorphisms $\widetilde{\mathfrak{L}} = \left\{D_{(x_1,\cdots,x_n)} | \forall x_1, \cdots, x_n \in \mathcal{A}, n \geq 2\right\}$ is just the Lie algebra $\mathfrak{L}$ endowed with the usual Lie bracket from an associative algebra of endomorphisms.

3. INFINITESIMAL GROUP ASSOCIATED WITH A HQDS

In what follows we consider the finite dimensional case, only.

If $E = \mathbb{R}^n$, then any HQDS has the form $\dot{x}^i = a^i_{jk} x^j x^k \quad i, j, k = 1, 2, \cdots, n,$ $a^i_{jk} \in \mathbb{R}$ and the binary operation defined on $\mathbb{R}^n$ is $x \cdot y = \left(a^1_{jk} x^j y^k, a^2_{jk} x^j y^k, \cdots, a^n_{jk} x^j y^k\right),\ \ \forall x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. It is clear that the structure constants of the associated algebra are just the coefficients that define the system. In this case, only a finite number of elements in $\widetilde{\mathfrak{L}}$ is a system of generators for $\mathfrak{L}$.

In what follows, we shall prove that $\widetilde{\mathfrak{L}}$ can be realized as a system of local vector fields. Indeed, if $\mathbf{B} = \{e_1, \cdots, e_m\}$ is a basis in E, then the structure constants of the h.s. are defined in the usual way by $[e_{i_1}, e_{i_2}, \cdots, e_{i_k}] = t^j_{i_1 i_2 \cdots i_k} e_j, \quad n \geq 2$. They satisfy the following conditions

(h.s.1)' $t^s_{iij_1\cdots j_k} = 0, \quad k \geq 1$

(h.s.2)' $t^s_{ijk} + t^s_{jki} + t^s_{kij} = 0,$

(h.s.3)' $t^s_{ijkp} + t^s_{jkip} + t^s_{kijp} = 0,$

(h.s.4)' $t^s_{i_1\cdots i_k pq} - t^s_{i_1\cdots i_k qp} = 0, \quad \forall k \geq 2$

(h.s.5)' $t^s_{i_1\cdots i_k p}t^p_{qrm} - t^s_{qrp}t^p_{i_1\cdots i_k m} = t^p_{i_1\cdots i_k q}t^s_{prm} - t^p_{i_1\cdots i_k r}t^s_{pqm} +$
. $+ t^s_{i_1\cdots i_k qrm} - t^s_{i_1\cdots i_k rqm}, \qquad k \geq 2$
(h.s.6)' $t^s_{i_1\cdots i_k p}t^p_{j_1\cdots j_q j_{q+1} m} - t^s_{j_1\cdots j_q j_{q+1} p}t^p_{i_1\cdots i_k m} =$
. $\widetilde{\mathfrak{D}}_{j_{q+1}}\left(t^s_{i_1\cdots i_k p}t^p_{j_1\cdots j_q m} - t^s_{j_1\cdots j_q p}t^p_{i_1\cdots i_k m}\right) -$
. $- t^s_{i_1\cdots i_k j_{q+1} p}t^p_{j_1\cdots j_q m} + t^s_{j_1\cdots j_q p}t^p_{i_1\cdots i_k j_{q+1} m} -$
. $- t^s_{j_1\cdots j_q j_{q+1}}t^s_{i_1\cdots i_k pm} + t^p_{i_1\cdots i_k j_{q+1}}t^s_{j_1\cdots j_q pm}, \quad \forall k,q \geq 2$

where $\widetilde{\mathfrak{D}}_{j_{q+1}}\left(t^s_{i_1\cdots i_k m}\right) = t^s_{i_1\cdots i_k j_{q+1} m}$.

We shall use the notation $D_{i_1 i_2\cdots i_n} = D_{e_{i_1}, e_{i_2}\cdots e_{i_n}}, \quad \forall n \geq 2$. The matrices of $D_{i_1 i_2\cdots i_n}$ in basis $\mathbf{B}$ are defined by $D_{i_1 i_2\cdots i_n}(e_k) = t^j_{i_1\cdots i_n k}e_j, \quad \forall n \geq 2 \ (i_p, k, j = 1, 2, \cdots, m)$. They generates a Lie subalgebra of $g\ell(n, \mathbb{R})$.

The main result of our paper is the following.

Theorem 3.1. *Let $A(\cdot)$ be the binary algebra associated with a HQDS defined on $\mathbb{R}^n$, let a^k_{ij} be the structure constants of $A(\cdot)$ and let $t^j_{i_1 i_2\cdots i_s}$ be the structure constants of the associated h.s.. Then, the following vector fields*

$$X_i = \sum_{j,k=1}^{m} a^k_{ij}x^k \frac{\partial}{\partial x^j}, \quad X_{i_1 i_2\cdots i_n} = \sum_{j,k=1}^{m} t_{i_1 i_2\cdots i_n k}x^j \frac{\partial}{\partial x^j}, \qquad \forall n \geq 2.$$

generates an infinitesimal group.

Proof. The axioms (h.s.3)'-(h.s.6)' lead to the equations

$$\begin{gathered} X_{ijk} + X_{jki} + X_{kij} = 0, \\ X_{i_1 i_2\cdots i_k js} = X_{i_1 i_2\cdots i_k sj}, \\ \left[X_{i_1 i_2\cdots i_n}, X_{jk}\right] = t^p_{i_1\cdots i_n j}X_{pk} - t^p_{i_1\cdots i_n k}X_{jp}, \qquad k \geq 2, \\ \left[X_{i_1 i_2\cdots i_k}, X_{j_1 j_2\cdots j_s j_{s+1}}\right] = \widetilde{\mathfrak{D}}_{j_{s+1}}\left(\left[X_{i_1 i_2\cdots i_k}, X_{j_1 j_2\cdots j_s}\right]\right) - \\ - \left[X_{i_1 i_2\cdots i_k j_{s+1}}, X_{j_1 j_2\cdots j_s}\right] - t^p_{i_1\cdots i_k j_{s+1}}X_{j_1 j_2\cdots j_s p}, \qquad k, s \geq 2. \end{gathered}$$

Moreover, the following identities hold

$$\left[X_i, X_j\right] = X_{ij}, \qquad \left[X_{i_1\cdots i_k}, X_j\right] = X_{i_1\cdots i_k j} + t^s_{i_1\cdots i_k j}X_s.$$

These identities assure us that the linear enveloping of the family $\left\{X_i, X_{i_1\cdots i_n} | \forall n \geq 2\right\}$ is a Lie algebra $\mathfrak{L}_{vf}$; further $\mathfrak{L}_{vf1} = \left\{X_{i_1\cdots i_n} | \forall n \geq 2\right\}$ is a Lie subalgebra of $\mathfrak{L}_{vf}$. The mapping

$$L_i \to X_i, \qquad D_{i_1 i_2 \cdot i_n} \to X_{i_1 i_2 \cdot i_n}, \qquad \forall n \geq 2$$

is just a Lie algebra isomorphism. Since $dim\mathfrak{L} = dim\mathfrak{L}_{vf} \leq m^2$, it follows that we have a finite dimensional infinitesimal (local) Lie group G_A whose composition can be locally given by means of Campbell-Hausdorff formula. On E it induces the local multiplication

$$x \star y = x + y + \frac{1}{12}[x, y, y] + \frac{1}{12}[y, x, x] + \frac{1}{24}[y, x, y, x] + \frac{1}{24}[x, y, x, y] + ...$$

4. PARTICULAR CASES

It is well known that the structure of the associated binary algebra is responsible for certain structural properties of the corresponding HQDS. Indeed, if $E(\cdot)$ is an algebra with associative powers, then the Cauchy problem $\dot{x} = x^2, x(t_0) = x_0$ has the solution $x(t) = (I-(t-t_0)L_{x_0})^{-1}(x_0)$, where L_{x_0} denote the left multiplication by x_0. For example, this is the case when the algebra $E(\cdot)$ is a Jordan algebra.

If $E(\cdot)$ is a commutative algebra satisfying the identity [8]

$$2(yx \cdot x)x + y(x^2 \cdot x) = 3(y \cdot x^2) \cdot x$$

then, it implies $[y, x, x, x] = 0$ and, consequently, the h.s. associated with it is just a Lie triple system. It was proved [14] that $\Delta = det[t^s_{ijk}x^j x^k - \rho\delta^s_i]$ is an invariant of the local group generated by X_{ij} (i.e. $X_{ij}(\Delta) = 0$). This result can be extended to the case when the associated algebra satisfies the identity $[y, x, x, x, x] = 0$. This time, the corresponding h.s. is a general Lie triple system [14] and the linear enveloping of $\{X_{ijk}\}$ is a Lie subalgebra of $\mathfrak{L}_{vf}$. Then it can proved the following proposition.

Proposition 4.1. *The functions*

$$\Delta_1 = det[t^s_{ijk}x^j x^k - \rho\delta^s_i],$$
$$\Delta_2 = det[t^s_{ijkp}x^j x^k x^p - \rho\delta^s_i]$$

are invariant under the action of the local group generated by X_{ijk}.

Proof. It is suffices to prove $X_{ijk}(\Delta_1) = 0, X_{ijk}(\Delta_2) = 0$. This is get by straightforward computations and using the axioms of general triple systems.

Remark 4.1. Δ_1 *and* Δ_2 *are the characteristic polynomials of the endomorphisms* $Y \to [Y, X, X]$, $Y \to [Y, X, X, X]$, *respectively.*

References

[1] Burdujan, I., *Sur un théorème de K. Yamaguti*, Proc. of Inst. of Math. Iasi, Ed. Acad. Rom., 1974, 23-30.

[2] Burdujan, I., *Quadratic dynamical systems in ecology*, Sci. Ann. of USAMV Iasi, Sect. Hort., (2002), 7-22, Proc. of the annual Symposium on Mathematics applied in Biology and Biophysics.

[3] Burdujan, I., *On homogeneous systems associated to a binary algebra*, An. St. Univ. Ovidius Constantza, **2** (1994), 31-38.

[4] Coppel, W. A., *A survey of quadratic systems*, J. Diff. Eqs., **2** (1966), 293-304.

[5] Kinyon, M. K., Sagle, A. A., *Quadratic dynamical systems and algebras*, J. of Diff. Eqs., **117** (1995), 67-126.

[6] Markus, L., *Quadratic differential equations and non-associative algebras*, in Contributions to the theory of nonlinear oscillations, Annals of Math. Studies, **45**, Princeton Univ, Press, Princeton N.Y.,(1960).

[7] Micali, A., *Algèbres non associatives et equations differentielles*, Seminaire d'Analyse, Univ. Clermont II, 1988-1989.

[8] Osnborn, J. M.,*Commutative algebras satisfying an identity of degree four*, Proc. Amer. Math. Soc., **16**, 5 (1965), 1114-1120.

[9] Popa, M., *Methods with algebras for differential systems*, Piteşti Univ. Press, 2004.

[10] Röhrl, H.,*Algebras and differential equations*, Nagoya Math. J., **68** (1977), 59-122.

[11] Sibirskij, C. S., *Introduction to the algebraic theory of invariants for differential equations*, Ştiinţa Press, Chişinău, 1982, (Russian).

[12] Vulpe, N. I., Sibirskij, C. S.,*Geometric classification of quadratic differential systems*, Diff. Uravn., **13**, 5, 803-814, (Russian).

[13] West, B. J., *An essay on the importance of being nonlinear*, LN in Biomathematics **62**, 1985.

[14] Yamaguti, K., *A note on a theorem of N. Jacobson*, J. Sci. Hiroshima Univ., Ser. A, **22**, 3(1958), 187-190.

ON FRATTINI P-SUBALGEBRAS AND P-IDEALS OF A LIE P-ALGEBRAS

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Abstract A class of Lie algebras, defined in [6], namely the class of finite dimensional **Lie p-algebras**, is studied. Some properties of their **Frattini *p-subalgebras* and *p-ideals*** are presented.

1. INTRODUCTION

First recall some definitions and notation, assuming that L denotes a finite dimensional Lie algebra over a field k of characteristic $p > 0$, and $[x, y]$ stands for the Lie bracket of $x, y \in L$. The mapping

$$ady : L \to L,\ x(ady) = [x, y],\ for\ x,\ y \in L \text{ (a fixed } y),$$

is called the *inner derivation* defined by y or the *adjoint mapping* associated with $y \in L$. For mappings we use the right hand notation .

Definition 1.1. *A **Lie p-algebra** or a **restricted Lie algebra** is a Lie algebra L over a field k of characteristic p¿0, having a p-mapping* $x \to x^{[p]}$ *defined on it, such that the following conditions are fulfilled:*

(i) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ *for all* $\alpha \in k, x \in L$;

(ii) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, *for all* $x, y \in L$, *where* $is_i(x, y)is$ *the coefficient of* λ^{i-1}*in the expression* $x(ad(\lambda x + y))^{p-1}$;

(iii) $[x, y^{[p]}] = x(ady^{[p]}) = x(ady)^p$,*for all* $x, y \in L, i.e. ady^{[p]} = (ady)^p$.

The notion of restricted algebra and first examples appear in the [6], [12], [4], [5], [9], [10], [1], [2], [3]. A classification of the restricted simple Lie p-algebras has been made in [1].

Example 1.1. *(i) If A is an associative algebra over k, char k = p, then by letting* $[x, y] = xy - -yx$ *and* $x^{[p]} = x^p$, *we endow A with a structure of Lie p-algebra,* A_L.

(ii) If L is a Lie algebra over k, char k=p, and D(L) is the algebra of derivations of L, then $(xy)D^p = (xD^p)y + x(yD^p)$, *therefore* $D^p \in D(L)$ *and*

$D(L)$ is a Lie p-algebra with the Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$, and the p-mapping $D \to D^p$.

(iii) If A is an associative algebra over k, char $k = p$, with an antiautomorphism $a \to \overline{a}$, and we denote $L = \{a \in A/\overline{a} = -a\}$, then: $\overline{a-b} = \overline{a} - \overline{b} = -a - (-b) = -(a-b)$, $\overline{[a,b]} = \overline{ab - ba} = \overline{b}\overline{a} - \overline{a}\overline{b} = -[a,b]$, $\overline{a^p} = \overline{a}^p = (-a)^p = -a^p$, for all $a, b \in L$, and L is a Lie p-algebra.

The next proposition describes the possibility of endowing a Lie algebra with a structure of Lie p-algebra over a field of characteristic p. Obviously, we suppose that Lie algebra L has a supplementary property.

Proposition 1.1. [2]. *If L is a Lie algebra of characteristic p¿0 having an ordered basis $\{u_i/i \in J\}$, such that for all $u_i, (adu_i)^p$ is a inner derivation (i.e $(adu_i)^p = adu_i^{[p]}$, for an element $u_i^{[p]}$ in L), then there exists an unique mapping $x \to x^{[p]}$ from L to L such that L is a Lie p-algebra.*

Definition 1.2. *Let L and M be two Lie p-algebra. A mapping $\varphi : L \to M$, $x\varphi = x^\varphi$ is called **a morphism of Lie p-algebras** if: $(x+y)^\varphi = x^\varphi + y^\varphi$, $(\alpha x)^\varphi = \alpha x^\varphi$, $[x,y]^\varphi = [x^\varphi, y^\varphi]$, $(x^{[p]})^p = (x^p)^{[p]}$, for all $x, y \in L$ and $\alpha \in k$.*

Definition 1.3. *A subalgebra (resp. an ideal) of a Lie p-algebra L is a **p-subalgebra** (resp. a **p-ideal**) of L if it is closed under the p-mapping on L.*

Some properties of p-subalgebras of a Lie p-algebra L are given in the next proposition.

Denote $x^p := x^{[p]}$.

Proposition 1.2. [10] *If A is a p-ideal and B is a p-subalgebra of the Lie p-algebra L, then: A+B is a p-subalgebra of L;*

$(B)_p = (\{x^{p^n}/x \in B, n \in N\}), B$ being only a subalgebra of L, the subalgebra generated by the p^n-powers of the elements of B, is a p-subalgebra which is the smallest p-subalgebra of L including B. Moreover, $(B)_p^{(1)} \subseteq B^{(1)}$, where $^{(1)}$ denotes the derived algebra of B.

$(A)_p \subseteq C_L(A) := \{x \in L/[x, L] \subseteq A\}$ and $(A)_p$ is a p-ideal of L.

Denote by $\Phi(L)$ (resp. $\Phi_p(L)$) the intersection of all maximal subalgebras (resp. the maximal p-subalgebras) of L. It is called the **Frattini subalgebra** (resp. **Frattini p-subalgebra**) of L.

Denote by $F(L)(and, F_p(L))$ the greatest ideal included in $\Phi(L)$ (and the greatest p-ideal included in $\Phi_p(L)$). We call it the (p-) **Frattini ideal of L.**

2. FRATTINI THEORY FOR LIE P-ALGEBRAS

Some properties of the Frattini subalgebra and the Frattini ideal of Lie p-algebra are given in the next propositions.

Proposition 2.1. *If A is a p-subalgebra of L such that $A + \Phi_p\ (L)$=L, then A=L.*

The proof uses only the fact that there exists a maximal p-subalgebra containing A, if $A \neq L$.

Proposition 2.2. *([2], Lemma II.2.7.,II2.8.)*
If $L = \overset{n}{\underset{i=1}{\oplus}} L_i$, where L_i is a p-ideal of L, $i = \overline{1,n}$, then $F_p(L) = \overset{n}{\underset{i=1}{\oplus}} F_p(L_i)$.
If I is a p-ideal of L, then $(\Phi_p(L)+I)/I \subset \Phi_p(L/I), (F_p(L)+I)/I \subset F_p(L/I)$. If $I \subset \Phi_p(L)$, then we have $(F_p(L) + I)/I = F_p(L/I)$and $(\Phi_p(L) + I)/I = \Phi_p(L/I)$. In addition, if $F_p\ (L/I) = 0$, then $F_p\ (L) \subset I$.

Let $S_{ap}(L) = \sum \{I/I$ is a minimal Abelian p-ideal of $L\}$ be the Abelian p-socle of L and let $S_a(L) = \sum \{I/I$ is a minimal Abelian ideal of $L\}$ be the Abelian socle of L. One can easily see that $S_{ap}(L)$ is a p-ideal of L and if $F_p(L) = 0$, then $L = S_{ap}(L) \dot{+} A$, where A is a p-subalgebra of L and the sum is a direct sum of vector spaces.

Which are the relationships between $\Phi_p(L)$ and $F_p(L)$? We have found such relationships for special cases of Lie p-algebras.

Relation. 1. $\Phi_p(L) \subseteq \Phi(L)$. If $L = L^{(1)}$, then the equality takes place.

2. $F_p(L) = \Phi_p(L) \cap N(L)$, where *N(L)* is the nilradical of L.

3. $F_p(L) = 0$, if and only if $L = S_{ap}(L) \dot{+} A$,where A is a p-subalgebra of L.

4. If $F_p(L) = 0$, then $S_{ap}(L) = N\ (L)$.

For a subalgebra A of L, denote by A^p the subalgebra of L generated by $\{a^p/a \in A\}$. The Lie p-algebra L is *nil* if $L^{p^n} = 0$, and it is *toral* if L is Abelian and $L = L^p$.

5. If L is toral, then $F_p(L) = 0$.

6. If L is an Abelian Lie p-algebra, then $x^{p^n} \in \Phi_p(L)$ implies $x^{p^{n-1}} \in \Phi_p(L)$ for all $n \geq 2$ and $\Phi_p(L) \subseteq L^p$.

7. If L is nilpotent and $L_0 := \{x \in L/x^{p^n} = 0$, for all $n \in N\}$, then

$$L_0^p + L^{(1)} \subseteq \Phi_p(L) = F_pL) \subset L^p + L^{(1)},$$

while if L is nil, then

$$\Phi_p(L) = F_p(L) = L^p + L^{(1)}.$$

8. If L is solvable, then $\Phi_p(L)$ is a p-ideal of L and $\Phi_p(L) = F_p(L)$. (The proof can be found in [3].)

If $F_p(L) = 0$, and S is a p-subalgebra of L, it is not sure that $F_p(S) = 0$.

However, we may state the following proposition

Proposition 2.3. *If $F_p(L) = 0$ and $L^{(1)}$ is nilpotent, then: (i) for any p-subalgebra S of L including the Abelian p-socle of L, $F_p(S) = 0$; (ii) for any p-ideal I of L, $F_p(L) = 0$.*

Proof (i) Write $L = I \dot{+} A$. Then $S = I \dot{+} (A \cap S)$ since $I = S_p(L) \subseteq S$. Now A acts completely reducible on $[A, I]$, and hence so does $A \cap S$. It follows that $A \cap S$ acts completely reducible on $[A \cap S, I]$. Moreover, Z(S) is completely reducible under the p-map, and we may conclude that $F_p(S) = 0$

(ii) It suffices to show this for maximal ideals. By (i) we may assume that $I_1 \not\subset I$, where $S_p(L) = I_1 \oplus ... \oplus I_n$, with $I_1, ..., I_n$-minimal Abelian p-ideals. Then $L = I + I_1$, since I is maximal, and $I \cap I_1 = 0$. Thus $L = I \oplus I_1$, $I \cong L/I_1 \cong A \dot{+} (I_2 \oplus ... \oplus I_n)$, and$I_1 \subseteq Z(L)$. Hence $C_A(I_2 \oplus ... \oplus I_n) = C_A$(I)=0 and it is clear that all of the conditions hold , thus $F_p(L) = 0$. ■

By applying the theory for toral Lie p-algebras and other Lie p-algebras, in the future, we shall study thoroughly these properties for Lie p-algebras.

References

[1] R. E. Block, R. L. Wilson, *Classification of the restricted simple Lie algebras,* J. Algebra **114** (1998), 115-259.

[2] C. Ciobanu, *On a class of Lie p-algebras,* Ann. St. Univ. Ovidius, Constantza, **7**,2 (1999), 9-16.

[3] C. Ciobanu, *Cohomological study of Lie p-algebras,* Ph. D. Thesis, Ovidius University, Constantza,2001.

[4] S. P Demushkin, *Cartan subalgebras of the simple Lie p-algebras W_n and S_n*, Sibirsk. Math. Z., **11** (1970), 310-325. (Russian) (English trans. in Siberian Math. J., **11** (1970), 233-245).

[5] S. P Demushkin, *Cartan subalgebras of the simple nonclassical Lie p-algebras*, Izv. Akad. Nauk USSR, Ser. Math. **36** (1972), 915-932. (Russian) (English trans. in Math. USSR-Izv., **6** (1972), 905-924).

[6] N. Jacobson, *Lie algebras,* Interscience, New York,1962.

[7] J. Felovoos, *On the cohomology of restricted Lie algebras,* Commun. in Algebra, **19**, 10(1991), 2865-2906.

[8] J. Felovoos, *Theory der ebdlichen algebraischen Gruppen,* Lecture Notes in Mathematics, **592**, Springer, Berlin, 1977.

[9] M. I. Kuznetson, V. A. Yakovlev, *An elementary proof of Demushkin theorem on tori in Hamiltonian Lie p-algebras,* Commun. in Algebra, **27**, 6(1999), 2779-2784.

[10] M. Lincoln, D. A. Towers, *Frattini theory for restricted Lie algebras,* Arch. Math. **45** (1985), 451-457.

[11] M. Lincoln, D. A. Towers, *The Frattini p-subalgebra of a solvable Lie p-algebra,* Proc. of the Edinburgh Math. Soc., **40** (1997), 31-40.

[12] J. R. Schue, *Cartan decompositions for Lie algebras of prime characteristic,* J. Algebra, **11** (1969), 25-52.

[13] E. L. Stitzinger, *Frattini sbalgebras of a class of solvable Lie algebras,* Pacific J. Math., **34** (1970), 117-182.

[14] D. A. Towers, *A Frattini theory for algebras,* Proc. London Math. Soc., **27**, 3(1963), 440-462.

TRAVELING WAVE PATTERNS IN A BIOLOGICAL MODEL

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Abstract A Turing-Hopf bifurcation is investigated in a neural model that consists in a system of two integro-differential equations with three positive parameters. The corresponding normal form is constructed and then the existence of a stable spatio-temporal pattern in the system is proved. That pattern takes the form of a traveling wave.

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1. INTRODUCTION

The model we are interested in was introduced by Hansel and Sompolinsky [7] when studying feature selectivity in local cortical circuits. In that context, the network of neurons was assumed to code for a sensory or movement scalar feature x (for example the angle a bar is rotated in the subject receptor field, so that x can be taken in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$). The local cortical network consists of ensembles of neurons that respond (are tuned) to a particular feature of an external stimulus, and so are called 'feature columns', and that are interconnected by recurrent synaptic connections. In other words, each neuron in the network is selective, firing maximally when a feature ('preferred feature' of the neuron) with a particular value is present. The synaptic interactions between a presynaptic neuron y from the β-population and a postsynaptic neuron x from the α-population are denoted by a function $J^{\alpha\beta}(x-y) = j_0^{\alpha\beta} + j_2^{\alpha\beta}\cos(2(x-y))$, where α and β indices stand for E (excitatory) and/or I (inhibitory) population of neurons, depending on the context. We take $j_0^{\alpha E} \geq j_2^{\alpha E} \geq 0$ for input coming from the excitatory population, and $j_0^{\alpha E} \leq j_2^{\alpha E} \leq 0$ for input coming from the inhibitory population.

Hansel and Sompolinsky collapsed both excitatory and inhibitory populations into a single equivalent population. In this case the synaptic connectivity function J is defined as $J(x-y) = j_0 + j_2\cos(2(x-y))$ with no restrictions on the sign of coefficients, and the rate model has a single rate variable $m(x,t)$

that represents the activity of the population of neurons in the column x at time t. Moreover, the population is assumed to display adaptation.

The resulting model [7] is

$$\begin{cases} \tau_0 \frac{\partial m}{\partial t}(x,t) = -m(x,t) \\ \qquad\qquad\qquad + F\left(\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J(x-y)\, m(y,t)\, dy + I^0(x-x_0) - I_a(x,t) - T\right) \\ \tau_a \frac{\partial I_a}{\partial t}(x,t) = -I_a(x,t) + J_a\, m(x,t)\,. \end{cases}$$

I^0 stands for the synaptic currents from the external neurons, T is the neuronal threshold, I_a is the adaptation current, $\tau_a > \tau_0$ is its time constant, and J_a measures the strength of adaptation.

An additional assumption for (1) is that the stable state of the network is such that all the neurons are far from their saturation level, allowing the gain function F to be in a semilinear form $F(I) = I$ for $I > 0$, and zero otherwise.

In the next sections we investigate the Hopf bifurcation phenomenon in a system based on the above Hansel and Sompolinsky model, *a more general nonlinear sigma-shaped gain function* F being considered.

2. LINEAR STABILITY ANALYSIS IN A RATE MODEL

2.1. MATHEMATICAL RATE MODEL

Under the assumptions considered in the previous section, the mathematical model we will consider is

$$\begin{aligned} \frac{\partial u}{\partial t} &= -u(x,t) + F\left(\alpha\, J * u\,(x,t) - g\, v(x,t)\right), \\ \tau\, \frac{\partial v}{\partial t} &= -v(x,t) + u(x,t), \end{aligned} \tag{2}$$

with $x \in \mathbb{R}$ the one-dimensional spatial coordinate, and α, g and τ positive parameters.

The variables u and v represent the neuronal activity and adaptation respectively, τ and g correspond to the time constant and the strength of adaptation, and α is a parameter that controls the strength of the synaptic coupling J.

J is a continuous and even function, $J(-x) = J(x)$, $\forall x \in \mathbb{R}$, and absolutely integrable on $\mathbb{R}$ with $\lim_{x\to-\infty} J(x) = \lim_{x\to\infty} J(x) = 0$.

Then the operator $J * u$ is defined as

$$J * u\,(x,t) = \int_{-\infty}^{\infty} J(x-y)\, u(y,t)\, dy\,. \tag{3}$$

There is an operator associated to J, which is defined on the frequency space, and that is

$$\hat{J}(k) = \int_{-\infty}^{\infty} J(x)\, e^{ikx}\, dx\,. \tag{4}$$

For an infinite neural network the function J is typically defined as

$$J(x) = \frac{1}{\sqrt{\pi}} \left[A\sqrt{a}\, e^{-ax^2} - B\sqrt{b}\, e^{-bx^2} \right]\,,\ x \in \mathbb{R} \tag{5}$$

where $A \geq B > 0$, $a > b > 0$. Then $\hat{J}(k) = A\, e^{-k^2/4a} - B\, e^{-k^2/4b}\,,\ k \in \mathbb{R}$.

The firing rate F in (2) is a sigmoid function assumed to satisfy

$$F(0) = 0\,,\ F'(0) = 1\,.$$

The first condition translates the steady state to the origin $\bar{u} = 0$, $\bar{v} = 0$. The second condition brings additional simplifications to our calculations. A typical expression for F is then $F(u) = K\left[\frac{1}{1+e^{-r(u-\theta)}} - \frac{1}{1+e^{r\theta}}\right]$ with r and θ positive parameters, and $K = (1+e^{r\theta})^2\, e^{-r\theta}/r$, i.e.

$$F(u) = \frac{1+e^{r\theta}}{r} \cdot \frac{1-e^{-ru}}{1+e^{-r(u-\theta)}}\,. \tag{6}$$

The condition $F'(0) = 1$ is not essential. As long as $F'(0)$ is nonzero and positive, the results proved in the following sections remain valid. To see this, let us assume that $F'(0) \neq 1$. Then, by the change of variables $u_{\text{new}} = u/F'(0)$, $v_{\text{new}} = v/F'(0)$, the change of parameters $\alpha_{\text{new}} = F'(0)\,\alpha$, $g_{\text{new}} = F'(0)\, g$, and the change of function $F_{\text{new}} = F/F'(0)$ we obtain a system topologically equivalent to (2) where F_{new} satisfies the constraints $F_{\text{new}}(0) = 0$ and $F'_{\text{new}}(0) = 1$.

2.2. ASSOCIATED LINEARIZED EQUATION

In the following we investigate the possible spatial and spatio-temporal patterns that can occur in the neuronal system with adaptation (2), as a dependence on the parameters α, g and τ.

Based on the hypotheses $F(0) = 0$, $F'(0) = 1$, the expansion of (2) in linear and higher order terms becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= -u + (\alpha J * u - gv) + \frac{F''(0)}{2}(\alpha J * u - gv)^2 + \frac{F'''(0)}{6}(\alpha J * u - gv)^3 + \ldots \\ \frac{\partial v}{\partial t} &= (-v+u)/\tau\,, \end{aligned} \tag{7}$$

and then the linear operator is

$$L_0 U = \frac{\partial}{\partial t} U - \begin{pmatrix} -1 + \alpha J * (\cdot) & -g \\ 1/\tau & -1/\tau \end{pmatrix} U \tag{8}$$

where $U = (u, v)^T$. We are looking for solutions of $L_0 U = 0$ that are bounded and have the form $\xi(t)\, e^{ikx}$ with $k \in \mathbb{R}$.

According to (3) and (4), the equation (8) can be written as

$$\left[\frac{d\xi}{dt} - \hat{L}(k)\xi(t)\right] e^{ikx} = 0,$$

where

$$\hat{L}(k) = \begin{pmatrix} -1 + \alpha \hat{J}(k) & -g \\ 1/\tau & -1/\tau \end{pmatrix}. \tag{9}$$

Since we work on an infinite domain and J is symmetric, this statement is true for all values of $k \in \mathbb{R}$. Moreover, we have $\hat{J}(-k) = \hat{J}(k)$.

The equation to be solved now is the ODE $\frac{d\xi}{dt} = \hat{L}(k)\xi$ which has two independent solutions $\xi_{1k}\, e^{\lambda_{1k} t}$, $\xi_{2k}\, e^{\lambda_{2k} t}$ where $\xi_{1,2\,k}$ are two-dimensional complex vectors. Therefore the eigenfunctions of L_0 have the form $\xi_{1,2\,k}\, e^{\lambda_{1,2\,k} t \pm ikx}$ and $\overline{\xi}_{1,2\,k}\, e^{\overline{\lambda}_{1,2\,k} t \mp ikx}$, where $\lambda_{1,2\,k}$ are the eigenvalues defined by

$$\lambda_{1,2\,k} = \frac{1}{2}\left[Tr(\hat{L}(k)) \pm \sqrt{Tr(\hat{L}(k))^2 - 4\, det(\hat{L}(k))}\right].$$

If $det(\hat{L}(k)) > 0$ and $Tr(\hat{L}(k)) < 0$ for all k, i.e. $\alpha\, \hat{J}(k) < g+1$ and $\alpha\, \hat{J}(k) < 1/\tau + 1$, then all eigenfunctions of L_0 correspond to the stable manifold , and they decay exponentially in time to zero. The trivial solution is asymptotically stable.

Remark 2.1. *The eigenvalues k represent a measure of the wave-like pattern that can occur in the system. That is why k are called* wavenumbers*, or* modes *of the system, and $2\pi/k$ are called* wavelengths*.*

We consider k_0 to be *the most unstable mode*, defined as

$$\hat{J}(k_0) = \max_{k \geq 0} \hat{J}(k) = \max_{k \geq 0} \left(\int_{-\infty}^{\infty} J(x)\, e^{ikx}\, dx\right) \tag{10}$$

and assume that

$$k_0 \neq 0 \quad \text{and} \quad \hat{J}(k_0) > 0\,, \tag{11}$$

$$\hat{J}(k_0) \neq \hat{J}(k)\,, \forall k \neq \pm k_0\,. \tag{12}$$

There are only two ways the trivial solution can lose its stability: either when the determinant, or the trace becomes zero. We notice that (10), with additional conditions (11), (12), implies that $Tr(\hat{L}(k)) < Tr(\hat{L}(k_0))$ and $det(\hat{L}(k)) > det(\hat{L}(k_0))$ for $k \neq \pm k_0$. Therefore k_0 is the first eigenvalue where the system may lose its stability, that is k_0 is the most unstable mode of the system (2). For all $k \neq \pm k_0$ the eigenfunctions belong to the stable manifold. On the other hand, the eigenfunctions with $\pm k_0$ wavenumber may form a basis for the center manifold that becomes our point of interest.

The wavenumber k_0 determines then the mechanism that generates the emerged pattern. There are basically two possible cases. At $\alpha\,\hat{J}(k_0) = g + 1$, $g < 1/\tau$ the determinant becomes zero and a spatial pattern (steady state SS) bifurcates. At $\alpha\,\hat{J}(k_0) = 1 + 1/\tau$, $g > 1/\tau$ the trace becomes zero and a spatio-temporal pattern (traveling wave TW /standing wave SW) bifurcates. That last case is the one we investigate in our paper.

3. HOPF BIFURCATION AND TRAVELING WAVE SOLUTIONS

In the case of $Tr(\hat{L}(k_0)) = 0$ and $det(\hat{L}(k_0)) > 0$, at the most unstable mode k_0 defined by (10) with conditions (11), (12), the eigenvalues of the associated ODE $\frac{d\xi}{dt} = \hat{L}(k_0)\xi$ are complex with zero real part. This happens when the parameters of the system (2) satisfy

$$g > 1/\tau \quad \text{and} \quad \alpha^* = \frac{1 + 1/\tau}{\hat{J}(k_0)}\,. \tag{13}$$

The matrix $\hat{L}(k_0)$ has purely imaginary eigenvalues $\pm i\omega_0$ with corresponding eigenvectors Φ_0 and $\overline{\Phi}_0$ such that

$$\omega_0 = \frac{1}{\tau}\sqrt{g\tau - 1}\,, \tag{14}$$

$$\hat{L}(k_0)\Phi_0 = i\omega_0\Phi_0 \quad \text{with} \quad \Phi_0 = \left(\phi\,, \frac{\phi}{1 + i\sqrt{g\tau - 1}}\right)^T. \tag{15}$$

Based on the general theory [6], in the case of a pair of purely imaginary eigenvalues that arises at the most unstable mode k_0, the solution U of the nonlinear system (2) can be approximated by

$$U(x,t) \approx 2\mathrm{Re}\left[z(t)\,\Phi_0\,e^{i(\omega_0 t + k_0 x)} + w(t)\,\Phi_0\,e^{i(\omega_0 t - k_0 x)}\right]\,, \tag{16}$$

where z, w are time-dependent functions that satisfy the ODE system

$$\begin{cases} z' = z(a + bz\overline{z} + cw\overline{w})\,, \\ w' = w(a + bw\overline{w} + cz\overline{z})\,, \end{cases} \tag{17}$$

called *the normal form for the Turing-Hopf bifurcation* in the time-and-space-variable case, with $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ complex coefficients.

This equation has a *stable traveling wave solution* TW if and only if $a_1 > 0$, $b_1 < 0$ and $c_1 - b_1 < 0$ [6].

The construction of the normal form uses a singular perturbation approach with a proper scaling of the variables, parameters, and time with respect to ϵ, the small pertubation quantity. The Fredholm alternative method is then used to identify solutions for the functional equations obtained from the ϵ-power series expansion.

In the case of a pair of purely imaginary eigenvalues, a good scaling for the bifurcation parameter α and the solution U we are seeking, is

$$\begin{aligned} \alpha - \alpha^* &= \epsilon^2 \gamma \,, \; \gamma \in \mathbb{R} \,, \\ U(x,t) &= \epsilon \, U_0(x,t) + \epsilon^2 \, U_1(x,t) + \epsilon^3 \, U_2(x,t) + \cdots \\ &= \epsilon \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \epsilon^3 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \cdots \,. \end{aligned} \tag{18}$$

The system (7) can be written in the equivalent form

$$L_0 U = (\alpha - \alpha^*) \, (J * u \,, \, 0)^T + B(U,U) + C(U,U,U) + \ldots \tag{19}$$

with $B(U,U) = \left(\frac{F''(0)}{2} (\alpha \, J * u - gv)^2 \,, \, 0 \right)^T$ and

$C(U,U,U) = \left(\frac{F'''(0)}{6} (\alpha \, J * u - gv)^3 \,, \, 0 \right)^T$.

With the notation $\boldsymbol{E} = (1,0)^T$, (18) and (19) imply

$$\begin{aligned} \epsilon L_0 U_0 + \epsilon^2 L_0 U_1 + \epsilon^3 L_0 U_2 + \mathcal{O}(\epsilon^4) = \epsilon^2 \boldsymbol{E} \, \frac{F''(0)}{2} [\alpha^* \, J * u_0 - g v_0]^2 + \\ \epsilon^3 \boldsymbol{E} \Bigg[\gamma \, (J * u_0) + F''(0) [\alpha^* \, J * u_0 - g v_0] [\alpha^* \, J * u_1 - g v_1] + \\ \frac{F'''(0)}{6} [\alpha^* \, J * u_0 - g v_0]^3 \Bigg] + \mathcal{O}(\epsilon^4) \,. \end{aligned} \tag{20}$$

The main steps in the calculation of the norml form are the following:

The first equation to be solved is $L_0 U_0 = \mathbf{0}$. The nullspace of L_0 corresponding to the center manifold is four-dimensional and it has the basis $\left\{ \Phi_0 \, e^{i(\omega_0 t \pm k_0 x)}, \overline{\Phi}_0 \, e^{-i(\omega_0 t \pm k_0 x)} \right\}$, therefore U_0 can be written as

$$U_0 = z \, \Phi_0 \, e^{i(\omega_0 t + k_0 x)} + w \, \Phi_0 \, e^{i(\omega_0 t - k_0 x)} + \overline{z} \, \overline{\Phi}_0 \, e^{-i(\omega_0 t + k_0 x)} + \overline{w} \, \overline{\Phi}_0 \, e^{-i(\omega_0 t - k_0 x)} \,.$$

Since in (20), $L_0 U_0 = \mathcal{O}(\epsilon)$, we have z and w as ϵ-dependent. By considering $z = z(\epsilon^2 t)$ and $w = w(\epsilon^2 t)$ and expanding them as $z = z(0) + z'(0) \epsilon^2 t + \mathcal{O}(\epsilon^4)$ and $w = w(0) + w'(0) \epsilon^2 t + \mathcal{O}(\epsilon^4)$ as $\epsilon \to 0$, we then obtain

$$\begin{aligned} U_0 = \Bigg[z(0) \, \Phi_0 \, e^{i(\omega_0 t + k_0 x)} + w(0) \, \Phi_0 \, e^{i(\omega_0 t - k_0 x)} + \overline{z}(0) \, \overline{\Phi}_0 \, e^{-i(\omega_0 t + k_0 x)} \\ + \overline{w}(0) \, \overline{\Phi}_0 \, e^{-i(\omega_0 t - k_0 x)} \Bigg] + \mathcal{O}(\epsilon^2) \,. \end{aligned}$$

The equation that defines U_1 is

$$L_0 U_1 = \Big\{ \Big[A^2 z(0)^2\, e^{2i(\omega_0 t + k_0 x)} + A^2 w(0)^2\, e^{2i(\omega_0 t - k_0 x)}$$
$$+ 2A^2 z(0) w(0)\, e^{2i\,\omega_0 t} + cc \Big] + 2A\,\overline{A}\, \Big[z(0)\overline{z}(0) + w(0)\overline{w}(0) + z(0)\overline{w}(0)\, e^{2ik_0 x}$$
$$+ \overline{z}(0) w(0)\, e^{-2ik_0 x} \Big] \Big\} \frac{F''(0)}{2} \boldsymbol{E} \,,$$

where cc denotes the complex conjugation of the previous expression, and

$$A = \Phi_0^T \cdot (1 + 1/\tau\,,\, -g) = \phi\,(1 + i\omega_0)\,. \tag{21}$$

Therefore U_1 can be constructed as

$$U_1 = \Big(\xi_1\, z^2\, e^{2i(\omega_0 t + k_0 x)} + \xi_2\, w^2\, e^{2i(\omega_0 t - k_0 x)} + \xi_3\, z\, w\, e^{2i\omega_0 t}$$
$$+ \xi_4\, z\,\overline{w}\, e^{2ik_0 x} + cc \Big) + \xi_5\, z\,\overline{z} + \xi_6\, w\,\overline{w} \tag{22}$$

with ξ_i, $i = 1, \ldots, 6$, vectors in $\mathbb{C}^2$ that depend on g, τ, $\hat{J}(k_0)$, $\hat{J}(0)$, $\hat{J}(2k_0)$, and $F''(0)$. Then we obtain the equation for U_2

$$L_0 U_2 = \boldsymbol{Q}_{(1)} \boldsymbol{E} - \Big(z'(0)\, \Phi_0\, e^{i(\omega_0 t + k_0 x)} + w'(0)\, \Phi_0\, e^{i(\omega_0 t - k_0 x)} + cc \Big)\,,$$

that provides the explicit normal form for the Hopf bifurcation

$$\begin{cases} z'(0) = z(0)\, \big(\, \tilde{a} + b\, z(0)\overline{z}(0) + c\, w(0)\overline{w}(0)\, \big)\,, \\ w'(0) = w(0)\, \big(\, \tilde{a} + b\, w(0)\overline{w}(0) + c\, z(0)\overline{z}(0)\, \big)\,, \end{cases} \tag{23}$$

where the time variable is $\epsilon^2 t$, and $\tilde{a} = a/\epsilon^2$, with a, b, c of order 1.

As a consequence we obtain the following result.

Theorem 3.1. *Let us assume that the firing rate function F is such that $F(0) = 0$, $F'(0) > 0$, $F''(0) = 0$, $F'''(0) < 0$, and that the most unstable mode k_0 of the system (2) satisfies the conditions (11), (12) and at k_0 a pair of purely imaginary eigenvalues appears.*

If $g > 1/\tau$, in the neighborhood of the bifurcation value $\alpha^* = \frac{1+1/\tau}{\hat{J}(k_0)}$, the system (2) has the normal form (17) with $a_1 = Re(a) = \frac{1}{2}\left[\alpha\,\hat{J}(k_0) - (1 + \frac{1}{\tau})\right]$, and $b_1 = Re(b)$, $c_1 = Re(c)$ satisfying the equations

$$b_1 = \frac{\tau + 1}{4\tau}\, |A|^2 F'''(0)\,,$$
$$c_1 = \frac{\tau + 1}{2\tau}\, |A|^2 F'''(0)\,.$$

Thus for $\alpha > \alpha^*$, α close to α^*, the system (2) possesses a stable traveling wave solution.

Proof. First we notice that there exist sigmoid functions F that satisfy the theorem hypotheses. For example, if $\theta = 0$, we have from (6), $F(u) = \frac{2}{r}\tanh\left(\frac{ru}{2}\right)$ and so $F(0) = 0$, $F'(0) = 1$, $F''(0) = 0$, $F'''(0) = -\frac{r^2}{2} < 0$.

The normal form (17) is obtained directly from (23) as a result of the scaling $\epsilon\, z(0) \leftrightarrow z$, $\epsilon\, w(0) \leftrightarrow w$, and $\epsilon^2 t \leftrightarrow t$.

In this case $b_1 < 0$, $c_1 = 2b_1$, and therefore $c_1 + b_1 < 0$ and $c_1 - b_1 < 0$. Both standing and traveling waves bifurcate from the trivial solution at $\alpha = \alpha^*$, but only the TW is stable. □

References

[1] Arrowsmith, D.K., Place, C.M., *An introduction to dynamical systems*, Cambridge University Press, 1990.

[2] Curtu, R., *Waves and oscillations in model neuronal networks*, Ph.D. Dissertation, University of Pittsburgh, USA, 2003.

[3] Curtu, R., Ermentrout, B., *Pattern formation in a network of excitatory and inhibitory cells with adaptation*, SIAM J. on Appl. Dyn. Syst. **3**(3), 191-231 (2004).

[4] Curtu, R., Ermentrout, B., *Oscillations in a refractory neural net*, J. Math.Biol. **43**, 81-100 (2001).

[5] Dayan, P., Abbott, L.F., *Theoretical neuroscience. Computational and mathematical modeling of neural systems*, MIT Press, 2001.

[6] Ermentrout, G.B., *Symmetry breaking in homogeneous, isotropic, stationary neuronal nets*, Ph.D. Dissertation, The University of Chicago, 1979.

[7] Hansel, D., Sompolinsky, H., *Modeling feature selectivity in local cortical circuits*, in Methods in Neuronal Modeling (Koch C., Segev I. eds), 2-nd ed., 1998.

LIE ALGEBRAS FOR N-DIMENSIONAL ($N = 3,4,5$) AFFINE DIFFERENTIAL FACTORSYSTEMS

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Abstract The results of the present paper continue the researches started in [2] and concern the construction of Lie algebras of operators admitted by n-dimensional ($n = 3,4,5$) affine differential factorsystems.

Consider the affine differential system

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha \ (j,\alpha = \overline{1,n};\ n > 2), \tag{1}$$

and the group of linear non-degenerate transformations $GL(n,\mathbb{R})$

$$y^r = q^r_\alpha x^\alpha,\ det(q^r_\alpha) \neq 0 \ \ (r,\alpha = \overline{1,n};\ n > 2) \tag{2}$$

with real coefficients and variables. The complete convolution [1] takes place in lower indices in (1) and (2). Following [2], in according to [1,3], the contravariants, mixed comitants and invariants for system (1) with respect to group $GL(n,\mathbb{R})$ are

$$R_1 = a^{\alpha_1}u_{\alpha_1},\ R_2 = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}u_{\alpha_1},\ R_3 = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}u_{\alpha_2},\ ...,$$

$$R_n = a^{\alpha_1}_{\alpha_n}a^{\alpha_2}_{\alpha_1}...a^{\alpha_{n-1}}_{\alpha_{n-2}}a^{\alpha_n}u_{\alpha_{n-1}},\ R = det\left(\frac{\partial S_i}{\partial x^j}\right)_{i,j=\overline{1,n}},$$

$$S_1 = u_\alpha x^\alpha,\ S_2 = a^{\alpha_1}_{\alpha_2}x^{\alpha_2}u_{\alpha_1},\ S_3 = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}x^{\alpha_3}u_{\alpha_2},\ ...,$$

$$S_n = a^{\alpha_1}_{\alpha_n}a^{\alpha_2}_{\alpha_1}...a^{\alpha_{n-1}}_{\alpha_{n-2}}x^{\alpha_n}u_{\alpha_{n-1}},$$

$$T_1 = a^{\alpha_1}_{\alpha_1},\ T_2 = a^{\alpha_1}_{\alpha_2}a^{\alpha_2}_{\alpha_1},\ T_3 = a^{\alpha_1}_{\alpha_3}a^{\alpha_2}_{\alpha_1}a^{\alpha_3}_{\alpha_2},\ ...,\ T_n = a^{\alpha_1}_{\alpha_n}a^{\alpha_2}_{\alpha_1}...a^{\alpha_n}_{\alpha_{n-1}}, \tag{3}$$

where the complete convolutions takes place in indices $\alpha_1,\ \alpha_2,\ ...,\ \alpha_n$. According to [3] we will call vectors $x = (x^1, x^2, ..., x^n)$ and $u = (u_1, u_2, ..., u_n)$

contravariant and covariant, and as follows from (3) their inner product is a comitant of system (1) with respect to group $GL(2,\mathbb{R})$.

If $R \not\equiv 0$, from (1),(2), for $n = 3,4,5$, with the aid of centro-affine transformation $y^r = S_r$ $(r = \overline{1,n})$, where R and S_r are given by (3), we obtain a factorsystem [4]

$$\frac{dy^1}{dt} = R_1 + y^2,\ \frac{dy^2}{dt} = R_2 + y^3,\ ...,\ \frac{dy^{n-1}}{dt} = R_{n-1} + y^n,$$
$$\frac{dy^n}{dt} = R_n - L_i y^i \quad (i = \overline{1,n}). \tag{4}$$

In the last equation the complete convolution takes place in i. In (4) the absolute terms R_i are contravariants, and coefficients L_i $(i = \overline{1,n})$ can be expressed polynomially by invariants $T_1,\ T_2,\ ...,\ T_n$ from (3), moreover, they appear as coefficients in characteristic equation of matrix $A = (a^j_\alpha)_{j,\alpha=\overline{1,n}}$, which can be written as $|A - \lambda E| = 0$. For any n the last equality is

$$\lambda^n + L_n\lambda^{n-1} + L_{n-1}\lambda^{n-2} + ... + L_1 = 0.$$

Using (1)-(4) one can verify the following lemmas holds.

Lemma 1. *For $n = 3$ and $R = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u_\beta u_r \varepsilon^{pqr} \not\equiv 0$ system (1) admits factorsystem*

$$\frac{dy^1}{dt} = R_1 + y^2,\ \frac{dy^2}{dt} = R_2 + y^3,\ \frac{dy^3}{dt} = R_3 - L_1y^1 - L_2y^2 - L_3y^3, \tag{5}$$

where

$$L_1 = \frac{1}{6}\left(3T_1T_2 - 2T_3 - T_1^3\right),\ L_2 = \frac{1}{2}\left(T_1^2 - T_2\right),\ L_3 = -T_1, \tag{6}$$

and $R_1,\ R_2,\ R_3,\ T_1,\ T_2,\ T_3$ are taken from (3).

Lemma 2. *For $n = 4$ and $R = a^\alpha_p a^\beta_q a^\gamma_\beta a^\delta_r a^\mu_\delta a^\nu_\mu u_s u_\alpha u_\gamma u_\nu \varepsilon^{pqrs} \not\equiv 0$ system (1) admits factorsystem*

$$\frac{dy^1}{dt} = R_1+y^2,\quad \frac{dy^2}{dt} = R_2+y^3,\ \frac{dy^3}{dt} = R_3+y^4,\quad \frac{dy^4}{dt} = R_4-L_1y^1-L_2y^2-L_3y^3-L_4y^4, \tag{7}$$

where

$$L_1 = \frac{1}{24}\left(8T_1T_3 - 6T_4 - 6T_1^2T_2 + 3T_2^2 + T_1^4\right),\ L_2 = \frac{1}{6}\left(3T_1T_2 - 2T_3 - T_1^3\right),$$
$$L_3 = \frac{1}{2}\left(T_1^2 - T_2\right),\quad L_4 = -T_1, \tag{8}$$

and $R_1,\ R_2,\ R_3,\ R_4,\ T_1,\ T_2,\ T_3,\ T_4$ are taken from (3).

Lemma 3. *For $n = 5$ and $R = a_p^\alpha a_q^\beta a_\beta^\gamma a_r^\delta a_\delta^\mu a_\mu^\nu a_s^\phi a_\phi^\psi a_\psi^\theta a_\theta^\sigma u_t u_\alpha u_\gamma u_\nu u_\sigma \varepsilon^{pqrst} \not\equiv 0$ system (1) admits factorsystem*

$$\frac{dy^1}{dt} = R_1 + y^2, \quad \frac{dy^2}{dt} = R_2 + y^3, \ \frac{dy^3}{dt} = R_3 + y^4, \quad \frac{dy^4}{dt} = R_4 + y^5,$$

$$\frac{dy^5}{dt} = R_5 - L_1 y^1 - L_2 y^2 - L_3 y^3 - L_4 y^4 - L_5 y^5, \tag{9}$$

where

$$L_1 = -\frac{1}{120}\left(T_1^5 - 10T_1^3T_2 + 20T_1^2T_3 + 15T_1T_2^2 - 30T_1T_4 - 20T_2T_3 + 24T_5\right),$$

$$L_2 = \frac{1}{24}\left(8T_1T_3 - 6T_4 - 6T_1^2T_2 + 3T_2^2 + T_1^4\right),$$

$$L_3 = \frac{1}{6}\left(3T_1T_2 - 2T_3 - T_1^3\right), \quad L_4 = \frac{1}{2}\left(T_1^2 - T_2\right), \quad L_5 = -T_1, \tag{10}$$

and $R_1, \ R_2, \ R_3, \ R_4, \ R_5, \ T_1, \ T_2, \ T_3, \ T_4, \ T_5$ are taken from (3).

Remark 1. *If in system (4) we have $R_1^2 + R_2^2 + ... + R_n^2 \not\equiv 0$ ($n = 3, 4, 5$) and invariant $L_1 \neq 0$, then with the aid of change of variables*

$$z^1 = y^1 - \frac{1}{L_1}(R_n + \sum_{i=2}^{n} L_i R_{i-1}), \ z^2 = y^2 + R_1, \ z^3 = y^3 + R_2, \ ..., \ z^n = y^n + R_{n-1}$$

system (4) reduces to linear factorsystem

$$\frac{dz^1}{dt} = z^2, \ \frac{dz^2}{dt} = z^3, \ ..., \ \frac{dz^{n-1}}{dt} = z^n, \ \frac{dz^n}{dt} = -L_i z^i \ (i = \overline{1, n}), \tag{11}$$

where the complete convolution takes place in i.

Following Remark 1 we underline that for system (4) it is important to investigate the corresponding linear system (5) admitting Lie algebras of operators [5].

In accordance to [5], we will investigate system (11) admitting linear Lie operator

$$Y = \xi^1(z^1, z^2, ..., z^{n-1}, z^n)\frac{\partial}{\partial z^1} + \xi^2(z^1, z^2, ..., z^{n-1}, z^n)\frac{\partial}{\partial z^2} + ... +$$

$$+ \xi^{n-1}(z^1, z^2, ..., z^{n-1}, z^n)\frac{\partial}{\partial z^{n-1}} + \xi^n(z^1, z^2, ..., z^{n-1}, z^n)\frac{\partial}{\partial z^n}, \quad (n = 3, 4, 5) \tag{12}$$

where

$$\xi^1 = A_{1,1}z^1 + ... + A_{1,n-1}z^{n-1} + A_{1,n}z^n, \quad \xi^2 = A_{2,1}z^1 + ... + A_{2,n-1}z^{n-1} + A_{2,n}z^n,$$

$$\ldots$$

$$\xi^{n-1} = A_{n-1,1}z^1 + A_{n-1,2}z^2 + ... + A_{n-1,n-1}z^{n-1} + A_{n-1,n}z^n,$$

$$\xi^n = A_{n,1}z^1 + A_{n,2}z^2 + ... + A_{n,n-1}z^{n-1} + A_{n,n}z^n. \quad (n = 3, 4, 5) \tag{13}$$

Following [5] and taking into account (12)-(13), the equations defining the Lie algebra of operators admitted by system (11) are

$$\xi^1_{z^1}z^2 + \xi^1_{z^2}z^3 + ... + \xi^1_{z^n}(-L_i z^i) = \xi^2, \quad \xi^2_{z^1}z^2 + \xi^2_{z^2}z^3 + ... + \xi^2_{z^n}(-L_i z^i) = \xi^3,$$

$$\ldots$$

$$\xi^{n-1}_{z^1}z^2 + \xi^{n-1}_{z^2}z^3 + ... + \xi^{n-1}_{z^n}(-L_i z^i) = \xi^n, \quad \xi^n_{z^1}z^2 + \xi^n_{z^2}z^3 + ... + \xi^n_{z^n}(-L_i z^i) = -\xi^i L_i,$$

$$(i = \overline{1,n}) \quad (n = 3, 4, 5), \tag{14}$$

where the complete convolution takes place in i.

With the aid of (12)-(14) one can show that the following theorems hold.

Theorem 1. *The Lie algebra of operators admitted by system (11) for $n = 3$ and corresponding to system (5) is the two-dimensional commutative algebra with operators*

$$Y_1 = z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + z^3 \frac{\partial}{\partial z^3},$$

$$Y_2 = z^3 \frac{\partial}{\partial z^1} - \left(L_1 z^1 + L_2 z^2 + L_3 z^3\right) \frac{\partial}{\partial z^2} +$$

$$\left[L_1 L_3 z^1 + \left(L_2 L_3 - L_1\right) z^2 + \left(L_3^2 - L_2\right) z^3\right] \frac{\partial}{\partial z^3}, \tag{15}$$

where $L_1,\ L_2,\ L_3$ are given in (6).

Theorem 2. *The Lie algebra of operators admitted by system (11) for $n = 4$ and corresponding to system (6) is the three-dimensional commutative algebra with operators*

$$Z_1 = z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} + z^3 \frac{\partial}{\partial z^3} + z^4 \frac{\partial}{\partial z^4},$$

$$Z_2 = z^3 \frac{\partial}{\partial z^1} + z^4 \frac{\partial}{\partial z^2} - \left(L_1 z^1 + L_2 z^2 + L_3 z^3 + L_4 z^4\right) \frac{\partial}{\partial z^3} +$$

$$+ \left[L_1 L_4 z^1 + \left(L_2 L_4 - L_1\right) z^2 + \left(L_3 L_4 - L_2\right) z^3 + \left(L_4^2 - L_3\right) z^4\right] \frac{\partial}{\partial z^4},$$

$$Z_3 = \left(L_2 z^1 + L_3 z^2 + L_4 z^3 + z^4\right) \frac{\partial}{\partial z^1} - L_1 z^1 \frac{\partial}{\partial z^2} - L_1 z^2 \frac{\partial}{\partial z^3} - L_1 z^3 \frac{\partial}{\partial z^4}, \tag{16}$$

where $L_1,\ L_2,\ L_3,\ L_4$ are taken from (8).

Theorem 3. *The Lie algebra of operators admitted by system (11) for $n = 5$ and corresponding to system (9) is the four-dimensional commutative algebra with operators*

$$U_1 = z^1\frac{\partial}{\partial z^1} + z^2\frac{\partial}{\partial z^2} + z^3\frac{\partial}{\partial z^3} + z^4\frac{\partial}{\partial z^4} + z^5\frac{\partial}{\partial z^5},$$

$$U_2 = z^3\frac{\partial}{\partial z^1} + z^4\frac{\partial}{\partial z^2} + z^5\frac{\partial}{\partial z^3} - \left(L_1z^1 + L_2z^2 + L_3z^3 + L_4z^4 - L_5z^5\right)\frac{\partial}{\partial z^4} +$$

$$+ \left[L_1L_5z^1 + (L_2L_5 - L_1)\, z^2 + (L_3L_5 - L_2)\, z^3 + (L_4L_5 - L_3)\, z^4 + \left(L_5^2 - L_4\right) z^5\right]\frac{\partial}{\partial z^5},$$

$$U_3 = z^4\frac{\partial}{\partial z^1} + z^5\frac{\partial}{\partial z^2} - (L_1z^1 + L_2z^2 + L_3z^3 + L_4z^4 + L_5x^5)\frac{\partial}{\partial z^3} +$$

$$+[L_1L_5z^1 + (L_2L_5 - L_1)z^2 + (L_3L_5 - L_2)z^3 + (L_4L_5 - L_3)z^4 + (L_5^2 - L_4)z^5]\frac{\partial}{\partial z^4} +$$

$$+\{L_1(L_4 - L_5^2)z^1 + [L_2(L_4 - L_5^2) + L_1L_5]z^2 + [L_3(L_4 - L_5^2) - L_1 + L_2L_5]z^3 +$$

$$+[L_4(L_4 - L_5^2) - L_2 + L_3L_5]z^4 + [L_5(2L_4 - L_5^2) - L_3]z^5\}\frac{\partial}{\partial z^5},$$

$$U_4 = (L_3z^2 + L_4z^3 + z^5)\frac{\partial}{\partial z^1} - (L_1z^1 + L_2z^2 + L_5z^5)\frac{\partial}{\partial z^2} + [L_1L_5z^1 + (L_2L_5 - L_1)z^2 +$$

$$+(L_3L_5 - L_2)z^3 + L_4L_5z^4 + L_5^2z^5]\frac{\partial}{\partial z^3} + [-L_1L_5^2z^1 + (L_1 - L_2L_5)L_5z^2 +$$

$$+(L_2L_5 - L_1 - L_3L_5^2)z^3 + (L_3L_5 - L_2 - L_4L_5^2)z^4 + (L_4 - L_5^2)L_5z^5]\frac{\partial}{\partial z^4} +$$

$$+[(L_1L_5^2 - L_1L_4)L_5z^1 + (L_2L_5^2 - L_2L_4 - L_1L_5)L_5z^2 + (L_1 - L_3L_4 - L_2L_5 + L_3L_5^2)L_5z^3 +$$

$$+(L_2L_5 - L_1 - L_4^2L_5 - L_3L_5^2 + L_4L_5^3)z^4 + (L_3L_5 - L_2 - 2L_4L_5^2 + L_5^4)z^5]\frac{\partial}{\partial z^5}, \quad (17)$$

where L_1, L_2, L_3, L_4, L_5 are taken from (10).

Remark 2. *Operators Y_1 from (15), Z_1 from (16), U_1 from (17) are the tension operators.*

Remark 3. *One can show the truth of the Theorems 1-3 by substituting the coordinates of obtained operators in (14) for the corresponding factorsystem.*

References

[1] Sibirsky K. S., *Introduction to the algebraic theory of invariants of differential equations*, Kishinev, Ştiinţa, 1982. English transl. Manchester University Press, Manchester, 1988.

[2] Cherstega N. N., Popa M. N., *Mixed comitants and* $GL(3,\mathbb{R})$*-orbit's dimensions for the three-dimensional differential systems*, Bul. Şt., Univ. Piteşti, seria Mat. Info., **9** (2003), 163-164.

[3] Gurevich G. B., *Foundations of the theory of algebraic invariants*, GITTL, Moscow, 1948. English transl. Noordhoff, 1964.

[4] Ovsyannikov L. V., *Group analysis of differential equations*, Nauka, Moscow, 1978. English transl. by Academic Press, 1982.

[5] Popa M. N., Application of algebraic methods to differential systems, Series in Applied and Industrial Mathematics of Univ. Piteşti, The Flower Power Ed., Piteşti, 2004. (Romanian)

APPROXIMATE STABILITY SURFACES IN A CONVECTION PROBLEM FOR A MICROPOLAR FLUID. NUMERICAL RESULTS FOR THE HYDRODYNAMIC CASE

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Abstract In [2] the stability problem of thermal convection in a heat conducting micropolar fluid layer between rigid boundaries was treated theoretically using direct and variational methods. In this paper, for the same problem, we investigate the approximate numerical values of the Rayleigh number in the hydrodynamic case for different values of the physical parameters.

Keywords: stability, micropolar fluid, thermal convection, Rayleigh number

2000 MSC: 65L15,34K20,34K28 **Introduction**. We present a numerical study of the onset of thermal convection in a heat conducting micropolar fluid layer between two rigid boundaries. This particular problem of stability was solved theoretically in [2] using two direct and variational methods. We recall that these methods are based on the fact that the space $L^2(a,b)$ is a separable Hilbert space [5].

As it was stated in [2], assuming that the exchange of stability principle holds [9], the linear stability against normal mode perturbations is governed by the two-point problem

$$\begin{cases} (1+R)\Big[(D^2-a^2)^2-QD^2\Big]W+R(D^2-a^2)Z-Ra\cdot a^2\Theta=0, \\ \Big[A(D^2-a^2)-2R\Big]Z-R(D^2-a^2)W=0, \\ (D^2-a^2)\Theta+W-\overline{\delta}Z=0, \end{cases} \tag{1}$$

$$W=DW=Z=\Theta=0 \text{ at } z=\pm 0.5. \tag{2}$$

The micropolar parameters are $R=\dfrac{k}{\mu}$, $A=\dfrac{\gamma}{\mu d^2}$, $\overline{\delta}=\dfrac{\delta}{\varrho_0 c_v d^2}$, Q is the intensity of the magnetic field, Ra stands for the Rayleigh number and $a>0$ is the wave number. The numbers μ, k, α, β, γ and δ are material constants. The functions W, Θ, $Z:[-0.5,0.5]\to\mathbb{R}$ characterize the amplitude of the perturbation of the vertical component of the velocity, temperature and the vertical component of the spin vorticity, respectively.

One of the methods used in [2] is the Budiansky-DiPrima method, based on the expansion of the unknown functions upon total sets of functions which do not satisfy all boundary conditions of the problem.

Each of the unknown functions from (1) can be written as a sum of an odd function and an even function. In this way the problem splits into a problem with even functions and one problem with odd functions. Let us consider here the even part of the problem. In this case, the unknown functions W, Θ, Z are even functions, so we expand them upon the total set $\{E_{2n-1}\}_{n\in\mathbb{N}}$, $E_{2n-1}(z) = \sqrt{2}\cos(2n-1)\pi z,\ n \in \mathbb{N}$, namely

$$W = \sum_{n=1}^{\infty} W_{2n-1}E_{2n-1}(z), Z = \sum_{n=1}^{\infty} Z_{2n-1}E_{2n-1}(z), \Theta = \sum_{n=1}^{\infty} \Theta_{2n-1}E_{2n-1}(z). \quad (3)$$

The condition $DW = 0$ at $z = \pm 0.5$ is not satisfied such that it introduces a constraint for the problem (1)-(2).

The expressions of the derivatives occuring in (1) are obtained by the backward integration technique [5]. Substitute these expressions in (1), impose the condition that the obtained equations be orthogonal to E_{2m-1}, $m = 1, 2, ...$ to get the system

$$\begin{cases} (1+R)[A_n^2 + Q(2n-1)^2\pi^2]W_{2n-1} - RA_nZ_{2n-1} - Ra \cdot a^2\Theta_{2n-1} = \\ = 2\sqrt{2}(-1)^n(1+R)(2n-1)\pi\alpha \\ \\ RA_nW_{2n-1} - (AA_n + 2R)Z_{2n-1} = 0, W_{2n-1} - \overline{\delta}Z_{2n-1} - A_n\Theta_{2n-1} = 0, \end{cases} \quad (4)$$

with the constraint

$$\sum_{n=1}^{\infty}(-1)^n\sqrt{2}(2n-1)\pi W_{2n-1} = 0, \quad (5)$$

where $A_n = (2n-1)^2\pi^2 + a^2$.

The secular equation is obtained by solving the system (4) and replacing the obtained expression for W_{2n-1} in (5). **Numerical results**. The numerical study is done only for the hydrodynamic case, i.e. $Q = 0$. In this two cases ($Q = 0, \overline{\delta} = 0$), ($Q = 0, \overline{\delta} \neq 0$) we perform numerical computations and we obtain approximate values of the Rayleigh number for various values of the physical parameters. Also the neutral cureves and surfaces are drawn in various parameter spaces.

Case $Q = 0$**,** $\overline{\delta} = 0$. In this case the secular equation has the form

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2A_nD_n}{A_n^3\Big[D_n(1+R) - R^2\Big] - Ra \cdot a^2D_n} = 0. \quad (6)$$

Since this series is convergent, performing numerical evaluations for (6) we obtained values for the Rayleigh number. For different values of the micropolar parameters R, A and of the wave number a some of them are presented in Table 1. The results show that as R increases, the value of the Rayleigh number increases rapidly. Also, the growth of the wave number a implies a steep growth of the Rayleigh number. When the parameter A modify, the changes in the values of the Rayleigh number are not significant. From the numerical results it seems that the biggest influence on the values of the Rayleigh number has the parameter R.

A	R	a	Ra
0.001	0.001	3.117	1850.624086
0.001	0.5	3.117	2302.200180
0.001	0.5	5.00	3213.376953
0.001	1.00	5.00	3860.174964
0.002	1.00	5.00	3911.264998
0.001	2.00	6.70	9015.560744
0.001	2.00	14.00	93743.02414
0.001	2.00	9.50	24618.15747
0.001	4.00	9.50	36480.43048
0.001	6.00	9.50	48513.80933
0.001	8.00	9.50	60537.15202

Table1 : Approximate Rayleigh number for $Q = 0, \overline{\delta} = 0$.

When the micropolar parameters R and A are very small (close to zero), we found the classical case treated by Chandrasekar [1] (fig.1). The neutral curve is also similar to the one in [1].

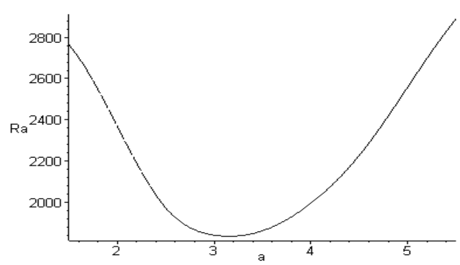


Fig.1 : The approximate Rayleigh number at which instability sets in for $Q = 0, \overline{\delta} = 0, A = 0.001$.

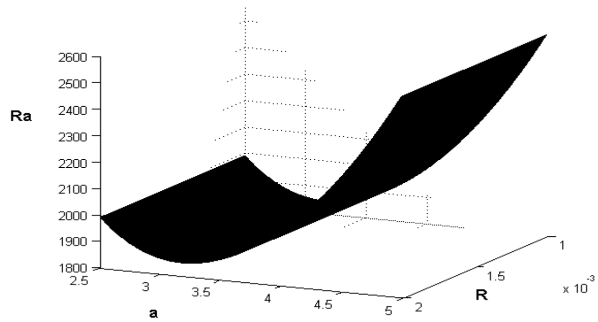


Fig.2 : The approximate neutral surface in the space (R, a, Ra) for small values of the parameters R and A.

Case $Q = 0$, $\overline{\delta} \neq 0$. If the intensity of the magnetic field is zero, but the micropolar parameter $\overline{\delta} \neq 0$, the secular equation has the form

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2 A_n D_n}{A_n^3 \Big[D_n(1+R) - R^2\Big] + Ra \cdot a^2(\delta R A_n - D_n)} = 0. \tag{7}$$

This series is also convergent.

To verify the obtained numerical values, we selected, at the beginning, the same values for the micropolar parameters and for the wave number as in [11]. The differences between the two numerical evaluations are found to be small. In carrying out the numerical computations, we have taken more values for the parameters than in [11], so that we noticed the different influences that the parameters have on the Rayleigh number.

A	R	$\overline{\delta}$	a	Ra
0.001	2	0.1	6.90	-5204.902
0.001	4	0.1	6.80	-7723.6289
0.001	6	0.1	6.85	-10259.840
0.001	8	0.1	6.85	-12785.852
0.001	2	0.05	9.00	-16641.5305
0.001	2	0.02	14.00	-97263.438
0.005	2	0.1	6.75	-5863.3438
0.01	2	0.1	6.70	-6723.4570
0.05	2	0.1	6.70	-17143.020

a) Critical values of $R_a$obtained in [8].

A	R	$\overline{\delta}$	a	Ra
0.001	2	0.1	6.90	−5218.031767
0.001	4	0.1	6.80	−7778.615586
0.001	6	0.1	6.85	−10271.62894
0.001	8	0.1	6.85	−12836.01081
0.001	2	0.05	9.00	−16674.77845
0.001	2	0.02	14.00	−97384.66469
0.005	2	0.1	6.75	−5909.801271
0.01	2	0.1	6.70	−6763.317816
0.05	2	0.1	6.70	−17344.22361
0.01	2	0.05	6.70	−44526.37592
0.05	2	0.05	6.70	59310.64812
0.05	4	0.05	6.70	−870390.2727

b) Critical values of $R_a$obtained by us.

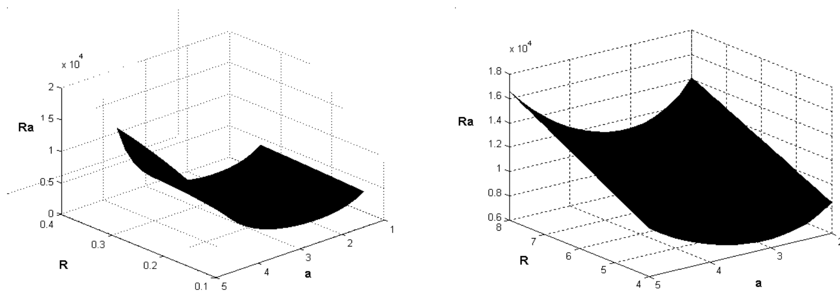
Table 2: Critical values of the Rayleigh number in the case $Q = 0$, $\overline{\delta} \neq 0$.

a) $A = 0.001,\ \overline{\delta} = 0.05$ *b*) $A = 0.001,\ \overline{\delta} = 0.01$

Fig.3 : The approximate neutral curve in the parameter space (a, Ra) in the case $Q = 0,\ \overline{\delta} \neq 0$.

Conclusions. In this paper we approximates the values for the Rayleigh number on the neutral surface. The evaluations showed that when the micropolar parameter $\overline{\delta}$ is not null, the viscosity parameter has a stabilizing influence on the flow. When A and $\overline{\delta}$ increases, large values of the wave number seems to have a stabilizing effect on the fluid.

When $\overline{\delta} = 0$, we can also treat the problem using the variational Budiansky-DiPrima method [2]. In this case, the operator is selfadjoint, so the evaluations are simplified since the number of derivatives is half of the derivatives occuring when we use the direct Budiansky-DiPrima method. The secular equation obtained in this case is the same. The direct Budiansky-DiPrima method was chosen to perform the numerical evaluations for the Rayleigh number so that we avoid very difficult numerical evaluations.



$a)$ $A = 0.001,\ \overline{\delta} = 0.05$ $A = 0.001,\ \overline{\delta} = 0.001$

Fig.4 : The approximate neutral surface in the parameters space(a, R, Ra) in the case $Q = 0,\ \overline{\delta} \neq 0$.

1. REFERENCES

1 Chandrasekar, S., *Hydrodynamic and hydromagnetic stability*, Oxford Univ. Press, 1961.

2 Dragomirescu, I., *Stability surfaces in a convection problem for a micropolar fluid. Theoretical results*, accepted for publication.

3 Georgescu, A., *Hydrodynamic stability theory*, Kluwer, Dordrecht, 1985.

4 Georgescu, A., Gavrilescu, M., Palese, L., *Neutral thermal hydrodynamic and hydromagnetic stability hypersurface for a micropolar fluid layer*, Indian J. Pure Appl. Math., **29**, 6 (1998), 575-582.

5 Georgescu, A., *Characteristic equations for some eigenvalue problems in hydromagnetic stability theory*, Mathematica, **24** (**47**), 1-2 (1982), 31-41.

6 Georgescu, A., Oprea I., Paşca, D., *Metode de determinare a curbei neutrale în stabilitatea Bénard*, Studii şi Cercetări de Mecanică Aplicată, **52**, 4 (1993), 267-276.

7 Georgescu, A., Setelecan, A., *Comparative study of the analytic methods used to solve some problems in hydrodynamic stability theory*, Analele Univ. Bucureşti, **38**, 1 (1989), 15-20.

8 Georgescu, A., *Variational formulation of some non-selfadjoint problems occuring in Bénard instability*, Preprints Series in Maths., **35**, 1977, Inst. of Maths., Bucharest.

9 Rama Rao, K.V., *Numerical solution of the thermal instability of a micropolar fluid layer between rigid boundaries*, Acta Mecanica, 32, 79-88 (1979).

A DIRECT METHOD AND ITS APPLICATION TO A LINEAR HYDROMAGNETIC STABILITY PROBLEM

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Abstract The main steps (Section 1), advantages and drawbacks (Section 3) of the direct method are discussed. A few tricks leading to easier computations and some open problems are revealed too (Section 3). The secular manifolds, the characteristic manifolds and their bifurcation sets, called the false neutral manifolds, are described (Section 2). The survey of the existing results obtained by applying the direct method to a particular Couette flow and a few new results are presented (Section 4).

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1. THE DIRECT METHOD BASED ON THE CHARACTERISTIC EQUATION

After more than hundred years of its existence, the linear theory of hydrodynamic and hydromagnetic stability is still of interest mainly due to two facts: this theory provides the necessary conditions for instability and it is much simpler than the non linear theory.

The eigenvalue problems governing the linear stability of certain fluid flows, and consisting in two points problems for systems of ordinary differential equations (ode's) with constant coefficients, were solved by the direct method and four methods based on Fourier series. Here we deal with the simplest direct method based on the characteristic equation. We point out its main steps, the advantage over other methods, and the main tricks used to simplify the computations.

The eigenvalue problems we deal with are either of the form of an ode

$$\sum_{k=0}^{n} a_{n-k} D^k u = 0, \quad x \in (-0.5, 0.5) \tag{1}$$

and n homogeneous boundary conditions

$$B_r u = 0, \quad r = \overline{1,n}, \quad at \quad x = \pm 0.5 \tag{2}$$

where $D = \frac{d}{dx}$, $u : [-0.5, 0.5] \to \mathbf{R}$, $u \in C^\infty[-0.5, 0.5]$ is the unknown function and the constant coefficients a_i depend on m physical parameters $\mathcal{R}_{\overline{1,m}}$, or of the form of a system of ode's

$$\mathbf{AU} = \mathbf{0}, \quad x \in (-0.5, 0.5) \tag{3}$$

and the boundary conditions

$$B_r U_i = 0, \quad r = \overline{1,n}, \quad i = \overline{1,s} \quad at \quad x = \pm 0.5 \tag{4}$$

where $\mathbf{A}$ is a $s \times s$ differential matrix the entries of which are polynomials, with constant coefficients depending on $\mathcal{R}_{\overline{1,m}}$, in the derivative D. The order of this system is n.

By an eigenvalue of the problem (1), (2) or (3), (4) we understand one value of the chosen parameter, say $\mathcal{R}_1$, to which nontrivial solutions (called eigenvectors or eigenfunctions) u of the problem correspond, each eigenvalue is a root of the secular equation, obtained by replacing the general solution of (1) into (2) or (3) into (4). In this way the eigenvalue depends on all other parameters. Therefore, the secular equation defines some manifolds. The most convenient (physically) *secular manifold* is called the *neutral manifold* (NM). In $\mathbf{R}^m$ it separates the domain of linear and nonlinear stability. Consequently, first our aim is to determine the secular equation.

First, consider the problem (1), (2). In order to determine the general solution of (1) we formally look for it in the form $u = e^{\lambda x}$ and replace this in (1) to obtain the characteristic equation

$$f(\lambda) = 0, \tag{5}$$

where $f(\lambda)$ is a n degree polynomial in λ, the coefficients of which depend on $\mathcal{R}_{\overline{1,m}}$. Up to a null measure set, for the points $(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) \in \mathbf{R}^m$ the roots of (5) are multiple. Denote by

$$g(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{6}$$

the equations defining some manifolds the points of which correspond to multiple roots of (5). These manifolds are referred to as *false neutral manifolds* (FNM) and they have various topological dimensions smaller than m.

Let $\lambda_{1,\cdots,p}$ be the roots of (5) and let m_j, $j = 1, \cdots, p$ be their multiplicity, such that $\sum_{j=1}^{p} m_j = n$. These multiplicities are deduced by taking into account the Viète relations and the fact that most physical parameters are real and positive. Corresponding to $\lambda_{1,\cdots,p}$, a basis for the vector space consisting of the nontrivial solutions of (1) is $e^{\lambda_1 x}$, $xe^{\lambda_1 x}$, $\cdots$, $x^{m_1-1}e^{\lambda_1 x}$, $e^{\lambda_2 x}$, $xe^{\lambda_2 x}$, $\cdots$, $x^{m_2-2}e^{\lambda_2 x}$, $\cdots$, $e^{\lambda_p x}$, $xe^{\lambda_p x}$, $\cdots$, $x^{m_p-1}e^{\lambda_p x}$. Thus, in the case of the multiple roots of (5), the general solution of (1) reads

$$u(x) = \sum_{j=1}^{p} \sum_{k=0}^{m_j-1} A_k^{(j)} x^k e^{\lambda_j x}; \tag{7}$$

in the case of the simple roots of (5), i. e. $m_1 = \cdots = m_p = 1$, $p = n$, it is

$$u(x) = \sum_{i=1}^{n} A_i e^{\lambda_i x}. \tag{8}$$

Introducing (8) and (7) into (2) we obtain the secular equation

$$F^*(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{9}$$

for the case of multiple roots of (5), and

$$F(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{10}$$

for the case of simple roots of (5).

2. CHARACTERISTIC MANIFOLDS AND THEIR BIFURCATION SETS. SECULAR AND NEUTRAL MANIFOLDS

Involving only $sinh(\lambda_i/2)$ and $cosh(\lambda_i/2)$, $i = 1, \cdots, p$, and powers of λ_i, in general, the secular equation is trascendent. It yields the dependence of $\mathcal{R}_1$ on $\mathcal{R}_2, \cdots, \mathcal{R}_m$. This represents the end of the method.

The equation (10) defines a (secular) manifold in the m-dimensional space of parameters. This manifold can have an infinity of *sheets*. From the physical point of view, the most convenient sheet is just the neutral manifold.

In (10) F is a determinant the columns of which have the same form in λ_i, i corresponding to the i-th column. Formally, (10) is satisfied on FNM because for each point of (6) at least two roots λ_i and λ_j coincide, and, therefore, the columns i and j of (10) are identical. Thus, formally, every FNM is a secular manifold. In fact, this is not true because (10) is not defined on FNM and, therefore, (10) is not entitled to serve as a secular equation for the points of FNM. Concrete examples [6] show that FNM could be physically more convenient (if it would be a secular manifold) than the true neutral manifold given by (10). Whence the name of *false* bearing by the manifolds defined by (6). In these cases the direct numerical computations are invalid. In other examples, parts of FNM proved to be limits of the secular manifolds of (10), or even of the neutral manifolds defined by (10). This is the reason why, apart from (10) we must solve all secular equations (9), corresponding to all multiplicities m_j and, so, to all manifolds (6). It is only in this way that we can deduce which points of $\mathbf{R}^m$ are secular points, indeed. Formally, the secular equations (9) are deduced from (10), namely writing the column j for λ_j while the column $j+k$ $k = 1, \cdots, m_j - 1$ are obtained by differentiating k times the $j+k$-th column of (10) with respect to λ_{j+k} and then replacing λ_{j+k} by λ_j.

The equations (9) are valid only on the manifold (6). Consequently, if some manifold defined by (6) is q dimensional, then the secular manifold of (9), when it exists, is $q-1$ dimensional. In this way, the secular manifolds are: those of dimension $m-1$ (corresponding to (10)) and those of smaller dimension (corresponding to (9) and situated on the manifolds (6)). Thus the complicated problem concerning the relative position, intersection and geometric structure of these manifolds arises [11].

Only seldom the roots of the secular equations (9) exist.

The false neutral manifolds defined by (6) are bifurcation sets for the characteristic manifold (10), the dimension of which is $m+1$ if $\lambda_{1,\cdots,n} \in \mathbf{R}$ and $m+2$ if some $\lambda_i \in \mathbf{C}$. Some among the FNM consist of bifurcarion points for some other FNM or are the bifurcation sets of these ones. We recall that the bifurcation set B of a manifold M is the projection on the parameter space of the set B_M of the bifurcation points of M, corresponding to the points where at least two sheets of M coalesce. Hence B is the set of bifurcation values corresponding to B_M.

3. ADVANTAGE AND DRAWBACKS OF THE DIRECT METHOD AND TRICKS TO ITS EASIER APPLICATION. OPEN PROBLEMS

Among the advantages of the direct method we quote :1) the very simple general form (7) of the solution of (1) and the corresponding secular equations have only a finite number of terms, therefore these solutions are *exact*. The method based on Fourier and asymptotic series involve an infinite number of terms of the solution representation and of the secular equations. Therefore, they are approximate, even if they are called exact; 2) among all methods we know, it is the only one which provides the *false secular points*. It shows how dangerous is to apply numerical methods without a theoretical support; 3) this method is the *simplest* among all methods used for problems (1), (2); 4) the direct method applies *irrespective the form of the boundary conditions*. In the case of the problem (3), (4) the coefficients of various functions are related, such that the boundary conditions can be easily written for a single component of the solution $\mathbf{U}$, which in other methods generally is impossible

[12]; 5) in the direct method, in order to get a simpler form of the secular equation, the columns of F in (10) are divided by $cosh(\lambda_i/2)$. If λ_i are purely imaginary, then this division is forbidden but the condition $cosh(\lambda_i/2) = 0$ yields secular points valid for *every boundary conditions*. This is a striking property with basic implications in applications.

When applying the Fourier series based methods the expressions of coefficients have as denominator just the expression $f(\lambda)$ from (5). Therefore they ceased to be valid for $\lambda = \lambda_i$ and, so, the direct method must be applied in order to complete this study.

Often, instead of the given problem (1), (2) or (3), (4), the problems for the even and odd part of the solution are solved. In spite of the fact that the trascendent secular equation has a finite number of terms containing the powers of λ_i and hyperbolic sine and cosine of $\lambda_i/2$, the solution of this equation is pratically impossible to obtain. In the same way, for higher-order ode's of the governing eigenvalue problems no closed-form solutions of the characteristic equation (5) are known. This is why, in carring out numerical computations, the equations (5) and (10) or (5) and (9) must be solved simultaneously [1].

In the case of ode's containing only even-order derivatives, a suitable change of variables, e. g. $\mu = \lambda^2 - a^2$, leads to a characteristic equation the degree of which is half of the initial degree. For the equation in the closed-form solution is immediate.

The direct method was applied in hydrodynamic stability theory only by us and our collaborators starting with the year 1977. Apart from very simple situations, a systematical theoretical investigation of the bifurcation of the involved manifolds is a difficult open problem. This is the case especially when more than three parameters occur. This was also remarked by Collatz in [11]. It is also in these cases that the determination of the multiplicity of the characteristic roots is another open problem. The separate numerical solution of the characteristic equation and of the secular equations require numerical methods specific to bifurcation theory. Therefore, we can avoid this by numerically solving both these equations simultaneously [1].

4. APPLICATION OF THE DIRECT METHOD TO THE CASE OF A PARTICULAR COUETTE FLOW

Consider the Couette flow of a fluid situated between two rotating coaxial cylinders situated at a very small distance. The fluid is electrically conducting and subject to an axial magnetic field. The eigenvalue problem governing the linear stability of this flow against normal mode perturbation reads [6]

$$\left\{(D^2 - a^2)^2 + Qa^2\right\}^2 v = -Ta^2(D^2 - a^2)v, \quad -0.5 < z < 0.5 \qquad (1')$$

$$Dv = (D^2 - a^2)v = \Big\{(D^2 - a^2)^2 + Qa^2\Big\}v = D\Big\{(D^2 - a^2)^2 + Qa^2\Big\}v = 0, \quad (2')$$

where T, $Q > 0$ are the dimensionless Taylor and Chandrasekhar numbers respectively, a is the positive wavenumber, z is the vertical coordinate, $D = \frac{d}{dz}$ and v is the unknown stream function. The associated characteristic equation is

$$(\lambda^2 - a^2)^4 + 2Qa^2(\lambda^2 - a^2)^2 + Ta^2(\lambda^2 - a^2) + Q^2a^4 = 0. \qquad (5')$$

With the notation $\mu = \lambda^2 - a^2$ it becomes

$$\mu^4 + 2Qa^2\mu^2 + Ta^2\mu + Q^2a^4 = 0. \qquad (5'')$$

Introduce the surfaces

$$\mathcal{C} : T = T^* \equiv 16aQ\sqrt{Q}/(3\sqrt{3}), \qquad (6_1')$$

$$\mathcal{C}_1 : T = T^{**} \equiv (Q^2 + a^2)^2, \qquad (6_2')$$

and denote by $\mathcal{C}^*$ their intersection, i. e.

$$\mathcal{C}^* : Q = 3a^2, \quad T = 16a^4. \qquad (6_3')$$

The projection of $\mathcal{C}^*$ on the plane (a, Q) is

$$\mathcal{C}^*_* : Q = 3a^2. \qquad (6_4')$$

Consider $a, Q, T > 0$. Then, for $(a, Q, T) \in \mathbf{R}^3 \setminus (C \cup C_1)$, $(5')$ has eight mutually disjoint roots $\lambda_{1,\cdots,8}$ and $(5'')$ has four mutually distinct roots $\mu_{1,\cdots 4}$, related to $\lambda_{1,\cdots,8}$ by the relations: $\lambda_{1,5} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{2,6} = \pm\sqrt{\mu_2 + a^2}$, $\lambda_{3,7} = \pm\sqrt{\mu_3 + a^2}$, $\lambda_{4,8} = \pm\sqrt{\mu_4 + a^2}$, for $\mu_i > -a^2$; $\lambda_{1,5} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{2,6} = \overline{\lambda_{1,5}}$, $\lambda_{3,7} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{4,8} = \overline{\lambda_{3,7}}$ for $\mu_i \in \mathbf{C}$ or $\mu_i < -a^2$. Since it is much easier to study $(5'')$ than $(5')$, let us relate the multiplicities of the roots of these two characteristic equations. Thus, $\mu_1 = \mu_2$ implies either $\lambda_1 = \lambda_2 > 0$, $\lambda_5 = \lambda_6$ if $\mu_1 = \mu_2 \in \mathbf{R}$, or $\lambda_1 = \lambda_2 = 0$ if $\mu_1 = \mu_2 = -a^2$, or λ_1 is purely imaginary and $\lambda_1 = -\lambda_2$, $\lambda_5 = -\lambda_6$, if $\mu_{1,2} < -a^2$. Therefore, almost everywhere in $\mathbf{R}^3$, more exactly for $(a, b, Q) \in \mathbf{R}^3 \setminus (C \cup C_1)$, $(5'')$ and $(5')$ have mutually distinct roots; for $(a, b, Q) \in C \setminus C^*$, $(5'')$ has two equal real roots $\mu_1 = \mu_2 \neq -a^2$ and $(5')$ has two pairs of equal and nontrivial roots $\lambda_1 = \lambda_2$, $\lambda_5 = \lambda_6$; for $(a, b, Q) \in C^*$, $(5'')$ has two roots equal to $-a^2$, i. e. $\mu_1 = \mu_2 = -a^2$ and $(5')$ has four trivial roots $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$; for $(a, b, Q) \in C_1 \setminus C^*$, $(5'')$ has one root equal to $-a^2$ and the others different, i. e. $\mu_1 = -a^2$ μ_2, μ_3, $\mu_4 \neq -a^2$. In this case $\lambda_1 = \lambda_5 = 0$, all other roots of $(5')$ are nonvanishing and mutually disjoint [2],[4].

In the space (μ, a, Q, T) the characteristic manifold defined by $(5'')$ has four sheets which coalesce along the surface $\mu_1 = \mu_2 = -a\sqrt{Q/3}$, $T = T^*$. In the space (λ, a, Q, T) the characteristic manifold defined by $(5')$ has eight sheets, two of them coalescing along the surface $\lambda_1 = \lambda_5 = 0$, $T = T^{**}$, $Q \neq 3a^2$;two pairs of them coalescing along the surfaces $\lambda_1 = \lambda_2 = \sqrt{a^2 - a\sqrt{Q/3}}$, $T = T^*$, $Q \neq 3a^2$ and $\lambda_5 = \lambda_6 = -\sqrt{a^2 - a\sqrt{Q/3}}$, $T = T^*$, $Q \neq 3a^2$; four of them coalesce along the curve $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$, $T = 16a^4$, $Q = 3a^2$. These results suggested us the following

Theorem 1. *The surface $\mathcal{C}$ is a bifurcation set for the characteristic manifolds $(5')$ and $(5'')$, the surface C_1 for $(5')$ but not for $(5'')$, C^* for $(5')$ C^*_* for $(5')$ taken on C. In addition, C^*_* is the bifurcation set for $C \cup C_1$.*

Proof. If the characteristic manifold defined by $(5')$ has bifurcation points, then the equation $(5')$ has multiple roots. For the case of a double root, this root satisfies the equation $(5')$ and the equation obtained by differentiating $(5')$ with respect to λ, i. e.

$$2\lambda[4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2] = 0. \qquad (5')_1$$

If $\lambda = 0$ is the double root, $(5')$ implies $T = T^{**}$, henceC_1 is the bifurcation set for $(5')$. If the double root is one of the roots of $(5')$ and the equation

$$4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2 = 0, \qquad (5')_2$$

then, by the Euler algorithm, it follows that the double root can be either $\lambda = \sqrt{a^2 - \frac{4aQ}{3\sqrt{T}}}$ or $\lambda = -\sqrt{a^2 - \frac{4aQ}{3\sqrt{T}}}$, both leading to $T = T^*$. Whence C is the bifurcation set for $(5')$.

If the characteristic manifold defined by $(5'')$ has bifurcation points, then the equation $(5'')$ has multiple roots. For the case of a double root, this root satisfies the equation $(5'')$ and the equation obtained by differentiating $(5'')$ with respect to μ, i. e.

$$4\mu^3 + 4Qa^2\mu + Ta^2 = 0. \qquad (5'')_1$$

Then, by the Euler algorithm, it follows that the double root is $\mu = -\frac{16}{9}\frac{Q^2a^2}{T}$, which introduced into $(5'')$ or $(5'')_1$ leads to $T = T^*$. Hence C is the single bifurcation set for $(5'')$.

As expected, C_1 is not a bifurcation set for $(5'')$ because for the points of C_1 the equation $(5'')$ has mutually disjoint roots, one of which being equal to $-a^2$ and leading to two equal solutions of $(5')$.

Assume that $(5')$ has a multiple root of multiplicity equal to 4. Then it must satisfy $(5')$, $(5')_1$ and

$$4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2 + 2\lambda^2[12(\lambda^2 - a^2)^2 + 4Qa^2] = 0, \qquad (5')_3$$

$$\lambda[12(\lambda^2 - a^2)^2 + 4Qa^2] + 16\lambda^3(\lambda^2 - a^2) = 0. \qquad (5')_4$$

Since the multiple root satisfies $(5')_2$, from $(5')_3$ it follows that it must satisfy the equation $12(\lambda^2 - a^2)^2 + 4Qa^2 = 0$ and from $(5')_4$ it follows that it satisfies the equation $16\lambda^3(\lambda^2 - a^2) = 0$. Supposing that this root is nontrivial, it follows that it can be either $\lambda_1 = a$ or $\lambda_2 = -a$. In both these cases from $12(\lambda^2 - a^2)^2 + 4Qa^2 = 0$ we have $Qa^2 = 0$, which contradicts the assumption $a, Q, T > 0$. Therefore the single root of $(5')$ which can have multiplicity equal to four is $\lambda_1 = 0$ for which from $(5')_3$ it follows $-4a^6 - 4Qa^4 + Ta^2 = 0$, while from $(5')$ we have $a^8 + 2Qa^6 - Ta^4 + Q^2a^4 = 0$. These two relations imply $Q = 3a^2$, $T = 16a^4$, hence C^* is a bifurcation set for $(5')$ (of a type different from those of $C \setminus C^*$ and $C_1 \setminus C^*$).

For $(5'')$, C^* is just part of the bifurcation set.

The surface $C \cup C_1$ has two sheets but for C^*, where the two sheets coalesce. The projection of C^* on the (a, Q) plane is C^*_*, therefore C^*_* is the bifurcation set for $C \cup C_1$. This can be seen also considering the equation

$$[T - 16aQ\sqrt{Q}/(3\sqrt{3})][T - (Q + a^2)^2] = 0, \qquad (5')_5$$

which defines $C \cup C_1$ and which possesses a double root for $Q = 3a^2$.

C^*_* is a bifurcation set for $(5')$ taken on C, i. e. for

$$(\lambda^2 - a^2)^4 + 2Qa^2(\lambda^2 - a^2)^2 + 16a^3Q\sqrt{Q}/(3\sqrt{3})(\lambda^2 - a^2) + Q^2a^4 = 0, \quad (5')_1$$

because differentiating this equation with respect to λ and imposing to the solution $\lambda = 0$ of the obtained equation to satisfy $(5')_1$ we obtain $Q = 3a^2$ defining C^*_*. Whence, Theorem 1. The detailed geometrical structure of the characteristic manifolds and bifurcation manifolds is given in [4], [3].

The secular equations must be written separately for each of the bifurcation sets, because for each of them the form of the general solution of $(1')$ $(2')$ is different. We write them only for an even function v [2] (physical reasons show that odd v is not realistic). We have the following types of general even solution for $(1')$, $(2')$ corresponding to various multiplicities of λ_i

$$v(z) = \sum_{i=1}^{4} A_i cosh(\lambda_i z), \quad \text{for} \quad (a, Q, T) \in \mathbf{R}^3 \setminus (C \cup C_1) \qquad (8')$$

$$v(z) = A_1 cosh(\lambda_1 z) + B_2 z sinh(\lambda_1 z) + \sum_{i=1}^{3} A_i cosh(\lambda_i z), \quad \text{for} \quad (a, Q, T) \in C \setminus C^* \qquad (7')_1$$

$$v(z) = A_1 + \sum_{i=2}^{4} A_i cosh(\lambda_i z), \quad \text{for} \quad (a, Q, T) \in C_1 \setminus C^* \qquad (7')_2$$

$$v(z) = A_1 + A_2 z^2 + \sum_{i=1}^{3} A_i cosh(\lambda_i z), \quad \text{for} \quad (a, Q, T) \in C^* \qquad (7')_3$$

Since $(7')_2$ can be, simply, obtained from $(8')$ for $\lambda_1 = 0$, we consider it no longer. In this case no secular points exist [4] and, so, the entire surface $C_1 \setminus C^*$ is a false secular manifold indeed.

The secular equation corresponding to $(8')$ reads

$$\begin{vmatrix} \lambda_1 sinh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ \mu_1 cosh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ (\mu_1^2 + Qa^2) cosh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ \lambda_1(\mu_1^2 + Qa^2) sinh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \end{vmatrix} = 0, \quad \text{for} \quad (a, Q, T) \in \mathbf{R}^3 \setminus (C \cup C_1). \quad (10')$$

The $i-$th lacking columns in $(10')$ is identical to the first column but with λ_1 and μ_1 replaced by λ_i and μ_i.

Formally, in the secular equation for $(a, Q, T) \in C \setminus C^*$ the first, third and columns are identical with those from $(10')$ while the second column is obtained by differentiating the second column in $(10')$ with respect to λ_2 and then replacing λ_2 by λ_1. Similarly, in the secular equation for $(a, Q, T) \in C^*$, the second column is obtained by differentiating twice the second column in $(10')$ and then replacing λ_2 by λ_1 [2], [4], the remaining columns being identical with those from $(10')$.

For an easier solution of the secular equations the notation $t_i = \lambda_i tanh(\lambda_i/2)$ is introduced. As a consequence, the new form of the secular equations will contain the product of $cosh(\lambda_i/2)$, $i = 1 \cdots 4$. If some of λ_i are purely imaginary then the corresponding $cosh(\lambda_i/2)$ vanish, the equality $\prod_{i=1}^{4} cosh(\lambda_i/2) = 0$ representing additional secular equations which do not depend on the boundary conditions. For the points of C^*, in [4] it was shown that the corresponding secular equation (in the old form) has no solution, therefore no secular points belong to C^*. Moreover, since in this case $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$ and $\lambda_{3,4,7,8}$ are complex not purely imaginary, we have neither additional secular points. For the points of $C \setminus C^*$ we have $\mu_1 = \mu_2 = -a\sqrt{Q/3}$, hence $\lambda_{1,2} = \sqrt{a^2 - a\sqrt{Q/3}} = -\lambda_{5,6}$. Consequently, taking into account that $C \setminus C^*$ is a FNM indeed, we have

Theorem 2. *No point of C_1 is secular. For $(a, Q, T) \in C \setminus C^*$, $a < \sqrt{Q/3}$, there exists the additional secular equation $cos\sqrt{a\sqrt{(Q/3)} - a^2} = 0$, which are independent of the boundary conditions $(2')$.*

In the last case the secular curves are $Q = 3[a + (2k + 1)^2\pi^2/(4a)]^2$, $T = 16a[a + (2k + 1)^2\pi^2/(4a)]^3$, $k \in \mathbf{N}$, and they are situated on $C \setminus C^*$.

There are a lot of theoretical open problems for $(1')$ and $(2')$. The first is the existence of solutions of $(10')$ and of the secular equation for points of $C \setminus C^*$. However, computations show that there exist infinitely many secular surfaces (sheets) defined by $(10')$ and an infinity of spatial curves situated on $C \setminus C^*$

which consist of secular points. No secular points exist on C_1, but additional secular curves (Theorem 2) exist on C. Thus, even for the very simple case of $(1')$, $(2')$, the geometry of the set of the secular points is complicated, this set consisting of surfaces and curves separated by C_1 and by C^*. A heuristical reasoning [4] shows that these curves (all of them belonging to $C \setminus C^*$) are limits for the secular surfaces of $(10')$. A few numerical results [4] suggest *Conjecture. The curve* $T = T^*$, $Q = const$ *and the surface* $C \setminus C^*$, *except for the secular curves situated on* $C \setminus C^*$ *are false secular manifolds. When they exist, the secular curves situated on* $C \setminus C^*$ *are limit sets of the secular surfaces defined by* $(10')$ *and have some extremality properties.*

References

[1] Georgescu, A., Cardos,V., Neutral stability cueves for a thermal convection problem. Acta Mechanica, **37**, (1980), 165-168.

[2] Georgescu, A., Characteristic equations for some eigenvalue problems in hydrodynamic stability theory. Mathematica **24** (47), 1-2 (1982), 31-41.

[3] Georgescu, A., Bifurcation (catastrophe) surfaces for a problem in hydromagnetic stability, Rev. Roum. Math. Pures et Appl. **27**, 3 (1982), 335-337.

[4] Georgescu, A., Oprea, I., Oprea, C., Bifurcation manifolds in a multiparametric eigenvalue problem from linear hydromagnetic stability theory, Mathematica **18**, 2 (1989), 123-138.

[5] Oprea, I., Georgescu, A., Bifurcating neutral surfaces in the problem of the inibition of convection by a magnetic field, St. Cerc. Mec. Apl. **42**, 3 (1989), 263-278.

[6] Chandrasekhar, S., The stability of viscous flow between rotating cylinders in the presence of a magnetic field, Proc. Roy. Soc. **A 216** (1953), 293-309.

[7] Georgescu, A., Oprea, I., Pasca, D., Methods to determine the neutral curve in the Bénard stability, St. Cerc. Mec. Apl. **52**, 4 (1993), 267-276. (Romanian)

[8] Georgescu, A., Pasca, D., Gradinaru, S., Gavrilescu, M., Bifurcation manifolds in multiparametric linear stability of continua, ZAMM, **73**, 7/8, (1993) T767-T769.

[9] Georgescu, A., Pasca, D., Complements to the determination of the neutral curve in the Bénard convection, Proceedings of the Conference preparing the Congress of the Romanian Mathematiciens All Over the World, vol. II, Univ. of Bucharest, 223-224, 1993.

[10] Georgescu, A., Nica, D., Excepted cases in the convection subject to the action of the dielectrophoretic forces, Proceedings of the Conference preparing the Congress of the Romanian Mathematiciens All Over the World, vol. II, Univ. of Bucharest (1993),218-219.

[11] Collatz, L., Remark on bifurcation problems with several parameters, LNM **846**, Springer, Berlin, 82-87, 1981.

[12] Georgescu, A., Variational formulation of some nonselfadjoint problems occurring in Bénard instability theory, Preprint Series in Math., **35**/1977, Inst. of Math., Bucharest.

CONTINUOUS DYNAMICAL SYSTEM ASSOCIATED WITH COMPETING SPECIES

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Abstract A mathematical model associated with two competing species is analyzed. Dynamics and bifurcation results for this model, existing in the literature or obtained by us, are presented too.

1. THE MODEL OF TWO COMPETING SPECIES

Two similar species of animals compete with each other in an environment where their common food supply is limited. The system of ordinary differential equations (SODE) associated with the model who represents two species being in competition is

$$\left\{\begin{array}{lcl}\dot{x} & = & r_1 x(1 - x/K_1 - p_{12}y/K_1), \\ \dot{y} & = & r_2 y(1 - y/K_2 - p_{21}x/K_2),\end{array}\right. \tag{1}$$

where x, y represent the two species, r_1, r_2 - the growth rates of these species, K_1, K_2 -the carryings capacity of every species, $p_{12} > 0$ - the action of the second population and $p_{21} > 0$ - the action of the first population. In this study we consider r_1, r_2, K_1 and K_2 as fixed, such that in (1) only two parameters occur: p_{12} and p_{21}.

Due to physical reasons, the phase space must be the first quadrant (without axes of coordinates). However, for mathematical reasons we consider also these half-axes.

2. THE EQUILIBRIUM POINTS

The two populations are at an equilibrium point when they coexist without affecting one other. The system (1) has the following equilibrium points:

- (0,0) (both species become extinct);
- $(K_1, 0)$ (species x survives and species y becomes extinct);
- $(0, K_2)$ (species y survives and species x becomes extinct);
- $\left(\dfrac{K_2 p_{12} - K_1}{p_{12}p_{21} - 1}, \dfrac{K_1 p_{21} - K_2}{p_{12}p_{21} - 1}\right)$ (the two species coexist) if $p_{12}p_{21} \neq 1$.

If $p_{12}p_{21} = 1$ then only the equilibria $(0, 0)$, $(K_1, 0)$ and $(0, K_2)$ exist.

For $p_{12}p_{21} \neq 1$, $\dfrac{K_2p_{12}-K_1}{p_{12}p_{21}-1}$ and $\dfrac{K_1p_{21}-K_2}{p_{12}p_{21}-1}$ must be greater than zero. Therefore, for $\dfrac{K_2p_{12}-K_1}{p_{12}p_{21}-1} > 0$ we must have either

$$\begin{cases} K_2p_{12}-K_1 > 0, \\ p_{12}p_{21}-1 > 0, \end{cases} \Leftrightarrow \begin{cases} K_1/p_{12} < K_2, \\ p_{12}p_{21} > 1, \end{cases} \tag{2}$$

or

$$\begin{cases} K_2p_{12}-K_1 < 0, \\ p_{12}p_{21}-1 < 0, \end{cases} \Leftrightarrow \begin{cases} K_1/p_{12} > K_2, \\ p_{12}p_{21} < 1, \end{cases} \tag{3}$$

while for $\dfrac{K_1p_{21}-K_2}{p_{12}p_{21}-1}$ we must have either

$$\begin{cases} K_1p_{21}-K_2 > 0, \\ p_{12}p_{21}-1 > 0, \end{cases} \Leftrightarrow \begin{cases} K_2/p_{21} < K_1, \\ p_{12}p_{21} > 1, \end{cases} \tag{4}$$

or

$$\begin{cases} K_1p_{21}-K_2 < 0, \\ p_{12}p_{21}-1 < 0, \end{cases} \Leftrightarrow \begin{cases} K_2/p_{21} > K_1, \\ p_{12}p_{21} < 1. \end{cases} \tag{5}$$

The number and the multiplicity of the equilibrium points depend on the values of the parameters $p_{12},\ p_{21}$.

The attractivity of the equilibrium point (x^*, y^*) is determined by the eigenvalues of the matrix

$$\left.\begin{pmatrix} r_1(1-2x/K_1-p_{12}y/K_1) & -r_1p_{12}x/K_1 \\ -r_2p_{21}y/K_2 & r_2(1-2y/K_2-p_{21}x/K_2) \end{pmatrix}\right|_{(x^*,y^*)}. \tag{6}$$

Let us analyze the attractivity and the nature of the equilibrium points for various values of the parameters p_{12} and p_{21}.

For $\mathbf{p_{12} = p_{21} = 0}$ (1) becomes

$$\begin{cases} \dot{x} &= r_1x(1-x/K_1), \\ \dot{y} &= r_2y(1-y/K_2). \end{cases} \tag{7}$$

and the matrix (6) reads

$$\left.\begin{pmatrix} r_1(1-2x/K_1) & 0 \\ 0 & r_2(1-2y/K_2) \end{pmatrix}\right|_{(x^*,y^*)}. \tag{8}$$

In this case there are four equilibrium points: (0,0), $(K_1, 0)$, $(0, K_1)$ and (K_1, K_2). The equilibrium point (0,0) is a repulsive node since the eigenvalues of (6) are $\lambda_1 = r_1 > 0,\ \lambda_2 = r_2 > 0$. The equilibrium point $(K_1, 0)$ is a saddle, because the eigenvalues of (8) are $\lambda_1 = -r_1 < 0,\ \lambda_2 = r_2 > 0$. For $(0, K_2)$ the eigenvalues of (8) are $\lambda_1 = r_1 > 0,\ \lambda_2 = -r_2 < 0$, so $(0, K_2)$ is a saddle too. For (K_1, K_2) the eigenvalues of (8) are $\lambda_1 = -r_1 < 0,\ \lambda_2 = -r_2 < 0$, so, this point is an attractive node.

For $\mathbf{p_{12} \neq K_1/K_2},\ \mathbf{p_{21} \neq K_2/K_1}$ there are three or four equilibrium points. (0,0) is a repulsive node since the eigenvalues of (6) are $\lambda_1 = r_1 >$

0, $\lambda_2 = r_2 > 0$. For the equilibrium point $(K_1, 0)$ the eigenvalues of (6) are $\lambda_1 = -r_1 < 0$, $\lambda_2 = r_2(1 - p_{21}K_1/K_2)$. Therefore $(K_1, 0)$ is an attractive node if $\lambda_2 < 0$, i.e. $p_{21} > K_2/K_1$, and a saddle if $\lambda_2 > 0$, i.e. $p_{21} < K_2/K_1$. Similarly, for the equilibrium point $(0, K_2)$ the eigenvalues of (6) are $\lambda_1 = -r_2 < 0$, $\lambda_2 = r_1(1 - p_{12}K_2/K_1)$. Therefore $(0, K_2)$ is an attractive node if $\lambda_2 < 0$, i.e. $p_{12} > K_1/K_2$ and a saddle if $\lambda_2 > 0$, i.e. $p_{12} < K_1/K_2$.

Let us see what happens with the other equilibrium points. The eigenvalues of (6) for the point $\left(\dfrac{K_2p_{12} - K_1}{p_{12}p_{21} - 1}, \dfrac{K_1p_{21} - K_2}{p_{12}p_{21} - 1}\right)$ are

$$\begin{aligned}\lambda_{1,2} \quad = \quad & \frac{1}{2(p_{12}p_{21} - 1)} \{-[r_1(1 - p_{12}K_2/K_1) + r_2(1 - p_{21}K_1/K_2)] \pm \\ & \big\{[r_1(1 - p_{12}K_2/K_1) + r_2(1 - p_{21}K_1/K_2)]^2 - \\ & -4r_1r_2(1 - p_{12}K_2/K_1)(1 - p_{21}K_1/K_2)(1 - p_{12}p_{21})\}^{1/2}\big\}\end{aligned}$$

and they are the roots of the characteristic equation

$$\lambda^2 - \mathrm{tr}\ \mathbf{A}\ \lambda + \det \mathbf{A} = 0, \tag{9}$$

where

$$\mathbf{A} = \frac{1}{p_{12}p_{21} - 1} \begin{pmatrix} r_1 (1 - p_{12}K_2/K_1) & r_1p_{12} (1 - p_{12}K_2/K_1) \\ r_2p_{21} (1 - p_{21}K_1/K_2) & r_2 (1 - p_{21}K_1/K_2) \end{pmatrix}. \tag{10}$$

We have

$$\det \mathbf{A} = \frac{r_1r_2}{p_{12}p_{21} - 1} (1 - p_{12}K_2/K_1) (1 - p_{21}K_1/K_2). \tag{11}$$

If $\det \mathbf{A} < 0$, then λ_1, λ_2 have different signs, while if $\det \mathbf{A} > 0$ then λ_1, λ_2 have the same signs. Hence, in order to determine the sign of $\det \mathbf{A}$ we have two possibilities.

i) $p_{12}p_{21} - 1 > 0$. From (2) and (4) we have that $K_1/p_{12} < K_2$ and $K_2/p_{21} < K_1$. It follows that $1 - p_{12}K_2/K_1 < 0$, $1 - p_{21}K_1/K_2 < 0$, so $\det \mathbf{A} < 0$. Thus λ_1, λ_2 have different signs , so the fourth equilibrium point is a saddle.

ii) $p_{12}p_{21} - 1 < 0$. From (3) and (5) we have that $K_1/p_{12} > K_2$ and $K_2/p_{21} > K_1$. It follows that $1 - p_{12}K_2/K_1 > 0$, $1 - p_{21}K_1/K_2 > 0$, therefore $\det \mathbf{A} > 0$. Thus λ_1, λ_2 have the same signs , so the fourth equilibrium point is a node. On the other hand we have

$$\mathrm{tr}\mathbf{A} = \frac{1}{p_{12}p_{21} - 1} [r_1(1 - p_{12}K_2/K_1) + r_2(1 - p_{21}K_1/K_2)] < 0; \tag{12}$$

it follows that $\lambda_1, \lambda_2 < 0$, thus this node is attractive.

If only one parameter p_{12} or p_{21} is zero, the nature of the equilibrium point is the same like in the previous case.

For $\mathbf{p_{21} = K_2/K_1}$, $\mathbf{p_{12} \neq K_1/K_2}$, (1) becomes

$$\begin{cases} \dot{x} &= r_1x (1 - x/K_1 - p_{12}y/K_1), \\ \dot{y} &= r_2y (1 - y/K_2 - x/K_1). \end{cases} \tag{13}$$

The equilibrium points are (0,0), $(K_1, 0)$, which is a double point, and $(0, K_2)$. The equilibrium (0,0) is a repulsive node, $(K_1, 0)$ is a saddle-node and $(0, K_2)$ is a saddle if $p_{12} < K_1/K_2$ and an attractive node if $p_{12} > K_1/K_2$.

For $\mathbf{p_{12} = K_1/K_2,\ p_{21} \neq K_2/K_1}$, (1) becomes

$$\begin{cases} \dot{x} &= r_1 x\,(1 - x/K_1 - y/K_2)\,, \\ \dot{y} &= r_2 y\,(1 - y/K_2 - p_{21}x/K_2)\,. \end{cases} \tag{14}$$

The equilibrium points are (0,0), $(K_1, 0)$ and $(0, K_2)$, which is a double point. The equilibrium (0,0) is a repulsive node, $(K_1, 0)$ is a saddle if $p_{21} < K_2/K_1$ and it is an attractive node if $p_{21} > K_2/K_1$. The equilibrium point $(0, K_2)$ is a saddle-node.

For $\mathbf{p_{12} = K_1/K_2,\ p_{21} = K_2/K_1}$ we have $p_{12}p_{21} = 1$. In this case (1) becomes

$$\begin{cases} \dot{x} &= r_1 x\,(1 - x/K_1 - y/K_2)\,, \\ \dot{y} &= r_2 y\,(1 - y/K_2 - x/K_1)\,. \end{cases} \tag{15}$$

and there are the following equilibrium points (0,0), $(K_1, 0)$, $(0, K_2)$ and the set of the points situated on the straight line $1 - x/K_1 - y/K_2 = 0$. The point (0,0) is a repulsive node, while $(K_1, 0)$, $(0, K_2)$ are saddle-nodes. The points situated on the straight line $1 - x/K_1 - y/K_2 = 0$ have the form $(\alpha, K_2(1 - \alpha/K_1))$ with $\alpha \in [0, K_1]$. In this case, at $(\alpha, K_2(1 - \alpha/K_1))$

$$\mathbf{A} = \begin{pmatrix} -r_1\alpha/K_1 & -r_1\alpha/K_2 \\ -r_2K_2(1-\alpha/K_1)/K_1 & -r_2(1-\alpha/K_1) \end{pmatrix}. \tag{16}$$

We have

$$tr\ \mathbf{A} = -r_1\alpha/K_1 - r_2(1-\alpha/K_1) < 0 \qquad \text{and} \qquad \det \mathbf{A} = 0.$$

Thus the equilibrium point $(\alpha, K_2(1-\alpha/K_1))$ is a saddle-node (degenerated).

3. THE PARAMETRIC PORTRAIT AND THE PHASE PORTRAIT

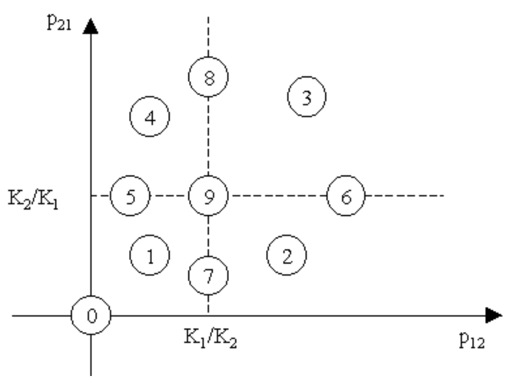

Fig. 1.

The discussion in Section 2 shows that in the parameter space there are ten regions corresponding to topologically equivalent dynamical systems. In fig. 1. we represent this parametric portrait.

In fig. 2. it is represented the phase portrait corresponding to each of the ten cases. This shows that,in spite of their unrealistic significance for the population dynamics, the equilibria (0,0), $(K_1, 0)$, $(0, K_2)$ heavily contribute to the phase portrait and to the dynamic bifurcation diagram. On the other hand, the topologic type of the nonhyperbolic equilibria was not investigated. All these will be studied elsewhere.

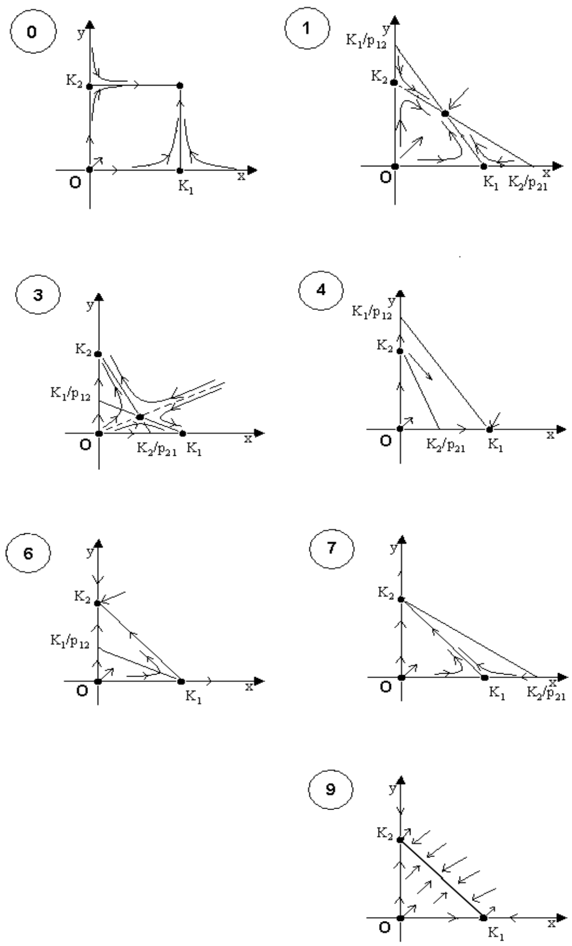

Fig.2.

References

[1] Arrowsmith, D.K., Place, C.M., *Ordinary differential equations*, Chapman and Hall, London, 1982.

[2] Giurgiţeanu, N., *Dinamică economică şi biologică computaţională*, Europa, Craiova, 1997.

[3] Murray, J.D., *Mathematical biology, 2nd corr. ed.* Springer, Berlin, 1993 (first ed. 1989).

SETS GOVERNING THE PHASE PORTRAIT (APPROXIMATION OF THE ASYMPTOTIC DYNAMICS)

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Abstract The most important sets governing the phase-portrait (ω-limit sets, unstable manifolds, centre manifolds, inertial manifolds, inertial sets and approximate inertial manifolds) are described.

Examples are worked out and the relationships (of strict inclusion or coincidence) between different governing sets are shown.

1. ω -LIMIT SETS. ATTRACTORS

Let M be a set and let (M, ϕ) be a continuous dynamical system. We say that $p \in M$ is an *ω-limit point of* $u \in M$ if there exists $0 < t_1 < t_2 < \ldots < t_n \to \infty$ such that $\lim\limits_{n\to\infty} \phi_{t_n}(u) = p$ [4]. Similarly, $q \in M$ is called an *α-limit point of* $u \in M$ if there exists $0 > t_1 > t_2 > \ldots > t_n \to -\infty$ such that $\lim\limits_{n\to\infty} \phi_{t_n}(u) = q$ [4].

The set of all ω-limit points of u is called *ω-limit set of* u and we write $\omega(u)$ [4]. The set of all α-limit points of u is called *α-limit set of* u and we write $\alpha(u)$ [4].

Further, we show that the ω-limit sets can be defined not only for a point $u \in M$, but also for a set $\mathcal{A} \subset M$.

These sets are invariant through the dynamics ϕ.

Let $M = H$ be a metric space and assume that the time t runs over $\mathbb{R}^+$. Thus, the forward evolution of the points of phase space is described by a family of operators $S(t)$, $t \geq 0$, $S(t) : H \to H$ enjoying the properties

$$\begin{aligned} S(t+s) &= S(t) \cdot S(s), \\ S(0) &= I \,(\text{identity in H}). \end{aligned} \tag{1}$$

and being associated with a semidynamical system (H, S).

Assume that

$$S(t) \text{ is a continuous (nonlinear) operator from } H \text{ into itself, for all } t \geq 0. \tag{2}$$

For $u \in H$ (or for $\mathcal{A} \subset H$) the *ω-limit set of u* (or $\mathcal{A}$) is defined [11] as

$$\omega(u) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u}\,, \quad \omega(\mathcal{A}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{A}}\,.$$

Notice that $\varphi \in \omega(\mathcal{A})$ if and only if there exists a sequence of elements $\varphi_n \in \mathcal{A}$ and a sequence $t_n \to \infty$ such that

$$S(t_n)\varphi_n \to \varphi \text{ as } n \to \infty. \tag{3}$$

Let $\mathcal{B}$ be a subset of H and let $\mathcal{U}$ be an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is *absorbing* in $\mathcal{U}$ if the orbit of any bounded subset of $\mathcal{U}$ enters into $\mathcal{B}$ after a certain time.

Let A, B be two sets from the phase space M. The invariant set A is an *attractive set* for B if the distance between A and $\phi_t(B)(= \bigcup_{u \in B} \phi_t(u))$ tends to zero for $t \to \infty$ [6].

An *attractor* is an invariant, closed, attractive set for an entire neighborhood [6].

A *global attractor* is the union of all the attractors of the system.

The attractor is included in the absorbing domain and it is also included in the ω -limit set.

For a long time it was understood that the attractors are the most important sets of the phase space. Lately, in the absence of attractors, it was found that some other invariant sets may be of primary importance, for instance, the unstable manifolds, central manifolds etc.

Hypothesis:

$$\text{For } t \text{ large, the operators } S(t) \text{ are uniformly compact.} \tag{4}$$

Alternatively, if H is a Banach space, we may assume that

$$\begin{aligned} &S(t) \text{ is the perturbation of an operator satisfying (4) by} \\ &\qquad\qquad \text{an operator which converges to 0 as } t \to \infty. \end{aligned} \tag{5}$$

The following theorem shows that, with a few hypotheses, the ω-limit set is the attractor.

Theorem 1.1. *[11] Assume that H is a metric space and that the operators $S(t)$ are given and satisfy (1), (2) and either (4) or (5). We also suppose that there exists an open set $\mathcal{U}$ and a bounded set $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the ω-limit set of $\mathcal{B}$ is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$.*

Example 1.1. Consider the two-dimensional dynamical system generated by the Cauchy problem for

$$\begin{cases} \dot{x} = \lambda_1 x, \\ \dot{y} = \lambda_2 y, \end{cases} \tag{6}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}^*$, $|\lambda_2| > |\lambda_1|$ parameters of the same sign.

The solution of this problem reads $x(t) = x_0 e^{\lambda_1 t}$, $y(t) = y_0 e^{\lambda_2 t}$, for all $(x_0, y_0) \in \mathbb{R}^2$. If $\lambda_1, \lambda_2 < 0$, we find that $(0, 0)$ is an ω-limit point for all points of $\mathbb{R}^2$ (fig. 1a)). The sequence $t_n = n$ is strictly ascending and $t_n \to \infty$. $\phi_{t_n}(x_0, y_0) = \phi_n(x_0, y_0) = (x_0 e^{\lambda_1 n}, y_0 e^{\lambda_2 n})$ has the limit $(0, 0)$ for $n \to \infty$, for all (x_0, y_0). If $\lambda_1, \lambda_2 > 0$, $(0, 0)$ is an α-limit point for all points of the phase plane, $\mathbb{R}^2$ (fig. 1b)).

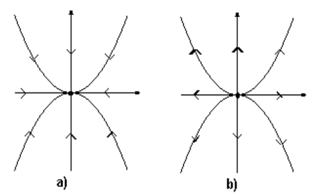

Fig. 1.

The two-dimensional dynamical systems have as ω-limit sets or α-limit sets only the fixed points, homoclinic orbits or limit cycles [4]. In $\mathbb{R}^3$ there exist ω-limit sets which are invariant tori, so the dynamics is not necessary tending to a steady state or a periodic dynamics, but to a quasiperiodic dynamics [4].

2. UNSTABLE MANIFOLDS

Let $(\phi_t)_{t \in \mathbb{R}}$ be the dynamical system generated by the Cauchy problem for the equation $\dot{x} = f(x)$, $x \in M$. Let $x_0 \in M$ be an equilibrium point and let $U \subset M$ be a neighborhood of x_0. By definition the *local unstable manifold* of x_0 [4] is

$$W^u_{loc}(x_0) = \{x \in U \mid \phi_t(x) \in U, \, \forall t \leq 0 \text{ and } \lim_{t \to -\infty} \phi_t(x) = x_0\} \tag{7}$$

while *the unstable manifold* [4] is

$$W^u(x_0) = \bigcup_{t \geq 0} \phi_t(W^u_{loc}(x_0)). \tag{8}$$

$W^u_{loc}(x_0)$ is a differentiable manifold, while, generally, $W^u(x_0)$ is not. However, $W^u(x_0)$ is an invariant set.

Let $\gamma \subset M$ be a limit cycle described by $x(t + T) = x(t)$ and let $U \subset M$ be a neighborhood of γ. The local unstable manifold of the limit cycle is the set

$$W^u_{loc}(\gamma) = \{x \in U \mid \phi_t(x) \in U, \, \forall t \leq 0 \text{ and } \lim_{t \to -\infty} |\phi_t(x) - \gamma| = 0\}. \tag{9}$$

Assume that $M = E$ is a Banach space. Consider a semigroup $\{S(t)\}_{t\geq 0}$ which satisfies the properties (1) and suppose that the mapping

$$(t, u_0) \to S(t)u_0 \text{ from } \mathbb{R}_+ \times E \text{ into } E \text{ is continuous.} \tag{10}$$

Let $X \subset E$ be a subset of E (not necessarily a limit cycle).

The *unstable set* of X is the (possibly empty) set of points u which belong to a complete orbit $\{u(t),\, t \in \mathbb{R}\}$ and such that $d(u(t), X) \to 0$ as $t \to -\infty$ [11].

If X is invariant then $W^u(X)$ is invariant too.

Theorem 2.1. *[11] Let E be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a semigroup of operators satisfying (1) and (10) which possessing a global attractor $\mathcal{A}$. Let $X \subset E$ be a compact set invariant through $S(t)$. Then $W^u(X) \subset \mathcal{A}$. For $X = \mathcal{A}$, $W^u(\mathcal{A}) = \mathcal{A}$.*

Again we remark the importance, from the point of view of asymptotic property of attractivity, of the unstable manifold, and not of the stable manifold as expected.

Example 2.1. Consider the system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y. \end{cases} \tag{11}$$

The unique equilibrium point is $(0, 0)$ and it is a saddle point. For every (x_0, y_0) initial condition, the solution of the Cauchy problem is $x(t) = x_0 e^t, \quad y(t) = y_0 e^{-t}$, i.e. $\phi_t(x_0, y_0) = (x_0 e^t, y_0 e^{-t})$.

Let $U \subset \mathbb{R}^2$ be a neighborhood of $(0, 0)$. We choose U to be the disk of radius 1. Assume first that the initial points $(x_0, y_0) \in U \cap 0x$. Since they satisfy $|x_0| < 1, y_0 = 0$, we have $|x_0 e^t| < 1$, $y_0 e^{-t} = 0$, $\forall\, t \leq 0$, therefore $\phi_t(x_0, y_0) \in U \cap Ox$ too. In addition, $\phi_t(x_0, 0) = (x_0 e^t, 0) \to (0, 0)$ as $t \to -\infty$. Hence, the points of $U \cap Ox$ belong to the unstable manifold of $(0, 0)$.

Let now $(x_0, y_0) \in U$, $y_0 \neq 0$, $|y_0 e^{-t}| \to +\infty$ for $t \to -\infty$, i.e. $\phi_t(x_0, y_0)$ is not in U, for $t < 0$ sufficiently small. Therefore this (x_0, y_0) does not belong to the unstable manifold of $(0, 0)$.

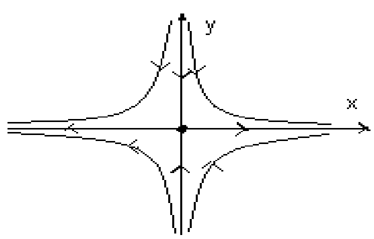

Fig. 2a.

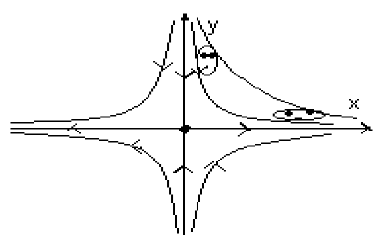

Fig. 2b.

Hence $W^u_{loc}((0,0)) = [-1,1] \times \{0\}$ and $W^u((0,0)) = \bigcup_{t \geq 0} \phi_t(W^u_{loc}((0,0))) = \mathbb{R} \times \{0\}$. Remark that all trajectories of the dynamical system generated by (11) approach exponentially to the trajectory $x(t) = x_0 e^t$, $y(t) = 0$ from W^u.

$$d((x_0 e^t, y_0 e^{-t}), (x_0 e^t, 0)) = \sqrt{(y_0 e^{-t})^2} = y_0 e^{-t} \to 0.$$

Therefore, the distance between the current points of these two trajectories is equal to the distance between their ordinates in the stable manifold direction and, of course, this tends to zero. It is interesting that this decay to zero is exponential.

As a consequence, as $t \to \infty$, a figure in the phase plane will be contracted in the Oy direction (i.e. of the stable manifold) and magnified in the W^u-direction (fig.2b).

The approach to the unstable manifold is more important than the approach to the stable manifold (because the last one is contracting) and it leads to the notion of axiom A attractor.

The action of the dynamics generated by (11) on the phase space is contractant in one direction and expanding in another one. In the case of finite dynamical systems, if the associated linearized operator around the hyperbolic equilibrium has eigenvalues with n_1 positive real parts and n_2 negative real parts, the effect of the dynamics is the "flattening " in the n_2 directions and "magnification" in the n_1 directions. In the infinite dimensional case the presence of inertial manifolds is associated with "contraction" in an infinity of directions and "magnification" in a finite number of directions. In addition, similarly to the case of the two-dimensional saddle, on the centre manifold there are trajectories towards which the phase space trajectories approach exponentially. In this sense, the asymptotic dynamics (i.e. for $t \to +\infty$) is more similar to the dynamics on the centre manifold. In other words, the study of the dynamics generated by, say, a partial differential equation is better and better approximated by the dynamics on the centre manifold.

3. CENTRE MANIFOLD

In order to study the bifurcation for a dynamical system, one preliminary step is to simplify the problem as much as possible without changing the topological properties of the dynamics of the original system. The linearization principles of Hartman-Grobman type provide conditions under which the stability of an equilibrium point, as well as the behaviour of the solutions around the equilibrium of a nonlinear system are described (up to a homeomorphism) by those of the zero equilibrium of the associated linearized system $\dot{x} = \mathbf{A}x$.

Unfortunatelly, these priciples are stated only for hyperbolic equilibrium points. When the equilibrium point is not hyperbolic, and, therefore, the Hartman-Grobman theorem does not apply, the reduction theory of the centre manifold is applicable: the dimension of the problem can be reduced, by using a convenient decomposition of the phase space into a direct sum of the stable, unstable and center manifold. Further, we discuss only the finite-dimensional theory of centre manifolds for equilibrium points. The infinite-dimensional case is dealt with in [6].

Consider the particular system

$$\begin{cases} \dot{x} = Ax + f(x, y), \\ \dot{y} = By + g(x, y), \end{cases} \tag{12}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A and B are constant matrices such that the eigenvalues of A have zero real parts, the eigenvalues of B have negative real parts and f and g are functions of class C^2, with $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$ (here f' is the Jacobian matrix of f). It is understood that $(0,0)$ is a nonhyperbolic equilibrium of (12).

If $y = h(x)$ is an invariant manifold for (12) and h is smooth, then it is called a *centre manifold* if $h(0) = 0$, $h'(0) = 0$ [1].

Theorem 3.1. *[1](existence of centre manifolds) Equation (12) has a local centre manifold $y = h(x)$, $|x| < \delta$, where h is of class C^2.*

The dynamics on the centre manifold is governed by the (reduced) n-dimensional system

$$\dot{u} = Au + f(u, h(u)) \tag{13}$$

The following theorem relates the asymptotic behaviour of small solutions of (12), i.e. near the equilibrium $(0,0)$, to solutions of (13).

Theorem 3.2. *[1](reduction principle) Suppose that the zero solution of (13) is stable (asymptotically stable)(unstable). Then the zero solution of (12) is stable (asymptotically stable)(unstable).*

Suppose that the zero solution of (12) is stable. Let $(x(t), y(t))$ be a solution of (12) with $(x(0), y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (13) such that, as $t \to \infty$, $x(t) = u(t) + O(e^{-\gamma t})$, $y(t) = h(u(t)) + O(e^{-\gamma t})$, where $\gamma > 0$ is a constant depending only on B.

In other words, just as remarked in Example 2.1, with a trajectory of the phase space we associate a trajectory on the centre manifold exponentially approaching one to each other as $t \to \infty$.

For functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin let us define

$$(M\phi)(x) = \phi'(x)[Ax + f(x, \phi(x))] - B\phi(x) - g(x, \phi(x)).$$

Theorem 3.3. *[1](approximation of the centre manifold) Suppose that $\phi(0) = 0$, $\phi'(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$ where $q > 1$. Then as $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.*

4. INERTIAL MANIFOLDS

So far we already saw two invariant manifolds on which the dynamics approximates exponentially, asymptotically for $t \to \infty$, the given dynamics. These manifolds were: some unstable manifolds and all centre manifolds.

Now, for an infinite dynamical system, we introduce another invariant manifold, namely the inertial manifold, which is of finite dimension but it describes the large-time behaviour of dynamical systems. The concept of inertial manifold was introduced by Peter Constantin and Ciprian-Ilie Foiaş [2], [3]. It permits the reduction of the infinite-dimensional case to the finite-dimensional one. In addition, the phase space trajectories tend exponentially to the corresponding trajectories situated on the inertial manifold. Moreover, this manifold is Lipschitz.

Consider a Cauchy problem $u(0) = u_0$ for the evolution equation

$$u' = F(u), \tag{14}$$

in a finite or infinite-dimensional Hilbert space H, with which we can associate the semigroup $\{S(t)\}_{t\geq 0}$, where $S(t) : u_0 \to u(t)$.

An *inertial manifold* of this system is a finite-dimensional Lipschitz manifold $\mathcal{M}$, positively invariant and which attracts exponentially all the orbits of (14) [11].

Let A be a given linear closed unbounded positive self-adjoint operator in H, with the domain $D(A) \subset H$. Assume that A^{-1} is compact in H. Let us see in which conditions the global attractor of a Cauchy problem for the p.d.e.

$$\frac{du}{dt} + Au + R(u) = 0, \tag{15}$$

is included in a smooth, finite-dimensional, manifold of solutions, $\mathcal{M}$. If $\mathcal{M}$ would be positively invariant (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$), the equation could be restricted to this manifold if the asymptotic behaviour is intended to be obtained. There can be constructed examples with the ratio of attraction arbitrary large. An exponentially attractive manifold would lead to a more complicated asymptotic behaviuor and would make the description of the dynamics on the manifold more relevant for the dynamics in the phase space. In order to satisfy these, there has been introduced the concept of inertial manifold. All inertial manifolds obtained so far are the graphs of some function in a finite-dimensional subspace of H.

Let A^s, $s \in \mathbb{R}$, be the powers of A, defined on $D(A^s)$, and assume that the eigenvalues of A are λ_j and the corresponding eigenvectors w_j of A form an orthonormal basis of H. Define a projector $P_n u = \sum\limits_{i=1}^{n} (u, w_i) w_i$ and let Q_n be its orthogonal complement, i.e. $Q_n = I - P_n$, $Q_n u = \sum\limits_{i=n+1}^{\infty} (u, w_i) w_i$. There exists a function $\phi : P_nH \to Q_nH$ such that on the inertial manifold $\mathcal{M}$ to have $q \equiv Q_nu = \phi(p)$, where $p \equiv P_nu$.

$\mathcal{M}$ can be defined as

$$\mathcal{M} = \{p + \phi(p),\, p \in P_nH\}.$$

Since on the inertial manifold $q = \phi(p)$, restricting (15) to $\mathcal{M}$ we obtain a finite-dimensional e.d.o.

$$\frac{dp}{dt} + Ap + P_nR(p + \phi(p)) = 0, \tag{16}$$

which is called *the inertial form*.

We can state that the dynamics on the inertial manifold is finite-dimensional because every trajectory from $\mathcal{M}$ is given by $u(t) = p(t) + \phi(p(t))$ with $p(t)$ solution of (16).

The theory of inertial manifolds and the theory of centre manifold are particular cases of the reduction principle, which, for a system of equations

$$\dot{x} = F(x, y), \quad \dot{y} = G(x, y),$$

where $F : P \times Q \to P$, $G : P \times Q \to Q$, $F, G \in C^1$ and P, Q are Banach spaces becomes as follows. Suppose that the surface M, which is the graph of φ, is an invariant and exponentially stable set for this system and $\varphi : P \to Q$ is a Lipschitz continuous function. Then, for large times, the dynamics of the system is completely described by the solutions of the reduced system

$$\dot{x} = F(x, \varphi(x)),\, x \in P.$$

The theory which approximates the dynamics for large time is called *the approximate dynamics* [6].

As noticed in Example 2.1, $W^u((0,0))$ is a finite-dimensional invariant manifold which attracts exponentially all the orbits of (11), thus $W^u((0,0))$ is also an inertial manifold. Of course, this was a particular case of unstable manifold. In the general case we do not know if it attracts exponentially the phase trajectories, while the centre manifold *is* an inertial manifold.

5. APPROXIMATE INERTIAL MANIFOLDS

The theorem of existence of an inertial manifold [11] ensures the existence of a function ϕ having as a graph just the inertial manifold. Since the proof of this theorem is not constructive, in applications we do not know the form of this function. Moreover, there exists important equations, e.g. the two-dimensional Navier-Stokes equations, for which the existence of the inertial manifold can not be proved by standard methods. Thus, for numerical reasons, it is of interest and can be found approximate inertial manifolds, which can be described explicitly. The class of constructed approximate inertial manifolds is larger than that of exact inertial manifold.

As the inertial manifold is given by the exact asymptotic relation $q = \phi(p)$, an approximate inertial manifold is given by an approximate relation $q \approx \psi(p)$ and it is related to the nonlinear Galerkin-Faedo-Hopf method; basic applications in the theory of Navier-Stokes equations were carried out by Foiaş and Prodi, whence the fundamental contribution of Foiaş to inertial manifolds and approximate inertial manifolds.

Analyse the equation

$$\frac{du}{dt} + Au + R(u) = 0, \tag{17}$$

and consider the finite-dimensional functions u_m, $u_m \in P_m H$, solutions of the truncated (finite-dimensional) equation.

$$\frac{du_m}{dt} + Au_m + P_m R(u_m) = 0. \tag{18}$$

The solutions u_m of (18) converge to the solution u a lui (17), namely they are uniformly convergent on bounded intervals of time and on compact sets from H, as $m \to \infty$. In the Navier-Stokes case the nonstationary problem (18) is reduced to a stationary one and u_m are the eigenvectors of the corresponding linearized operator. This reasoning was frequently considered by Foiaş and Prodi.

The nonlinear Galerkin method extends (18) by including the neglected terms in the approximation of the inertial manifold, such that (18) becomes

$$\frac{du_m}{dt} + Au_m + P_m R(u_m + P_{2m}\psi(u_m)) = 0,$$

where ψ is the function defining the approximate inertial manifold. The term P_{2m} had to be included because otherwise ψ would have produced an infinite-dimensional term. The standard Galerkin approximation stands now for the approximate inertial manifold $\psi = 0$ [9], [11].

6. INERTIAL SETS

Inertial sets, also called *exponential attractors*, are sets somehow intermediate between the attractors and inertial manifolds. All three objects describe the behaviour for $t \to \infty$ of semidynamical systems. The global attractor is the smallest set from the phase space governing the large time dynamics. Usually, the attractor is sensitive to perturbations and attracts the orbits with a small speed. In most applications the global attractor does not exist (e.g. in Example 2.1 no local or global attractor exist). Inertial manifolds, when they exist, are smooth finite-dimensional manifolds which attract all orbits at an exponential rate; they are stable with respect to perturbations.

Exponential attractors attract all orbits at exponential rate, they are stable with respect to perturbations and they exist for a broad class of evolutionary equations.

In general, the attractors are not manifolds and can have a very complex geometric structure. This is why their characterization is complicated too. Thus, let X be a compact connected subset of a Hilbert space H, let S be a Lipschitz continuous map from X into itself and let us denote the Lipschitz constant of S on X by $Lip_X(S) = L$. If S is restricted to X, then it possesses an universal attractor $\mathcal{A}$ which is a compact connected set given by $\mathcal{A} = \bigcap\limits_{n=1}^{\infty} S^n X$.

A compact set M is called an *exponential fractal attractor* for (S, X) if $\mathcal{A} \subseteq M \subseteq X$ and

i) $SM \subset M$;

ii) M has finite fractal dimension, d_F;

iii) there exist positive constants c_0 and c_1 such that

$$h(S^n X, M) \leq c_0 \exp(-c_1 n), \quad \forall\, n \geq 1,$$

where A, B are compact sets and $h(A, B) = \max\limits_{a} \min\limits_{b} |a - b|_H$, is the standard asymmetric Hausdorff pseudodistance.

Exponential attractors share properties of "good" attractors with those of the inertial manifolds.

If a continuous dynamical system has an inertial manifold $\mathcal{M}$, then $\mathcal{M} \bigcap X$ is an exponential fractal attractor.

7. EXAMPLES

7.1. AN INERTIAL MANIFOLD IN THE DYNAMICS OF GAS BUBBLES

In [7] the existence of the inertial manifold for the dynamical system generated by the Cauchy problem for the equation of small oscillations of the radius of a spherical gas bubble is proved.

The small variations of the radius of a spherical gas bubble, surrounded by an incompressible fluid are governed by the following Cauchy problem

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\alpha\varepsilon cos(\omega t), \tag{19}$$

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{20}$$

where β, ω_0, α, ε and ω are constant real parameters.

The phase space trajectories are

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C\cos(\omega t) + D\sin(\omega t), \tag{21}$$

where

$$A = \frac{\lambda_2 x_0 - \dot{x}_0 + \frac{\alpha\lambda_1}{\lambda_1^2+\omega^2}}{2\sqrt{\beta^2-\omega_0^2}}, \qquad B = \frac{\dot{x}_0 - \lambda_1 x_0 - \frac{\alpha\lambda_2}{\lambda_2^2+\omega^2}}{2\sqrt{\beta^2-\omega_0^2}},$$

$$C = -\alpha\frac{\frac{\alpha\lambda_1}{\lambda_1^2+\omega_0^2} - \frac{\lambda_2}{\lambda_2^2+\omega_0^2}}{2\sqrt{\beta^2-\omega_0^2}}, \quad D = \frac{-\alpha\omega(\lambda_1^2-\lambda_2^2)}{2(\lambda_1^2+\omega_0^2)(\lambda_2^2+\omega_0^2)\sqrt{\beta^2-\omega_0^2}},$$

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2-\omega_0^2}. \tag{22}$$

The attractor of the dynamical system associated with (19), (20) is the ellipse

$$\frac{\dot{x}^2}{\omega^2(C^2+D^2)} + \frac{x^2}{C^2+D^2} = 1, \tag{23}$$

from the plane

$$\ddot{x} + \omega^2 x = 0. \tag{24}$$

The related trajectories on this invariant manifold are $x(t) = C\cos(\omega t) + D\sin(\omega t)$. The distance between the phase space trajectories and the corresponding ones on the invariant manifold (24) is

$$\begin{aligned}
\rho(C,I) &= \sqrt{(x_C-x_I)^2 + (\dot{x}_C-\dot{x}_I)^2 + (\ddot{x}_C-\ddot{x}_I)^2} = \\
&= \sqrt{(Ae^{\lambda_1 t}+Be^{\lambda_2 t})^2 + (A\lambda_1 e^{\lambda_1 t}+B\lambda_2 e^{\lambda_2 t})^2 + (A\lambda_1^2 e^{\lambda_1 t}+B\lambda_2^2 e^{\lambda_2 t})^2} = \\
&= \sqrt{A^2e^{2\lambda_1 t}(1+\lambda_1^2+\lambda_1^4) + 2AB(1+\lambda_1\lambda_2+\lambda_1^2\lambda_2^2) + B^2e^{2\lambda_2 t}(1+\lambda_2^2+\lambda_2^4)} \\
&= \sqrt{A^2e^{2\lambda_1 t}(1+\lambda_1^2+\lambda_1^4) + o(e^{2\lambda_1 t})} \approx Ae^{\lambda_1 t}\sqrt{1+\lambda_1^2+\lambda_1^4} = \mathrm{Ord}(e^{\lambda_1 t}),
\end{aligned}$$

and it tends exponentially to zero.

The plane ellipse (23), (24) is proved to be the inertial manifold [7].

7.2. A CENTRE MANIFOLD

In [10] a local analysis of the Cauchy problem for the Gierer-Meinhardt activator-inhibitor normalized system is provided. For a particular case, where only one parameter is variable, a centre manifold is found.

The Gierer-Meinhardt activator-inhibitor model is a Cauchy problem for the system

$$\begin{cases} \dot{a} = \frac{c\rho a^2}{h} - \mu a + \rho\rho_0, \\ \dot{h} = c'\rho a^2 - \gamma h. \end{cases} \tag{25}$$

The normalized form of (25) reads

$$\begin{cases} \frac{dA}{d\tau} = \frac{\rho A^2}{H} - A + \rho\rho_0', \\ \frac{dH}{d\tau} = F(\rho A^2 - H). \end{cases} \tag{26}$$

In order to apply the center manifold theory, in [10] is performed a translation and (25) is linearized around the origin. The obtained system is

$$\begin{cases} A_1' = [\frac{2}{1+\rho\rho_0'} - 1]A_1 - [\frac{1}{\rho}(1+\rho\rho_0')^2]H_1, \\ H_1' = 2\rho F(1+\rho\rho_0')A_1 - FH_1. \end{cases} \tag{27}$$

For an appropriate choice of the parameters, this system is a particular case of the system (12). The system (27) corresponds to the case $m = n = 1$, $A = \frac{2}{1+\rho\rho_0'} - 1$, $B = -F$, $f(A_1, H_1) = -\frac{H_1}{\rho(1+\rho\rho_0')^2}$, $g(A_1, H_1) = 2\rho F(1+\rho\rho_0')A_1$. The condition $f(0,0) = f'(0,0) = g(0,0) = g'(0,0) = 0$ is fulfilled if $\frac{2}{1+\rho\rho_0'} - 1 = 0$, involving $\rho\rho_0' = 1$.

In this case, introducing the notation $A_1 = x, H_1 = y$, the system (27) becomes

$$\begin{cases} x' = [\frac{2}{1+\rho\rho_0'} - 1]x - \frac{y}{\rho}(1+\rho\rho_0')^2, \\ y' = -Fy + 2\rho F(1+\rho\rho_0')x. \end{cases} \tag{28}$$

The equation defining the centre manifold reads [10]

$$h(x) = -2\rho(x^3 - x^2(1+F) + 2Fx).$$

References

[1] Carr, J., *Applications of the centre manifold theory*, Appl. Maths. Sci., **35**, Springer, New York, 1981.

[2] Constantin, P., Foiaş, C., *Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equation*, Comm. Pure Appl. Math. XXXVIII (1985), 1-27.

[3] Constantin, P., Foiaş, C., Temam, R., *Attractors representing turbulent flows*, Memoirs of AMS **314**, AMS, Providence, RI, USA, 1985.

[4] Curtu, R., *Introducere în teoria sistemelor dinamice*, Ed. Infomarket, Braşov, 2000.

[5] Eden, A., Foiaş, C., Nicolaenko, B., Temam, R., *Exponential attractors for dissipative evolution equations*, Research in Applied Mathematics, Paris, 1994.

[6] Georgescu, A., Moroianu, M. , Oprea, I., *Teoria bifurcaţiei, principii şi aplicaţii*, Ed. Universităţii Piteşti, 1999.

[7] Georgescu, A., Nicolescu, B., *On an inertial manifold in the dynamics of gas bubbles*, Rev. Roum. Sci. Techn. - Méc. Appl., **44**, 6, (1999), 629-631.

[8] Hale, J., K., *Asymptotic behaviour of dissipative systems*, Mathematical Surveys and Monographs **25**, AMS, Providence, Rhode Island, 1988.

[9] Ion, A., *Atractori globali si varietăţi inerţiale pentru două probleme din mecanica fluidelor*, Ed. Universităţii Piteşti, 2000.

[10] Ionescu, A., Georgescu, A., *Investigation of the normalized Gierer-Meinhardt system by center manifold theorem*, Buletin Ştiinţific, Univ. din Piteşti, Seria Matematică şi Informatică, **3** (1999), 277-283.

[11] Temam, R., *Infinite-dimensional dynamical systems in mechanis and physics*, Applied Mathematical Sciences **68**, Springer, Berlin, 1997.

SOME EXAMPLES IN MODULES

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Abstract The problem of when the direct product and direct sum of modules are isomorphic is discussed. A series of examples where the product and coproduct of an infinite family of modules are isomorphic is given. One may see that if we require that the isomorphism of $\prod\limits_I$ and $\bigoplus\limits_I$ be a natural (functorial) one, then this can only be done for finite sets I. If this is the case for modules, we show that for comodules over a coalgebra the product and coproduct of a family of comodules can be isomorphic even via the canonical morphism.

2000 MSC: primary 18A30; secondary 20K25, 16A.

Keywords: product, coproduct.

1. INTRODUCTION

Given a family $(M_i)_{i\in I}$ of (left) R modules, we consider the problem of when the direct sum and direct product of this family are isomorphic. It is obvious that if we require that the isomorphism is the canonic isomorphism, then we can easily see that the set must be finite, unless all but a finite part of the modules are 0. Nevertheless we may ask wether the direct product and direct sum of a module can be isomorphic via other isomorphisms. We produce a large class of examples that show that this is possible, so the direct product, by the categorical point of view (classification of modules) is not necessarily very different of the direct sum. We also show that in the case of categories other than categories of modules, namely for comodules over a coalgebra, the direct sum and direct product of a family of objects can be isomorphic even through the canonical morphism.

We can ask the more general question of when the two functors $\prod\limits_{i\in I}$ and $\coprod\limits_{i\in I}$ from the direct product category ${}_R\mathcal{M}^I$ to ${}_R\mathcal{M}$ are isomorphic. It can be shown (not very difficult) that this is only possible for finite sets I.

2. EXAMPLES AND RESULTS

Example 2.1. *Let K be a field and let $(V_i)_{i\in I}$ be vector spaces over K and $V = \prod\limits_{i\in I} V_i$. Then the direct product and direct sum of the family $\{V\} \cup \{V_i \mid i \in I\}$ are isomorphic.*

Proof Denote by A_i a set consisting of a basis for V_i for each $i \in I$ and let A be a basis for V. Then $\bigsqcup\limits_{i\in I} A_i \sqcup A$ is a basis for the coproduct $\bigoplus\limits_{i\in I} V_i \oplus V$ and $A \sqcup A$ is a basis for the direct product $\prod\limits_{i\in I} V_i \times V = V \times V$. But $\mathrm{card}(\bigsqcup\limits_{i\in I} A_i) \leq \mathrm{card}(A)$ because of the natural inclusion of vector spaces $\bigoplus\limits_{i\in I} A_i \hookrightarrow \prod\limits_{i\in I} A_i$ so we have

$$\mathrm{card}(A) \leq \mathrm{card}(\bigsqcup_{i\in I} A_i \sqcup A) \leq \mathrm{card}(A \sqcup A) = \mathrm{card}(A),$$

which shows the desired isomorphism. ■

Example 2.2. *Let A be a simple Artinian ring, that is, $A \simeq M_n(\Delta)$, with Δ a skewfield. If $(N_i)_{i\in I}$ is a family of A modules and $N = \prod\limits_{i\in I} N_i$, then the direct product and direct sum of the family $(N) \cup (N_i)_{i\in I}$ are isomorphic.*

Proof Let S denote a simple module. As any module is semisimple isomorphic to a direct sum of copies of S (A is semisimple with a single type of simple module), in order for two modules $N \simeq S^{(\alpha)}$ and $M \simeq S^{(\beta)}$ to be isomorphic it is necessary and sufficient for the sets α and β be of the same (infinite) cardinal (by Krull-Remak-Schmidt-Azumaya theorem). Let A_i and A be sets such that $N_i \simeq S^{(A_i)}$ $(\forall i)$ and $N \simeq S^{(A)}$. Then $N \oplus \bigoplus\limits_{i\in I} N_i \simeq S^{(A)} \oplus \bigoplus\limits_{i\in I} S^{(A_i)} \simeq S^{(\alpha)}$ with $\alpha = A \sqcup \bigsqcup\limits_{i\in I} A_i$ and $N \times \prod\limits_{i\in I} N_i \simeq N \times N \simeq N^{(\beta)}$ with $\beta = A \sqcup A$. Using an argument similar to the one in Example 2.1 we obtain that α and β are of the same cardinal and so $N \oplus \bigoplus\limits_{i\in I} N_i \simeq N \times \prod\limits_{i\in I} N_i$. ■

A ring is said to have finite representation type if there are only finitely many non-isomorphic indecomposable modules. It is known that any module over an Artinian finite representation type ring is a direct sum of indecomposable modules. For algebras the converse is also true; in fact for an Artin algebra (a finite length algebra over a commutative Artinian ring) the following are equivalent:
-every module is a direct sum of finitely generated indecomposable modules;
-there is only a finite number of nonisomorphic finitely generated indecomposable modules;
-every indecomposable module is finitely generated.
Moreover, these statements are left right symetric, that is, the statement for

left modules is equivalent to the one for right modules. We refer to [8], [1], [5], [3] for these facts. For modules over such algebras we can prove a result that gives a large class of examples of isomorphic direct sum and direct product of modules.

Theorem 2.1. *Let A be a (left) Artinian ring with the property that every module decomposes as a direct sum of indecomposable finitely generated modules (for example, A a finite representation type Artin algebra). Then for every family of (left) A modules $(M_n)_{n\in\mathbb{N}}$ there is a module M such that the direct product and direct sum of the family $(M)\cup(M_n)_{n\in\mathbb{N}}$ are isomorphic.*

Proof Let $\{H_j \mid j\in J\}$ be a set of representatives of indecomposable finitely generated A modules (one can see that actually this is a set, not a class!). For every A module M we have a unique decomposition in the sense of Krull-Schmidt decomposition theorem $M=\bigoplus\limits_k M_k$ where all M_k are isomorphic to one of the H_j's (as the generalized Krull-Remak-Schmidt-Azumaya theorem applies, because the endomorphism rings of finitely generated modules over Artinian rings - which are finite length modules - are local). Denote by $\alpha_j(M)$ the 'exponent' of H_j in M, that is a set (cardinal) such that $M\simeq H_j^{\alpha_j(M)}\oplus\bigoplus\limits_{l\in L} M_l$ and M_l not isomorphic to H_j, $\forall\, l\in L$. Then $M\simeq N$ iff $\alpha_j(M)\sim\alpha_j(N)$, $\forall\, j\in J$. By Krull-Schmidt theorem, $\alpha_j(\bigoplus\limits_{l\in L} M_l)\sim\bigsqcup\limits_{l\in L}\alpha_j(M_l)$. For every family $(M_n)_{n\in\mathbb{N}}$ of A modules, let $K=\{j\in J\mid \mathrm{card}(\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n))<\aleph_0\}$ and $M'=\bigoplus\limits_{j\in K} H_j^{(\mathbb{N})}$. Take $M=\prod\limits_{n\in\mathbb{N}} M_n\times M'$. We have $\mathrm{card}(\alpha_j(M_n))\le\mathrm{card}(\alpha_j(M))$ as each M_n is a direct summand in M (by Krull-Schmidt). Notice that $\alpha_j(M)$ is infinite for all j: if $\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n)$ is infinite then $\mathrm{card}(\alpha_j(M_n))$ is nonzero for infinitely many n's, say for all $n\in P$ and so for every finite set $F\subset P$, $\bigoplus\limits_{n\in F} M_n$ is a direct summand in M, showing that $\mathrm{card}(\alpha_j(M))\ge\mathrm{card}(\alpha_j(\bigoplus\limits_{n\in F} M_n))\ge\mathrm{card}(F)$. If $\alpha_j(\bigoplus\limits_{n\in\mathbb{N}} M_n)$ is finite then M contains $H_j^{(\mathbb{N})}$ from M'. Then

$$\mathrm{card}(\alpha_j(M))\le\mathrm{card}(\alpha_j(\bigoplus_{n\in\mathbb{N}} M_n\oplus M))=\mathrm{card}(\bigsqcup_{n\in\mathbb{N}}\alpha_j(M_n)\sqcup\alpha_j(M))$$

$$\le\mathrm{card}(\bigsqcup_{n\in\mathbb{N}}\alpha_j(M)\sqcup\alpha_j(M))=\mathrm{card}(\alpha_j(M)\times\mathbb{N})=\mathrm{card}(\alpha_j(M)).$$

On the other hand we have

$$\mathrm{card}(\alpha_j(M))\le\mathrm{card}(\alpha_j(\prod_{n\in\mathbb{N}} M_n\times M))\le\mathrm{card}(\alpha_j(M\oplus M))$$

$$= \mathrm{card}(\alpha_j(M) \sqcup \alpha_j(M)) = \mathrm{card}(\alpha_j(M)).$$

■

Thus we obtain $\alpha_j(\bigoplus_{n\in\mathbb{N}} M_n \oplus M) \sim \alpha_j(\prod_{n\in\mathbb{N}} M_n \times M)$, so the theorem is proved. Here we have used some well known facts from set theory, such as $a+a=a$ and $a \times \aleph_0 = \aleph_0$ for every transfinite cardinal a, which can be found, for example, in [11].

We provide now an example from comodule theory, where the canonical isomorphism from the direct sum to the direct product will be an isomorphism, but with infinite index set. We refer to [6] for basic facts about coalgebras and comodules over coalgebras.

Proposition 2.1. *Let $C = \bigoplus_{i\in I} C_i$ be a cosemisimple coalgebra, with C_i simple coalgebras. Then the canonical morphism from $\bigoplus_{i\in I} C_i$ to $\prod^{C}_{i\in I} C_i$ is an isomorphism (here C_i are right C comodules, and $\prod^{C}$ denotes the direct product in the category $\mathcal{M}^C$ of right comodules).*

Proof It is known (and easy to see) that the direct product of the family $(M_l)_{l\in L}$ of comodules is $\mathrm{Rat}^C(\prod_{l\in L} M_l)$, where $\prod$ represents the direct product of left C^* modules. We also have that $\mathrm{Rat}^C(C^*) = \mathrm{Rat}^C(\prod_{i\in I} C_i^*) = \bigoplus_{i\in I} C_i^*$ (this is true in a more general setting, for a left and right semiperfect coalgebra; see [6], Chapter III). If we denote by S_i the left simple comodule type associated to C_i (that is, a simple left comodule included in C_i) and T_i a right simple C_i module, then we have $T_i \simeq S_i^*$ in $\mathcal{M}^{C_i}$ and then also in $\mathcal{M}^C$ (because there is only a single type of simple left(right) comodule). Also $C_i \simeq S_i^n$ in ${}^{C_i}\mathcal{M}$ and $C_i \simeq T_i^m$ in $\mathcal{M}^{C_i}$ with $m = n$ because $T_i \simeq S_i^*$ implies that S_i and T_i have the same (finite!) dimension. We obtain that $C_i^* \simeq (S_i^*)^n \simeq T_i^n \simeq C_i$ in $\mathcal{M}^{C_i}$ and also in $\mathcal{M}^C$. Therefore we obtain

$$\prod^{C}_{i\in I} C_i = \mathrm{Rat}^C(\prod_{i\in I} C_i) \simeq \mathrm{Rat}^C(\prod_{i\in I} C_i^*) = \bigoplus_{i\in I} C_i^* \ (\mathrm{in}\, \mathcal{M}^C) = \bigoplus_{i\in I} C_i$$

and it is easy to see that the isomorphism is the canonical morphism from the direct sum into the direct product. ■

Remark 2.1. *The above example can be generalized to a more general case, namely for comodules over for (left and right) co-Frobenius coalgebras.*

References

[1] M. Auslander, *Large modules over Artin algebras*, Algebra, topology and categoty theory (a colection of papers in honour of Samuel Eilemberg), Academic, New York, 1976.

[2] E. Abe, *Hopf algebras*, Cambridge University Press, 1977.

[3] F. W. Anderson, K. R. Fuller, *Rings and categories of modules*, GTM 13, 2-nd ed., Springer, 1992.

[4] T. Brzeziński, R. Wisbauer, *Corings and comodules*, London Math. Soc. Lect. Notes Ser. **309**, Cambridge University Press, Cambridge, 2003.

[5] C. W. Curtis, I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.

[6] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, *Hopf algebras: an introduction*, Monographs Textbooks in Pure Appl. Math. **235**, Marcel Dekker, New York, 2001.

[7] S. Dăscălescu, C. Năstăsescu, Ş.Raianu, *Algebre Hopf*, Ed. Universităţii Bucureşti, 1998.

[8] C. Faith, *Rings and things and a fine array of twentieth century algebra*, AMS, Mathematical Surveys and Monographs, **65**, 1999.

[9] I. D. Ion, N. Radu, *Algebra*, Ed. Didactică şi Pedagogică, Bucureşti, 1975.

[10] N. Jacobson, *Basic algebra I,II*, 2nd ed., W. H. Freeman, New York, 1989.

[11] C. Năstăsescu, *Introducere in teoria multimilor*, Ed. Didactică şi Pedagogică, 1975.

[12] C. Năstăsescu, *Inele, module, categorii*, Editura Academiei, Bucureşti, 1975.

GEOMETRICAL ASPECTS OF SEMI-DIRECT PRODUCTS OF GROUPS

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Abstract The mixed transformations of geometrical "indexed" figures whose "indexes" that are added to points of the figure are of scalar nature or homogeneous oriented magnitudes are analyzed. In accordance with global nature or local nature of the rule for transformation of "indexes" four types of mixed transformations are obtained. The mixed transformations form subgroups of semi-direct products of various types (left or right) of the group of permutations P of "indexes" or of the Cartesian product of isomorphic copies of the initial group P by the discrete symmetry group of the initial geometrical figure.

MSC: 20H15, 20E22, 51F15, 51M20.

1. The right semi-direct products of the group P by the group G of operators [1] can be derived from P and G by the following steps: 1) to find in G all invariant subgroups H and in $AutP$ all subgroups Φ for which there is the isomorphism φ of factor-group G/H and Φ by the rule $\varphi(gH) =\overrightarrow{\varphi_g}$, where $\overrightarrow{\varphi_g}\,(p) = gpg^{-1}$; 2) to combine pairwise each g of G with each p of P: $\tilde{g} = pg$; 3) to introduce into the set of all these pairs the operation

$$\tilde{g}_i \star \tilde{g}_j = \tilde{g}_k,$$

where $\tilde{g}_i = p_i g_i$, $\tilde{g}_j = p_j g_j$, $\tilde{g}_k = p_k g_k$, $p_k = p_i\ \overrightarrow{\varphi_{g_i}}\,(p_j)$, $g_k = g_i g_j$. Denote the right semi-direct product $\tilde{G}$ of the group P by the group G of operators, accompanied with the homomorphism $\varphi : G \to AutP$, by the symbol

$$\tilde{G} = P \overrightarrow{\leftthreetimes}_{\varphi H(\Phi)} G,$$

where $H = Ker\varphi$ and $\Phi = Im\varphi$. If $Ker\varphi = 1$, then the symbol is $\tilde{G} = P \overrightarrow{\leftthreetimes} G$. We note that in the case when $Ker\varphi = G$ the right semi-direct product $\tilde{G}$ of the group P with the group G coincides with the direct product of the groups P and G: $\tilde{G} = P \overrightarrow{\leftthreetimes}_{\varphi G(i)} G = P \times G$.

The left semi-direct products of the group P by the group G can be derived from groups P and G by the analogous method. We denote the left semi-direct product $\tilde{G}$ of the group P by the group G, accompanied with the homomorphism $\varphi : G \to AutP$ by the symbol

$$\tilde{G} = G \overleftarrow{\leftthreetimes}_{\varphi H(\Phi)} P,$$

where $H = Ker\varphi$, $\Phi = Im\varphi$, $\varphi(g) = \overleftarrow{\varphi_g}$ and $\overleftarrow{\varphi_g}\ (p) = g^{-1}pg$. In the case when $Ker\varphi = 1$ we have $\tilde{G} = G \overleftarrow{\leftthreetimes}_{\varphi 1(\Phi)} P$=$G \overleftarrow{\leftthreetimes} P$.

2. Let P be the finite transitive group of permutations on the set $N = \{1, 2, ..., m\}$ and let G be the discrete symmetry group of a geometrical figure F. The "indexes" r of the set N have a non-geometrical nature. We consider the intersection of the figure F with the decomposition $\{G_i\}$ of space $F \cap \{G_i\} = \{F_i\}$, where $i \in I, |I| = |G|$ and G_i is a fundamental domain of the group G (the Dirihlet domain of a point A_i of general position with respect to the orbit $G : A_i$). Obviously $F = \cup F_i$. We call F_i the FG-domain.

Ascribe to each point M of the FG-domain F_i (to each fixed $i \in I$) the same indexes $r_1, r_2, ..., r_k$ from the set N, where k is a divisor of $m = |N|$. We obtain one "indexed" geometrical figure $F^{(N)}$ which is a subset of the Cartesian product of the sets F and N: $F^{(N)} \subseteq \{(M, r)|M \in F, r \in N\} = \tilde{F}$.

Let each "index" r from the set N have a scalar nature (colour, temperature, pressure, density). The mixed transformation $\tilde{g}$ of the "indexed" geometrical figure $F^{(N)}$ is composed of two independent components g and p: $\tilde{g} = gp$. The isometric geometrical component g operates only on points $M \in F$: $g(M) = M' \in F$. The "indexes" r, ascribed to points M, are transformed only by the permutation

$$p = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & m \\ k_1 & k_2 & \cdots & k_r & \cdots & k_m \end{pmatrix}.$$

If we analyze the concrete operation of the mixed transformation $\tilde{g} = gp$ concerning the distinct "indexed" points (M, r) of $F^{(N)}$, then we find that two different cases are possible: 1) the rule p which describes the transformation of the "indexes" r is the same for every "indexed" point of the figure $F^{(N)}$; 2) the "indexes" r_i and r_j ascribed to the points M_i and M_j which belong to distinct FG-domains are transformed, in general, by different permutations.

In conditions of the first case the mixed transformation $\tilde{g} = gp$ of the "indexed" geometrical figure $F^{(N)}$ is exactly a transformation of P-symmetry [2] if $p \in P$. In the case when $p \notin P$ we remain within the framework of the scheme of P-symmetry if we consider another defining group P' (one subgroup of symmetrical group of degree m: $P' \leq S_m$).

Note that the set of transformations of P-symmetry of any "indexed" geometrical figure $F^{(N)}$ forms a group. Moreover, the groups $G^{(P)}$ of P-symmetry are subgroups of the direct products of the defining group P with their generating discrete groups G of classical symmetry: $G^{(P)} \leq G \times P$. The theory of P-symmetry groups was elaborated and developed within the framework of Prof. A. Zamorzaev's scientific school on discrete geometry and mathematical crystallography from Chisinau [2-4].

In conditions of the second case the complex rule w which describes the transformation of the "indexes" r, ascribed to the points M of distinct FG-domains, needs to incorporate the respective information for each FG-domain F_i. This matter is possible only when the respective rule w is composed exactly from $|G|$ components-permutations p from defining group P: $w =< ..., p^{g_s}, ... >$. Therefore, in this case the mixed transformation $\tilde{g} = gw$ of the "indexed" geometrical figure $F^{(N)}$ is a transformation of W_p-symmetry [5,4].

The set of transformations of W_p-symmetry of the given "indexed" geometrical figure $F^{(N)}$ forms a group. The groups $G^{(W_p)}$ of W_p-symmetry are subgroups of the left semi-direct product of the group W by the group G of classical symmetry, accompanied with the isomorphism $\varphi : G \to AutW$, i.e. $G^{(W_p)} \leq G \overset{\leftarrow}{\leftthreetimes} W = \tilde{G}$. Note that the group W is the Cartesian product of isomorphic copies of the group P of permutations which are indexed by elements of the group G. Moreover, the accompanying automorphism $\overleftarrow{\varphi}_g = \overleftarrow{g}$ makes the left g-translation of the components in $w \in W$, i.e. $\overleftarrow{g} : w \mapsto w^g$. In other words, the group $\tilde{G} = G \overset{\leftarrow}{\leftthreetimes} W$ is a left standard Cartesian wreath product [1] of the group P of permutations by the discrete groups G of classical symmetry: $\tilde{G} = \overleftarrow{G} \,\overline{\wr}\, P$. The general theory of W_p-symmetry groups was elaborated at the end of the XX-th century [6-9].

3. Let the "indexes" r_i from the set N be homogeneous oriented magnitudes (vectors, tensors) and they are rigidly connected with the points. The mixed isometric transformation $\tilde{g}$ of the "indexed" geometrical figure $F^{(N)}$, which maps the "indexed" point (M, r) onto the "indexed" point (M', k_r), is composed also of two components p and g. The isometric transformation g maps the point M onto the point $M' = g(M)$ and the "indexes" r onto "indexes" s_r by the given rule. As for the component-permutation p it is only a compensating permutation of "indexes" (p maps the "index" s_r onto the "index" k_r). In other words, the mixed transformation $\tilde{g} = pg$ maps the "indexed" geometrical figure $F^{(N)}$ onto itself and $p \in P \leq S_m$.

Analysing the concrete operation of mixed transformation $\tilde{g} = pg$ about the distinct "indexed" points (M, r) of $F^{(N)}$ we remark that two different cases exist: 1) the component-permutation p is the same for every "indexed" point of figure $F^{(N)}$; 2) the "indexes" ascribed to the points which belong to distinct FG-domains are transformed, in general, by different permutations.

Assume the conditions of the first case. The mixed transformation $\tilde{g} = pg$ of the "indexed" geometrical figure $F^{(N)}$ is exactly the transformation of

$\overline{P}$-symmetry [10]. Note that the set of transformations of $\overline{P}$-symmetry of any "indexed" geometrical figure $F^{(N)}$ forms a group. The major group $\tilde{G}$ of $\overline{P}$-symmetry from the subfamily with the generating group G of classical symmetry, the kernel H of accompanying homomorphism $\varphi : G \to AutP$ and $Im\varphi = \Phi$ coincides with the right semi-direct product: $\tilde{G} = P \overset{\rightharpoonup}{\leftthreetimes}_{\varphi H(\Phi)} G$.

Obviously: 1) the major group $\tilde{G}$ is the group of mixed transformations by the "indexed" geometrical figure $F^{(N)} = \tilde{F}$ (in this case each point M of initial figure F is "indexed" by every element from the set N); 2) every automorphism $\overset{\rightharpoonup}{\varphi}_h$, which corresponds to the element h of the kernel H of accompanying homomorphism $\varphi : G \to AutP$, operates only on the element p of the group P, by the multiplication of the elements of group $\tilde{G}$, exactly like the identical automorphism of the group P. In other words, the influence of h concerning the oriented "index"-quality is not effective; 3) the groups of $\overline{P}$-symmetry with the generating group G and $Ker\varphi = H$, where $\varphi : G \to AutP$, are subgroups of the major group $\tilde{G}$: $G^{(\overline{P})} \leq \tilde{G} = P \overset{\rightharpoonup}{\leftthreetimes}_{\varphi H(\Phi)} G$.

The general theory of $\overline{P}$-symmetry groups was elaborated also within the framework of scientific school of discrete geometry and mathematical crystallography from Chisinau [10-13,4].

Assume the conditions of the second case, i.e. the oriented "indexes" ascribed to the points which belong to distinct FG-domains F_i are transformed, in general, by different permutations. In this case the complex rule w which describes the compensating permutation of "indexes", ascribed to the points $M \in F_i$ (for different i), needs to incorporate exactly $|G|$ components-permutations p from initial defining group $P \leq S_m$: $w =< ..., p^{g_s}, ... >$. Consequently, in this case the mixed transformation $\tilde{g} = wg$ of the "indexed" geometrical figure $F^{(N)}$ is a transformation of W_q-symmetry [14]. The set of transformations of W_q-symmetry of the given "indexed" geometrical figure $F^{(N)}$ forms a group.

Note that the major group $\tilde{G}$ of W_q-symmetry from the subfamily with the generating group G of classical symmetry, the kernel H of accompanying homomorphism $\varphi : G \to AutW$ (the group W is the Cartesian product of isomorphic copies of the group P of permutations which are indexed by elements of group G) and $Im\varphi = \Phi$ coincides with the crossed standard Cartesian wreath product [15,16] of the group P and discrete group G of classical symmetry: $\tilde{G} = P \overset{\rightharpoonup}{\bar{\wr}} \overset{\leftharpoonup}{G}$. Moreover, the crossed standard Cartesian wreath product $P \overset{\rightharpoonup}{\bar{\wr}} \overset{\leftharpoonup}{G}$ is accompanied also with the isomorphism $\alpha : G \to AutW$ by the rule $\alpha(g) = \overset{\leftharpoonup}{g}$ (where the automorphism $\overset{\leftharpoonup}{g}$ makes the left g-translation of the components in $w \in W$, i.e. $\overset{\leftharpoonup}{g} : w \mapsto w^g$.

The groups $G^{(W_q)}$ of W_q-symmetry are subgroups of the respective major group $\tilde{G}$ from the same subfamily. The foundations of the general theory of W_q-symmetry groups were also elaborated in Chisinau [14-17].

References

[1] A. Lungu, *Aplicaţii cvasiomomorfe şi produse semidirecte de grupuri.* Conferinţa ştiinţifică jubiliară "50 de ani ai USM", Chişinău, 1996, 22-24.

[2] A. M. Zamorzaev, *On the groups of quasi-symmetry (P-symmetry)*, Soviet. Phys. Crystallogr., **12**, 5(1967), 819-825.

[3] A. M. Zamorzaev, E. I. Galyarskii, A. F. Palistrant, *Colored symmetry, its generalizations and applications*, Chişinău: Ştiinţa, 1978, 275p. (Russian)

[4] A. M. Zamorzaev, Yu. S. Karpova, A. P. Lungu, A. F. Palistrant, *P-symmetry and its further development*, Chişinău: Ştiinţa, 1986, 156p. (Russian)

[5] V. A. Koptsik, I. N. Kotsev, *To the theory and classification of colour symmetry groups.* II. *W-symmetry*, OIYI Reports, R4-8068, Dubna, 1974, 17p. (Russian)

[6] A.P.Lungu, *On the theory of W-symmetry groups*, Izv. Akad. Nauk Respub. Moldova. Mat. 3(1992), 72-81.(Russian)

[7] A. P. Lungu, *Methods for deriving semi-minor and pseudo-minor groups of W-symmetry*, Izv. Akad. Nauk Respub. Moldova. Mat. 2(1994), 29-39. (Russian)

[8] A. P. Lungu, *Methods for deriving middle groups of W-symmetry.* (Russian). Izv. Akad. Nauk Respub. Moldova. Mat., 1 (1996), p. 35-39.

[9] A. P. Lungu, *The discrete groups of generalized symmetry and the quasi-homomorphic mappings*, Scientific Annals of the Faculty of Mathematics and Informatics, State University of Moldova, (1999), 115-124.

[10] A. P. Lungu, *On the theory of $\overline{P}$-symmetry*, Manuscript stored in VINITI archives, Moscow, 1709-78Dep.(1978), 16p.(Russian)

[11] A. P. Lungu, *On the methods for deriving minor groups of $\overline{P}$-symmetry*, Manuscript stored in VINITI archives, Moscow, 1587-79Dep.(1979), 22p.(Russian)

[12] A. P. Lungu, *To the deriving of Q-symmetry ($\overline{P}$-symmetry) groups*, Soviet. Phys. Crystallogr., **25**, 5(1980), 1051-1053.

[13] A. P. Lungu, *Universal method for deriving groups of $\overline{P}$-symmetry (Q-symmetry)*, Manuscript stored in MoldNIINTI archives, Chişinău, 308M-D83(1983), 14p. (Russian)

[14] A. P. Lungu, *Foundations of the general theory of W_q-symmetry*, Izv. Akad. Nauk Respub. Moldova. Mat., 3(1996), 94-100. (Russian)

[15] A. P. Lungu, *The theory of generalized and colored symmetry using extensions and wreath products of groups*, Dissertation of doctor habilitat in Physics and Mathematics, Chişinău, 1997, 309p. (Russian)

[16] A. Lungu, *W_q-simetria şi impletiri incrucişate de grupuri.* Analele Ştiinţifice ale USM, Ştiinţe fizico-matematice, Chişinău, (1999), 237-242.

[17] A. Lungu, *Discrete groups of W_q-symmetry.* Proceedings of 2nd Inter. Conf. on Symmetry and Antisymmetry in Mathematics, Formal Languages and Computer Science, Satellite Conf. of 3ECM, Brasov (2000), Romania, 175-184.

GENERATE THE COMPLEX TRAJECTORY THROUGH DIRECT FUNCTION COMPUTATION INTERPOLATING ALGORITHMS

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Abstract Formulas and implements methods of those formulas to generate the points co-ordinates from a complex trajectory are analyzed. The interpolation algorithms analyzed are named „direct function computation". Those algorithms can implement a feedback command to traverse the complex trajectory considering the current position of the robot to determine the next position. The implemented programmes have a real time evolution.

1. INTRODUCTION

There are industrial robots with trajectory that is important all time movement, e.g. industrial robots for painting or welding. Therefore it is necessary a compromise between the numbers of external information and required internal information to traverse the trajectory. The compromise is accomplish by decomposing the trajectory in simple, circular or linear segments. The co-ordinates of the segment extremity are the external information. The intermediary points co-ordinates are computed by interpolating along traversing the trajectory.

2. PROGRAMME LANGUAGE MOVEMENT COMMANDS FOR INDUSTRIAL ROBOTS

The interpolating module is called when the trajectory is specified all time movement. The types of movement for an industrial robot can be classified as: 1 point to point - unimportant trajectory, only the final position is important, such point welding or manipulation of the objects; 2 with articulation interpolating - this type command the movement for each articulation, the resulted trajectory is unknown in the working space of the industrial robot; 3 with Cartesian interpolating - the trajectory is exact and it is defined in a Cartesian space Oxyz with three axes. The trajectory co-ordinates are converted in movement for each articulation using the geometric model of the industrial robot.

The programming language of the industrial robots VAL II has instruction that select one of this type of movement described. So [1] there are the instructions:

$$MOVE \ < location >$$

to define a point to point movement with articulation interpolating between current position and final position specified in the instruction by the argument;

$$MOVES \ < location >$$

the movement trajectory is a linear one, to the final position specified in the instruction by the argument;

$$APPRO \ < location >, \ < distance >$$

the programmed movement is with articulation interpolating to the specified position, the Oz axis movement is specified by argument distance;

$$APPROS \ < location >, \ < distance >$$

is the Cartesian version of the previous instruction;

$$DEPART \ < distance >$$

in Oz axis direction movement with an articulation interpolating;

$$DEPARTS \ < distance >$$

is the Cartesian version of the previous instruction. The language LM defines two types of movement: the free movement with a non-defined trajectory and the Cartesian movement with Cartesian interpolating. Such instructions are [1]:

$$MOVE\ CUBE\ VIA\ POS1\ CARTESIAN,\ POS2,\ POS3\ TO$$

$$PUT\_CUBE\ CARTESIAN\ WITH\ SPEED = 0.8;$$

Those successive specifications command a Cartesian interpolating movement from the initial position, named $POS1$, then point to point movement without defined trajectory, and, at the end, a Cartesian interpolating movement from $POS3$ to PUT_CUBE.

Other programming language, also final effect movement oriented, contain diverse instructions regarding the movement. The $ROBEX - M$ language contains an interesting instruction that programmes a movement with the same orientation of the tool [1]:

$$COMOVE$$

constant orientation $MOVE$.

The language $ROBEX$ defines relative movement regarding the current position of linear three axes Oxyz or circular axes associates with the linear axes [1], so there are instructions:

$$GODLTA/\ dx,\ dy,\ dz\ [,EVENT,\ a[,\ ELSE,\ m]]$$

go to a specified distance,

$$TURN/XYROT, ww1, [XYROT, ww2, [XYROT, ww3]][, EVENT, a[ELSE, m]]$$

$$TURN/YZROT, ww1, [YZROT, ww2, [YZROT, ww3]][, EVENT, a[ELSE, m]]$$

$$TURN/ZXROT, ww1, [ZXROT, ww2, [ZXROT, ww3]][, EVENT, a[ELSE, m]]$$

Language ACL programmes a movement according to linear or circular trajectory through $MOVEL$ and $MOVEC$ instructions.

3. INTERPOLATING ALGORITHMS FOR DIRECT FUNCTION COMPUTATION

Some of the most efficient interpolating algorithms are the direct function computation algorithms [3]. Those algorithms work with trajectory analytical expression, the line analytical expression for a line segment and the circle analytical expression for a circle arc. Algorithms are based on the trajectory propriety that the points on one side of trajectory give a sign to analytical expression and the points on other side give the opposite signs to the same trajectory analytical expression. The programmed movement must give alternative sign to the trajectory analytical expression.

For a Cartesian space trajectory we must analyze the sign of the expression $sign(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz)$, where the trajectory is defined by the equation $z - f(x, y) = 0$ *or* $F(x, y, z) = 0$.

It is easy the analyze the algorithms for a Cartesian plane movement but the principle is the same for a Cartesian space movement.

The formulas, for current point $P(x_i, y_i)$ deviation computation, are:

- for a step δx on Ox direction $F(x_i + \delta x, y_i) - F(x_i, y_i) = \Delta F(x)$;
- for a step δy on Oy direction $F(x_i, y_i) - F(x_i, y_i + \delta y) = \Delta F(y)$;
- for a step on Ox and Oy directions $F(x_i + \delta x, y_i) - F(x_i, y_i + \delta y) = \Delta F(x, y)$.

OCTANT	1	2	3	4	5	6	7	8
direction	u x	y	y	-x	-x	-y	-y	x
direction u+v	x+y	x+y	-x+y	-x+y	-x-y	-x-y	x-y	x-y

Table 1 The change of variable for linear interpolating octant algorithm.

The sign analysis is necessary in order to determine next point co-ordinates $P(x_i + 1, y_i + 1)$. For a linear segment with analytical trajectory equation $ax + by + c = 0$ the steps deviation can be determined by formulas

$$\Delta F(x) = a\delta x,\ \Delta F(y) = b\delta y,\ \Delta F(x, y) = \Delta F(x) + \Delta F(y).$$

For space trajectory movement it can be considered a simplification that commands a linear or circular plane Oxy trajectory movement and a constant movement Oz-axis direction. For a circle segment, an arc with analytical equations $x_2 + y^2 - R^2 = 0$ the step deviation are

$$\Delta F(x) = 2x_i\delta x + \delta x^2,\ \Delta F(y) = 2y_i\delta y + \delta y^2,\ \Delta F(x, y) = \Delta F(x) + \Delta F(y).$$

The direct function algorithm is: for a linear rising analytical trajectory function, a positive deviation demand a step to Ox-axis direction, a negative deviation a step to Oy-axis direction. For a circle segment, trigonometric sense, a positive deviation demand a step to Ox-axis direction. The step dimension is defined by the speed value. For any kind of analytical trajectory function the algorithm can work. It must determine the sign of the function for points on one side of the trajectory by considering one example of co-ordinates point. The alternating sign principle determines the step demanded for going to the side with opposite sign. This method is very useful, it implements a real time feedback of the movement command. The next command is computed based on the current point co-ordinates, irrespewctive the fact the current point co-ordinates are the precedent computed point co-ordinates or not. The algorithm can be improved by considering the sign of a simplified expression, named a „discriminant". For a linear segment, with the final point $F(x_F, y_F)$ the algorithm must analyze the sign of the expression $D_D = x_F y_i + y_F x_i$. For a circular segment the „discriminant" is $D_C = x_i^2 + y_i^2 - (x_0^2 + y_0^2)$, where (x_0, y_0) are the co-ordinates of the initial point considering the circle center as the origin of the axes. In order to simplify the computation it is recommended to make a change of variables such that for a linear segment we consider a new origin on the beginning segment point and for circular segment a new origin on the circle center.

Working performance have named the interpolating algorithm method of octants [2]. For linear interpolating the algorithm considers two movement

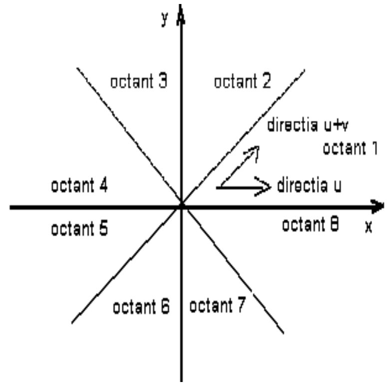

Figure 1 Octants definition.

OCTANT	1	2	3	4	5	6	7	8
direction v	y	-x	-x	-y	-y	x	x	y
direction u+v	-x+y	-x+y	-x-y	-x-y	x-y	x-y	x+y	x+y

Table 2 The changes of variables for circular interpolation, octants methods.

direction u and u+v according with the change of variable presented n Table 1, depending on the number of octant where the linear segment is placed.

The algorithm analyzes the sign of expression $F(u, v) = u_F v_i - v_F u_i$. The algorithm is: positive deviation demand direction u movement command. For opposite sign the movement command is to u+v direction. The sense and the step value are defined according to the definition of the octant, (fig. 1). There are defined eight octants. Circular interpolation, trigonometric sense works with change of variables presented in Table 1.

4. COMPUTATION EXAMPLES

The step value for each sample period can be computed depending speed value. The speed variation at the beginning and the end of the movement is programmed according to inertia reasons. Table 3 shows movement commands for a linear segment $A(0, 0)$ $B(5, 6)$, „discriminant" method. The step value is 1.

The same linear segment can be interpolated with octant method, the commands are described in Table 4. The linear segment is in octant 2, so $u_F = y_F = 6$, $v_F = x_F = 5$. For the two methods see fig. 2.

A circular interpolation example, method "discriminant" is analyzed in Table 5. The beginning point is $A(13, 0)$ and the end point is $B(5, 12)$.

NR.	D=	C-DA U = Y	C-DA V = X	OX	OY
1	0	*		0	0
2	-6		*	1	0
3	-1		*	1	1
4	4	*		1	2
5	-2		*	2	2
6	3	*		2	3
7	-3		*	3	3
8	2	*		3	4
9	-4		*	4	4
10	1	*		4	5
11	-5		*	5	5
12	0	stop		5	6

Table 3 Computation example, linear interpolation, "discriminant" method.

NR.	D=	C-DA U = Y	C-DA V = X	OX	OY
1	0	*		0	0
2	-5	*	*	0	1
3	-4	*	*	1	2
4	-3	*	*	2	3
5	-2	*	*	3	4
6	-1	*	*	4	5
7	0	stop		5	6

Table 4 Computation example, linear interpolation, octants method.

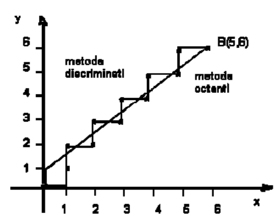

Figure 2 Linear interpolation, examples.

NR.	D=	C-DA OX	C-DA OY	OX	OY
1	0	*		13	0
2	-25		*	12	0
3	-24		*	12	1
4	-21		*	12	2
5	-16		*	12	3
6	-9		*	12	4
7	0	*		12	5
8	-23		*	11	5
9	-12		*	11	6
10	1	*		11	7
11	-20		*	10	7
12	-5		*	10	7
13	12	*		10	9
14	-7		*	9	9
15	12	*		9	10
16	-5		*	8	10
17	16	*		8	11
18	1	*		7	11
19	-12		*	6	11
20	11	*		6	12
21	0	stop		5	12

Table 5 Computation example, circular interpolation, "discriminant" method.

The same circular segment can be generated by octants method. At the beginning, the arch is in the second octant.

Beginning with the 11th step, the arch is in the first octant, the variables must be changed.

In these examples, the current co-ordinate was considered the computed coordinate, while in real mode utilization, the current coordinate must be read by transducers. The analyzed function sign is computed with the aid of these current values of the coordinates.

5. CONCLUSION

The direct function computation interpolating methods are simple methods, that have the advantage to implement a feed-back adjust of the movement on a complex trajectory. That adjust is the effect of the direct function sign computation using the current point coordinates to determine the next step in the movement. From all these algorithms the octants algorithm is more indicated because it determines the speed variation less than 9% .

NR.	D=	C-DA	C-DA Y = V	OX	OY
		X = U	Y = V		
1	0		*	13	0
2	1	*	*	13	1
3	-21		*	12	2
4	-16		*	12	3
5	-9		*	12	4
6	0	*	*	12	5
7	-12		*	11	6
8	1	*	*	11	7
9	-5		*	10	8
10	12	*	*	10	9
		X = V	Y = U		
11	12	*		9	10
12	-5	*	*	8	10
13	1	*		7	11
14	-12	*	*	6	11
15	0	stop		5	12

Table 6 Circular interpolation, octants method.

References

[1] Lecocq, H., *Langage de programmation de la robotique*, Univ. Liege, 1998.

[2] Matica, L. M., Abrudan-Purece, A., *Asupra implementării generării traiectoriilor complexe prin algoritmi de interpolare*, Al XV-lea Simpozion Naţional de Robotică, ROBOTICA 2000, Oradea, 2000, 225-228.

[3] Matica, L. M., *Sisteme informatice industriale*, Universitatea Oradea, 2001.

[4] Stoicu-Tivadar, L., *Programarea roboţilor industriali şi a maşinilor unelte cu comandă numerică*, Universitatea ,,Politehnica" Timişoara, 1996.

MANAGEMENT OF PREPARING AND FORECASTING OF PERSONNEL IN EDUCATIONAL SPHERE BASED IN FREE SOFTWARE

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1. INTRODUCTION

Automatization of the personnel management is a well-known and widespread problem. Obviously this problem can be solved using different ways – a creation of enterprise advanced applications with great economic and business investment (The solution based on expensive Shareware software and technologies, for example - Microsoft SQL Server, Microsoft Windows NT, Visual Studio .NET and others) and creation of inexpensive system based on free software like a Apache Web Server, script language PHP, database server MySQL, free OS Unix (Linux) and other. In this paper we consider a specially developed system for automatization of personnel and personnel management which is based on free software.

2. SYSTEM DESCRIPTION

An automatization system of preparing and forecasting of professional personnel was created in connection with developing computer technologies. This system contains a large set of functions of the entering data, editing, and automatic receiving of the results and forecasting of professional personnel in different spheres. System also includes multi-level access for different users and groups of users. Each user have a determinate level of access to this system – access of viewing individual data or editing of personal data or viewing data of other group of persons or editing data of other group of persons or access of different groups of administrators for editing or updating or creating, erasing data of other persons, groups of persons and even groups of other administrators. All of listed rights of access can be used together or separately.

System consists of next parts: entering data, automatic receiving of the results and information's belonging all activity of personnel by using simple and complex multilevel queries, viewing data, statistical data manipulation and elements of forecasting. In future developments expert system of different levels will be used for forecasting.

One of the basic advantages in using this system is the easiness of the development and using. User does not need any knowledge in programming. It is sufficient to have the basic computer knowledge. This system differs from other existing databases and information systems, because it has a complete web-interface application. Therefore this system can be used in the local network and Internet.

3. METHODS AND SELECTION OF TECHNICAL TOOLS AND SOFTWARE

The environment of the development is an union of next technologies: script language of programming PHP, the goal of which is to allow web developers to write dynamically generates pages; a very fast multi-threaded, multi-user and robust SQL database server named MySQL; HTML-for creating static web pages and helping for creating dynamically generating pages, Java Script language; web server Apache, which can be used on Windows, Linux and other platforms. The information transfer in the local computation networks, WAN, Internet and telephone lines is one of the advantages of these technologies.

Union of these technologies is widely used in the network of Internet for creating complex, multi-level web portals with complex structure. This decision is very popular among web-developers because these technologies have the following advantages:

PHP support many kinds and sorts of databases, such Oracle, Sybase, dBase, others and MySQL, which was used for creating this system; good integration with different Operation Systems as Linux, Mac, Windows and others. We note that we are using this system on Windows platform. But we have not any problems for using this system on the platform Linux, which is popular hosting system in Internet also. Causes of using MySQL are described below:

MySQL is most popular open source SQL database. MySQL was originally developed to handle large databases much faster than existing solutions and has been successfully used in highly demanding production environments for several years. The connectivity, speed, and, security make MySQL highly suited for accessing databases on the Internet.

4. DISCUSSION

Now we consider a main idea of the automatization personnel system, which was developed for our country needs. We only use open-source free software. The idea behind Open Source software is rather simple: when programmers can read, distribute and change code, the code will mature. People can adapt it, fix it, debug it, and they can do it at a speed that dwarfs the performance of software developers at conventional companies. This software will be more flexible and of a better quality than software that has been developed using the conventional channels, because more people have tested it in more different conditions than the closed software developer ever can.

4.1 System properties and features

System of the management of preparing and forecasting of professional personnel was created for helping in management of personnel in the country. In particular, this system was developed for management of preparing and forecasting of professional personnel in education sphere of the country. Education sphere of the Republic consist of set of education organizations, institutes, universities and other such organizations. The system must contain all personal data on each member of all education organizations in the Republic. This personal data must be entered by special administrators of this organizations and institutes, who are responsible for the entering of personal data of officials (workers) of organizations and institutes. These administrators must have an access to Internet for entering data, viewing and editing. They should open web page of this system, enter their login and password, which they have obtained from main administrator early, and must work with this system by methods, described above.

Consider this problem for an example of one university, which consists of faculties and departments. Each faculty is divided on chairs. In this case we have several levels for access to data on the web-portal of the system. If you are a simple user and do not work in education sphere, you can use this system for only viewing data of any teacher, lector, professor, doctor and other specialists of any educational institutes, organizations or universities of the Republic. In addition, it has been a possibility of multifunctional search of persons on different parameters (place of work, specialty of person, nation of person and other parameters).

If user works in education sphere – he has access of simple user plus possibility of editing self-personal data. If user responses for the enter data of persons of his organization, in other words, he has access as administrator of his educational organization, then he has access of simple user plus possibility of editing, updating and deleting personal data of any officials of organization. If he works on the faculty and has access of administrator of his faculty, he can change any data of officials of his faculty and give rights of administrating to determine person on the chairs of faculty. We consider this below.

4.2 System components (based on free software) properties and features

We are using a MySQL database in developing.

MySQL is a most popular Open Source SQL database. We want MySQL to be:

1. The best and the most used database in the world.
2. Available and affordable for all.
3. Easy to use.
4. Continuously improved while remaining fast and safe.
5. Fun to use and improve.
6. Free from bugs.

This database contains more than a great number of tables and each table contains determined set of data. Three tables of that database called "The main tables". There are three main tables in our database – table of educational organizations, universities, institutes, faculties and chairs, called "Table of departments"; table of users and table of rights of access to data. Table of users contains a personal data of all officials of any educational organizations or their divisions with login and password, which are distinct for each registered user (all officials of educational organizations are registered users). Table of rights of access to the personal data contain data of access for different levels of administrators. Other tables are information tables, named by inquiry tables. Structure of inquiry tables is described below.

There is a several inquiry tables in database. Consider one of them. Other inquiry tables were created through the same method. Our considered table is a table of the scientific specialties in the Republic. Table consists of two fields (columns) – field, named "ID" and field, named "Name". "Id" is an autoincreament field and this field is primary key of the table. This column contains an ordinal numbers of the specialities. Column "Name" contains all scientific specialities immediately. This structure have another inquiry tables as table of domains of the Republic, table of the nationalities, table of scientific degrees, table of government rewards and others.

We consider the first of the main tables. It is a table of the educational organizations of the Republic and their divisions. Structure of this table is described below.

The table of educational organizations consists of 4 fields. The first field is on autoincreament field "ID" (not that id, which was used in inquiry tables).

Second field is a name of organization or section of organization. Third field is a brief name of organization or section of organization and fourth field called "SUPER_ID", special field, values of which can be repeated.

In the top of hierarchy of the educational organizational is Universities, institutes, or organization, but not sections of the organizations or faculties or chairs. All this organizations have SUPER_ID is equals zero.

Any faculty or section of organization will have SUPER_ID, equals ID of parent organization. Consider this in the next example.

We have University with many faculties and chairs in the faculties. In this case SUPER_ID of university is equal to zero, SUPER_ID of the faculty is equal to ID of the University (not SUPER_ID), and SUPER_ID of a chair of the determined faculty is equal to ID of this faculty.

The table of scientists or users containts the following fields: ID, Login, Password, place of birthday, specialty, day of birthday, family position, social origin, list and names of scientific works and publications and other fields of individual character, and plus field "ID_ORG" and "ID_RIGHT".

The field "ID_ORG" is connected with field "SUPER_ID" from the table of the educational organizations. The fields "ID_RIGHT" are connected with field "ID" from the table of rights, described below.

Table of rights consists of the autoincreament field "ID" and field "RIGHT".

In the field "RIGHT" can be contained different levels of administrating, such as local administrator of organization, local administrator or faculty, local administrator of chair or global administrator.

We considered a structure of our database, and we consider structure of PHP scripts, used in web-developing of this system now.

4.3 Programming tools architecture

Architecture of PHP scripts is divided in three parts: parts of representations, part of business-logic and data stage.

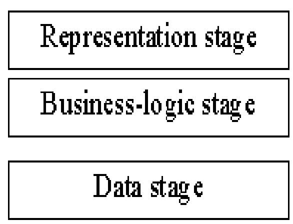

Fig 1. System architecture.

Part of representations contains php-scripts and templates, composing user interface of system. This type of representation is very comfortable comparatively with usual one-level architecture. First, representation of data is strongly separated from the logic of functionality. Therefore, we separated work of designer and web-developer. Second, such representation simplifies the structure and possibility of comfort reading of codes. Third, the application will be more powerful, because in order to add new function of the system we must just add new methods in class.

There were created many classes with determined functions and methods. Each class performs determine "functions". Example, class "login" is for the authentification of users and others.

There were performed some basic classes, which were included in another classes- PageControl class (In this class were created functions and methods for the control of number of data on the page), Db class (for the connection with database using host or ip-address of machine, where database is placed) and others. Listed classes are basic classes of the system or a kernel of the system. They were used in almost all other classes. Files of templates are responsable for placing data on the web page. Templates are transforming on determined class.

Structure of templates contains web-design of the automatically generating pages, CSS schemes for the nicely printing data to the web page.

4.4 Operation System selection

System can be installed on many platforms such Windows, Mac, Linux and other operation systems. But we used OS Linux because:

1. Linux is free;
2. Linux is portable to any hardware platform;
3. Linux was made to keep on running;
4. Linux is secure and versatile;
5. Linux is scalable;
6. The Linux OS and Linux applications have very short debug-times.

5. RESULTS

The practice of using this system shows that it is very powerful tool for the simplifying work with the personnel of the educational sphere.

This system may be used not only in the educational sphere, but in any other sphere of management systems. Using this system in other sphere of management requires only a little modifications of the system.

References

[1] L. Algerich, W. Choi , J. Coggesball, K. Egervari and others, Professional PHP 4. Wrox Press Ltd, 2003.

[2] Actual problem of mathematical physics and information technologies,vol.I, Tashkent, 2003.

[3] M. Fowler, Patterns of enterprise application architecture, Addison-Wesley, Boston, 2003.

[4] H. Garsia-Molina, J. Ullman, J. Widom, Database systems: The Complete Book, Prentice-Hall, New Jersey.

ON ARNOLDI'S METHOD RECURSIONS

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Abstract

In 2000 Jia introduced some recursions based on the Arnoldi's method to improve inverse Hessenberg matrices computation. Working with these recursions particular forms are obtained in order to be more useful in computation. Illustrative examples are included.

Introduction. The incomplete orthogonalization method which derives from Arnoldi's method, is used for solving large non Hermitian linear systems. Jia [1] developed some recursions to compute the inverse of the Hessenberg matrices. These updating recursions is useful for any type of nonsingular matrices in special conditions. Starting from these recursions particular forms are obtained that are cheaper in computation.

Recursions. Assume that $A = (a_{i,j})_{i,j=\overline{1,n}}$, $\quad |A_j| \neq 0, \quad j = 1, ..., n,$ with $A_j = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j} \\ \cdots & \cdots & \cdots \\ a_{j,1} & \cdots & a_{j,j} \end{pmatrix}$ and define $d_j = a_{j+1,j} e_j^{(j)}$,

$c_j = (a_{1,j+1}, a_{2,j+1}, \ldots, a_{j,j+1})^T$, $u_{j+1,j+1} = a_{j+1,j+1} - d_j^T A_j^{-1} c_j$, $j = 1, 2, \ldots, n$.

Theorem 1 [1].

$$A_{j+1}^{-1} = \begin{pmatrix} A_j^{-1} + \frac{A_j^{-1} c_j d_j^T A_j^{-1}}{u_{j+1,j+1}} & -\frac{A_j^{-1} c_j}{u_{j+1,j+1}} \\ -\frac{d_j^T A_j^{-1}}{u_{j+1,j+1}} & \frac{1}{u_{j+1,j+1}} \end{pmatrix} \tag{1}$$

If A is a symmetric matrix then $A_{j+1} = \begin{pmatrix} A_j & d_j \\ d_j^T & a_{j+1,j+1} \end{pmatrix}$. Based on (1) we obtain

$$A_{j+1}^{-1} = \begin{pmatrix} A_j^{-1} + \frac{A_j^{-1} d_j d_j^T A_j^{-1}}{u_{j+1,j+1}} & -\frac{A_j^{-1} d_j}{u_{j+1,j+1}} \\ -\frac{d_j^T A_j^{-1}}{u_{j+1,j+1}} & \frac{1}{u_{j+1,j+1}} \end{pmatrix} = \begin{pmatrix} A_j^{-1} + \frac{A_j^{-1} d_j (A_j^{-1} d_j)^T}{u_{j+1,j+1}} & -\frac{A_j^{-1} c_j}{u_{j+1,j+1}} \\ -\frac{(A_j^{-1} d_j)^T}{u_{j+1,j+1}} & \frac{1}{u_{j+1,j+1}} \end{pmatrix}$$

Recursions for Hessenberg matrices. Assume that H is a Hessenberg matrix, $h_{i,j} = 0$, for $i > j + 1$, $i, j = 1, \ldots, n$ and denote $K_j = H_j^{-1}$. Thus

$$d_j^T H_j^{-1} = d_j^T K_j = (0, \dots, 0, h_{j+1,j}) \begin{pmatrix} k_{1,1} & \cdots & k_{1,j} \\ \cdots & \cdots & \cdots \\ k_{j,1} & \cdots & k_{j,j} \end{pmatrix}$$

$$= (h_{j+1,j}k_{j,1},\ h_{j+1,j}k_{j,2}, \dots,\ h_{j+1,j}k_{j,j}) = s_j^T.$$

We obtain

$$K_{j+1} = H_{j+1}^{-1} \quad = \quad \begin{pmatrix} K_j + \frac{K_j c_j s_j^T}{u_{j+1,j+1}} & -\frac{K_j c_j}{u_{j+1,j+1}} \\ -\frac{s_j^T}{u_{j+1,j+1}} & \frac{1}{u_{j+1,j+1}} \end{pmatrix} \tag{2}$$

which is cheaper then (1) because we avoid unuseful multiplications with zero.

Results. The implementation of these updating recursions also demonstrate that for large Hessenberg matrices (2) is cheaper than (1). N represents the dimension of the tested matrix. Values in the second and third column represent the number of miliseconds necessary to compute the inverse of the Hessenberg matrix using relation (1), respectively (2).

N	milisecs - (1)	milisecs - (2)
10	94	94
20	594	578
30	1937	1922
40	4579	4531
50	8969	8875
60	15219	15390
70	24250	24453
80	37547	36250
90	53140	51969
100	73110	70875

References

[1] Z. Jia, *Some recursions on Arnoldi's method and IOM for large non-Hermitian linear systems*, Computers and Mathematics with Applications **39**(2000), 125-129.

[2] Y. Saad, *Iterative methods for sparse linear systems*, 2000.

LINEAR STABILITY OF DYNAMICAL SYSTEMS IN PHYSIOLOGY

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Abstract The SODE-modelled calcium variations in certain cell-types are investigated. The structural stability of hepatocyte physiology time-delayed flow is studied for three distinct cases (bursting, chaotic and quasiperiodic behavior), for the uniform and exponential distributions.

2000MSC: 37G10, 37M20, 37N25, 92C45.
Keywords: dynamical system, time delay, stationary point, bifurcation.

1. INTRODUCTION

The SODE which describes the intra-cell calcium variation in time exhibits a very rich and complex dynamical behavior. An illustrative fact is the case when a time-delay imposed to one of the state variables leads to structural instability and Hopf bifurcations. In the present work, we investigate the dynamics of a mathematic biological model which describes the calcium variations in time in the living cell, while one of the state variables is delayed in time. The applicative biological aspects represent an important open question in the field, and are subject of further research.

The variables used in the present model for calcium variations are:

Z - the concentration of free Ca^{2+} in the cytosol;

Y - the concentration of free Ca^{2+} in the internal pool;

A - the $InsP_3$ concentration.

The time evolution of these variables is governed by the following SODE

$$\begin{cases} \dfrac{dZ}{dt} = & -k \cdot Z + V_0 + \beta \cdot V_1 + T \\ \dfrac{dY}{dt} = & -T \\ \dfrac{dA}{dt} = & \beta \cdot V_{M_4} - V_{M_5} \cdot \dfrac{A^p}{k_5^p + A^p} \cdot \dfrac{Z^n}{k_d^n + Z^n} - \varepsilon \cdot A, \end{cases} \tag{1}$$

where $T = k_f \cdot Y - V_{M_2} \cdot \dfrac{Z^2}{k_2^2 + Z^2} + V_{M_3} \cdot \dfrac{Z^m}{k_Z^m + Z^m} \cdot \dfrac{Y^2}{k_Y^2 + Y^2} \cdot \dfrac{A^4}{k_A^4 + A^4}$.

The parameters involved are: V_0, V_1, β, V_2, V_3, k_2, k_Y, k_Z, k_A, k_f, k, V_{M4}, V_{M_5}, m, n, p and ε, and are described in detail in [7], [6].

From biological point of view, this SODE is based on the mechanism of calcium release induced by calcium influenced by the inozitol 1,4,5-triphosphate (IP_3) degradation by a 3-kynase. This model may exhibit various types of variations as: explosion, chaos, quasi-periodicity, depending on the values assigned to the parameters.

2. THE TIME-DELAYED EVOLUTION FLOW

We shall study the biological flow when one variable coordinate is subject to time-delay. In our case, we assume this to be A - which denotes the concentration of inozitol, leaving still open the question of biological interpretations to their full extent.

In order to obtain the dynamical system with delayed argument in the dependent variable $A(t)$, it is known that for a given probability density $f : \mathbb{R} \to \mathbb{R}_+$ obeying $\int_0^\infty f(s)ds = 1$, the transformation (perturbation) of the state variable $A(t) \in \mathbb{R}$ dependent on f is the new variable $\tilde{A}(t)$ defined by

$$\tilde{A}(t) = \int_0^\infty A(t-s)f(s)ds = \int_{-\infty}^t A(s)f(t-s)ds. \tag{2}$$

Applying the time-delay process to A, one changes the system (1) into the new SODE

$$\begin{cases} \dfrac{dZ}{dt} = & -k \cdot Z(t) + V_0 + \beta \cdot V_1 + \tilde{T}(t), \\ \dfrac{dY}{dt} = & -\tilde{T}(t), \\ \dfrac{dA}{dt} = & \beta \cdot V_{M_4} - V_{M_5} \cdot \dfrac{\tilde{A}(t)}{k_5 + \tilde{A}(t)} \cdot \dfrac{Z(t)^2}{k_d^2 + Z(t)^2} - \varepsilon \cdot \tilde{A}(t), \end{cases} \tag{3}$$

where

$$\begin{aligned} \tilde{T}(t) = \quad & k_f \cdot Y(t) - V_{M_2} \cdot \frac{Z(t)^2}{k_2^2 + Z(t)^2} + \\ & + V_{M_3} \cdot \frac{Z(t)^4}{k_Z(t)^4 + Z(t)^4} \cdot \frac{Y(t)^2}{k_Y^2 + Y(t)^2} \cdot \frac{\tilde{A}(t)^4}{k_A^4 + \tilde{A}(t)^4}, \end{aligned}$$

with $Z(0) = Z_0, Y(0) = Y_0, A(\theta) = \varphi(\theta), \theta \in [-\tau, 0], \tau \geq 0$, where the transform $\tilde{A}(t)$ is defined by (2) and $\varphi : [-\tau, 0] \to \mathbb{R}$ is a differentiable function which describes the behavior of the flow in the O direction. In other words, the initial SODE is replaced by a differential-functional system.

The equilibrium points of the system (1) are obtained when the right side of the differential system (1) is set to zero. The resulting nonlinear system has in general several solutions; from physiological point of view only the positive ones can be accepted. We denote such a solution as (Z^*, Y^*, A^*). Regarding the linearization of the SODE (3) we have the following statement [7]

Proposition 1. *The following assertions hold:*

a) The linearized SODE of the differential autonomous system with delayed argument (3) at its equilibrium point (Z^*, Y^*, A^*) *is* $\dot{V}(t) = M_1 V(t) + M_2 V(t-\tau)$, *where we have denoted by "dot" the* t*-differentation,* $V(t) = {}^t(Z(t), Y(t), A(t))$ *and*

$$M_1 = \left.\begin{pmatrix} \frac{\partial f_1}{\partial Z} & \frac{\partial f_1}{\partial Y} & 0 \\ \frac{\partial f_2}{\partial Z} & \frac{\partial f_2}{\partial Y} & 0 \\ \frac{\partial f_3}{\partial Z} & \frac{\partial f_3}{\partial Y} & 0 \end{pmatrix}\right|_{(Z^*,Y^*,A^*)}, \qquad M_2 = \left.\begin{pmatrix} 0 & 0 & \frac{\partial f_1}{\partial A} \\ 0 & 0 & \frac{\partial f_2}{\partial A} \\ 0 & 0 & \frac{\partial f_3}{\partial A} \end{pmatrix}\right|_{(Z^*,Y^*,A^*)}$$

and (f_1, f_2, f_3) *are the components of the field which provides the SODE (1).*

b) The characteristic equation of the differential autonomous system with delayed argument (3) is

$$\det\left(\lambda I - M_1 - \int_0^\infty e^{-\lambda s} f(s) ds \cdot M_2\right) = 0. \tag{4}$$

Besides the Dirac distribution case, extensively studied in [7], two more notable distributions are worthy to consider: the uniform distribution and the gamma distribution. In these cases, the delayed A-component of the system has respectively the following forms:

1. If f is the *uniform distribution* of $\tau > 0$, i.e., $f(s) = \begin{cases} \frac{1}{\tau}, & 0 \le s \le \tau \\ 0, & s > 0 \end{cases}$, then $\tilde{A}(t) = \frac{1}{\tau}\int_{-\tau}^0 A(t+s)ds$.

2. If f is the *gamma distribution* of $\tau > 0$, i.e., $f(s) = \frac{d^m}{\Gamma(m)} s^{m-1} e^{-ds}, s \ge 0, d > 0$, then $\tilde{A}(t) = \frac{d^m}{\Gamma(m)} \int_{-\infty}^t A(s)(t-s)^{m-1} e^{-d(t-s)} ds$. For $m = 1$, we obtain the *exponential distribution* and the delayed function respectively

$$f(s) = \frac{d}{\Gamma(1)} e^{-ds}, s \ge 0, d > 0, \text{ and } \widetilde{A}(t) = \frac{d}{\Gamma(1)} \int_{-\infty}^t A(s) e^{-d(t-s)} ds.$$

3. STABILITY RESULTS REGARDING THE DELAYED FLOW

Further we study the subcases when the SODE leads to explosion, chaos and quasiperiodicity. The first one was thoroughly investigated in [7], in the case of the Dirac distribution. For the three cases of parameter sets and for the two types of distributions described above, we further develop the basic results regarding the stability and bifurcation of the considered delayed SODE.

The three sets of values for parameters, corresponding to the subcases (explosion, chaos and quasiperiodicity) are provided in detail in the works [6], [7].

a) Explosion. In this case is known the following [7]

Proposition 2. *i) The only non-negative equilibrium point is*

$$(Z^*, Y^*, A^*) = (0.2916496701;\ 0.2344675015;\ 0.1989819160).$$

ii) The eigenvalues of the Jacobian matrix of the field at this point are

$$\{-0.07285104555, 0.02709536609 \pm i\ 0.2468748453\}; \tag{5}$$

iii) The constitutive matrices of the linearized delayed SODE (4) are

$$M_1 = \begin{pmatrix} 0.3886052798 & 0.3240919299 & 0 \\ -0.5553052794 & -0.3240919299 & 0 \\ -0.08796881783 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0.103140906 \\ 0 & 0 & -0.103140906 \\ 0 & 0 & -0.08317366327 \end{pmatrix}.$$

b) Chaos. In this case we obtain:

Proposition 3. *i) The only non-negative equilibrium point is*

$$(Z^*, Y^*, A^*) = (0.3296040792; 0.7830862038; 0.1365437815);$$

ii) The eigenvalues of the Jacobian matrix of the field at this point are

$$\{-0.1767271957, 0.2753920311 \pm i0.9217492250\};$$

iii) The constitutive matrices of the linearized delayed SODE (4) are

$$M_1 = \begin{pmatrix} 0.1523424612 & 0.0423568326 & 0 \\ -0.3190424612 & -0.0423568326 & 0 \\ -0.03491470092 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0.513718989 \\ 0 & 0 & -0.513718989 \\ 0 & 0 & -0.2316344181 \end{pmatrix};$$

c) Quasiperiodicity. We obtain

Proposition 4. *i) The only equilibrium point of the SODE is*

$$(Z^*, Y^*, A^*) = (0.3016376725; 0.6476260612; -0.5077053483);$$

We obtained a negative solution which can not be acceptable from a physiological point of view.

ii) The eigenvalues of the Jacobian matrix of the field at this point are

$$\{-0.1767271957, 0.2753920311 \pm i0.9217492250\}.$$

In the following we examine the two distribution cases with the parameter sets corresponding to the subcases a) and b).

1-a) The uniform distribution-explosion subcase. In this case $\widetilde{A}(t) = \frac{1}{\tau}\int_{-\tau}^{0} A(t+s)ds$, and the equation (4) becomes

$$\det\left(\lambda I - M_1 - \frac{1-e^{-\lambda\tau}}{\lambda\tau}M_2\right) = 0. \qquad (6)$$

Following the same steps like in the case of Dirac distribution, we obtain that there exist no solutions for the bifurcation parameter τ_0.

1-b) The uniform distribution-chaos subcase. In this case $\widetilde{A}(t) = \frac{1}{\tau}\int_{-\tau}^{0} A(t+s)ds$ and the equation (4) becomes (6). Following the same steps like in the case of the Dirac distribution, we obtain that there exists no solution for the equation (6) for $u = 0$ in terms of τ_0 and ω_0.

2-a) The exponential distribution-explosion subcase. In this case the equation (4) becomes

$$\det\left(\lambda I - M_1 - \frac{d}{\lambda + d}M_2\right) = 0. \qquad (7)$$

Using the graphic package Maple 8, we obtain that there exists no solution for the equation (7) for $u = 0$ in terms of τ_0 and ω_0.

2-b) The case of exponential distribution-chaos subcase. In this case the equation (4) becomes (7). Using the graphic package Maple 8, we obtain that there exists no solution for the equation (6) for $u = 0$ in terms of τ_0 and ω_0.

In the cases pointed out by Propositions 2 and 3, the initial dynamical SODE becomes subject to the Hopf bifurcation theorem ([13]). Further considerations on this issue can be found in [6] and [7].

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References

[1] M. Berridge, P. Lipp, M. Bootman, *Calcium signalling*, Curr. Biol. **9** (1999), R157-R159.

[2] J. A. M. Borghans, G. Dupont, A. Goldbeter, *Complex intracellular calcium oscillations: A theoretical exploration of possible mechanisms*, Biophys. Chem., **66** (1997), 25-41.

[3] G. Houart, G. Dupont, A. Goldbeter, *Bursting, chaos, birhytmicity originating from self-modulation of the inositol 1,4,5-triphosphate signal in a model for intracellular Ca^{2+} oscillations*, Bull. of Math. Biology, **61** (1999), 507-530.

[4] G. Mircea, M. Neamţu, D. Opriş, *Dynamical systems in economy, mechanics, biology described by differential equations with time-delay*, Mirton Press, Timişoara 2003, (Romanian).

[5] M. Neagu, I. R. Nicola, *Geometric dynamics of calcium oscillations ODE systems*, BJGA **9**, 2(2004), 36-67.

[6] I. R. Nicola, C. Udrişte, V. Balan, *Linear stability and Hopf bifurcations for time-delayed intra-cell calcium variation models*, Proc. of the 3-rd International Colloquium "Mathematics in Engineering and Numerical Physics" (MENP-3), 7-9 October 2004, Bucharest, Romania, to appear.

[7] I. R. Nicola, C. Udrişte, V.Balan, *Time-delayed flow of hepatocyte physiology*, Annals of the University of Bucharest, to appear.

[8] D. Opriş, C. Udrişte, *Pole shifts explained by a Dirac delay in a Stefanescu magnetic flow*, An. Univ. Bucureşti, **LIII**, 1 (2004), 115-144.

[9] C. Robinson, *Dynamical systems: stability, symbolic dynamics and chaos*, CRC Press, 1995.

[10] J. A. Rottingen, J. G. Iversen, *Rulled by waves ? Intracellular and intercellular calcium signalling*, Acta Physiol. Scand. **169**(2000), 203-219.

[11] G. Stefan, *Retarded dynamical systems. Stability and characteristic functions*, Longman Sci & Technical, England 1989.

[12] N. V. Tu, *Introductory optimization dynamics*, Springer, Berlin, 1991.

[13] C. Udrişte, Algebra, geometry and differential equations, Editura Didactică şi Pedagogică, Bucharest, 2002(Romanian).

AVERAGE VELOCITY OF THE TURBULENT FLOW NEAR A SMOOTH WALL

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Abstract

The universal law concerning the distribution of the turbulent flow average velocity field near a rigid smooth wall is established by considering the field in the transition regime from the laminar motion in the laminar sub-layer, to the turbulent motion in the fully turbulent layer, subsequently passing on the application of the effects of molecular viscosity and turbulent viscosity. Consequently, the expression of the universal function of the correlation between the longitudinal and transversal components of the turbulent flow velocity pulsation near the wall is deduced.

1. Determination of the universal law concerning the average velocity of the turbulent flow near a smooth wall

The general equations of a turbulent flow determined by O. Reynolds are

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} = \bar{X}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\upsilon \frac{\partial \bar{u}_i}{\partial x_k} - \overline{u_i' u_k'}\right),\ i = 1,\, 2,\, 3. \tag{1}$$

We investigate the plane-parallel stationary turbulent motion of a fluid along the Ox axis in the semi-space $z > 0$, assuming there is no gradient of the average pressure. In this case $(1)_1$ has the form

$$\upsilon \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{d}{dz}\left(\overline{u'w'}\right) = 0. \tag{2}$$

This equation is integrated to have

$$\eta \frac{\partial \bar{u}}{\partial z} - \rho\left(\overline{u'w'}\right) = \tau_0, \tag{3}$$

where $\eta = \rho\upsilon$ is the dynamic coefficient of viscosity, and τ_0 is the friction tension at the wall.

The first term in equations (2) and (3), represents the contribution of the molecular viscosity, while the second one the contribution of the velocity pulsation when the friction tension τ_0 is formed. We admit the classical model of the turbulent motion near a wall i.e. the flow field is divided in three domains: the viscous sub-layer, the transition region and the region of fully developed turbulence. The notation of the dynamic coefficient of turbulent viscosity is η'= $\rho\upsilon$'. It was introduced for the first time by J. Boussinesq. We have

$$\tau' = -\rho(\overline{u'w'}) = \eta' \frac{d\bar{u}}{dz}. \tag{4}$$

Therefore the equation of motion near a rigid wall is written as

$$\left(\eta + \eta^{'}\right) \frac{d\bar{u}}{dz} = \tau_0. \tag{5}$$

Assume that this equation is available everywhere near the wall (which means that the principle of superposition of both the molecular viscosity and of the turbulent viscosity is supposed to hold), specifying that in the viscous sub-layer $\eta >> \eta'$, in the transition region η and η' have the same order of magnitude and in the field of full turbulence $\eta \ll \eta^{'}$.

For the turbulent viscosity we propose an expression of the type $\eta = \rho l u_*$ [1], where the mixing length for distances larger than the thickness d of the laminar sub-layer is expressed by the relation $l = \kappa\,(z-d)$where κ is the constant of Kármán and $u_* = \sqrt{\tau_0/\rho}$ the friction velocity. It follows that

$$\eta' = \kappa\rho\,(z-d)\,u_*, \tag{6}$$

where $\kappa = 0$ for $z < d$ and $k \neq 0$ for $z \geq d$. Introducing the dimensionless values $\zeta = zu_*/\upsilon$ and $\delta = \frac{du*}{\upsilon}$, the equation (5) is written in the dimensionless form

$$\left[1 + \kappa\,(\zeta - \delta)\right] \frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) = 1, \tag{7}$$

where $\kappa = 0$ for $\zeta < \delta$, $\kappa \neq 0$ for $\zeta \geq \delta$. The equation (7) is integrated, taking into account the value of κ and the evident condition of the adherence to the wall, $\bar{u} = 0$ for $\zeta = 0$. The integration constant is determinated provided that the transition from the laminar profile to the turbulent profile $\bar{u}/u_* = \delta$ occur. The following universal function is obtained for the profile of the turbulent velocities near a rigid smooth plane wall

$$\frac{\bar{u}}{u_k} = \begin{cases} \zeta, & \text{for} \quad 0 \leq \zeta \leq \delta, \\ \frac{1}{\kappa}\ln\left[1 + \kappa\,(\zeta - \delta)\right], & \text{for} \quad \zeta \geq \delta. \end{cases} \tag{8}$$

The constant δ introduces the influence of the molecular viscosity, while the constant κ introduces the influence of the turbulent viscosity when forming the average velocity profile. The numerical values of these constants are $\kappa = 0.4$ and $\delta = 7.5$ [2].

In Table 1 we present comparatively some characteristic elements of the classical theories of the turbulence and of our theory presented herein. In Table 2 we reproduced the expressions of the universal profile of the average velocity $u_+ = \bar{u}/u_*$ of the turbulent flow near a wall.

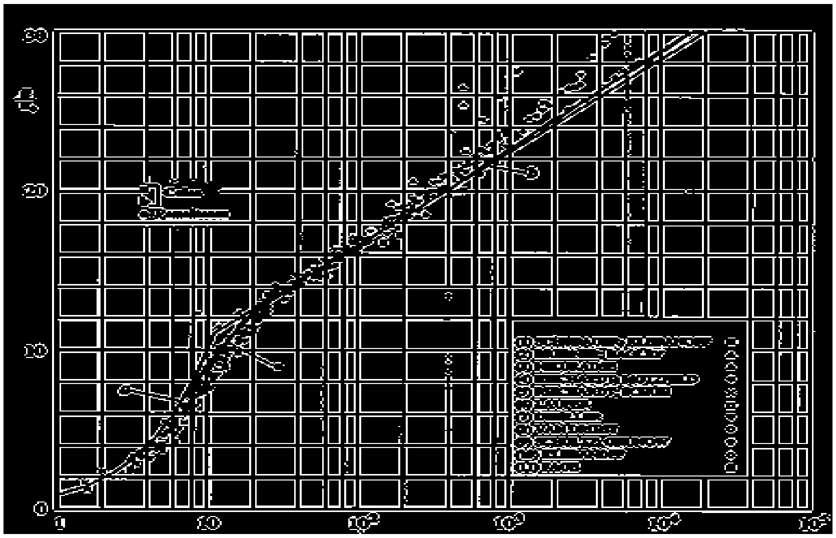
Table 1

Mixing length	Dynamic coefficient of turbulent viscosity	Authors
$l = \kappa z$	$\mu' = \kappa^2 \rho z^2 \dfrac{d\bar{u}}{dz}$	Prandtl
$l = \kappa\left(\dfrac{d\bar{u}}{dz}\right)\Big/\left(\dfrac{d^2\bar{u}}{dz^2}\right)$	$\mu' = \kappa^2 \rho\left(\dfrac{d\bar{u}}{dz}\right)^3\Big/\left(\dfrac{d^2\bar{u}}{dz^2}\right)^2$	Kármán
$l = \kappa(z-d)$	$\mu' = \kappa\rho(z-d)u_*$	Panaitescu

Table 2

Formula $\bar{u}_+ = \dfrac{\bar{u}}{u_*}$	Range	Authors
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 11.5$ $\zeta > 11.5$	Prandtl and Taylor
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 2.5\ln\zeta + 5.1$	$0 \le \zeta \le 11.1$ $\zeta > 11.1$	Coles
$\bar{u}_+ = \zeta$ $\bar{u}_+ = 5.0\ln\zeta - 3.05$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 5$ $5 < \zeta \le 30$ $\zeta > 30$	Kármán
$\bar{u}_+ = 14.53\,\mathrm{th}\,(\zeta/14.53)$ $\bar{u}_+ = 2.5\ln\zeta + 5.5$	$0 \le \zeta \le 27.5$ $\zeta > 27.5$	Rannie
$\dfrac{d\bar{u}_+}{d\zeta} = \dfrac{1}{1 - n^2\bar{u}_*\zeta\left[1 - \exp\left(-n^2\bar{u}_*\zeta\right)\right]}$ $\bar{u}_+ = 2.78\ln\zeta + 3.8$	$0 \le \zeta \le 26$ $n = 0.124$ $\zeta > 26$	Deissler
$\zeta = \bar{u}_+ + A\left[\exp B\bar{u}_+ - 1 - B\bar{u}_+ - \dfrac{1}{2}(B\bar{u}_+)^2 - \dfrac{1}{6}(B\bar{u}_+)^3 - \dfrac{1}{24}(B\bar{u}_+)^4 - \mathrm{K}\right]$	for any ζ $A = 0.1108$ $B = 0.4$	Spalding
$\dfrac{d\bar{u}_+}{d\zeta} = \dfrac{1}{1 + \left\{1 + 4\kappa^2\zeta\left[1 - \exp\left(-\dfrac{\zeta}{A}\right)\right]^2\right\}^{1/2}}$	for any ζ $A = 26$	Van Driest
$u_+ = \begin{cases} \zeta & 0 \le \zeta \le \delta \\ \dfrac{1}{\kappa}\ln\left[1 + \kappa(\zeta - \delta)\right] + \delta & \zeta > \delta \end{cases}$	for any ζ $\kappa = 0.4$ $\delta = 7.5$	Panaitescu

Fig. 1. Universal velocity profiles.



Comparing the profile (8) proposed by us with the experimental data of a large number of research workers [3], [4] a very good agreement is obtained (fig. 1).

2. Deduction of the expression of universal function of correlation between the longitudinal and transversal components of the pulsation of turbulent velocity near a wall

The equation of motion near a wall (3) is transcribed in the dimensionless form as

$$\frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) - \frac{\overline{u'w'}}{u_*^2} = 1. \tag{9}$$

For the universal law of distribution of average velocities (8) it follows

$$-\frac{\overline{u'w'}}{u_*^2} = 1 - \frac{d}{d\zeta}\left(\frac{\bar{u}}{u_*}\right) = \frac{\kappa\,(\zeta-\delta)}{1+\kappa\,(\zeta-\delta)}. \tag{10}$$

We specify that the function (10) makes sense for $\zeta \geq \delta$ only therefore except for the viscous sub-layer, which is very well confirmed experimentally [1], [4].

3. Conclusions

The form achieved for the average velocity profile and the comparison with other results of the literature lead to the folowing conclusions:

- the average velocity profile of the turbulent flow near a smooth wall is expressed by a continuous function together with its first order derivative on the whole field of definition, i.e., for any distance from the wall (other formulae, e.g. the ones proposed by Prandtl and Taylor, Coles, Kármán, Rannie, Deissler are represented by functions that have angular points at the common points of the three regions, which is inconsistent with the physical reality);

- the logarithmic form obtained for the average velocity profile (8) is analogous to the classic one of Prandtl – Kármán, but it is more general in the sense that it is valid for the region of transition from the laminar motion in the viscous sub-layer, to the turbulent motion in the fully developed turbulent layer;

- we estimate that our form (8), for the average velocity profile, is simpler (e.g. as compared to the formulae proposed by Diessler, Spalding, Van Driest for the description of the distribution of the average velocity over all distances from the wall, therefore in the region of transition as well);

- the constants κ and δ, occurring in the expression of the universal function of the average velocity profile, have a physical importance well determinated, previously specified;

- the universal function of the correlation between the longitudinal and transversal components of the velocity pulsations (10) is obtained analytically and is confirmed by experimental results.

References

[1] V. Panaitescu, Method of building the general profile of the turbulent flow average velocity near a rigid wall, St. Cerc. Mec. Apl., **33**, 4 (1974), 679 - 694 (Romanian)

[2] V. Panaitescu, Sur les lois du mouvement turbulent dans les conduites lisses et rugueuses, Bul. Inst. Politech., Bucureşti, **49**, 2 (1977), 57 -62.

[3] V. Panaitescu, Heat and mass transfer in smooth pipes, St. Cerc. Mec. Apl., **36**, 6 (1977), 811 - 818 (Romanian)

[4] J. Kestin, P. D. Richardson, Heat and transfer across turbulent incompressible boundary layers, Intern J. Heat Mass Trans., (1963), 147 - 189.

CONSTRUCTION OF THE DYNAMICAL ABSORBERS FOR SEVERAL FINITE DEGREES OF FREEDOM SYSTEMS

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Abstract The general formulae for a several finite degrees of freedom dynamical absorber computation are established. A numerical example is also presented.

1. INTRODUCTION

It is known the role of the dynamical absorber which being attached to a system with one degree of freedom, called the *main system*, is excited at resonance, making this to stay at rest, and the absorber vibrates with limited amplitude at the resonance frequency of the main system.

In [4] the existence of a dynamical absorber for the elastically hanged rigid excited at resonance at one of its six eigenfrequencies was studied.

In this paper the existence of a dynamical absorber for a system with n degrees of freedom excited at the resonance at one of the eigenfrequencies will be studied.

2. GENERAL EQUATIONS

Consider the system with n degrees of freedom (fig. 1), called the *main system*, which is acted by the exciting force $F_0 \cos(\omega t)$ and let p_k, $k = 1, 2, ..., n$ be the eigenpulsations of this system. In order to avoid the resonance in the case $\omega = p_k$, one adds the secondary system formed by the mass m_{n+1} and the spring of stiffness k_{n+1}. In order to dimension this system as a dynamical absorber one writes the equations of motion

$$\left\{\begin{array}{c} m_1\ddot{x}_1 + (k_1 + k_2) - k_2x_2 = 0, \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3) - k_3x_3 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m_n\ddot{x}_n - k_nx_{n-1} + (k_n + k_{n+1})\,x_n - k_{n+1}x_{n+1} = F_0\cos(\omega t)\,, \\ m_{n+1}\ddot{x}_{n+1} - k_{n+1}x_n + k_{n+1}x_{n+1} = 0. \end{array}\right. \tag{1}$$

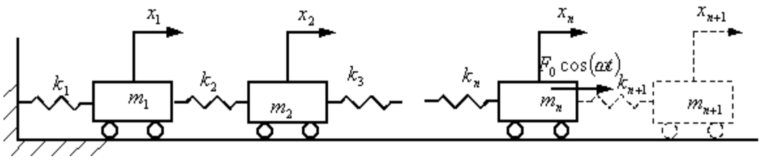

Fig. 1. The studied model.

The stationary solution of the system (1) being of the form

$$x_j = a_j cos\left(\omega t\right); \quad j = 1, 2, ..., n, \tag{2}$$

one obtains the algebraic system

$$\begin{cases} \left(-m_1\omega^2 + k_1 + k_2\right) a_1 - k_2 a_2 = 0, \\ -k_2 a_1 + \left(-m_2\omega^2 + k_2 + k_3\right) a_2 - k_3 a_3 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -k_n a_{n-1} + \left(-m_n\omega^2 + k_n + k_{n+1}\right) a_n - k_{n+1} a_{n+1} = F_0, \\ -k_{n+1} a_n + \left(-m_{n+1}\omega^2 + k_{n+1}\right) a_{n+1} = 0. \end{cases} \tag{3}$$

Using the notation

$$\delta_0 = 1; \quad \delta_1 = k_1 + k2 - m_1\omega^2, \tag{4}$$

$$\delta_j = \left(k_j + k_{j+1} - m_j\omega^2\right)\delta_{j-1} - k_j^2\delta_{j-2}; \quad j = 2, 3, ..., n, \tag{5}$$

$$\delta_{n+1} = \left(k_{n+1} - m_{n+1}\omega^2\right)\delta_n - k_{n+1}^2\delta_{n-1}, \tag{6}$$

$$\Delta_j = F_0\left(k_{n+1} - m_{n+1}\omega^2\right)\delta_{j-1}\prod_{i=j+1}^{n} k_i; \quad j = 1, 2, ..., n-1 \ , \tag{7}$$

$$\Delta_n = F_0\left(k_{n+1} - m_{n+1}\omega^2\right)\delta_{n-1}; \quad \Delta_{n+1} = F_0 k_{n+1}\delta_{n-1}, \tag{8}$$

from the system (2) one obtains the amplitudes

$$a_j = \frac{\Delta_j}{\delta_{n+1}}; \quad j = 1, 2, ..., n, n+1. \tag{9}$$

3. DIMENSIONING THE DYNAMICAL ABSORBER

From the relations (5)-(9) it follows that, at resonance $\omega = p_k$ if, in addition, the condition $p_k^2 = \frac{k_{n+1}}{m_{n+1}}$ is fulfilled, the amplitudes a_j, $j = 1, 2, ..., n$ in the main system are zero. Then, from the relation (5) one deduces the equality $\delta_{n+1} = k_{n+1}^2\delta_{n-1}$ and the amplitude a_{n+1} of the dynamical absorber is $a_{n+1} = -\frac{F_0}{k_{n+1}}$.

4. THE TRANSIENT REGIME

From the equation $\delta_{n+1} = 0$ one deduces the $n+1$ eigenpulsations of the system obtained from the main system at which we added the dynamical absorber. It follows that in the transient regime the new system has $n+1$ resonances. Moreover, we obtain that the amplitudes a_j, $j = 1, 2, ..., n+1$ become equal to zero at the points which are the zeros of the equations $\delta_{j-1} = 0$, i.e. the amplitudes a_j, when the pulsation ω varies become equal to zero at $j-1$ points, too. These points, as it follows from the relations (7), (8), coincide for the amplitudes a_n, a_{n+1}.

5. THE DAMPING ABSORBER

In order to avoid the resonances in the transient regime, the absorber is endowed with the damping c. Using the notation

$$[\mathbf{M}] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & m_{n+1} \end{bmatrix}, \ [\mathbf{C}] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & c & -c \\ 0 & 0 & \cdots & 0 & -c & c \end{bmatrix},$$

$$[\mathbf{K}] = \begin{bmatrix} k_1 + k2 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & k_2 + k3 & -k_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & k_n & k_n + k_{n+1} & -k_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n+1} & k_{n+1} \end{bmatrix},$$

$\{\mathbf{X}\} = \{\ x_1 \quad x_2 \quad \cdots \quad x_n \quad x_{n+1}\ \}^T$, $\{\mathbf{J}\} = \{\ 0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0\ \}^T$,
the matrix differential equation of such a system's vibrations, reads

$$[\mathbf{M}]\left\{\ddot{\mathbf{X}}\right\} + [\mathbf{C}]\left\{\dot{\mathbf{X}}\right\} + [\mathbf{K}]\{\mathbf{X}\} = F_0\{\mathbf{J}\}\cos(\omega t). \tag{11}$$

In this case, the variations of the amplitudes are limited and they are deduced by solving the equation (11) by complex numbers method.

6. NUMERICAL EXAMPLE

Consider that the main system has three masses: $m_1 = 3m$, $m_2 = 2m$ and $m_3 = m$. All the stiffness of the coupling springs are equal to k. The absorber has the mass m_4 and it is coupled to the main system by a spring of stiffness k_4. The damping between the absorber and the main system is supposed to be c. The main system is acted upon by an exciting force, say $F_0 \cos(\omega t)$. The equations of motion read

$$\left\{\begin{array}{c} 3m\ddot{x}_1 = -kx_1 + k(x_2 - x_1), \\ 2m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2), \\ m\ddot{x}_3 = -k(x_3 - x_2) + k_4(x_4 - x_3) + c(\dot{x}_4 - \dot{x}_3) + F_0\cos(\omega t), \\ m_4\ddot{x}_4 = -k_4(x_4 - x_3) - c(\dot{x}_4 - \dot{x}_3). \end{array}\right. \tag{12}$$

The motions for the main system are given by the following system

$$\left\{\begin{array}{c} 3m\ddot{x}_1 = -kx_1 + k\left(x_2 - x_1\right), \\ 2m\ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right), \\ m\ddot{x}_3 = -k\left(x_3 - x_2\right) + F_0\cos\left(\omega t\right). \end{array}\right. \tag{13}$$

The solution for the system (13) being of the form (2), where $j = 1, 2, 3$, one obtains the following algebraic system

$$\left\{\begin{array}{c} -\left(3m\omega^2 + 2k\right)a_1 - ka_2 = 0, \\ -ka_1 + \left(-2m\omega^2 + 2k\right)a_2 - ka_3 = 0, \\ -ka_2 + \left(-m\omega^2 + k\right)a_3 = F_0. \end{array}\right.$$

The resonance pulsations of the main system, follow from the equation

$$\begin{vmatrix} 2k - 3m\omega^2 & -k & 0 \\ -k & 2k - 2m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0,$$

or equivalently, $6u^3 - 16u^2 + 10u - 1 = 0$, where $u = \dfrac{m\omega^2}{k}$.

The solutions for (6) are $u_1 = 0.1231$, $u_2 = 0.7580$ and $u_3 = 1.7855$, which lead to $\omega_1 = 0.35085\sqrt{\dfrac{k}{m}}$, $\omega_2 = 0.87063\sqrt{\dfrac{k}{m}}$ and $\omega_3 = 1.33623\sqrt{\dfrac{k}{m}}$.

Assume that $m = 1\,[\mathrm{kg}]$, $F_0 = 100\,[\mathrm{N}]$ and $k = 100\left[\dfrac{\mathrm{N}}{\mathrm{m}}\right]$. In these conditions, we obtain $\omega_1 = 3.5085\left[\mathrm{s}^{-1}\right]$, $\omega_2 = 8.7063\left[\mathrm{s}^{-1}\right]$ and $\omega_3 = 13.3623\left[\mathrm{s}^{-1}\right]$. In order to avoid the third resonance, which corresponds to ω_3, we select an absorber for which $k_4 = m_4\omega_3^2$. Taking $k_4 = 50\left[\dfrac{\mathrm{N}}{\mathrm{m}}\right]$, it follows $m_4 = 0.28\,[\mathrm{kg}]$. We shall write that $k_4 = \mu_1 k$ and $m_4 = \mu_2 m$, with $\mu_1 = 0.5$ and $\mu_2 = 0.28$.

In the case of no damping absorber the equations of motion lead to the following algebraic system

$$\left\{\begin{array}{c} \left(-3m\omega^2 + 2k\right)a_1 - ka_2 = 0, \\ -ka_1 + \left(-2m\omega^2 + 2k\right)a_2 - ka_3 = 0, \\ -ka_2 + \left[k\left(1 + \mu_1\right) - m\omega^2\right]a_3 - \mu_1 ka_4 = F_0, \\ -\mu_1 ka_3 + \left(\mu_1 k - \mu_2 m\omega^2\right)a_4 = 0. \end{array}\right. \tag{16}$$

From the first relation (16) we have $a_2 = \frac{2k - 3m\omega^2}{k}a_1$ and from the second relation (16) taking into account the expression of a_2 we obtain $a_3 = \frac{3k^2 - 10km\omega^2 + 6m^2\omega^4}{k^2}a_1$. The fourth relation(16) with the account of these relations for a_2 and a_3 yields $a_4 = \frac{\mu_1}{\mu_1 k - \mu_2 m\omega^2}\frac{3k^2 - 10km\omega^2 + 6m^2\omega^4}{k}a_1$. Replacing these values in the third equation (16), we have $a_1 = F_0\frac{k^2\left(\mu_1 k - \mu_2 m\omega^2\right)}{A}$, where $A = 6\mu_2 m^4\omega^8 - \left(6\mu_1 + 16\mu_2 + 6\mu_1\mu_2\right)km^3\omega^6 + (16\mu_1 + 10\mu_2 +$ $+10\mu_1\mu 2)\,k^2m^2\omega^4 - \left(10\mu_1 + \mu_2 + 3\mu_1\mu_2\right)k^3m\omega^2 + \mu_1 k^4$. The resonance appears when $A = 0$. Denoting again $u = \dfrac{m\omega^2}{k}$, one obtains the equation

$$6\mu_2 u^4 - (6\mu_1 + 16\mu_2 + 6\mu_1\mu_2)\, u^3 + (16\mu_1 + 10\mu_2 + \\ +10\mu_1\mu 2)\, u^2 - (10\mu_1 + \mu_2 + 3\mu_1\mu_2)\, u + \mu_1 k^4 = 0$$

with the roots: $u_1 = 0.1129$, $u_2 = 0.6742$, $u_3 = 1.4265$ and $u_4 = 2.7385$. At the end, the resonance pulsations are: $\omega_1 = 3.3601\,[\mathrm{s}^{-1}]$, $\omega_2 = 8.211\,[\mathrm{s}^{-1}]$, $\omega_3 = 11.9436\,[\mathrm{s}^{-1}]$ and $\omega_4 = 16.5484\,[\mathrm{s}^{-1}]$.

In fig. 2 we represented the diagrams $a_i = a_i(\omega)$. It is easy to note that the equation $a_1 = 0$ has exactly one root, say $\omega = \sqrt{\frac{50}{28}} \approx 13.363\,[\mathrm{s}^{-1}]$. The equation $a_2 = 0$ has two roots: $\omega = \sqrt{\frac{50}{28}} \approx 13.363\,[\mathrm{s}^{-1}]$ and $\omega = \sqrt{\frac{200}{3}} \approx 8.165\,[\mathrm{s}^{-1}]$. The equations $a_3 = 0$ and $a_4 = 0$ have the same roots: $\omega = \sqrt{\frac{1000+\sqrt{28000}}{12}} \approx 11.288\,[\mathrm{s}^{-1}]$ and $\omega = \sqrt{\frac{1000-\sqrt{28000}}{12}} \approx 6.264\,[\mathrm{s}^{-1}]$.

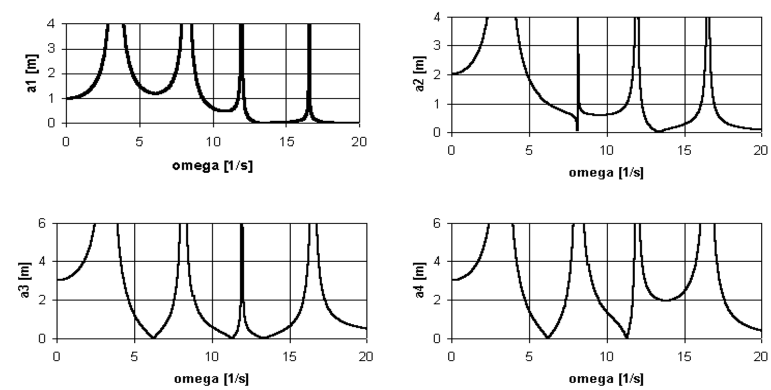

Fig. 2. The diagrams $a_i = a_i(\omega)$ for $0 \le \omega \le 20\,[\mathrm{s}^{-1}]$.

For the **damping absorber** the solution of (12) is of the form $x_i = B_i \cos(\omega t) + D_i \sin(\omega t)\,;\ i = \overline{1,4}$, therefore one obtains the algebraic system

$$\begin{cases} \left(2k - 3m\omega^2\right) B_1 - kB_2 = 0, \\ -kB_1 + \left(2k - 3m\omega^2\right) B_2 - kB_3 = 0, \\ -kB_2 + \left(k + k_4 - m\omega^2\right) B_3 - k_4 B_4 + c\omega D_3 - c\omega D_4 = F_0, \\ -k_4 B_3 + \left(k_4 - m_4\omega^2\right) - c\omega D_3 + c\omega D_4 = 0, \\ \left(2k - 3m\omega^2\right) D_1 - kD_2 = 0, \\ -kD_1 + \left(2k - 2m\omega^2\right) D_2 - kD_3 = 0, \\ -c\omega B_3 + c\omega B_4 - kD_2 + \left(k + k_4 - m\omega^2\right) - k_4 D_4 = 0, \\ c\omega B_3 - c\omega B_4 - k_4 D_3 + \left(k_4 - m_4\omega^2\right) D_4 = 0. \end{cases}$$

The amplitudes of the vibrations are $a_i = \sqrt{B_i^2 + D_i^2};\quad i = \overline{1,4}$.

In fig. 3 we presented the diagrams $a_i = a_i(\omega)$ for four cases of damping: $c = 0\left[\frac{\mathrm{Ns}}{\mathrm{m}}\right]$, $c = 0.1\left[\frac{\mathrm{Ns}}{\mathrm{m}}\right]$, $c = 1\left[\frac{\mathrm{Ns}}{\mathrm{m}}\right]$ and $c = 10\left[\frac{\mathrm{Ns}}{\mathrm{m}}\right]$.

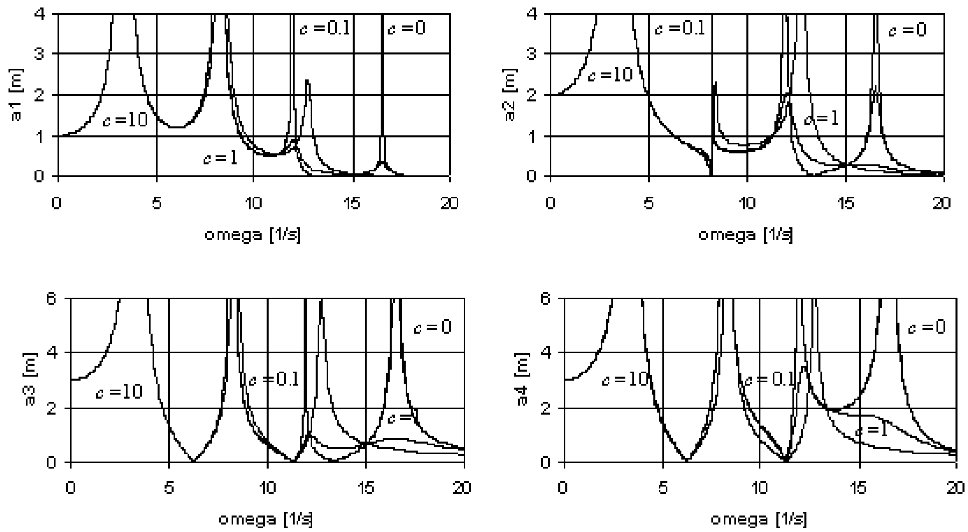

Fig. 3. The diagrams $a_i = a_i(\omega)$ for $0 \leq \omega \leq 20\left[\mathrm{s}^{-1}\right]$.

7. CONCLUSIONS

Adding a dynamical absorber to the main system, the resonance pulsation becomes non-dangerous, and the amplitudes for these pulsations are finite.

Remark the existence of two common points for the diagrams of a_1, a_2 and a_3, no matter the value for the parameter c, and the existence of only one common point for the diagram a_4 in the same conditions.

The two common points for the diagrams a_1, a_2 and a_3 have the coordinates (12.25 0.45) and (15 0.057), respectively. The common point for the diagram a_4 has the coordinates (13.46 1.975).

These results are very similar to those known from the one degree of freedom systems.

References

[1] Dimentberg, F., Teorya vintov i ee prilozhenia, Nauka, Moskwa, 1978.

[2] Pandrea, N., Pandrea, M., Determination of the elastic center by the method of the relative displacement, St. Cerc. Mec. Apl., **47**, 5 (1987), 409-425. (Romanian)

[3] Pandrea, N., Vibrations of the rigid with elastic links, St. Cerc. Mec. Apl., **47**, 6 (1987), 499-511. (Romanian)

[4] Pandrea, N., Pandrea, M., Bădău, C., A complex dynamic damper, 9th World Congress on the Theory of Machines and Mechanisms **2**, Milano, 1995.

[5] Pandrea, N., Elements of mechanics of solids în Plückerian coordinates, Ed. Academiei Române, Bucureşti, 2000. (Romanian)

SPREADING OF ACTIVE WORMS USING RANDOM SCANNING

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Abstract Active worms have been a persistent security threat on the Internet since the Morris worm in 1988. Automatically spreaded worms or active worms, like Code Red and Nimda, can flood large part of Internet hosts in a very short amount of time. Modeling the spread of such worms can help understand how they multiply and also help monitoring and defending effectively against the propagation of worms. In this paper, we present a mathematical model which characterizes the propagation of worms that are using random scanning in wide and local area networks. We also present Code Red v2 worm as an example, giving a quantitative analysis for monitoring, detecting and defending against worms of this type.

1. INTRODUCTION

Active worms have been a persistent security threat on the Internet since the Morris worm arose in 1988. The Code Red and Nimda worms infected hundreds of thousands of systems, and cost both the public and private sectors millions of dollars. In this paper, we present a model known as the Analytical Active Worm Propagation (AAWP) model, which characterizes the propagation of worms that employ random scanning. We take advantage of a discrete time model and deterministic approximation to describe the spread of active worms. We also compare AAWP with epidemiological model and Weaver's simulator, then present an extended AAWP model (LAAWP) to characterize the spread of a worm that employs the localized scanning strategy, which is used by the Code Red II and Nimda worms.

2. SPREAD OF ACTIVE WORMS

Briefly, active worm propagation can be described as folowing:

1. attacker releases the worm into Internet;
2. worm starts probing for vulnerable machines;

3. identified vulnerable machines become infected and start spreading the worm as well;

4. as soon as worm outbreak is detected, sysadmins are aplying patches which repairs security holes in order to slow down or stop further spreading.

Worm's authors goal is to perform first three steps as quick as possible and to infect as many machines as possible before step 4. To speed up the spread of active worms, Weaver presented the "hitlist" idea. Long before an attacker releases the worm, he/she gathers a list of potentially vulnerable machines with good network connections. After the worm has been fired onto an initial machine on this list, it begins scanning down the list. Hence, the worm will first start infecting the machines on this list which will become infected very soon, then will start to scan for randomly choosed machines. In this paper we do not consider the amount of time it takes a worm to infect the hitlist since the hitlist can be acquired well before a worm is released and be infected in a very short period of time. Table I shows the parameters involved in the spread of active worms. There are several different scanning mechanisms that active worms employ, such as random, local subnet, permutation and topological scanning.

Parameters	**Notation**	**Description**
Number of vulnerable machines	N	The number of vulnerable machines
Size of hitlist	h	The number of infected machines at the beginning of the spread of active worm
Scanning rate	s	The average number of machines scanned by an infected machine per unit time
Death rate	d	The rate at which an infection is detected on a machine and eliminated without patching
Patching rate	p	The rate at which an infected or vulnerable machine becomes invulnerable

Table 1. The parameters for spreading of active worms.

In this paper we focus on two mechanisms, random scanning and local subnet scanning. In random scanning, it is assumed that every computer in the Internet is just as likely to infect or be infected by other computers. Such a network can be pictured as a fully-connected graph in which the nodes represent computers and the arcs represent connections (neighboring-relationships) between pairs of nodes. This topology is called "homogeneous mixing" in the theoretical epidemiology. AAWP model is used to model random scans. In local subnet scanning, computers also connect to each other directly, forming "homogeneous mixing". However, instead of selecting targets randomly, the worms preferentially scan for hosts on the "local" address space. For example, the Nimda worm selects target IP addresses as follows:

- 50% of the time, an address with the same first two octets will be chosen.

- 25% of the time, an address with the same first octet will be chosen.
- 25% of the time, a random address will be chosen.

3. MODELING SPREAD OF WORMS THAT ARE USING RANDOM SCANNING

AAWP is using the discrete time and continuous state deterministic approximation model. In this section, we first describe in detail the AAWP model , then compare it to the epidemiological model and Weaver's simulator, finally use it to simulate the Code Red v2 worm.

3.1. DETERMINISTIC APPROXIMATION MODELING

We assume that worms can simultaneously scan many machines and will not re-infect a machine that is already infected. We also assume that the machines on the hitlist are already infected at the start of the worm's propagation. Suppose that an active worm takes one time tick to complete infection. That is, when one scan hits a machine, regardless of whether this machine is vulnerable, invulnerable, infected or with an unused IP address, the time it takes for the worm to finish communicating with this machine is one time tick. This assumption might not be realistic, but it can simplify the model without significantly affecting the results.

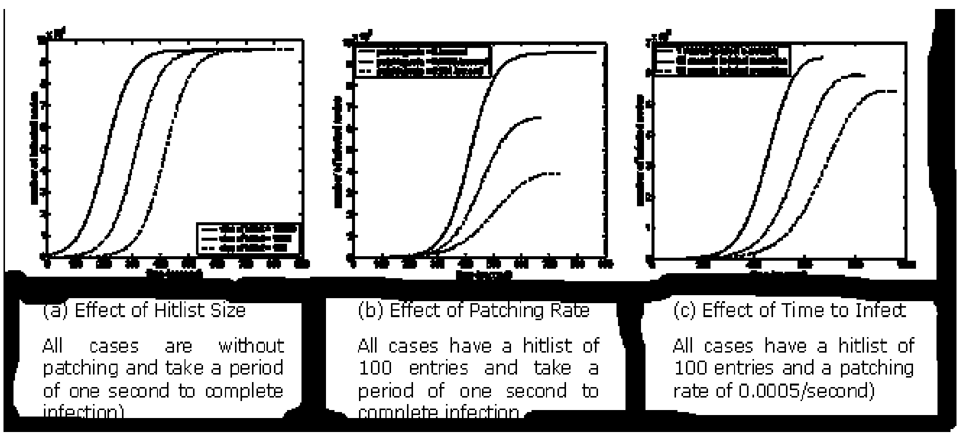


Fig. 1. Modeling the spread of worms that use random scanning (for 1,000,000 vulnerable machines, a scanning rate of 100 scans/second, and a death rate of 0.0001/second).

Although Internet address space is not completely connected, active worms always scan all the 2^{32} addesses. So for random scanning, the probability that any computer is hit by one scan is 2^{-32} .

Let m_i and n_i denote the total number of vulnerable machines (including the infected ones) and the number of infected machines at time tick i ($i \geq 0$) respectively. Before the active worms spread (at $i = 0$), we have $m_0 = N$ and $n_0 = h$.

Theorem 3.1. *If there are m_i vulnerable machines (including the infected ones), and n_i infected computers, then on average, the next time tick will have $(m_i - n_i)[1 - (1 - 2^{-32})^{sn_i}]$ newly infected machines, where s is the scanning rate.*

Proof Let e_i denote the number of newly infected machines at time tick i ($i \geq 0$). Then n_i infected machines can generate sn_i scans in an attempt to infect other machines. So, if we can prove that $E\{e_{i+1}/k\} = (m_i - n_i)[1-(1-2^{-32})^k]$ for any k ($k > 0$) scans, then the equation also holds when $k = sn_i$. We prove the above equation by induction on k. When $k = 1$, since there are $(m_i - n_i)$ vulnerable machines that have not yet been infected, the probability that one scan can add a newly infected machine is $(m_i - n_i)2^{-32}$, which is equivalent to $(m_i - n_i)[1 - (1 - 2^{-32})^1]$. Suppose that the theorem is true for $k = j$, i.e. $E\{e_{i+1}/k = j\} = (m_i - n_i)[1 - (1 - 2^{-32})^j]$. Then, when $k = j + 1$, we divide $j + 1$ scans into two parts: the first j scans and the last scan. There are two possibilities for the last scan: adding a newly infected machine or not. Let $Y = 1$ if the last scan hits a vulnerable machine that has not yet been infected and let $Y = 0$ otherwise. Then,

$$E\{e_{i+1}/k = j+1\} = (E\{e_{i+1}/k = j\}+1)P(Y = 1)+E\{e_{i+1}/k = j\}\cdot P(Y = 0) =$$

$$(E\{e_{i+1}/k = j\} + 1)(m_i - n_i - E\{e_{i+1}/k = j\})2^{-32} + E\{e_{i+1}/k = j\}$$

$$[1 - (m_i - n_i - E\{e_{i+1}/k = j\})2^{-32})] = (m_i - n_i)2^{-32} + (1 - 2^{-32})$$

$E\{e_{i+1}/k = j\} = (m_i-n_i)[1-(1-2^{-32})^{j+1}]$, which means it is also true for $k = j+1$. Therefore, when $k = sn_i$, $E\{e_{i+1}/k = sn_i\} = (m_i-n_i)[1-(1-2^{-32})^{sn_i}]$. That is, on the next time tick there will be $(m_i - n_i)[1 - (1 - 2^{-32})^{sn_i}]$ new infected machines. Given the death rate d and the patching rate p, on the next tick there will be $dn_i + pn_i$ infected machines that will change to either vulnerable machines without being infected or invulnerable machines, and the total number of vulnerable machines (including the infected ones) will be reduced to $(1 - p)m_i$. Therefore, on the next time tick the number of total infected machines will be $n_{i+1} = n_i+(m_i-n_i)[1-(1-2^{-32})^{sn_i}]-(d+p)n_i$. At the same time, $mi + 1 = (1-p)m_i$, which means $m_i = (1-p)^i m_0 = (1-p)^i N$. That is

$$n_{i+1} = (1 - d - p)n_i + [(1 - p)^i N - n_i][1 - (1 - 2^{-32})^{sn_i}], \tag{1}$$

where $i \geq 0$ and $n_0 = h$. The recursion process will stop when there are no more vulnerable machines left or when the worm cannot increase the total number of infected machines. ■

Using equation (1), we can find the characteristics of the active worms' spreading. For example, fig. 1(a) shows the propagation of the active worms with different hitlist sizes. As the size of the hitlist increases, it takes the worms less time to spread. Fig. 1(b) depicts another example. As the patching rate grows, the spread of active worms slows down. It should be noted that because the patching rate $p > 0$, the two slower curves return to zero at the end. At the beginning, we assume that it takes the worms one time tick to infect a machine. To display the effect of the amount of time it takes to infect a machine on the worm propagation, we change the time unit. For example, in fig. 1(c) we first draw the curve with a time interval of one second, which is the amount of time required to complete infection. If the worm needs 30 seconds to infect a machine, we set the time unit to 30 seconds and change the corresponding parameters $s,\ d,\ p$ for this period of time. In this case, the parameters will become $30s,\ 30d,\ 30p$ for a period of 30 seconds. Then, we can use the AAWP model to get the result, but now, n_i $(i \geq 0)$ expresses the number of infected machines at $30i$ seconds $(i \geq 0)$. The figure shows the effect of the time to complete infection on the worm's propagation. The worm's propagation will be slowed down as the time required to infect a machine increases.

3.2. COMPARING AAWP MODEL TO THE EPIDEMIOLOGICAL MODEL AND WEAVER'S SIMULATOR

In the epidemiological model, a nonlinear ordinary differential equation is used to measure the virus population dynamics $\frac{dn}{dt} = \beta n(1-n) - dn$, where $n(t)$ is the fraction of infected nodes, β is the birth rate (the rate at which an infected machine infects other vulnerable machines) and d is the death rate. The solution to the above equation is

$$n(t) = \frac{n_0(1-\rho)}{n_0 + (1-\rho-n_0)e^{-(\beta-d)t}}, \tag{2}$$

where $\rho = d/\beta$ and $n_0 \equiv n(t=0) = size\_of\_hitlist/N = h/N$, where from here we deduce the relationship between the birth rate and the scanning rate $\beta = N_s 2^{-32}$.

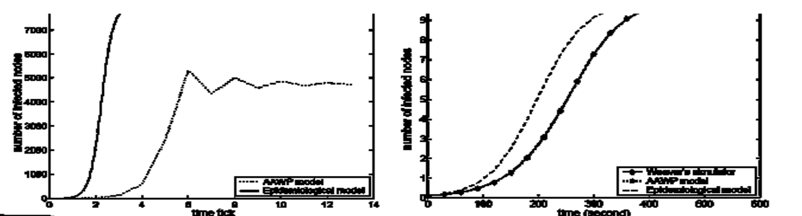
The differences between the AAWP model and the epidemiological model are:

1 the epidemiological model uses a continuous time differential equation, while the AAWP model is based on a discrete time model. AAWP may be considered more accurate because in this model, a computer cannot infect other machines before it is infected completely - compared with the epidemiological model, where a computer begins infecting other

machines even though only a "small part" of it is infected. Therefore, the speed that the worm can achieve and the number of machines that can be infected are totally different;

2 the epidemiological model considers neither the patching rate nor the time that it takes the worm to infect a machine, while the AAWP model does. During the propagation of the worm, it is possible nowadays to promptly patch the vulnerability on computers, assuming a reasonable patching rate. Moreover, different worms have different infection abilities which are reflected by the scanning rate (or the birth rate) and the time taken to infect a machine. The time required to infect a machine always depends on the size of the worm' copy, the degree of network congestion, the distance between source and destination, and the vulnerability that the worm exploit. From fig. 1(c), it can be seen that the time to infect a machine is an important factor for the spread of active worms;

3 AAWP model considers the case that the worm can infect the same destination at the same time, while the epidemiological model ignores this case. In fact, it is not uncommon for a vulnerable machine to be hit by two (or more) scans at the same time. Both models, however, try to get the expected number of infected machines, given the size of the hitlist, total number of vulnerable machines, scanning rate/birth rate and death rate.



(a) All cases are for 10,000 vulnerable machines, a hitlist with 1 entry, a scanning rate of 2147500 scans/time tick or a birth rate of 5 /time tick and a death rate of 1 /time tick. No patching and a time period of 1 time tick to complete infection for the AAWP model

(b) All cases are for 1,000,000 vulnerable machines, a hitlist with 10,000 entries and a scanning rate of 100 scans/second. A time period of 30 seconds to complete infection for Weaver's simulator and the AAWP model. A death rate of zero for both the AAWP model and the Epidemiological model. No patching for the AAWP model

Fig. 2. Comparing the AAWP model to the epidemiological model and weaver simulator.

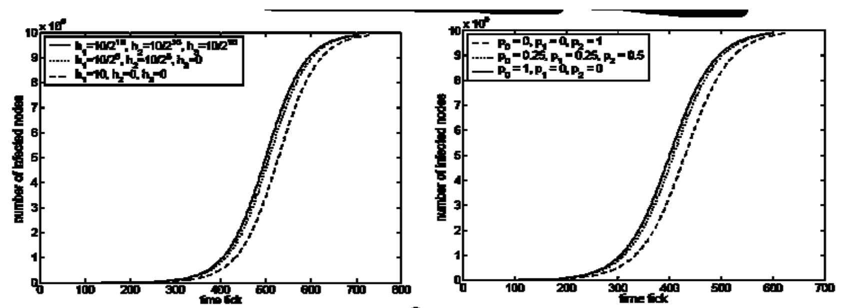
Fig. 2(a) shows the comparison between these two models with 10,000 vulnerable machines, a hitlist with 1 entry, a birth rate of 5 /time tick and a death rate of 1 /time tick. It takes the epidemiological model about 4 time ticks to enter an equilibrium stage, while the AAWP model needs about 10 time ticks. Moreover, after entering the equilibrium stage, the epidemiological model totally infects 8,000 vulnerable machines (occupying 80% of all vulnerable machines), while the AAWP model infects about 4,750 vulnerable machines (occupying 47.5% of all vulnerable machines). This difference may explain the low level of worm prevalence in attacks analized so far.

Weaver wrote a small, abstract simulator of a Warhol worm's spread. This simulator uses a 32-bit, 6-round variant of RC5 to generate all permutations and random numbers. For the assumption presented above, only one condition of the simulator was modified: all "newly" infected machines on a previous time tick will be activated at the same time on the current time tick, other than based on different clocks.

Fig. 2(b) shows the growing of infected nodes with time for the two models and Weaver's simulator, which have the following parameters: a total of 1,000,000 vulnerable machines, a hitlist of size 10,000, a scanning rate of 100 scans/second, a death rate of zero, no patching, and a time period of 30 seconds to infect one machine. This figure shows that the AAWP model and Weaver's simulator results overlap. While AAAWP model and Weaver's simulator take about 6 minutes to infect 90% of the vulnerable machines, the epidemiological model only takes about 5 minutes.

4. MODELING THE SPREAD OF ACTIVE WORMS THAT USE LOCAL SUBNET SCANNING

Instead of simply selecting destinations at random, the Code Red II and the Nimda worms preferentially search for targets on the "local" address space. Local AAWP (LAAWP) model extends AAWP to understand the characteristics of the spread of active worms that employ local subnet scanning.



(a) Local subnet scanning with different hitlist distributions. All cases are for $p_0 = 0.25, p_1 = 0.25, p_2 = 0.5$

(b) Local subnet scanning with an uneven hitlist distribution. All cases are for $h_1 = 10, h_2 = \frac{40}{2^8 - 1}, h3 = \frac{50}{2^{16} - 2^8}$

Fig. 3. Modeling the spread of active worms that employ local subnet scanning (All cases are for 1,000,000 vulnerable machines which are evenly

distributed to every subnet, a scanning rate of 100 scans/time tick and a time period of 1 time tick to complete infection).

As the AAWP model, the LAAWP model uses deterministic approximation. We focus on the active worms' scanning policy and ignore both the death rate and the patching rate to simplify the model. The function of firewalls is not considered, either. Suppose that a worm scans the Internet as follows:

- p_0 of the time, a random address will be chosen;
- p_1 of the time, an address with the same first octet will be chosen;
- p_2 of the time, an address with the same first two octets will be chosen, where $p_0 + p_1 + p_2 = 1$. We can regard random scanning as one special case of local subnet scanning, when $p_0 = 1,\ p_1 = 0,$ and $p_2 = 0$.

Assume that the vulnerable machines are evenly distributed in every subnet which is identified by the first two octets. The subnets can be classified into three different kinds of networks:

- a „special" subnet (denoted by Subnet type 1), which always has a larger hitlist size;
- 2^8-1 subnets having the same first octet as the "special" subnet (denoted by Subnet type 2);
- other $2^{16} - 2^8$ subnets (denoted by Subnet type 3).

Different kinds of networks have hitlists of different sizes. In the same type of subnet, all networks have the same hitlist size. Let $h_1,\ h_2,\ h_3$ denote the size of the hitlist in Subnet type 1, 2, and 3, respectively, then let $b_1,\ b_2,\ b_3$ denote the average number of infected machines in Subnet type 1, 2, and 3, respectively and let $k_1,\ k_2,\ k_3$ denote the average number of scans hitting Subnet type 1, 2, and 3, respectively. Then at some time tick, the relationship between the average number of scans hitting Subnet type and the average number of infected machines in different Subnets is:

$$k_1 = p_2 s b_1 + p_1 s[b_1 + (2^8 - 1)b_2]/2^8 + p_0 s[b_1 + (2^8 - 1)b_2 + (2^{16} - 2^8)b_3]/2^{16};$$

$$k_1 = p_2 s b_2 + p_1 s[b_1 + (2^8 - 1)b_2]/2^8 + p_0 s[b_1 + (2^8 - 1)b_2 + (2^{16} - 2^8)b_3]/2^{16};$$

$$k_1 = p_2 s b_3 + p_1 s b_3 + p_0 s[b_1 + (2^8 - 1)b_2 + (2^{16} - 2^8)b_3]/2^{16}.$$

For $k_i,\ (i = 1,\ 2,\ 3)$, the first item is the average number of scans coming from the local subnet (with the same first two octets). The second item is the average number of scans coming from neighboring subnets (with the same first octet). And the last item is the average number of scans coming from global subnets. In every subnet the scans will randomly hit targets, which can be modeled by the AAWP model. The total number of machines will be 2^{16} instead of 2^{32} and the total number of scans will be k_i. Thus, equation (1) becomes

$$b'_i = b_i + (N2^{-16} - b_i)[1 - (1 - 2^{-16})^{k_i}], \tag{3}$$

where $(i = 1,\ 2,\ 3)$ and b'_i is the number of infected machines on the next time tick. The recursion process will stop when there are no more vulnerable machines left. At some time tick, the total number of infected machines will be $b_1 + (2^8 - 1)b_2 + (2^{16} - 2^8)b_3$. Based on the above formulae, we can understand the characteristics of local subnet scanning and the effect of the hitlist's distribution. Different $p_1,\ p_2,\ p_3$ and $h_1,\ h_2,\ h_3$ can generate different patterns for the spread of worms.

Four cases are considered:

1) random scanning $p_1 = 1,\ p_2 = 0,\ p_3 = 0$. In this case $k_1 = k_2 = k_3 =$ (total number of infected machines)/2^{16} which means the distribution of the hitlist cannot effect the spread of active worms;

2) a hitlist with an even distribution $h_1 = h_2 = h_3 = 0$. This gives $k_1 = k_2 = k_3 = sb_1 = sb_2 = sb_3$. Local subnet scanning, therefore, cannot change the spread of active worms in this case;

3) similar to the Nimda worm ($p_1 = 0.25,\ p_2 = 0.25,\ p_3 = 0.5$). In this case, we select different distributions of the hitlist, just as in fig. 3(a). Evenly distributed hitlists give the best performance, while putting all hitlists together in one „special" subnet ($h_1 = 10,\ h_2 = h_3 = 0$) gives the worst performance. This figure shows that the hitlist's distribution can affect the spread of active worms;

4) local subnet scanning with a hitlist of uneven distribution (fix $h_1,\ h_2,\ h_3$ and $h_1 > h_2 > h_3$): This stands for a hitlist of uneven distribution and a centralization of more hitlist machines in the „special" subnet. However, fig. 3(b) shows that in this case local subnet scanning slows down the propagation of active worms. From the four cases above, we see that for local subnet scanning the hitlist's distribution can influence the spread of active worms, while the even distribution gives us the best performance. In addition when the hitlist is more concentrated in the „special" subnet, local subnet scanning slows down the spread of active worms. The LAAWP model implies that local subnet scanning may slow down the spread of active worms. There are two main reasons for this:

1) firewalls can protect vulnerable machines behind it. But local subnet scanning allows a single copy of a worm running behind the firewall to rapidly infect all the other local vulnerable machines;

2) one subnet always belongs to a company or organization and has a lot of similar machines. Therefore, it can be expected that if a machine has a

security hole, then there is a high probability that many other machines in the same network have the same security hole.

5. CONCLUSIONS

In this paper we present the AAWP model to analyze the characteristics of the spread of active worms. Even though the AAWP model also used deterministic approximation, it gives more realistic results when compared to the epidemiological model. AAWP model was extended to the LAAWP model to understand the spread of active worms using local subnet scanning. The distribution of the hitlist can affect the local subnet scanning policy. In particular, a worm using an evenly distributed hitlist spreads at the fastest rate. When the hitlist is concentrated in some subnet, the spread of active worms is slowed down. In the LAAWP model, the vulnerable machines are assumed to be evenly distributed in every subnet.

References

[1] R. Russell, A. Machie, *Code Red II Worm*, Incident Analysis, SecurityFocus, Tech. Rep., Aug. 2001.

[2] A. Machie, J. Roculan, R. Russell, and M. V. Velzen, *Nimda Worm Analysis*, Incident Analysis, SecurityFocus, Tech. Rep., Sept. 2001.

[3] CERT/CC, CERT Advisory CA-2001-26 Nimda Worm, http://www.cert.org/advisories/CA-2001-26.html, Sept. 2001.

[4] D. Song, R. Malan, and R. Stone, *A Snapshot of Global Internet Worm Activity*, http://research.arbornetworks.com/up media/up files/snapshot worm activity.pdf, Arbor Networks, Tech. Rep., Nov. 2001.

[5] S. Staniford, V. Paxson, and N. Weaver, *How to own the Internet in your spare time*, in Proc. of the 11th USENIX Security Symposium (Security '02), 2002.

[6] J. O. Kephart, *How topology affects population dynamics*, in C. Langton, ed., Artificial Life III. Studies in the Sciences of Complexity, 1994, pp. 447-463.

[7] J. O. Kephart, S. R. White, *Directed-graph epidemiological models of computer viruses*, in Proc. of the 1991 IEEE Computer Society Symposium on Research in Security and Privacy, May 1991, pp. 343-359.

GL(2,R)-ORBITELE SISTEMELOR POLINOMIALE DE ECUAŢII DIFERENŢIALE

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Abstract

În această lucrare, sunt studiate orbitele sistemului polinomial $\dot{x}_1 = P(x_1, x_2)$, $\dot{x}_2 = Q(x_1, x_2)$ la acţiunea grupului de transformări liniare GL(2,R). Se arată că nu există sisteme polinomiale cu dimensiunea GL-orbitei egală cu unu, iar GL-orbite de dimensiunea zero şi doi poate avea doar sistemul liniar. Luându-se ca bază dimensiunea GL-orbitelor, se face clasificarea sistemelor polinomiale pentru care originea de coordonate este punct singular cu rădăcinile ecuaţiei caracteristice reale şi distincte. Se demonstrează că pe GL-orbitele de dimensiunea trei aceste sisteme sunt Darboux integrabile.

1. **Transformări centroafine**

Să considerăm sistemul polinomial

$$\dot{x}_1 = \sum_{k=0}^{n} P_k(x_1, x_2), \quad \dot{x}_2 = \sum_{k=0}^{n} Q_k(x_1, x_2) \tag{1}$$

unde P_k,Q_k sunt polinoame omogene de gradul k

$$P_k = \sum_{i+j=k} a_{ij} x_1^i x_2^j, \quad Q_k = \sum_{i+j=k} b_{ij} x_1^i x_2^j. \tag{2}$$

Să notăm cu E spaţiul coeficienţilor

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, \ldots, a_{0n}; b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, b_{30}, \ldots, b_{0n})$$

ai sistemului (1) şi cu $GL(2,R)$ grupul transformărilor centroafine ale spaţiului fazic Ox, $x = (x_1, x_2)$. Efectuând în (1) transformarea $X = qx$, unde $X = (X_1, X_2)$, $q \in GL(2,R)$, i.e.

$$q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \quad \alpha,\beta,\gamma,\delta \in R, \quad \Delta = \det(q) \neq 0, \quad q^{-1} = \frac{1}{\Delta}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \tag{3}$$

obţinem sistemul

$$\dot{X}_1 = \sum_{k=0}^{n} P_k^*\left(X_1, X_2\right), \quad \dot{X}_2 = \sum_{k=0}^{n} Q_k^*\left(X_1, X_2\right), \tag{4}$$

unde

$$\begin{aligned} P_k^* &= \alpha \cdot P_k\left(q^{-1}x\right) + \beta \cdot Q_k\left(q^{-1}x\right) = \sum_{i+j=k} a_{ij}^* X_1^i X_2^j, \\ Q_k^* &= \gamma \cdot P_k\left(q^{-1}x\right) + \delta \cdot Q_k\left(q^{-1}x\right) = \sum_{i+j=k} b_{ij}^* X_1^i X_2^j. \end{aligned} \tag{5}$$

Remarca 1. *Din (5) se vede că fiecare transformare $q \in GL(2,R)$ acţionează separat asupra omogenităţilor de acelaşi grad din (1).*

Coeficienţii a^* ai sistemului (4) se exprimă liniar prin coeficienţii sistemului (1): $a^* = L_{(q)}(a)$, $\det L_{(q)} \neq 0$. Mulţimea $L = \left\{ L_{(q)} \middle| \, q \in GL(2,R) \right\}$ formează un grup 4-parametric în raport cu operaţia de compunere. L se numeşte *reprezentarea grupului* $GL\,(2,R)$ de transformări centroafine ale spaţiului fazic Ox în spaţiul coeficienţilor E ai sistemului (1).

Fie $a \in E$ fixat. Mulţimea $L(a) = \left\{ L_{(q)}(a) \middle| \, q \in GL(2,R) \right\}$ se numeşte *GL-orbită* a punctului a sau a sistemului de ecuaţii diferenţiale (1) corespunzător acestui punct.

2. Transformări uniparametrice

Să considerăm funcţia $g : R \times E \to E$ astfel încât oricare ar fi $\tau \in R$ transformarea $g^\tau : E \to E$, unde $g^\tau(a) = g(\tau, a),\ a \in E$, este un difeomorfism. Se spune că $(E, \{g^\tau\})$ este un *flux* (flow) *diferenţiabil*, dacă

1) $g^0 = id$; 2) $g^{\tau+s} = g^\tau g^s \ \forall \tau, s \in R$; 3) $\left(g^\tau\right)^{-1} = g^{-\tau} \ \forall \tau \in R$ şi 4) $g : R \times E \to E$ este funcţie diferenţiabilă.

Conform lui [1],[7], transformarea 4-parametrică q (vezi (3)) poate fi reprezentată ca produs a patru transformări uniparametrice

$$q^{\alpha_1^*} = \begin{pmatrix} \alpha_1^* & 0 \\ 0 & 1 \end{pmatrix}, \quad q^{\alpha_2} = \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}, \quad q^{\alpha_3} = \begin{pmatrix} 1 & 0 \\ \alpha_3 & 1 \end{pmatrix}, \quad q^{\alpha_4^*} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_4^* \end{pmatrix},$$

unde $\alpha_1^*, \alpha_4^* \in R \setminus \{0\}$; $\alpha_2, \alpha_3 \in R$. Notăm

$$q^{\alpha_1} = \begin{pmatrix} e^{\alpha_1} & 0 \\ 0 & 1 \end{pmatrix}, \quad q^{\alpha_4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha_4} \end{pmatrix}, \quad \alpha_1, \alpha_4 \in R; \quad L_l = L_{(q^{\alpha_l})},\ l = \overline{1,4};$$

$$L_1^* = L_{\left(q^{\alpha_1^*}\right)}, L_4^* = L_{\left(q^{\alpha_4^*}\right)}.$$

Fiecărui grup de transformări uniparametrice q_l, $l = \overline{1,4}$; $q^{\alpha_1^*}$,$q^{\alpha_4^*}$ ale spaţiului fazic Ox îi corespunde un sistem de forma (4) cu a_{ij}^*,b_{ij}^* respectivi.

Se verifică uşor că $(E, \{q^{\alpha_l}\})$, $l = \overline{1,4}$, sunt fluxuri diferenţiabile. Ele definesc în E sistemele de ecuaţii liniare

$$\frac{da}{d\alpha_l} = \left(\frac{dL_l(a)}{d\alpha_l}\right)\Bigg|_{\alpha_l=0}, \quad l = \overline{1,4}, \tag{6}$$

sau pe coordonate

$$q^{\alpha_l}: \begin{cases} \frac{da_{ij}}{d\alpha_l} = \left(\frac{da_{ij}^*}{d\alpha_l}\right)\Big|_{\alpha_l=0} \equiv A_{ij}^l(a), \\ \frac{db_{ij}}{d\alpha_l} = \left(\frac{db_{ij}^*}{d\alpha_l}\right)\Big|_{\alpha_l=0} \equiv B_{ij}^l(a), \\ i+j = \overline{0,n}, \quad l = \overline{1,4}. \end{cases} \tag{7}$$

În cazurile $l = 1$ şi $l = 4$ matricea coeficienţilor sistemului (7) este diagonală. Într-adevăr, în aceste cazuri avem

$$\begin{matrix} A_{ij}^1(a) = (1-i)a_{ij}, \ B_{ij}^1(a) = -ib_{ij}, \\ A_{ij}^4(a) = -ja_{ij}, \ B_{ij}^4(a) = (1-j)b_{ij}. \end{matrix} \tag{8}$$

Menţionăm că $\left(E, \left\{q^{\alpha_1^*}\right\}\right)$ şi $\left(E, \left\{q^{\alpha_4^*}\right\}\right)$ nu sunt fluxuri.

Să considerăm sistemele

$$q^{\alpha_l^*}: \frac{da}{d\alpha_l^*} = \left(\frac{dL_l^*(a)}{d\alpha_l^*}\right)\Bigg|_{\alpha_l^*=1}, \quad l = 1,4. \tag{9}$$

Remarca 2. *Sistemul* $((9), l = 1)(((9), l = 4))$ *coincide cu sistemul* $((6), l = 1) \quad (((6), l = 4))$.

Câmpurile vectoriale

$$V_l = \sum_{i+j=0}^{n} A_{ij}^l(a)\frac{\partial}{\partial a_{ij}} + B_{ij}^l(a)\frac{\partial}{\partial b_{ij}}, \quad l = \overline{1,4}$$

generează o algebră Lie. Potrivit [4,6,7] dimensiunea orbitei $O(a)$ este egală cu dimensiunea acestei algebre, i.e. cu rangul matricei M alcătuită din coordonatele vectorilor V_l, $l = \overline{1,4}$.

3. Orbite de dimensiune zero

Fie sistemul omogen

$$\dot{x}_1 = P_k(x_1, x_2), \quad \dot{x}_2 = Q_k(x_1, x_2), \tag{10}$$

unde $0 \leq k \leq n$ şi P_k, Q_k sunt definite în (2). Pentru (10) avem câmpurile vectoriale

$$W_l = \sum_{i+j=k} A^l_{ij}(a)\frac{\partial}{\partial a_{ij}} + B^l_{ij}(a)\frac{\partial}{\partial b_{ij}}, l = \overline{1,4}. \tag{11}$$

Să notăm cu M_k matricea de dimensiune $4 \times (2k+2)$ alcătuită din coordonatele vectorilor (11). De exemplu,

$$M_0 = \begin{pmatrix} a_{00} & 0 \\ b_{00} & 0 \\ 0 & b_{00} \\ 0 & a_{00} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01}-a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10}-b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}. \tag{12}$$

Avem $M = (M_0, M_1, \ldots, M_n)$, deci

$$rank\, M \geq rank\, M_k, \;\; k = \overline{0,n}. \tag{13}$$

Prin urmare, dimensiunea orbitelor sistemului (10) nu depăşeşte dimensiunea orbitelor corespunzătoare ale sistemului (1).

În [7], în fiecare din cazurile $k = 0$, 1, 2, 3 sunt clasificate sistemele (10) în dependenţă de dimensiunea orbitei $O(a)$. Astfel, se arată că dacă $k = 0$, 2 sau 3, atunci $\dim O(a) = 0$ dacă şi numai dacă $P_k \equiv 0$, $Q_k \equiv 0$, iar în cazul $k = 1$ dimensiunea orbitei $O(a)$ este egală cu zero atunci şi numai atunci, când

$$a_{10} - b_{01} = a_{01} = b_{10} = 0. \tag{14}$$

Lema 1. *În cazurile $k \neq 1$ dimensiunea orbitei $O(a)$ a sistemului (10) este egală cu zero atunci şi numai atunci, când $P_k \equiv 0, Q_k \equiv 0$.*

Demonstraţie. Să presupunem că $k \neq 1$. Orbita $O(a)$ a sistemului (10) are dimensiunea zero dacă şi numai dacă a este concomitent punct singular pentru sistemele (7), $l = \overline{1,4}$, i.e. $A^l_{ij}(a) = B^l_{ij}(a) = 0 \quad \forall\, i + j = k, l = \overline{1,4}$. De aici, $j = k - i$ şi (8) avem

$$(1-i)a_{i,k-i} = ib_{i,k-i} = 0, \;\; i = \overline{0,k}, \tag{15}$$

$$(k-i)a_{i,k-i} = (k-i-1)b_{i,k-i} = 0, \;\; i = \overline{0,k}. \tag{16}$$

Din (15) şi $k \neq 1$ rezultă că $a_{i,k-i} = 0 \forall i \neq 1$ şi $b_{i,k-i} = 0$, $\forall i \neq 0$, iar din (16) obţinem că şi $a_{1,k-1} = b_{0k} = 0$. Prin urmare, $P_k \equiv 0$, $Q_k \equiv 0$.

Din (13), lema 1 şi (14) urmează

Teorema 1. *Sistemul polinomial (1) are dimensiunea GL-orbitei egală cu zero atunci şi numai atunci, când el are forma $\dot{x}_1 = bx_1, \dot{x}_2 = bx_2, b = const.$*

4. Neexistenţa orbitelor de dimensiune unu

Fie sistemul (10). În [7] se arată că în cazurile $k = 0, 1, 2, 3$ orbitele sistemului (10) au dimensiuni diferite de unu. Dăm aici demonstraţia noastră a acestui fapt, stabilind totodată că de această proprietate se bucură orice sistem polinomial bidimensional. Având în vedere Teorema 1, presupunem că $P_k \not\equiv 0$ sau $Q_k \not\equiv 0$ şi dacă $k = 1$, atunci $a_{10} \neq b_{01}$ sau $|a_{01}| + |b_{10}| \neq 0$. Din aceste condiţii imediat rezultă că $rank\, M_0 = 2$ şi $rank\, M_1 \geq 2$ (vezi (12)).

În continuare să considerăm $k \geq 2$ şi $P_k \not\equiv 0$. Fie, de exemplu, $a_{\nu,k-\nu} \neq 0$, unde ν este egal cu unul din numerele $0, 1, 2, \ldots, k$. Vom arăta că matricea M_k are cel puţin un minor de ordinul doi diferit de zero. Să presupunem contrariul, i.e. toţi minorii de ordinul doi ai lui M_k sunt egali cu zero. Pentru început să examinăm următorii minori formaţi din coordonatele vectorilor W_1 şi W_2 (vezi (11), (8))

$$\begin{gathered}\Delta^1_{\nu,i} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-i)a_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (i-k)a_{i,k-i} \end{vmatrix} = (k-1)(\nu-i)a_{\nu,k-\nu}a_{i,k-i},\ i \neq \nu; \\ \Delta^2_{\nu,i} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & -ib_{i,k-i} \\ (\nu-k)a_{\nu,k-\nu} & (1-k+i)b_{i,k-i} \end{vmatrix} = (k-1)(\nu-i-1)a_{\nu,k-\nu}b_{i,k-i}, \\ i = \overline{0,k}. \end{gathered} \tag{17}$$

Din $\Delta^1_{\nu,i} = 0$ rezultă că $a_{i,k-i} = 0 \quad \forall i \neq \nu$, iar din $\Delta^2_{\nu,i} = 0$ avem că $b_{i,k-i} = 0 \forall i$, dacă $\nu = 0$, şi că $b_{i,k-i} = 0 \quad \forall i \neq \nu - 1$, dacă $\nu \geq 1$. Prin urmare, sistemul (10) poate avea una din formele

$$\dot{x}_1 = a_{0,k}x_2^k, \quad \dot{x}_2 = 0, \quad a_{0,k} \neq 0; \tag{18}$$

$$\dot{x}_1 = a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu}, \quad \dot{x}_2 = b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1}, \quad a_{\nu,k-1} \neq 0. \tag{19}$$

Pentru (18) avem $W_1 = a_{0,k}\frac{\partial}{\partial a_{0,k}}$. Să determinăm W_3. Pentru aceasta efectuăm în (18) transformarea de coordonate
$q^{\alpha_3}: \ X_1 = x_1, \quad X_2 = \alpha_3 x_1 + x_2$

$$\dot{X}_1 = \dot{x}_1 = a_{0,k}x_2^k = a_{0,k}\left(X_2 - \alpha_3 X_1\right)^k = a_{0,k}X_2^k - k\alpha_3 a_{0,k}X_1X_2^{k-1} + o\left(\alpha_3\right),$$

$$\dot{X}_2 = \alpha_3\dot{x}_1 + \dot{x}_2 = \alpha_3 a_{0,k}x_2^k = \alpha_3 a_{0,k}\left(X_2 - \alpha_3 X_1\right)^k = \alpha_3 a_{0,k}X_2^k + o\left(\alpha_3\right).$$

Deci, $W_3 = -ka_{0,k}\frac{\partial}{\partial a_{1,k-1}} + a_{0,k}\frac{\partial}{\partial b_{0,k}}$ şi minorul $\begin{vmatrix} a_{0,k} & 0 \\ 0 & a_{0,k} \end{vmatrix} \neq 0$. Ultima inegalitate contrazice presupunerea că toţi minorii de ordinul doi ai matricei M_k sunt egali cu zero.

Fie în (19) $\nu = 1$. Avem $W_4 = (1-k)\left(a_{1,k-1}\frac{\partial}{\partial a_{1,k-1}} + b_{0,k}\frac{\partial}{\partial b_{0,k}}\right)$. Să calculăm W_3:

$$\begin{aligned}\dot{X}_1 = \dot{x}_1 = a_{1,k-1}x_1x_2^{k-1} = a_{1,k-1}X_1\left(X_2-\alpha_3X_1\right)^{k-1} = a_{1,k-1}X_1X_2^{k-1}+\\ +(1-k)\alpha_3a_{1,k-1}X_1^2X_2^{k-2}+o\left(\alpha_3\right),\end{aligned}$$

$$\begin{aligned}\dot{X}_2 = \alpha_3\dot{x}_1+\dot{x}_2 = \alpha_3a_{1,k-1}x_1x_2^{k-1}+b_{0,k}x_2^k = \alpha_3a_{1,k-1}X_1\left(X_2-\alpha_3X_1\right)^{k-1}+\\ +b_{0,k}\left(X_2-\alpha_3X_1\right)^k = b_{0,k}X_2^k+\alpha_3\left(a_{1,k-1}-kb_{0,k}\right)X_1X_2^{k-1}+o\left(\alpha_3\right).\end{aligned}$$

Prin urmare, $W_3 = (1-k)a_{1,k-1}\frac{\partial}{\partial a_{2,k-2}} + \left(a_{1,k-1}-kb_{0,k}\right)\frac{\partial}{\partial b_{1,k-1}}$ şi $\begin{vmatrix}(1-k)a_{1,k-1} & 0\\ 0 & (1-k)a_{1,k-1}\end{vmatrix} \neq 0$. Contradicţie.

Să trecem acum la examinarea cazului când în (19) $\nu \geq 2$. Avem

$$W_1 = (1-\nu)a_{\nu,k-\nu}\frac{\partial}{\partial a_{\nu,k-\nu}} + (\nu-k-1)b_{\nu-1,k-\nu+1}\frac{\partial}{\partial b_{\nu-1,k-\nu+1}}. \tag{20}$$

Să efectuăm în (19) transformarea $q^{\alpha_2}: X_1 = x_1+\alpha_2x_2,\ X_2 = x_2:$

$$\begin{aligned}\dot{X}_1 = \dot{x}_1+\alpha_2\dot{x}_2 = a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu}+\alpha_2b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1} =\\ = \left(X_1-\alpha_2X_2\right)^{\nu-1}X_2^{k-\nu}\left[a_{\nu,k-\nu}X_1+\alpha_2\left(b_{\nu-1,k-\nu+1}-a_{\nu,k-\nu}\right)X_2\right] =\\ = a_{\nu,k-\nu}X_1^{\nu}X_2^{k-\nu}+\alpha_2\left(b_{\nu-1,k-\nu+1}-\nu a_{\nu,k-\nu}\right)X_1^{\nu-1}X_2^{k-\nu+1}+o\left(\alpha_2\right),\end{aligned}$$

$$\begin{aligned}\dot{X}_2 = \dot{x}_2 = b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1} = b_{\nu-1,k-\nu+1}\left(X_1-\alpha_2X_2\right)^{\nu-1}X_2^{k-\nu+1} =\\ = b_{\nu-1,k-\nu+1}X_1^{\nu-1}X_2^{k-\nu+1}+\alpha_2(1-\nu)b_{\nu-1,k-\nu+1}X_1^{\nu-2}X_2^{k-\nu+2}+o\left(\alpha_2\right).\end{aligned}$$

De aici rezultă că

$$W_2 = \left(b_{\nu-1,k-\nu+1}-\nu a_{\nu,k-\nu}\right)\frac{\partial}{\partial a_{\nu-1,k-\nu+1}} + (1-\nu)b_{\nu-1,k-\nu+1}\frac{\partial}{\partial b_{\nu-2,k-\nu+2}}.$$

Având în vedere că $\nu \geq 2$ şi că $a_{\nu,k-\nu} \neq 0$, următorii doi minori, formaţi din coordonatele vectorilor (20) şi W_2,

$$\begin{vmatrix}(1-\nu)a_{\nu,1} & 0\\ 0 & (1-\nu)b_{\nu-1,k-\nu+1}\end{vmatrix},\quad \begin{vmatrix}(1-\nu)a_{\nu,k-\nu} & 0\\ 0 & b_{\nu-1,k-\nu+1}-\nu\cdot a_{\nu,k-\nu}\end{vmatrix}$$

nu pot să fie concomitent egali cu zero.

Aşadar, am demonstrat că atunci când $P_k \not\equiv 0$ dimensiunea oricărei orbite a sistemului (10) nu poate fi egală cu unu. Cazul $Q_k \not\equiv 0$ se reduce la cazul $P_k \not\equiv 0$ dacă în (10) se schimbă rolurile variabilelor x_1 şi x_2.

Din Teorema 1, inegalitatea (13) şi din cele examinate mai sus în această secţiune urmează

Teorema 2. *Oricare ar fi sistemul polinomial (1) dimensiunea GL-orbitei lui este diferită de unu.*

Se verifică uşor că matricea M_1 din (12) nu poate avea rangul mai mare ca doi. Acest fapt, precum şi Teoremele 1 şi 2, ne conduc la

Teorema 3. *Dimensiunea GL-orbitei sistemului liniar* $\dot{x}_1 = a_{10}x_1 + a_{01}x_2, \dot{x}_2 = b_{10}x_1 + b_{01}x_2$ *este egală cu zero atunci şi numai atunci, când* $a_{10} - b_{01} = a_{01} = b_{10} = 0$ *şi cu doi în celelalte cazuri.*

Fie sistemul

$$\dot{x}_1 = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2, \quad \dot{x}_2 = b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2.$$

Pentru el matricea alcătuită din coordonatele vectorilor X_l, $l = \overline{1,4}$ este

$$M_2 = \begin{pmatrix} -a_{20} & 0 & a_{02} & -2b_{20} & -b_{11} & 0 \\ b_{20} & b_{11} - 2a_{20} & b_{02} - a_{11} & 0 & -2b_{20} & -b_{11} \\ -a_{11} & -2a_{02} & 0 & a_{20} - b_{11} & a_{11} - 2b_{02} & a_{02} \\ 0 & -a_{11} & -2a_{02} & b_{20} & 0 & -b_{02} \end{pmatrix}. \quad (21)$$

Se arată uşor că pentru sistemul $\dot{x}_1 = 0$, $\dot{x}_2 = x_1x_2$ rangul matricei M_2 este egal cu trei, iar pentru sistemul $\dot{x}_1 = x_2^2$, $\dot{x}_2 = x_1^2 + x_1x_2$ avem că $rank\, M_2 = 4$.

De aici, din Teoremele 1, 2, 3 şi din inegalitatea (13) rezultă

Lema 2. *Dacă părţile drepte ale sistemului (1) conţin cel puţin un termen neliniar, atunci dimensiunea GL-orbitei lui este egală cu doi, trei sau patru.*

În continuare, efectuăm clasificarea în dependenţă de dimensiunea a GL-orbitelor sistemului (1) pentru care originea este punct singular cu rădăcinile λ_1 şi λ_2 ale ecuaţiei caracteristice reale şi distincte, i.e.

$$\lambda_1, \lambda_2 \in R, \ \lambda_1 \neq \lambda_2. \quad (22)$$

În acest caz $P_0 \equiv 0$, $Q_0 \equiv 0$ şi, conform lui [2], printr-o transformare de coordonate $q \in GL\,(2, R)$ sistemul (1) poate fi adus la forma

$$\dot{x}_1 = \lambda_1 x_1 + \sum_{k=2}^{n} P_k(x_1, x_2), \quad \dot{x}_2 = \lambda_2 x_2 + \sum_{k=2}^{n} Q_k(x_1, x_2). \quad (23)$$

În (23) s-au păstrat notaţiile (2) ale omogenităţilor P_k, Q_k, $k = \overline{2,n}$. Din (12) avem că pentru (23) $rank\, M_1 = 2$. De aici şi (13) urmează că dimensiunea oricărei GL-orbite a sistemului (23) cu condiţiile (22) poate fi egală cu doi, trei sau patru.

5. GL-orbitele sistemului (23) de dimensiune doi

Fie sistemul

$$\dot{x}_1 = \lambda_1 x_1 + P_k(x_1, x_2), \quad \dot{x}_2 = \lambda_2 x_2 + Q_k(x_1, x_2). \tag{24}$$

unde λ_1,λ_2 verifică (22) şi $2 \le k \le n$. În (24) polinoamele P_k, Q_k coincid respectiv cu polinoameleP_k şi Q_k din (23). Evident, are loc

Remarca 3. *Dimensiunea oricărei GL-orbite a sistemului (23) nu este mai mică decât dimensiunea GL-orbitei corespunzătoare a sistemului (24).*

Din (12) şi (8) avem că pentru (24) matricea $M = (M_1, M_k)$ alcătuită din coordonatele vectorilor V_l, $l = \overline{1,4}$, după aplicarea unor transformări elementare, poate fi adusă la forma

$$M \sim \left(\begin{array}{cccc|cccccc} 0 & 0 & 0 & 0 & (1-k)a_{k,0} & (2-k)a_{k-1,1} & \cdot & \cdot & \cdot & -b_{1,k-1} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{k-1,1} & \cdot & \cdot & \cdot & (2-k)b_{1,k-1} & (1-k)b_{0,k} \end{array} \right). \tag{25}$$

Examinând minorii de ordinul trei ai matricei (25)

$$\begin{vmatrix} 0 & 0 & (1-i)a_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & -ib_{ij} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -ja_{ij} \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-j)b_{ij} \end{vmatrix},$$

$$i + j = k,$$

observăm că ei sunt concomitent egali cu zero atunci şi numai atunci, când $a_{ij} = b_{ij} = 0$, $\forall i + j = k$. De aici, din Remarca 3 şi Teorema 3 rezultă

Lema 3. *Dimensiunea GL-orbitei sistemului (23) cu condiţiile (22) este egală cu doi atunci şi numai atunci, când $P_k \equiv 0, Q_k \equiv 0 \quad \forall k \ge 2$.*

Ţinând seama de această lemă şi de Remarca 1, obţinem

Teorema 4. *Dacă originea de coordonate este pentru (1) punct singular cu rădăcinile ecuaţiei caracteristice reale şi distincte, atunci GL-orbita sistemului (1) are dimensiunea egală cu doi, dacă şi numai dacă $P_k \equiv 0, Q_k \equiv 0 \quad \forall k \ge 2$.*

6. GL-orbitele sistemului (23) de dimensiunea trei

În această secţiune vom evidenţia acele sisteme de forma (23), (22) pentru care au dimensiunea GL-orbitei este egală cu trei. Ca şi mai înainte, începem prin a examina sistemul (24). Din (25) avem că $rankM = 2 + rank\tilde{M}_k$, unde

$$\tilde{M}_k = \begin{pmatrix} (1-k)a_{k,0} & (2-k)a_{k-1,1} & \dots & -b_{1,k-1} & 0 \\ 0 & -a_{k-1,1} & \dots & (2-k)b_{1,k-1} & (1-k)b_{0,k} \end{pmatrix}. \tag{26}$$

Minorii de ordinul doi ai matricei $\tilde{M}_k$ sunt $\Delta^1_{\nu,i}$, $\Delta^2_{\nu,i}$ din (17) şi

$$\Delta^3_{\nu,i} = \begin{vmatrix} -\nu b_{\nu,k-\nu} & -ib_{i,k-i} \\ (1+\nu-k)b_{\nu,k-\nu} & (1+i-k)b_{i,k-i} \end{vmatrix} = (k-1)(\nu-i)b_{\nu,k-\nu}b_{i,k-i},\ i \neq \nu$$

(vezi (8)). Dacă $a_{0,k} \neq 0$ ($b_{k,0} \neq 0$), atunci din $\Delta^1_{0,i} = 0$, $i = \overline{1,k}$ ($\Delta^3_{k,i} = 0$, $i = \overline{0,k-1}$) rezultă că $a_{i,k-i} = 0$ ($b_{i,k-i} = 0$), iar din $\Delta^2_{0,i} = 0$ ($\Delta^2_{i,k} = 0$),$i = \overline{0,k}$, avem că $b_{i,k-i} = 0$ ($a_{i,k-i} = 0$). În aceste cazuri, sistemul (24) are forma

$$S_n\,(k:1): \quad \dot{x}_1 = \lambda_1 x_1 + a_{0,k}x_2^k, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{0,k} \neq 0 \quad (k \geq 2) \tag{27}$$

şi respectiv

$$S_n\,(1:k): \quad \dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{k,0}x_1^k, \quad b_{k,0} \neq 0 \quad (k \geq 2)\,. \tag{28}$$

Să presupunem acum că $a_{\nu,k-\nu} \neq 0$ ($b_{\nu-1,k-\nu+1} \neq 0$) pentru un oarecare $\nu \in \{1,2,\ldots,k\}$. Din $\Delta^1_{\nu,i} = 0$ ($\Delta^3_{\nu,i} = 0$), $i \neq \nu$ şi $\Delta^2_{\nu,i} = 0$ ($\Delta^2_{i,\nu} = 0$), $i \neq \nu - 1$, rezultă că $a_{i,k-i} = 0 \quad \forall i \neq \nu$ şi $b_{i,k-i} = 0 \quad \forall i \neq \nu - 1$. Aceste cazuri ne conduc la sistemele

$$\begin{cases} \dot{x}_1 = x_1\left(\lambda_1 + a_{\nu,k-\nu}x_1^{\nu-1}x_2^{k-\nu}\right), \\ \dot{x}_2 = x_2\left(\lambda_2 + b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu}\right), \quad |a_{\nu,k-\nu}| + |b_{\nu-1,k-\nu+1}| \neq 0,\ \nu = \overline{1,k}. \end{cases} \tag{29}$$

Astfel, s-a demonstrat

Lema 4. *GL-orbita sistemului (24) are dimensiunea egală cu trei atunci şi numai atunci când el are una din formele (27)-(29).*

În continuare, să examinăm sistemul (23). Ca de obicei, cu M vom nota matricea alcătuită din coordonatele vectorilor V_j, $j = \overline{1,4}$, corespunzători sistemului (23), iar cu $\tilde{M}$ matricea $\left(\tilde{M}_2, \tilde{M}_3, \ldots, \tilde{M}_n\right)$, unde $\tilde{M}_k$, $k = \overline{2,n}$, este definită în (26). Evident,

$$rank\,M = 2 + rank\,\tilde{M} \geq 2 + rank\,\tilde{M}_k, k = \overline{2,n}. \tag{30}$$

Dacă $rank\,M = 3$, atunci din (30) urmează că există k: $2 \leq k \leq n$ astfel încât $rank\,\tilde{M}_k = 1$. Deci

$$|P_k\,(x_1,x_2)| + |Q_k\,(x_1,x_2)| \not\equiv 0. \tag{31}$$

În cazul când $P_j \equiv 0$, $Q_j \equiv 0 \quad \forall j \neq k$, $2 \leq j \leq n$, aplicăm Lema 4. Să presupunem că, împreună cu omogenităţile de ordinul k, părţile drepte ale sistemului (23) conţin şi omogenităţi de alt ordin, să spunem, de ordinul l, unde $l \neq k, 2 \leq l \leq n$. Prin urmare,

$$|P_l(x_1, x_2)| + |Q_l(x_1, x_2)| \not\equiv 0. \tag{32}$$

Condiţia ca $rank\, \tilde{M}_k = rank\, \tilde{M}_l = 1$ implică faptul că atât P_k, Q_k, precum şi P_l, Q_l, sunt de forma părţilor drepte ale unuia din sistemele (27)-(29). În cazul P_l, Q_l în (27)-(29) k se înlocuieşte cu l.

Fie $P_k = a_{0,k} x_2^k$, $a_{0,k} \neq 0$ şi $Q_k \equiv 0$. Următorii minori ai matricei $\tilde{M}$

$$\begin{vmatrix} a_{0,k} & (1-\mu)a_{\mu,l-\mu} \\ -ka_{0,k} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix} = [1-l+(1-\mu)(k-1)]\, a_{0,k}a_{\mu,l-\mu},$$

$$\begin{vmatrix} a_{0,k} & -\mu b_{\mu,l-\mu} \\ -ka_{0,k} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix} = [1-l+\mu(1-k)]\, a_{0,k}b_{\mu,l-\mu},$$

$0 \leq \mu \leq l$,sunt concomitent egali cu zero atunci şi numai atunci, când $a_{\mu,l-\mu} = b_{\mu,l-\mu} = 0 \quad \forall \mu = \overline{0,l}$, i.e. când $P_l \equiv 0$, $Q_l \equiv 0$, ceea ce contrazice (32).

Similar, prin examinarea minorilor

$$\begin{vmatrix} -kb_{k,0} & (1-\mu)a_{\mu,l-\mu} \\ b_{k,0} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix}, \quad \begin{vmatrix} -kb_{k,0} & -\mu b_{\mu,l-\mu} \\ b_{k,0} & (1+\mu-l)b_{\mu,l-\mu} \end{vmatrix},$$

se arată că, în condiţia (32), cazul $P_k \equiv 0$, $Q_k = b_{k,0}x_1^k$, $b_{k,0} \neq 0$, nu poate avea loc.

Având în vedere Lemele 3, 4 şi condiţiile (31),(32), rămâne să investigăm cazul

$$P_k = a_{\nu,k-\nu}x_1^{\nu}x_2^{k-\nu}, \quad Q_k = b_{\nu-1,k-\nu+1}x_1^{\nu-1}x_2^{k-\nu+1}, \quad P_l = a_{\mu,l-\mu}x_1^{\mu}x_2^{l-\mu},$$

$$Q_l = b_{\mu-1,l-\mu+1}x_1^{\mu-1}x_2^{l-\mu+1}, \quad 1 \leq \nu \leq k, \quad 1 \leq \mu \leq l.$$

Fie minorii

$$\Omega^1_{\nu,\mu} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-\mu)a_{\mu,l-\mu} \\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)a_{\mu,l-\mu} \end{vmatrix} = \omega_{\nu,\mu}a_{\nu,k-\nu}a_{\mu,l-\mu},$$

$$\Omega^2_{\nu,\mu} = \begin{vmatrix} (1-\nu)a_{\nu,k-\nu} & (1-\mu)b_{\mu-1,l-\mu+1} \\ (\nu-k)a_{\nu,k-\nu} & (\mu-l)b_{\mu-1,l-\mu+1}\text{`} \end{vmatrix} = \omega_{\nu,\mu}a_{\nu,k-\nu}b_{\mu-1,l-\mu+1},$$

$$\Omega^3_{\nu,\mu} = \begin{vmatrix} (1-\nu)b_{\nu-1,k-\nu+1} & (1-\mu)b_{\mu-1,l-\mu+1} \\ (\nu-k)b_{\nu-1,k-\nu+1} & (\mu-l)b_{\mu-1,l-\mu+1}\text{`} \end{vmatrix} = \omega_{\nu,\mu}b_{\nu-1,k-\nu+1}b_{\mu-1,l-\mu+1},$$

unde $\omega_{\nu,\mu} = (\nu - 1)(l - 1) - (\mu - 1)(k - 1), \quad 1 \leq \nu \leq k$ şi $1 \leq \mu \leq l$. Evident, $\omega_{1,1} = \omega_{k,l} = 0$.

Dacă $\nu = 1 \quad (\nu = k)$, atunci din (31) şi (32) rezultă că egalităţile $\Omega^1_{1,\mu} = \Omega^2_{1,\mu} == \Omega^3_{1,\mu} = 0$ au loc atunci şi numai atunci când $\mu = 1 \quad (\mu = l)$. Prin urmare, dimensiunea GL-orbitei fiecăruia din sistemele

$$S_n\left(\lambda_1 : 0\right): \quad \dot{x}_1 = x_1\left(\lambda_1 + \sum_{j=1}^{n-1} a_{1,j} x_2^j\right), \quad \dot{x}_2 = x_2\left(\lambda_2 + \sum_{j=1}^{n-1} b_{0,j+1} x_2^j\right), \tag{33}$$

$$\sum_{j=1}^{n-1} |a_{1,j}| + |b_{0,j+1}| \neq 0\,;$$

$$S_n\left(0 : \lambda_2\right): \quad \dot{x}_1 = x_1\left(\lambda_1 + \sum_{j=1}^{n-1} a_{j+1,0} x_1^j\right), \quad \dot{x}_2 = x_2\left(\lambda_2 + \sum_{j=1}^{n-1} b_{j,1} x_1^j\right), \tag{34}$$

$$\sum_{j=1}^{n-1} |a_{j+1,0}| + |b_{j,1}| \neq 0,$$

este egală cu trei.

În continuare, să presupunem că $2 \leq \nu \leq k - 1,\ 2 \leq \mu \leq l - 1$. Din (31), (32) şi $\Omega^j_{\nu,\mu} = 0,\ j = \overline{1,3}$, rezultă că $\omega_{\nu,\mu} = 0$, de unde avem că $\frac{l-1}{\mu-1} = \frac{k-1}{\nu-1} > 1$. Deci, există numerele naturale pozitive p, q, i, j astfel încât

$$(p,q) = 1, \quad k = (p+q)\,i + 1, \quad \nu = qi + 1, \quad l = (p+q)\,j + 1, \quad \mu = qj + 1.$$

Prin urmare, oricare ar fi numerele naturale reciproc prime p şi q, sistemul

$$S_n\left(p : -q\right): \quad \begin{cases} \dot{x}_1 = x_1\left[\lambda_1 + \sum\limits_{i=1}^{n^*} a_{qi+1,pi}\left(x_1^q x_2^p\right)^i\right], \\ \dot{x}_2 = x_2\left[\lambda_2 + \sum\limits_{i=1}^{n^*} b_{qi,pi+1}\left(x_1^q x_2^p\right)^i\right], \\ \sum\limits_{i=1}^{n^*} |a_{qi+1,pi}| + |b_{qi,pi+1}| \neq 0, \ \ (p,q) = 1, \end{cases} \tag{35}$$

unde $n^* = \left[\frac{n-1}{p+q}\right]$, are dimensiunea GL-orbitei egală cu trei.

Deci, am demonstrat.

Teorema 5. *Dimensiunea GL-orbitei sistemului (23) cu condiţiile (22) este egală cu trei atunci şi numai atunci când el are una din formele (27), (28), (33), (34) sau (35).*

Corolarul 1. *Sistemul cubic* ($n = 3$) *de forma (23), (22) are dimensiunea GL-orbitei egală cu trei atunci şi numai atunci, când el are una din formele* $S_3\,(2:1)\,, S_3\,(3:1)\,, S_3\,(1:2)\,, S_3\,(1:3)\,, S_3\,(\lambda_1:0)\,, S_3\,(0:\lambda_2)$, $S_3\,(1:-1)$, *i.e.*

$$\dot{x}_1 = \lambda_1 x_1 + a_{02}x_2^2, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{02} \neq 0; \tag{36}$$

$$\dot{x}_1 = \lambda_1 x_1 + a_{03}x_2^3, \quad \dot{x}_2 = \lambda_2 x_2, \quad a_{03} \neq 0; \tag{37}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{20}x_1^2, \quad b_{20} \neq 0; \tag{38}$$

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{x}_2 = \lambda_2 x_2 + b_{30}x_1^3, \quad b_{30} \neq 0; \tag{39}$$

$$\dot{x}_1 = x_1\left(\lambda_1 + a_{11}x_2 + a_{12}x_2^2\right), \ \ \dot{x}_2 = x_2\left(\lambda_2 + b_{02}x_2 + b_{03}x_2^2\right),$$

$$|a_{11}| + |a_{12}| + |b_{02}| + |b_{03}| \neq 0; \tag{40}$$

$$\dot{x}_1 = x_1\left(\lambda_1 + a_{20}x_1 + a_{30}x_1^2\right), \ \ \dot{x}_2 = x_2\left(\lambda_2 + b_{11}x_1 + b_{21}x_1^2\right),$$

$$|a_{20}| + |a_{30}| + |b_{11}| + |b_{21}| \neq 0; \tag{41}$$

$$\dot{x}_1 = x_1\left(\lambda_1 + a_{21}x_1x_2\right), \ \ \dot{x}_2 = x_2\left(\lambda_2 + b_{12}x_1x_2\right), \quad |a_{21}| + |b_{12}| \neq 0. \tag{42}$$

Afirmaţia Corolarului 1 poate fi obţinută şi în mod direct, i.e. prin egalarea cu zero a tuturor minorilor de ordinul patru ai matricei $M = (M_1, M_2, M_3)$, unde M_1 coincide cu matricea M_1 din (12) dacă în ultima se pune $a_{01} = b_{10} = 0$, $a_{10} = \lambda_1$, $b_{01} = \lambda_2$; matricea M_2 este definită în (21), iar $M_3 =$
$$\begin{pmatrix} -2a_{30} & -2a_{21} & 0 & a_{03} & -3b_{30} & -2b_{21} & -b_{12} & 0 \\ b_{30} & b_{21}-3a_{30} & b_{12}-2a_{21} & b_{03}-a_{12} & 0 & -3b_{30} & -2b_{21} & -b_{12} \\ -a_{21} & -2a_{12} & -3a_{03} & 0 & a_{30}-b_{21} & a_{21}-2b_{12} & a_{12}-3b_{03} & a_{03} \\ 0 & -a_{21} & -2a_{12} & -3a_{03} & b_{30} & 0 & -b_{12} & -2b_{03} \end{pmatrix}$$
şi cerinţa ca cel puţin unul din minorii de ordinul trei ai lui M să fie diferit de zero.

7. Rezonanţa

Cu $\varphi\,(x_1, x_2)$ şi $\psi\,(x_1, x_2)$ vom nota respectiv nelinearităţile din partea dreaptă a fiecăreia din ecuaţiile sistemului (23), i.e.

$$\varphi(x_1,x_2)=\sum_{k=2}^{n}P_k(x_1,x_2),\quad \psi(x_1,x_2)=\sum_{k=2}^{n}Q_k(x_1,x_2),\tag{43}$$

unde polinoamele P_k şi Q_k, $k=\overline{2,n}$, sunt aduse în (2).

Fie λ_1 şi λ_2 două numere reale distincte. Dacă există numerele întregi nenegative m_1, m_2; $m_1+m_2\geq 2$ (n_1, n_2; $n_1+n_2\geq 2$) încât are loc relaţia

$$\lambda_1=m_1\lambda_1+m_2\lambda_2\tag{44}$$

sau

$$\lambda_2=n_1\lambda_1+n_2\lambda_2,\tag{45}$$

atunci *cuplul* de numere (λ_1,λ_2) se numeşte *rezonant.*

Având (44) ((45)), vom spune că $a_{m_1,m_2}x_1^{m_1}x_2^{m_2}$ ($b_{n_1,n_2}x_1^{n_1}x_2^{n_2}$) este *termen rezonant* al polinomului $\varphi(x_1,x_2)$ $(\psi(x_1,x_2))$ corespunzător cuplului rezonant (λ_1,λ_2). *Perechea* de polinoame (φ,ψ) vom numi-o *rezonantă*, dacă ele conţin numai termeni rezonanţi corespunzători unuia şi aceluiaşi cuplu rezonant de numere (λ_1,λ_2), considerând $\psi\equiv 0$ $(\varphi\equiv 0)$, dacă λ_1 şi λ_2 verifică (44) ((45)) şi nu verifică (45) ((44)) oricare ar fi numerele întregi $n_1,n_2\geq 0$, $n_1+n_2\geq 2$ ($m_1,m_2\geq 0$, $m_1+m_2\geq 2$).

În continuare, în această secţiune, vom descrie perechile de polinoame rezonante. Presupunem că (λ_1,λ_2) este un cuplu rezonant. Vom deosebi următoarele patru cazuri posibile: 1) $\lambda_1\cdot\lambda_2>0$, $\lambda_1\neq\lambda_2$; 2) $\lambda_1\neq 0$, $\lambda_2=0$; 3) $\lambda_1=0$, $\lambda_2\neq 0$ şi 4) $\lambda_1\cdot\lambda_2<0$.

1) $\lambda_1\cdot\lambda_2>0$, $\lambda_1\neq\lambda_2$. În acest caz egalităţile (44) şi (45) nu pot avea loc concomitent. Dacă are loc egalitatea (44), atunci ea arată astfel

$$\lambda_1=0\cdot\lambda_1+k\cdot\lambda_2,\tag{46}$$

unde k este unul din numerele 2,3,.... Cuplului (λ_1,λ_2) ce verifică (46) îi corespunde perechea rezonantă de polinoame

$$\varphi(x_1,x_2)=a_{0,k}x_2^k,\quad \psi(x_1,x_2)\equiv 0.$$

Similar, dacă are loc egalitatea (45), atunci ea arată astfel: $\lambda_2=k\cdot\lambda_1+0\cdot\lambda_2$ şi ne conduce la perechea rezonantă de polinoame

$$\varphi(x_1,x_2)\equiv 0,\quad \psi(x_1,x_2)=b_{k,0}x_1^k.$$

2) $\lambda_1\neq 0$, $\lambda_2=0$. În acest caz relaţia (44) are loc pentru $m_1=1$ şi orice $m_2\in\{1,2,3,\ldots\}$, iar relaţia (45) are loc pentru $n_1=0$ şi $n_2\in\{2,3,\ldots\}$. Cuplului rezonant (λ_1,λ_2) îi corespunde perechea rezonantă de polinoame

$$\varphi\left(x_1,x_2\right)=x_1\sum_{j=1}^{n-1}a_{1,j}x_2^j,\quad \psi\left(x_1,x_2\right)=x_2\sum_{j=1}^{n-1}b_{0,j+1}x_2^j.$$

3) $\lambda_1=0$, $\lambda_2\neq 0$. Egalitatea (44) are loc pentru $m_1\in\{2,3,\ldots\}$ şi $m_2=0$, iar (45) pentru $n_1\in\{1,2,3,\ldots\}$ şi $n_2=1$. Corespunzător, perechea rezonantă de polinoame este

$$\varphi\left(x_1,x_2\right)=x_1\sum_{j=1}^{n-1}a_{j+1,0}x_1^j,\quad \psi\left(x_1,x_2\right)=x_2\sum_{j=1}^{n-1}b_{j,1}x_1^j.$$

4) $\lambda_1\cdot\lambda_2<0$. Fiecare din relaţiile (44) şi (45) poate avea loc doar atunci când raportul λ_1/λ_2 reprezintă un număr raţional. Fie $\lambda_1:\lambda_2=p:(-q)$, unde p şi q sunt numere naturale pozitive reciproc prime, i.e. $(p,q)=1$. Notăm cu n^* partea întreagă a numărului $(n-1)/(p+q)$. În cazul dat, egalitatea (44) are loc pentru $m_1=qi+1$, $m_2=pi$, iar (45) pentru $n_1=qi$, $n_2=pi+1$, $i=\overline{1,n^*}$. Perechea rezonantă de polinoame (φ,ψ) corespunzătoare lui (λ_1,λ_2) este

$$\varphi\left(x_1,x_2\right)=x_1\sum_{j=1}^{n^*}a_{qi+1,pi}\left(x_1^qx_2^p\right)^i,\quad \psi\left(x_1,x_2\right)=x_2\sum_{j=1}^{n^*}b_{qi,pi+1}\left(x_1^qx_2^p\right)^i.$$

Din cele de mai sus şi Teorema 5 rezultă

Teorema 6. *Dimensiunea GL-orbitei sistemului (23) cu condiţiile (22) este egală cu trei atunci şi numai atunci, când polinoamele φ* şi *ψ din (43) nu sunt concomitent identice cu zero şi perechea (φ,ψ) este rezonantă.*

Având în vedere Teoremele 1, 2, 4 şi 6, obţinem următoarea caracterizare a sistemelor (23) cu dimensiunea orbitei egală cu patru.

Teorema 7. *Dimensiunea GL-orbitei sistemului (23) cu condiţiile (22) este egală cu patru atunci şi numai atunci, când $|\varphi\left(x_1,x_2\right)|+|\psi\left(x_1,x_2\right)|\not\equiv 0$ şi perechea de polinoame (φ,ψ) nu este rezonantă.*

8. Integrabilitatea pe GL-orbitele de dimensiunea trei ale sistemului (23)

Fie sistemul polinomial

$$\dot{x}_1=P\left(x_1,x_2\right),\quad \dot{x}_2=Q\left(x_1,x_2\right). \tag{47}$$

Fie $n=\max\{\deg P,\deg Q\}$ şi $D=P\partial/\partial x_1+Q\partial/\partial x_2$. Curba $f\left(x_1,x_2\right)=0$, $f\in C\left[x_1,x_2\right]$, (expresia $f=\exp\left[h\left(x_1,x_2\right)/g\left(x_1,x_2\right)\right]$, unde $h,g\in C\left[x_1,x_2\right]$) se numeşte *curbă algebrică invariantă* (*curbă exponenţială invariantă*) pentru (47), dacă există un polinom $K\in C\left[x_1,x_2\right]$ de grad nu mai mare ca $n-1$

astfel încât are loc identitatea $Df \equiv f \cdot K$. Polinomul $K(x_1, x_2)$ se numeşte *cofactorul* curbei invariante f. Conform lui [5], dacă $f = \exp(h/g)$ este o curbă exponenţială invariantă pentru sistemul (47), atunci $g(x_1, x_2) = 0$ este curbă algebrică invariantă pentru acelaşi sistem.

Fie $f_1, \ldots, f_s$ o multime de curbe algebrice invariante şi curbe exponenţial invariante ale sistemului (47) şi fie respectiv $K_1, \ldots, K_s$ cofactorii lor. Dacă există numerele $\beta_1, \beta_2, \ldots, \beta_s \in C$ astfel încât $F \equiv f_1^{\beta_1} f_2^{\beta_2} \ldots f_s^{\beta_s} = const$ $\left(\mu = f_1^{\beta_1} f_2^{\beta_2} \ldots f_s^{\beta_s}\right)$ este integrală primă (factor integrant) pentru (47), i.e. $DF \equiv 0 \quad \left(D(\mu) + \mu\left(P'_{x_1} + Q'_{x_2}\right) \equiv 0\right)$, atunci se spune că sistemul de ecuaţii diferenţiale (47) este *Darboux integrabil în sens generalizat.* Dacă printre $f_1, \ldots, f_s$ nu există curbe exponenţiale invariante, atunci vorbim, pur şi simplu, despre integrabilitatea Darboux a lui (47).

Se arată uşor că $F(\mu)$ este integrală primă (factor integrant) de tipul Darboux pentru (47) atunci şi numai atunci când are loc identitatea

$$\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv 0 \quad \left(\sum_{i=1}^{s} \beta_i K_i(x_1, x_2) \equiv -\left(P'_{x_1} + Q'_{x_2}\right)\right).$$

În continuare, să examinăm integrabilitatea sistemelor de forma (23), (22) care au dimensiunea GL-orbitei egală cu trei, i.e. sistemele (27), (28), (33)-(35). Deoarece sistemul (28) ((34)) se reduce la sistemul (27) ((33)) cu ajutorul substituţiei $x_1 \to x_2$, $x_2 \to x_1$, este suficient să studiem doar problema integrabilităţii sistemelor (27), (33) şi (35).

Conform lui [3], sistemele de forma normală se integrează prin cuadraturi. Scopul acestei secţiuni constă în a arăta că sistemele date sunt Darboux integrabile în sens generalizat.

Sistemul (27). a) Fie $\lambda_1 \neq k\lambda_2$. Se verifică uşor că curbele $f_1 = x_2$ şi $f_2 = (\lambda_1 - k\lambda_2)x_1 + a_{0,k}x_2^k$ sunt curbe algebrice invariante pentru (27) şi au cofactorii $K_1(x_1, x_2) = \lambda_2$ şi $K_2(x_1, x_2) = \lambda_1$ respectiv. Evident, identitatea $\beta_1 \cdot K_1 + \beta_2 \cdot K_2 \equiv 0$ are loc pentru $\beta_1 = \lambda_1$, $\beta_2 = -\lambda_2$ şi deci, $F = f_1^{\lambda_1} f_2^{-\lambda_2}$ este integrală primă a sistemului (27).
b) $\lambda_1 = k\lambda_2$. În acest caz, pe lângă curba invariantă $f_1 = x_2$ cu $K_1 = \lambda_2$, avem şi curba degenerată invariantă $f_2 = \exp\left(x_1/x_2^k\right)$ cu $K_2 = a_{0,k}$. Integrala primă este $F = f_1^{a_{0,k}} f_2^{-\lambda_2}$.

Sistemul (33). Fie

$$\tilde{\varphi} = \lambda_1 + \sum_{j=1}^{n-1} a_{1,j}x_2^j, \quad \tilde{\psi} = x_2\left(\lambda_2 + \sum_{j=1}^{n-1} b_{0,j+1}x_2^j\right).$$

Dacă $\tilde{\varphi} \equiv 0 \quad \left(\tilde{\psi} \equiv 0\right)$, atunci $F = x_1 \quad (F = x_2)$ este integrală primă a lui (33), iar dacă $\tilde{\psi} \not\equiv 0$ această integrală este

$$F = x_1 \exp\left[-\int \left(\tilde{\varphi}\big/\tilde{\psi}\right) dx_2\right].$$

Fie $\tilde{\varphi} \not\equiv 0$, $\tilde{\psi} \not\equiv 0$, $r = \deg \tilde{\psi}$, $s = \max\left\{0, \deg \tilde{\psi} - \deg \tilde{\varphi} + 1\right\}$, $\tilde{\psi} = b_{0,r}(x_2 - b_1)^{r_1} \dots (x_2 - b_m)^{r_m}$, unde $b_1 = 0$, $b_j \in C \setminus \{0\}$, $j = \overline{2, m}$, $r_1 + \dots + r_m = r$. Pentru sistemul (33) $f_0 \equiv x_1 = 0$, $f_i \equiv x_2 - b_i = 0$, $i = \overline{1, m}$, sunt drepte invariante, iar

$$f_{m+1} = \exp\frac{1}{x_2 - b_1}, \dots, f_{m+r_1-1} = \exp\frac{1}{(x_2 - b_1)^{r_1-1}}, \dots, f_r = \exp\frac{1}{(x_2 - b_m)^{r_m-1}},$$

$$f_{r+1} = \exp(x_2), \dots, f_{r+s} = \exp(x_2^s)$$

sunt curbe exponenţiale invariante. Deoarece

$$\int \frac{\tilde{\varphi}}{\tilde{\psi}} dx_2 = -\left[\beta_1 \ln|x_2 - b_1| + \dots + \beta_m \ln|x_2 - b_m| + \frac{\beta_{m+1}}{x_2 - b_1} + \dots\right.$$
$$\left. + \frac{\beta_r}{(x_2 - b_m)^{r_m-1}} + \beta_{r+1} x_2 + \dots + \beta_{r+s} x_2^s\right],$$

integrala F a lui (33) poate fi scrisă sub forma Darboux: $F = \prod_{i=0}^{r+s} f_i^{\beta_i}$.

În cazul examinat, este mai uşor de găsit factorul integrant. Acesta este $\mu = 1\big/\left(x_1 \tilde{\psi}\right)$.

Sistemul (35). Deoarece p şi q sunt reciproc prime, pentru ele pot fi găsite numerele întregi pozitive u şi v astfel încât $pu - qv = 1$. Transformarea $z_1 = x_1^u x_2^v$, $z_2 = x_1^q x_2^p$ [3] reduce (35) la un sistem similar sistemului (33)

$$\dot{z}_1 = z_1\left[u\lambda_1 + v\lambda_2 + \sum_{i=1}^{n^*} (u a_{qi+1,pi} + v b_{qi,pi+1}) z_2^i\right],$$

$$\dot{z}_2 = z_2\left[q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*} (q a_{qi+1,pi} + p b_{qi,pi+1}) z_2^i\right].$$

Cu toate acestea, vom efectua direct integrarea sistemului (35). Dacă

$$\lambda_1 : \lambda_2 = a_{qi+1,pi} : b_{qi,pi+1} = -p : q, \quad i = \overline{1, n^*}, \tag{48}$$

atunci părţile drepte ale lui (35) au factorul comun $\lambda_1 + \sum\limits_{i=1}^{n^*} a_{qi+1,pi}\left(x_1^q x_2^p\right)^i$. După simplificarea cu acesta, obţinem sistemul $\dot{x}_1 = x_1$, $\dot{x}_2 = \frac{\lambda_2}{\lambda_1} x_2$, care are integrala generală $x_1^{\lambda_2} x_2^{-\lambda_1} = const$. În cazul când (48) n-are loc avem factorul integrant

$$\mu = \left[x_1 x_2 \left(q\lambda_1 + p\lambda_2 + \sum_{i=1}^{n^*} \left(q a_{qi+1,pi} + p b_{qi,pi+1}\right)\left(x_1^q x_2^p\right)^i \right) \right]^{-1}.$$

Din cele de mai sus rezultă

Teorema 8. *Pe GL-orbitele de dimensiunea trei sistemul (23) cu condiţiile (22) are integrală primă Darboux generalizată (factor integrant Darboux).*

În cazul sistemelor cubice (36) şi (37) avem integrale prime

$x_2^{\lambda_1}\left[\left(\lambda_1 - j\lambda_2\right)x_1 + a_{0,j}x_2^j\right]^{-\lambda_2}$,dacă $\lambda_1 \neq j\lambda_2$,

şi

$x_2^{a_{0,j}} \exp\left(-\lambda_2 x_1 \big/ x_2^j\right)$,dacă $\lambda_1 = j\lambda_2$, $j = \overline{2,3}$.

Sistemul (40) are integrala primă $x_2 = c$, dacă $\lambda_2 = b_{02} = b_{03} = 0$ şi factorul integrant $\mu = \left[x_1 x_2\left(\lambda_2 + b_{02}x_2 + b_{03}x_2^2\right)\right]^{-1}$, dacă $|\lambda_2| + |b_{02}| + |b_{03}| \neq 0$. La rândul său, sistemul (42) are integrala primă $x_1^{\lambda_2} x_2^{-\lambda_1} = const$, dacă $\lambda_1 + \lambda_2 = a_{21} + b_{12} = 0$, şi factorul integrant $\mu = \left[x_1 x_2\left(\lambda_1 + \lambda_2 + \left(a_{21} + b_{12}\right)x_1 \cdot x_2\right)\right]^{-1}$ în celelalte cazuri. Sistemele cubice (38), (39) şi (41) se reduc la sistemele examinate mai sus cu ajutorul substituţiei $x_1 \to x_2$, $x_2 \to x_1$.

References

[1] Alexeev, V.G., *Teoria formelor binare raţionale*, Yuriev, 1989. (în limba rusă)

[2] Bautin, N.N., Leontovich, E.A., *Metode şi procedee ale studiului calitativ al sistemelor dinamice în plan*, Nauka, Moskow, 1990. (în limba rusă)

[3] Briuno, A.D., *Metodă locală de analiză neliniară a ecuaţiilor diferenţiale*, Nauka, Moskow, 1979. (în limba rusă)

[4] Vinberg, E.B., Popov, V.L., *Teoria invarianţilor*, Itoghi Nauki i Tehniki, Series Contemporary Problems in Mathematics, Moscow, **55** (1989), 137-313. (în limba rusă)

[5] Christopher, C., Llibre, J. *Algebraic aspects of integrability for polynomial systems. Qualitative theory of dynamical systems*, **1**, 1(1999), 71-95.

[6] Ovsyanikov L.V. *Group analysis of differential equations*, Nauka, Moscow, 1978. (English transl. by Academic, New York, 1982)

[7] Popa, M.N., *Aplicaţii of algebra to differential systems*, Chişinău, 2001. (în limba rusă)

COMPUTATIONAL ASPECTS OF THE MLS METHOD

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Abstract The present paper introduces the Moving Least Squares (MLS) method in electrostatics. It presents the equations of the linear electrostatics, and the main features of the MLS method. The MLS method is used to solve a one-dimensional problem in the context of linear electrostatics, and an algorithm for implementing numerically this method is proposed. Finally, a numerical example is proposed and the exact solution is compared with the approximate one.

Keywords: Meshless methods, electrostatics, numeric analysis

1. INTRODUCTION

Presently, there are some numerical methods such as: smooth particle hydrodynamics, reproducing kernel particle methods, hp-clouds, and element free Galerkin that are of great importance in numerical modelling of mechanical and electrical phenomena. Their main advantage consists in the fact that these methods are mesh free, i.e. they don't use a mesh in order to assemble the system of equations. Mesh free methods are of great interests in the study of problems that involves discontinuous fields, such as crack problems or phase changes, and adaptive refinement. The MLS method was implemented for the first time in thermoelasticity by R.Raducanu in [5].

Coupling methods EFG (Element Free Galerkin)-FEM (Finite Element Method) are also of great interest in applied electro-mechanics, because these methods can reduce considerably the computational cost. Pure FEM methods are primarily used by the engineers, because are more common, but the advantages of meshless methods are not to be negligible.

2. BASIC EQUATIONS

Let Ω be a bounded domain in the three dimensional Euclidian space. Suppose that the domain Ω is occupied by an isotropic and homogenous medium. As in [3], the basic equations of linear electrostatics are:

- Gauss Law:

$$div\mathbf{D} = \rho \tag{2.1}$$

- Electrostatic form of the Faraday law:

$$rotE = 0 \tag{2.2}$$

- Constitutive relation:

$$D = \varepsilon E \tag{2.3}$$

where E is the electric field intensity and D is the electric flux density or, alternatively the electric displacement, ε is the electrical permittivity and ρrepresents charge density.

We can see from (2.2) that E is a potential field, i.e. there exists an electric potential V(voltage) defined by:

$$E = -\nabla V \tag{2.4}$$

In this way, supposing that ε is constant, from the relations (2.1)-(2.4) we can write:

$$-\varepsilon \Delta V = \rho \tag{2.5}$$

To this equation we'll attach the following mixt boundary conditions:

$$V = V_1 \text{ on } \Gamma_1 \text{ and } V_{,x}\, n = V_2 \text{ on } \Gamma_2 \tag{2.6}$$

where V_1, V_2 are continuous functions given on the specified boundary parts, and $\bar{\Gamma}_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \Phi$. Thus the boundary value problem is to find V which satisfy (2.5) and the boundary conditions (2.6).

In order to impose essential boundary conditions, a couple of methods have been developed [1], [4]. In the following we will use the Lagrange multipliers method. We consider the following weak forms for our problem: let trial functions $V(x) \in H^1$ and Lagrange multipliers $l \in H^0$ for all test functions $\delta\varphi(x) \in H^1$ and $\delta l \in H^0$. If we have:

$$\int_0^1 \varepsilon\delta\varphi^T{}_{,x}\, \delta V_{,x}\, dx - \int_0^1 \delta\varphi^T \rho dx - \delta\varphi^T V_2|_{\Gamma_2}$$

$$-\delta l^T \left(V - V_1\right)|_{\Gamma_1} - \delta\varphi^T l|_{\Gamma_2} = 0 \tag{2.7}$$

then (2.5) is satisfied together with the boundary conditions (2.6), where H^0 and H^1 denote Hilbert spaces. A detailed discussion about these Hilbert spaces can be found in [2] and [6]. The next section presents the fundamentals of the MLS method for our particular one-dimensional case.

3. MLS APROXIMANTS

Let us consider the domain = [0, 1] discretized by a set of 11 evenly spaced nodes. Let's suppose that each node has a corresponding 'nodal parameter': V_I associated with it. It was shown that in general $V_I \neq V\left(x_I\right)$. Let's we will consider the approximations $V^h\left(x\right)$ as polynomials of order m with non-constant coefficients:

$$V^h\left(x\right) = \sum_{i=1}^{m} p_i\left(x\right) a_i\left(x\right) = \mathbf{p}^T\left(x\right) \mathbf{a}\left(x\right) \tag{3.1}$$

where m represents the number of terms in the base, $p_i\left(x\right)$ are the basis functions (usually monomials) and $a_i\left(x\right)$ are their coefficients. For example, in an one dimensional space:

$$\mathbf{p}^T\left(x\right) = \left(1, x\right) \tag{3.2}$$

As a remark, it is possible to introduce singular functions in the basis as well. It was shown [2] that any function included in the basis could be reproduced exactly by an MLS approximation. This fact is very useful in the study of domains with cracks.

The unknown parameters $a_i\left(x\right)$ at a given point, are to be determined by minimizing the differences between the local approximation at that point and the nodal parameters: V_I. Let the nodes whose support include x, be numbered locally from 1 to n. The functional to be minimized are the following weighted, discrete L_2 norm:

$$J=\sum_{I=1}^{n} w\left(x-x_I\right)\left[\mathbf{p}^T\left(x_I\right)\mathbf{a}\left(x\right)-V_I\right]^2 \tag{3.3}$$

where n is the number of nodes in the neighborhood of x for which the weight function and are nodal values at . In the calculus from the remainder of this paper we'll take as a cubic spline weight function:

$$w(x-x_I)=w(r)=\begin{matrix} \frac{2}{3}-4r^2+4r^3 & \text{for } r\leq\frac{1}{2} \\ \frac{4}{3}-4r+4r^2-\frac{4}{3}r^3 & \text{for } \frac{1}{2}<r\leq 1 \\ 0 & \text{for } r>0 \end{matrix} \tag{3.4}$$

More details about the choice of the weight function can be found in [8]. Next, we will review the main steps in the determining the functional forms for V. Minimizing the functional J with respect to $\mathbf{a}(x)$, we obtain the following set of linear equations:

$$\mathbf{A}(x)\mathbf{a}(x)=\mathbf{B}(x)\mathbf{V}(x) \text{ or } \mathbf{a}(x)=\mathbf{A}^{-1}(x)\mathbf{B}(x)\mathbf{V}(x), \tag{3.5}$$

where

$$\mathbf{A}(x)=\sum_{i=1}^{n} w(x-x_I)\mathbf{p}(x_I)\mathbf{p}^T(x_I) \tag{3.6}$$

$$B(x)=[w(x-x_1)\mathbf{p}(x_1),w(x-x_2)\mathbf{p}(x_2),...,w(x-x_n)\mathbf{p}(x_n)] \tag{3.7}$$

$$V^T(x)=[V_1,V_2,...,V_n] \tag{3.8}$$

Substituting (3.5) into (3.1), we obtain the following form for the MLS approximants:

$$V^h(x)=\sum_{I=1}^{n}\Phi_I(x)V_I \tag{3.9}$$

where the shape functions $\Phi_I(x)$ are:

$$\Phi_I(x)=\sum_{j=0}^{m} p_j(x)\left(A^{-1}(x)B(x)\right)_{jI} \tag{3.10}$$

As it was very well pointed out in [1], [6], the shape functions are not real interpolants, because the Kronecker's delta criterion is not satisfied: $\Phi_I\left(x_J\right)\neq\delta_{IJ}$.

4. NUMERICAL IMPLEMENTATION

Let's consider the approximate solution V and the test function $\delta\varphi$ of the form given in (3.9). After some elementary computations, we obtain the following system of linear algebraic equations:

$$\begin{pmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{l} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{r} \end{pmatrix} \tag{4.1}$$

where,

$$M_{IJ} = \int_0^1 \varepsilon \Phi_{I,x}^T \Phi_{J,x} dx \tag{4.2}$$

$$N_{IJ} = -\Phi_K |\Gamma_{1I} \tag{4.3}$$

$$\mathbf{b}_I = \Phi_I V_{2x} |\Gamma_2 + \int_0^1 \Phi_I \rho dx,\, \mathbf{r}_K = -V_{1K} \tag{4.4}$$

To assemble these equations, we should integrate over the domain using Gauss quadrature. First we will determine the quadrature points, and second, the domain of influence of the nodes is determined. Then, the shape functions are computed and the equation (4.1) is assembled.

5. NUMERICAL EXAMPLE

In this section we'll implement the MLS method: consider a one-dimensional bar of unit length subjected to a charge density of magnitude x. Let's suppose that the electric potential of the bar is null at the left end, and its normal derivative is null at the right end. The bar has constant cross sectional area of unit value. Thus, our problem can be written:

$$kV,_{xx} + x = 0, x \in (0,1) \tag{5.1}$$

$$V(0) = 0 \tag{5.2}$$

$$V,_x(1) = 0. \tag{5.3}$$

The exact solution to (5.1)-(5.3) is given by:

$$V(x) = \frac{1}{k}\left(\frac{1}{2}x - \frac{1}{6}x^3\right) \tag{5.4}$$

In order to obtain the MLS solution, we have to assembly the equation (4.1), computing the equations (4.2)-(4.4). In the fig.1 we can compare the exact solution with the MLS solution. One can see that the errors in the approximation are negligible.

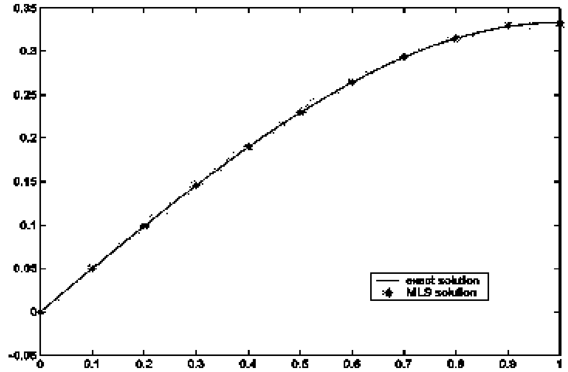

Figure 1 Electric potential vs. position

6. CONCLUSIONS

This paper proposes the implementation of MLS method in linear electrostatics. This numerical method has been implemented in elasticity [6] since 1977 and in thermoelasticity [5] since 2002. In 1995, the method was further developed by a great number of scientists, who proposed new meanings and interpretations. Meshless methods have to be developed in the future, especially regarding the computational cost which presently is too high. At this stage, the optimum way of implementing these methods is coupling with FEM. This paper represents the first step in implementing MLS in electrostatics. It is presented the case of linear electrostatics, and an algorithm for numerical implementation was proposed as well. Finally the exact solution was compared to the approximate one and the errors were discussed.

References

[1] T. Belytschko, Y.Y. Lu, and L. Gu, Element Free Galerkin Methods, International Journal for numerical methods in engineering, 37, 229-256, 1994.

[2] M. Fleming, Y.A. Chu, B. Moran, and T. Belytschko, Enriched Element-Free Galerkin Methods for Crack Tip Fields, International Journal for Numerical Methods in Engineering, 1997.

[3] F.Hantila, T.Leuca, C.Ifrim, Electrotehnica teoretica, Ed. Electra, Bucuresti, 2002.

[4] Y.Y. Lu, T. Belytschko and L. Gu, A New Implementation of the Element Free Galerkin Method, Computer Methods in Applied Mechanics and Engineering, 113, 397-414, 1994.

[5] R.Raducanu, MLS revisited, Proceedings of the Second French Roumanian Colloc. Mathematics in Engng. And Numerical Physics-invited paper, 2002.

[6] R.Raducanu, Probleme actuale in mecanica ruperii, Edit Dan 2003.

STOCHASTIC MODELLING FOR SOLVING PDE AND D-OPTIMALITY UNDER CORRELATION

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Abstract Solutions of the Laplace equation are important in many fields of science, notably the fields of electromagnetism, astronomy and fluid dynamics. The aim of this paper is to discuss the connection between elliptic equation, optimization problem $\max E_x(-\nabla^2 \log f)$ and D-optimal designs when the errors are correlated. We also discuss some simulation methods employed in effective evaluation of regular solution of the Dirichlet problem. Some examples are also provided.

Keywords: Wiener process, boundary PDE, Dirichlet problem, D-optimality, correlation.

1. INTRODUCTION

Solutions of the Laplace equation are important in many fields of science, notably the fields of electromagnetism, astronomy and fluid dynamics because they describe the behavior of gravitational, electric, and fluid potentials. It is often written as $\Delta u = 0$, or $\nabla^2 u = 0$. If the right-hand side is specified as a given function f, then the equation is called the Poisson equation. The Dirichlet problem for the Poisson equation consists in finding a solution $u \in H_0^1(G)$ on some domain G such that for $f \in L^2(G)$ and $\Gamma = \partial G$

$$-\Delta u = f, \text{ in } G, \tag{1}$$

$$u = \phi, \text{ in } \Gamma. \tag{2}$$

We say, that solution to (1) and (2) is regular, if $u \in C^2(G) \cap C(G \cup \Gamma)$.

We show that there is a correspondence between some Laplace equations with nonzero right-hand side and Fisher information on one parameter, which is to be maximized, when one seeks for the D-optimal designs. We also find and interpretation of the regularity loss under the correlation, when the interest parameter is the correlation one. To maintain the continuity of the explanation the proofs are included in the Appendix.

2. *D*-OPTIMAL CORRELATED DESIGN PDE

Consider an isotropic Gaussian random field $Y(x) \in \mathcal{Y} \subset \mathbb{R}^k$ with parametrized covariance function $\operatorname{cov}(Y(s), Y(t)) = c(||s-t||, r)$, measured on some compact design space $X \subset \mathbb{R}^s$, and parameter $r \in G \subset \mathbb{R}^m$. Such statistical models have many applications [6]. We use the notation ψ_n to denote the n-point design. Since all measurements are to be taken on one run of the process $Y(s)$ the replications are not allowed. The optimum n-point design is thus a solution of the maximization problem $\max_{\psi_n \subset X} \Phi(M)$, where Φ is the design criterion. For more see [7]. We can find employment of various criteria of design optimality in literature. Here we discuss D-optimality, which corresponds to the maximization of the determinant of a Fisher information matrix, i.e. we have $\Phi(M) = \det M$. Theoretical justifications for using the Fisher information in normal models with small variances of $Y(s)$ can be found in [9]. However there are also some asymptotical justifications, for instance for infill asymptotics see [1].

First, just for simplicity, let the only covariance parameter r be the parameter of interest. Let $h(y, r)$ be the density of the measurement y corresponding to the design $\psi_n = \{x_1, \ldots, x_n\}$. We define the abstract energy $E(r) = -\int_{\mathcal{Y}} |\nabla_r u|^2 e^u dy$, where $d\mu = e^u dy$ is some (unit) mass distribution over $\mathcal{Y}$. We say abstract energy, because the integration can be employed over space of dimension k, but the operator (stehlikLaplacian) is according to m-dimensional Cartesian coordinates $(r_1, \ldots, r_m)$ of the parameter r. This can correspond to the "classical" energy $-\int_{\mathcal{Y}} |\nabla_{r(y)} u|^2 d\mu$ under $m = k$, due to some diffeomorphism $r = r(y)$. Some interpretations of this abstract energies can be found in theoretical physics.

Theorem 2.1. *Let us fix the D-optimal design problem with correlated normal errors. Then there exists the Dirichlet problem for the Poisson equation, e.g. there exist sufficiently regular f and ϕ so that $\ln h$ be the solution of (1,2). Furthermore, there exists the abstract energy $E(r)$ which is equal to the Fisher information. If the correlation structure is collapsing, i.e. $\det \Sigma \to 0$, for $r \to r^\star$ then there exists no regular solution of the Dirichlet problem.*

However, when $m > 1$ the situation is more complex. Denote by $\mathcal{E}_m$ the functional of the D-optimal correlated design PDE. We have, $\mathcal{E}_1(u) = -\int_{\mathcal{Y}} \Delta_\vartheta u d\mu$, $\mathcal{E}_2(u) = \int_{\mathcal{Y}} \frac{\partial^2}{\partial \vartheta_1^2} u d\mu \int_{\mathcal{Y}} \frac{\partial^2}{\partial \vartheta_2^2} u d\mu - (\int_{\mathcal{Y}} \frac{\partial^2}{\partial \vartheta_1 \partial \vartheta_2} u d\mu)^2$, where $\vartheta = (\vartheta_1, \ldots \vartheta_m)$ is a general parameter. Generally $\mathcal{E}_m(u) = \det\{-\int_{\mathcal{Y}} \frac{\partial^2}{\partial \vartheta_i \partial \vartheta_j} u d\mu\}_{i,j=1}^m$.

2.1. CONVERGENCE OF ENERGIES AND THEIR OPTIMIZERS

Just for simplicity let $m = 1$. Consider two $(i = 1, 2)$ Laplace equations in a bounded domain $G \subset R^m : -\Delta u_i = f_i,\ r \in G$, and $u_i = \phi_i,\ r \in \Gamma$.

Further, let us have some convergences $\mathcal{T}_j$ under which $f_i \xrightarrow{\mathcal{T}_1} f$ and $\phi_i \xrightarrow{\mathcal{T}_2} \phi$. The following problems are interesting for the D-optimal design application:

- under which type of convergence we obtain (some type of) the convergence of energies E_i;
- when we define the multifunctions (energetic minimizers) $\mathcal{X}_i := \arg\sup E_i$, will we obtain some (set) convergence $\mathcal{T}$ under which $\mathcal{X}_i \xrightarrow{\mathcal{T}} \mathcal{X}$?

The following example illustrates some possibilities in such a situation.

Example 2.1. *Let* $\{x_1, x_2\}$, $x_1 < x_2$, *be the design in* $X = [-1,1]$, $m = 1, k = 2$, $G = (0, \frac{1}{3}), \mathcal{Y} = R^2, d = x_2 - x_1$, $f_1(r) = \exp(-rd)d^2(-2\exp(-rd) + 2\exp(-3rd) + 2y_1^2\exp(-3rd) + 2y_1^2\exp(-rd) - y_1y_2 - 6y_1y_2\exp(-2rd) - y_1y_2\exp(-4rd) + 2y_2^2\exp(-3rd) + 2y_2^2\exp(-rd))/(-1+\exp(-2rd))^3$ *and* $f_2(r) = (-4rd - 8y_1y_2 + 4y_1^2 + 4y_2^2 + 2y_1y_2r^3d^3 - 6y_1y_2r^2d^2 + 12y_1y_2rd + 3y_2^2r^2d^2 + 3y_1^2r^2d^2 - 6y_2^2rd - 6y_1^2rd - 4r^3d^3 + r^4d^4 + 6r^2d^2)/((rd-2)^3dr^3)$. *We have* $\Gamma = \{0, \frac{1}{3}\}$, $u_1(0) = u_2(0) = \infty$, $u_1(\frac{1}{3}) = -0.5\ln 2\pi - 0.5\ln\det\Sigma_1(\frac{1}{3}) + 0.5\frac{y_1^2 - 2y_1y_2\exp(-\frac{d}{3}) + y_2^2}{-1+\exp(-\frac{2}{3}d)}$ *and* $u_2(\frac{1}{3}) = -0.5\ln 2\pi - 0.5\ln\det\Sigma_2(\frac{1}{3}) + 0.5\frac{y_1^2+y_2^2+2y_1y_2(-1+\frac{d}{3})}{\frac{1}{9}d^2 - \frac{2}{3}rd}$, *where* $(\Sigma_1)_{k,l}(r) = \exp(-r|x_k - x_l|)$, $(\Sigma_2)_{k,l}(r) = 1 - r|x_k - x_l|$.

We obtain $d\mu_i(y) = (2\pi)^{-1}|\Sigma_i|^{-1/2}\exp(y^T\Sigma_i^{-1}y)dy_1dy_2$,

$$u_1(r) = -0.5\ln 2\pi - 0.5\ln\det\Sigma_1(r) + 0.5\frac{y_1^2 - 2y_1y_2\exp(-rd) + y_2^2}{-1+\exp(-2rd)}$$

and

$$u_2(r) = -0.5\ln 2\pi - 0.5\ln\det\Sigma_2(r) + 0.5\frac{y_1^2 + y_2^2 + 2y_1y_2(-1+rd)}{r^2d^2 - 2rd}.$$

Further, $E_1 = \frac{d^2\exp(-2rd)(1+\exp(-2rd))}{(1-\exp(-2rd))^2}$ *and* $E_2 = \frac{-2rd+r^2d^2+2}{r^2(rd-2)^2}$, $\mathcal{X}_1 = \{x, -1 \le x \le 1\}$ *(here collapsing effect occurs, e.g.* $x_1 = x_2 := x$ *and* $\mathcal{X}_2 = \{-1, 1\}$.

Notice, that $(\Sigma_1)_{k,l} = (\Sigma_2)_{k,l} + o(r|x_k - x_l|)$, *where* o *is a "little-o" Landau symbol. So we have* $\lim_{r\to 0}\Sigma_1 = \lim_{r\to 0}\Sigma_2$. *We have* $\lim_{r\to 0}E_1 = \lim_{r\to 0}E_2 = +\infty$, *but* $\lim_{r\to 0}\mathcal{X}_1 \ne \lim_{r\to 0}\mathcal{X}_2$.

We can conclude that from the statistical point of view the covariance structures has strong impact on the two-point D-optimal design (DOD) for covariance parameter r. In such situations the covariance misspecification effect is significant. As we have seen, under the exponential covariance structure Σ_1, *the distance between design points of DOD equals to* 0. *In other words, the two-point DOD for r is collapsing. As we can see in [13], when we misspecify the variogram and use the DOD for the linear one under exponential structure,* *we measure only* 15.48% *of the maximal information about r. The two-point DOD for covariance parameter r under the linear covariance structure Σ_2 is maximal distant. For simplicity, let us have $r = 1$. When the covariance structure is linear and we use the DOD for the exponential one (or an arbitrary design sufficiently different from DOD), we measure approximately* 0% *of the maximal information about r. This has also the following context: We reach a singular variance structure for $d \to 2$ and the limiting model is (with probability* 1*) a deterministic regular linear system. Although every observation possesses non-zero variance, zero-variance estimation is possible. Such phenomena has no correspondence in classical inference, but can exhibit under correlation. One possible regularization of such situations is considering of the non-zero nugget effect.*

When seeking for the regular solution, it can be sometimes useful to avoid such $r^\star \in G$ that $\lim_{r\to r^\star} \det \Sigma = 0$. On the other hand, such parameters can be natural in some sense. Then we can establish some convenient ε-perforation of the domain G and study the behavior of energies for $\varepsilon \to 0$. Further we assume that the domain G is bounded.

We can follow the [10] and establish the ε-perforation as follows: First we cover the $\mathbb{R}^m$ by cubes of size 2ε periodically arranged with period 2ε. The only finitely many of cubes, $C_i(\varepsilon), i = 1, \dots, n(\varepsilon)$ intersect the parameter domain G. Let T_i be closed balls of radius $0 < a_\varepsilon < \varepsilon$, centered at the center of the cubes $C_i(\varepsilon)$. Then the perforated domain G_ε is defined by $G_\varepsilon = G \setminus \cup C_i(\varepsilon)$.

Let $f \in L^2(G)$. The Dirichlet problem in G_ε is to find $u_\varepsilon \in H_0^1(G_\varepsilon)$ such that $-\Delta u_\varepsilon = f$ in G_ε and $u_\varepsilon = 0$ on ∂G_ε. The behavior of u_ε for $\varepsilon \to 0$ was studied in [3],[4]. Depending on the size of the holes, a_ε, various behavior is possible. It was shown, that there exists a critical size of holes, c_ε, such that

a) if $a_\varepsilon = c_\varepsilon$, then there exist the measure μ and $\bar{u}_\varepsilon \to u$ weakly in $H_0^1(G)$, where u is solution of $-\Delta u + \mu u = f$ in G and $u = 0$ on ∂G, where $^-$ denotes the extension by zero onto the holes,

b) if $a_\varepsilon << c_\varepsilon$, then $\mu = 0$ and $\bar{u}_\varepsilon \to u$ weakly in $H_0^1(G)$, where u is solution of the Poisson equation in the domain G,

c) if $a_\varepsilon >> c_\varepsilon$, then $\bar{u}_\varepsilon \to 0$ strongly in $H_0^1(G)$.

Example 2.2. *Here we illustrate the discussed matter on the lungs retention problem. The lungs retention is modelled by a function $E(Y(t)) = \eta(I, t, p)$, where I is the input to the system in time t_1, $t > t_1$ is time and p is Activity Median Aerodynamic Diameter. Here we discuss the D-optimal designs for bioassays following the ideas of [12]. We are interested in the limiting case ($r = \infty$) under the correlated observations $c(||s - t||, r) = \exp(-r|s - t|)$ and*

$$\eta(I, t, p) = I \frac{\gamma_1 e^{\alpha_1 p + \beta_1 t} + \gamma_2 e^{\alpha_2 p + \beta_2 t}}{1 + \gamma_3 e^{\alpha_3 p}}.$$

Let us have $x = (x_1, x_2) = (e^{t_1}, e^{t_2})$ *and* $\theta = e^p$. *Notice, that* $m = 2$.

Theorem 2.2. *For* $r \to \infty, t_1 \geq 0$, *we obtain*

$$\mathcal{E}_2(u) \sim g(\sigma, \gamma, \alpha, \beta)(-2x_1^{\beta_1+\beta_2} x_2^{\beta_1+\beta_2} + x_1^{2\beta_2} x_2^{2\beta_1} + x_2^{2\beta_2} x_1^{2\beta_1}). \tag{3}$$

where

$$g(\sigma, \gamma, \alpha, \beta) = -\gamma_1^2 \gamma_2^2 [-2\theta^{2\alpha_2+2\alpha_1+\alpha_3} \alpha_1^2 \gamma_3 - 4\theta^{2\alpha_2+2\alpha_1+\alpha_3} \alpha_2 \gamma_3 \alpha_1 - \theta^{2\alpha_2+2\alpha_1} \alpha_1^2 + 2\theta^{2\alpha_1+2\alpha_3+2\alpha_2} \alpha_2 \gamma_3^2 \alpha_1 - 2\theta^{2\alpha_2+2\alpha_1+\alpha_3} \alpha_2^2 \gamma_3 + 2\theta^{2\alpha_2+2\alpha_1} \alpha_2 \alpha_1 - \theta^{2\alpha_2+2\alpha_1} \alpha_2^2 - \theta^{2\alpha_1+2\alpha_3+2\alpha_2} \alpha_1^2 \gamma_3^2 - \theta^{2\alpha_1+2\alpha_3+2\alpha_2} \alpha_2^2 \gamma_3^2 + 8\theta^{2\alpha_2+2\alpha_1+\alpha_3} \alpha_1 \gamma_3 \alpha_2] / \left[\theta^2 (1 + \gamma_3 \theta^{\alpha_3})^6\right].$$

3. RANDOM WALKS ON BOUNDARY FOR SOLVING BOUNDARY PDE PROBLEMS

It is well-known that the random walk methods for boundary value problems (BVP) for high-dimensional domains with complex boundaries are quite efficient, especially if it is necessary to find the solution not at all points of a grid, but only at some marked points of interest.

Monte Carlo methods for solving PDEs are based:

1) *on classical probabilistic representations in the form of Wiener or diffusion paths integrals*; In this approach, diffusion processes generated by the relevant differential operator are simulated using numerical methods for solving ordinary stochastic differential equations.

2) *on probabilistic interpretation of integral equations equivalent to the original BVP which results in representations of the solutions as expectations over Markov chains.* For PDEs with constant coefficients, however, it is possible to use the strong Markov property of the Wiener process and create much more efficient algorithms first constructed for the Laplace equation [5] known as the walk on spheres method.

We now shortly present the first approach for constructing and justifying the walk on spheres algorithm. Let us start with a simple case, the Dirichlet problem (1), (2) for the Laplace equation in a bounded domain $G \subset R^3, f = 0$. We seek a regular solution to (1,2). Let $d^\star := \sup_{x \in G} d(x)$, where $d(x)$ is the largest radius of the spheres $S(x, d(x)) \subset clG$ centered at x. Let $W_x(t)$ be the Wiener process starting at the point $x \in G$, and let τ_Γ be the first passage time (the time of the first intersection of the process $W_x(t)$ with the boundary Γ). Suppose that the boundary Γ is regular so that (1) and (2) has a unique solution. Then

$$u(x) = E_x \phi(W_x(\tau_\Gamma)). \tag{4}$$

In (4), only the random points on the boundary are involved. We thus can formulate the following problem: how to find these points without explicit simulation of the Wiener process inside the domain G?

This problem was solved in [5] using the following considerations. In sphere $S(x, d(x))$ representation (4) gives $u(x) = E_x u(W_x(\tau_{S(x,d(x))}))$. The same representation is valid for all points $y \in S(x, d(x))$, so we can use the strong Markov property and write the conditional expectation
$u(x) = E_x\{E_y u(W_y(\tau_{S(y,d(y))}))/x = W(0), y = W(\tau_{S(x,d(x))})\}$. We can iterate this representation many times and remark that only random points lying on the spheres $S(x, d(x)), S(y, d(y)), \ldots,$ are involved. It is well-known, that the points $W_x(\tau_{S(x,d(x))})$ are uniformly distributed over the sphere $S(x, d(x))$. Thus we came to the definition of the walk on spheres process starting at x : it is defined as the homogeneous Markov chain $WS = WS\{x_0, x_1, \ldots, x_k, \ldots\}$ such that $x_0 = x$ and $x_k = x_{k-1} + d(x_{k-1})\omega_k,\ k = 1, 2 \ldots,$ where $\{\omega_k\}$ is a sequence of independent isotropic unit vectors. It is known [5], that $x_k \to y \in \Gamma$ as $k \to \infty$, however the number of steps of Markov chain WS is infinite with probability one. For more see [11].

4. APPENDIX

4.1. RANDOM WALK

In mathematics and physics, a random walk is a formalization of the intuitive idea of taking successive steps, each in a random direction. A random walk is a simple stochastic process. In this section we discuss some basic properties of the random walks.

Example 4.1. *The simplest random walk is a path constructed according to the following rules: 1) there is a starting point; 2) the distance from one point in the path to the next is a constant; 3) the direction from one point in the path to the next is chosen at random, and no direction is more probable than another.*

The average straight-line distance between start and finish points of a random walk of n steps is on the order of $\sqrt{n}$. In fact, if „average" is understood in the sense of root-mean-square, then the average distance after n steps is exactly $\sqrt{n}$ times the step length.

Definition 4.1. *Brownian motion is a continuous-time stochastic process $W(t)$ for $t \geq 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 \leq s < t$, and increments for nonoverlapping time intervals are independent. Brownian motion (i.e., random walk with random step sizes) is the most common example of a Wiener process.*

Brownian motion is the *scaling limit* of random walk in dimension 1. This means that if you take a random walk with very small steps you get an approximation to Brownian motion. To be more precise, if the step size is ε, one needs to take a walk of length L/ε^2 to approximate a Brownian motion of length L. As the size step tends to 0 (and the number of steps increased comparatively) random walk converges to Brownian motion in an appropriate sense. Formally, if B is the space of all paths of length L with the maximum topology, and if M is the space of measures over B with the norm topology, then the convergence is in the space M. Similarly, Brownian motion in several dimensions is the scaling limit of random walk in the same number of dimensions. A random walk is a discrete fractal, but Brownian motion is a true fractal, and there is a connection between the two. For more see [8].

4.2. PROOFS

Proof of Theorem 1 *Let us have* $f = -\Delta \ln h$, *in* G *and* $\phi = \ln h$, *in* Γ. *Then* $\ln h$ *solves the equation (1), (2) and from Lax-Milgram theorem we have the uniqueness. We have* $d\mu(y) = h(y, r)$ *and* $E(r)$ *equals to* r*-parameter Fisher information.*$\square$

Proof of Theorem 2 *Denote* $\eta(I, x, \theta) = I f_1(x, \theta)$. *Then for* $t_1 > 0$ *and* $t_2 > t_1$ *(D-optimal design is independent on* I *so we put* $I := 1$*) we have* $M = F^T F$ *where*

$$F = \begin{pmatrix} f_1(x_1, \theta), & \frac{\partial f_1(x_1,\theta)}{\partial \theta} \\ f_1(x_2, \theta), & \frac{\partial f_1(x_2,\theta)}{\partial \theta} \end{pmatrix}.$$

Let $t_1 = 0$. *Then we have* $\det M = g(\sigma, \gamma, \alpha, \beta)(x_2^{\beta_1} - x_2^{\beta_2})^2$. *Let us have* $t_1 > 0$. *Then we have* $\det M = g(\sigma, \gamma, \alpha, \beta)(-2x_1^{\beta_1+\beta_2} x_2^{\beta_1+\beta_2} + x_1^{2\beta_2} x_2^{2\beta_1} + x_2^{2\beta_2} x_1^{2\beta_1}).\square$

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References

[1] Abt M., Welch W. J., *Fisher information and maximum-likelihood estimation of covariance parameters in Gaussian stochastic processes*, The Canadian Journal of Statistics, **26**, 1(1998), 127 – 137.

[2] Cressie N. A. C., *Statistics for spatial Data*, Wiley, New York, 1993.

[3] Cioranescu D., Murat F. *A strange term brought from somewhere else*, Nonlinear differential equations and their applications, Collége de France Seminar, Vol II, 98-138, 389-390, (Paris, 1979/1980), Res. Notes in Math., **60**, Pitman, Boston, 1982.

[4] Cioranescu D., Murat F. *A strange term coming from nowhere*, Topics in Mathematical Modelling for composite materials, 45-93, Progr. in Nonlinear PDEs Appl.,**31**, Birkhäuser, Cherkaev A., Kohn R.(eds.), 1997.

[5] Müller M. E. *Some continuous Monte Carlo methods for the Dirichlet problem*, Ann. Math. Statistics, **27**, 3(1956), 569-589.

[6] Müller W. G., *Collecting spatial data*, 2nd ed., Physica Verlag (2000).

[7] Müller W. G., Stehlík M., *An example of D-optimal designs in the case of correlated errors*, Proceedings in Computational Statistics, Edited by J. Antoch, Physica-Verlag, (2004), 1543 – 1550.

[8] Feller W., *An introduction to probability theory and its applications* (volume 1, Ch. 3), 1968.

[9] Pázman A., *Correlated optimum design with parametrized covariance function: Justification of the use of the Fisher information matrix and of the method of virtual noise*, Report #5 of the Research Report Series of the Department of Statistics and Mathematics, Wirtschaftsuniversität Wien, Austria, free download from: http://epub.wu-wien.ac.at/, (2004).

[10] Rajesh M., *Convergence of some energies for the Dirichlet problem in perforated domains*, Rendiconti di Matematica, Roma, Serie VII, **21** (2001), 259-274.

[11] Sabelfeld K. K., N. A. Simonov N. A., *Random walks on boundary for solving PDEs*, VSP, Utrecht, 1994.

[12] Sánchez G., López-Fidalgo J., *Mathematical techniques for solving large compartmental systems*, Health Phys, **85**, 2(2003), 184-193.

[13] Stehlík M., *Covariance related properties of D-optimal correlated designs*, submitted to Proceedings: J. Antoch and G. Dohnal (eds.) ROBUST 2004, Praha.

SOME FIRST THOUGHTS ON THE STABILITY OF THE ASYNCHRONOUS SYSTEMS

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Abstract The (non-initialized, non-deterministic) asynchronous systems (in the input-output sense) are multi-valued functions from m-dimensional signals to sets of n-dimensional signals, the concept being inspired by the modeling of the asynchronous circuits. Our purpose is to state the problem of the their stability.

Keywords: signal, asynchronous system, stability.

1. INTRODUCTION

Let $\mathbf{B} = \{0,1\}$ be the binary Boole algebra. The function $x : \mathbf{R} \to \mathbf{B}^n$ has a limit as $t \to \infty$ if

$$\exists t_f, \forall t \geq t_f, x(t) = x(t_f). \tag{1}$$

The usual notation is $x(t_f) = \lim_{t\to\infty} x(t)$. x is called a (n-dimensional) signal if it is of the form

$$x(t) = x(t_0 - 0) \cdot \varphi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \varphi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \varphi_{[t_1,t_2)}(t) \oplus ... \tag{2}$$

where $t \in \mathbf{R}$. In (1.2) $\varphi_{(\quad)} : \mathbf{R} \to \mathbf{B}$ is the characteristic function and $t_0 < t_1 < t_2 < ...$ is some unbounded sequence. We denote $S^{(n)} = \{x|x : \mathbf{R} \to \mathbf{B}^n, x \text{ is signal}\}$, $P^*(S^{(n)}) = \{X|X \subset S^{(n)}, X \neq \emptyset\}$, $S_c^{(n)} = \{x|x \in S^{(n)}, \exists \lim_{t\to\infty} x(t)\}$. For the Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ we denote also $S_{F,c}^{(m)} = \{u|u \in S^{(m)}, \exists \lim_{t\to\infty} F(u(t))\}$. Any signal x has an initial time instant t_0, from the definition (1.2). It is not unique and it is precised by the condition $\forall t < t_0, x(t) = x(t_0 - 0)$, where (the unique) $x(t_0 - 0)$ is the initial value of x. In particular, the constant function x satisfies the property that any t_0 is an initial time instant and x coincides with its initial value. There exist signals without final time instant t_f and respectively without final value $\lim_{t\to\infty} x(t)$.If t_f exists, it is not unique and any $t'_f > t_f$ is a final time instant too. In particular, the constant function x satisfies the property that any t_f is a final time instant and x coincides with its final value.

When x is the state of a system, the problem of the existence of t_f, thus of the limit $\lim\limits_{t\to\infty} x(t)$ is the stability problem of that system.

2. ASYNCHRONOUS SYSTEMS

Definition We call (non-initialized, non-deterministic) *asynchronous system* (in the input-output sense) a function $f : U \to P^*(S^{(n)})$, where $U \in P^*(S^{(m)})$. The elements $u \in U$, respectively $x \in f(u)$, are called (admissible) *inputs*, respectively (possible) states, or *outputs*.

Remark The concept of asynchronous system has its origin in the modeling of the asynchronous circuits, where the multivalued association between the cause u and the effects $x \in f(u)$ is motivated by the changes in power supply, temperature, by the technologycal dispersion, by the errors of the measurement instruments etc.

Definition The system $g : V \to P^*(S^{(n)})$, $V \in P^*(S^{(m)})$ is called a *subsystem* of f if $V \subset U \quad and \quad \forall u \in V, g(u) \subset f(u)$.

Definition The system $f^* : U^* \to P^*(S^{(n)})$, $U^* \in P^*(S^{(m)})$ is called the *dual system* of f if $U^* = \{\overline{u} | u \in U\}$ and $\forall u \in U, f^*(\overline{u}) = \{\overline{x} | x \in f(u)\}$. We have denoted by $\overline{u}, \overline{x}$ the coordinatewise complements of these signals, for example $\overline{u}(t) = (\overline{u_1(t)}, ..., \overline{u_m(t)})$.

Definition Suppose that $U \cap V \neq \emptyset$ and that $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$. The system $f \cap g : U \cap V \to P^*(S^{(n)})$ is defined by

$$\forall u \in U \cap V, (f \cap g)(u) = f(u) \cap g(u).$$

Definition The system $f \cup g : U \cup V \to P^*(S^{(n)})$ is defined as

$$\forall u \in U \cup V, (f \cup g)(u) = \begin{cases} f(u), if \quad u \in U - V \\ g(u), if \quad u \in V - U \\ f(u) \cup g(u), if \quad u \in U \cap V \end{cases}$$

Definition Let the system $f' : U' \to P^*(S^{(n')}), U' \in P^*(S^{(m)})$. If $U \cap U' \neq \emptyset$, the parallel connection of f and f' is the system $(f, f') : U \cap U' \to P^*(S^{(n+n')})$ defined by

$$\forall u \in U \cap U', (f, f')(u) = \{z | z \in S^{(n+n')},$$

$$\forall i \in \{1, ..., n+n'\}, z_i = \begin{cases} x_i, \quad if \quad i \in \{1, ..., n\}, x \in f(u) \\ y_{i-n}, if \quad i \in \{n+1, ..., n+n'\}, y \in f'(u) \end{cases} \}$$

Definition The system $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ be given such that $\forall u \in U, f(u) \cap X \neq \emptyset$. The *serial connection* of h and f is the system $h \circ f : U \to P^*(S^{(p)})$ that is defined by

$$\forall u \in U, (h \circ f)(u) = \{y | \exists x \in f(u) \cap X, y \in h(x).\}$$

Definition The system f is called *non-anticipatory*, or *causal* if

$$\forall t_1 \in \mathbf{R}, \forall u \in U, \forall v \in U, u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)} \Longrightarrow$$
$$\Longrightarrow \{x_{|(-\infty,t_1)} | x \in f(u)\} = \{y_{|(-\infty,t_1)} | y \in f(v)\}$$

Definition The system f is *initialized* if

$$\exists w^0 \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = w^0$$

If so, the unique vector w^0 satisfying the previous property is called the initial state of f.

3. STEADY VALUES OF THE STATES

Definition Let the system $f : U \to P^*(S^{(n)})$, $U \subset S^{(m)}$. If

$$\exists u \in U, \exists x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

then the binary vector w is called the *steady value*, or the final value, or the limit when $t \to \infty$ of the state $x \in f(u)$. In the special case when

$$\exists u \in U, \exists x \in f(u), \exists w \in \mathbf{B}^n, \forall t \in \mathbf{R}, x(t) = w$$

is true, w is called a *point of equilibrium* of f.

Remark For any u and any $x \in f(u)$, if $w = \lim_{t\to\infty} x(t)$ exists, then it is unique.

Notation For $u \in U$, we note $\Sigma_f(u) = \{w | \exists x \in f(u), w = \lim_{t\to\infty} x(t)\}$.

4. INITIAL TIME AND FINAL TIME

Definition We say that the system f has an *initial time* (instant) t_0 which is:

a) *unbounded* if $\forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = x(t_0 - 0)$;

b) *bounded* if$\forall u \in U, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = x(t_0 - 0)$;

c) *fix* (or universal) if$\exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t < t_0, x(t) = x(t_0 - 0)$.

We say that the system f has a *final time* (instant) t_f which is:

a$'$) *unbounded* if $\forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f)$;

b$'$) *bounded* if $\forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f)$;

c$'$) *fix*(or universal) if $\exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f)$.

Remarks There are $3 \times 3 = 9$ possibilities of combining the initial time and the final time for a system.

The next implications are true: $t_0 \quad fix \Longrightarrow t_0 \quad bounded \Longrightarrow t_0 \quad unbounded$ and the next implications are true also:

$$t_f \quad fix \Longrightarrow t_f \quad bounded \Longrightarrow t_f \quad unbounded$$

5. ABSOLUTE STABILITY

Definition a) A system f that satisfies

$$\forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w,$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called *absolutely stable*.

b) If $\forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$, then f is called *absolutely race-free stable*, or *absolutely delay-insensitive*.

c) We say that f is *absolutely constantly stable* if it satisfies

$$\exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w.$$

Remarks The following implications are true:

$$f \quad abs \quad const \quad stable \Longrightarrow f \quad abs \quad race-free \quad stable \Longrightarrow f \quad abs \quad stable$$

On the other hand, if f is absolutely stable, then it defines the system $\lim f : U \to P^*(S^{(n)})$ by $\forall u \in U, \lim f(u) = \Sigma_f(u)$ and we have identified the binary vector with the constant vector function. In case of absolute race-free stability, this system is deterministic, i.e. $\forall u \in U$, the set $\lim f(u)$ has exactly one element. If the absolute constant stability of f is true also, then $\lim f$ is the constant univalued function.

Sometimes it will be useful to write the absolute stability condition under the form $\forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t-0) = w$ and similarly for the other two cases, showing the fact that x has reached its final value w sometimes before t_f.

Theorem *Suppose that $f : U \to P^*(S^{(n)})$, $U \subset S^{(m)}$ is an absolutely stable (an absolutely race-free stable, an absolutely constantly stable) system and let the systems $g : V \to P^*(S^{(n)})$, $V \subset S^{(m)}$, $f' : U' \to P^*(S^{(n')})$, $U' \subset S^{(m)}$. The next statements are true: a) if $g \subset f$, then g is absolutely stable (absolutely race-free stable, absolutely constantly stable); b) f^* is absolutely stable (absolutely race-free stable, absolutely constantly stable); c) if $U \cap V \neq \emptyset$ and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then $f \cap g$ is absolutely stable (absolutely race-free stable, absolutely constantly stable); d) if g is absolutely stable (absolutely race-free stable, absolutely constantly stable), then $f \cup g$ is absolutely stable (absolutely race-free stable, absolutely constantly stable); e) if f' is absolutely stable (absolutely race-free stable, absolutely constantly stable) and if $U \cap U' \neq \emptyset$, then (f, f') is absolutely stable (absolutely race-free stable, absolutely constantly stable).*

Theorem *Let the systems f and $h : X \to P^*(S^{(p)})$, $X \subset S^{(n)}$. Suppose that $\forall u \in U, f(u) \cap X \neq \emptyset$; then, if h is absolutely stable (absolutely constantly stable), we have that $h \circ f$ is absolutely stable (absolutely constantly stable).*

Remark In general, the statement of the previous theorem is false in the case of absolute race-free stability.

Theorem *The next properties are equivalent for the system f:*

a) *absolute stability with unbounded final time:*

$$\begin{cases} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w,$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute);

b) *absolute stability with bounded final time:*

$$\begin{cases} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \forall t \geq t_f, x(t) = w;$$

c) *absolute stability with fix final time:*

$$\begin{cases} \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \exists w \in \mathbf{B}^n, \forall t \geq t_f, x(t) = w;$$

d) *absolute race-free stability with unbounded final time:*

$$\begin{cases} \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w;$$

e) *absolute race-free stability with bounded final time:*

$$\begin{cases} \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \forall u \in U, \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t) = w,$$

where w and t_f depend on u only (thus $\exists w, \exists t_f$ commute);

f) *absolute race-free stability with fix final time:*

$$\begin{cases} \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists t_f \in \mathbf{R}, \forall u \in U, \exists w \in \mathbf{B}^n, \forall x \in f(u), \forall t \geq t_f, x(t) = w,$$

g) *absolute constant stability with unbounded final time:*

$$\begin{cases} \exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w;$$

h) *absolute constant stability with bounded final time:*

$$\begin{cases} \exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w \in \mathbf{B}^n, \forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t) = w;$$

i) *absolute constant stability with fix final time:*

$$\begin{cases} \exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w \\ \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \forall t \geq t_f, x(t) = x(t_f) \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t \geq t_f, x(t) = w,$$

where w and t_f are constant (thus $\exists w, \exists t_f$ commute).

Theorem *Let the system f having the property that it is non-anticipatory and with fix final time. a) If f is absolutely stable, then the set $\Sigma_f(u)$ depends on the restriction $u|_{(-\infty,t_f]}$ only. b) In the case that f is absolutely delay-insensitive, the limit $\lim\limits_{t\to\infty} x(t)$, that is the same for all $x \in f(u)$, depends on $u|_{(-\infty,t_f]}$ only. c) If f is absolutely constantly stable, $\lim\limits_{t\to\infty} x(t)$ is the same for all $x \in f(u)$ and all $u \in U$.*

6. RELATIVE STABILITY

Definition a) A system f that satisfies

$$\forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w,$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called *relatively stable.*

b) If the next property is true

$$\forall u \in U \cap S_c^{(m)}, \exists w \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w$$

then f is *called relatively race-free stable* or *relatively delay-insensitive.*

c) f is relatively constantly stable if

$$\exists w \in \mathbf{B}^n, \forall u \in U \cap S_c^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w.$$

If $U \cap S_c^{(m)} = \emptyset$ we say that the previous stability properties are *trivially fulfilled* and if $U \cap S_c^{(m)} \neq \emptyset$ that they are *non-trivially fulfilled*.

Remark Relative stability and absolute stability are analized similarly.

7. STABILITY RELATIVE TO A FUNCTION

Definition Let the Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$. a) A system f satisfying

$$\forall u \in U \cap S_{F,c}^{(m)}, \forall x \in f(u), \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w,$$

where w and t_f depend on x only (thus $\exists w, \exists t_f$ commute) is called $F-$*relatively stable* (or *stable relative to the function* F). b) If the next property holds

$$\forall u \in U \cap S_{F,c}^{(m)}, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = F(u(t_f))$$

then f is called $F-$*relatively race-free stable*, or $F-$ *relatively delay-insensitive* (*race-free stable relative to the function* F, *delay-insensitive relative to the function* F). c) f is $F-$*relatively constantly stable* if it is $F-$relatively race-free stable and the function F is constant

$$\exists w \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = w.$$

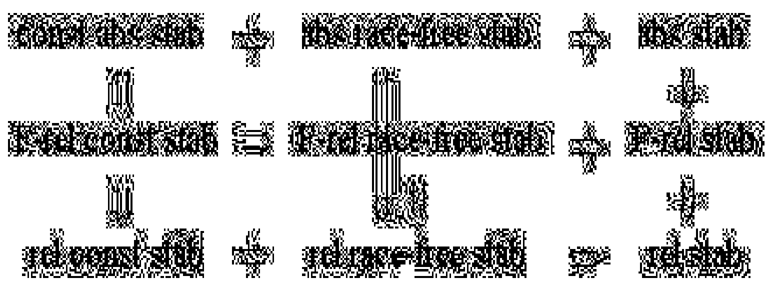

Fig. 1.

If $U \cap S_{F,c}^{(m)} = \emptyset$ the previous stability properties are trivial and if $U \cap S_{F,c}^{(m)} \neq \emptyset$ they are non-trivial.

Remarks The stability of a system relative to a Boolean function is similar with the other notions of stability. Remark that the notions of $F-$relative constant stability and respectively of absolutely constant stability coincide, being at the same time a special case of $F-$relative race-free stability.

In fig. 1 we give the existing connection between the nine previously defined types of stability.

8. SYNCHRONOUS-LIKE, MONOTONOUS AND HAZARD-FREE TRANSITIONS. THE FUNDAMENTAL MODE

Definition For $x \in S^{(n)}$ and the time instances $t' < t"$, the couple $(x(t'), x(t"))$ is called *transition*; we say that x has a transition in the interval $[t', t"]$ from the value $x(t')$ to the value $x(t")$.

Definition By the *transition* $(x(t'-0), x(t"-0))$ it is understood any of the transitions $(x(t'-\varepsilon), x(t"-\varepsilon))$, where $\varepsilon > 0$ is taken sufficiently small so that $\forall \xi \in (0, \varepsilon], x(t'-\xi) = x(t'-0)$, $\forall \xi \in (0, \varepsilon], x(t"-\xi) = x(t"-0)$. The interval on which this transition takes place is by definition any of the intervals $[t'-\xi, t"-\xi]$ with $\xi \in (0, \varepsilon]$.

Notations The usual notations for the transitions $(x(t'), x(t"))$ and $(x(t'-0), x(t"-0))$ are $x(t') \to x(t")$ and $x(t'-0) \to x(t"-0)$ respectively. The interval on which $x(t'-0) \to x(t"-0)$ takes place is denoted $[t'-0, t"-0]$.

Definition The next data are given: the system f, the input $u \in U$, the state $x \in f(u)$ and the instants $t' < t"$. In this case the transition $x(t') \to x(t")$ is also called *transfer* of x under the input u in the interval $[t', t"]$ from the value $x(t')$ to the value $x(t")$ and we say that f transfers x under the input u (it $u-$transfers x) in the interval $[t', t"]$ from $x(t')$ to $x(t")$.

Similarly for the transition $x(t'-0) \to x(t"-0)$.

Definition a) Suppose that $w, w' \in \mathbf{B}^n, t_0, t_f \in \mathbf{R}$ and $u \in U$ exist so that: a.i) $\forall x \in f(u), \forall t < t_0, x(t) = w$; a.ii) $\forall x \in f(u), \forall t \geq t_f, x(t-0) = w'$; a.iii) $t_0 < t_f$. Then $x(t_0 - 0) \to x(t_f - 0)$ is a synchronous-like transition (or transfer); we say that f *transfers synchronous-likely* (any) x under the input u in the interval $[t_0 - 0, t_f - 0]$ from the value w to the value w'. b) Suppose that $w, w' \in \mathbf{B}^n, t_f, t'_f \in \mathbf{R}$ and $u, v \in U$ exist so that b.i) $\forall x \in f(u), \forall t \geq t_f, x(t-0) = w$ b.ii) $\forall y \in f(v), \forall t \geq t'_f, y(t-0) = w'$; b.iii) $t_f < t'_f$; b.iv) $u_{|(-\infty, t_f)} = v_{|(-\infty, t_f)}$; b.v) $\{x_{|(-\infty, t_f)} | x \in f(u)\} = \{y_{|(-\infty, t_f)} | y \in f(v)\}$. If they are true, then $y(t_f - 0) \to y(t'_f - 0)$ is a *synchronous-like transition* (or transfer). We also say that the system f transfers synchronous-likely (any) y under the input $v = u \cdot \varphi_{(-\infty, t_f)} \oplus v \cdot \varphi_{[t_f, \infty)}$ in the interval $[t_f - 0, t'_f - 0]$ from the value w to the value w'.

c) All the synchronous-like transitions are these from a) and b).

Remarks The attribute 'synchronous-like' given to a transition $y(t_f - 0) \to y(t'_f - 0)$ implies the fact that $y(t_f - 0) = x(t_f - 0)$ is a steady value of $x \in f(u)$ and $y(t'_f - 0)$ is a steady value $y \in f(v)$. The initial value is the same for all $x \in f(u)$ and all $y \in f(v)$ and it is treated as a steady value. Things happen as if the unique state y switches with all the coordinates simultaneously (synchronously), in discrete time, in the manner $y(k) = w, y(k+1) = w', ...$ On the other hand, the 'composition' of the synchronous-like transitions is a synchronous-like transition: if $t_f < t'_f < t''_f$ and if $y(t_f - 0) \rightarrow y(t'_f - 0)$, $y(t'_f - 0) \rightarrow y(t''_f - 0)$ are synchronous-like transitions, then $y(t_f - 0) \rightarrow y(t''_f - 0)$ is synchronous-like too.

Definition Let the system f and the input u have the property of existence of an unbounded sequence $t_0 < t_1 < t_2 < ...$ so that $x(t_k - 0) \rightarrow x(t_{k+1} - 0)$ be synchronous-like for all $k \in \mathbf{N}$ and all $x \in f(u)$. We say that f is, under the input u, in the *fundamental (operating) mode.*

Definition The non-empty set $U \subset S^{(m)}$ is called $\sigma-$*closed* if for any sequence $u^k \in U, k \in \mathbf{N}$ of inputs and any unbounded sequence $t_0 < t_1 < t_2 < ...$ of real numbers we have $u^0 \cdot \varphi_{(-\infty,t_0)} \oplus u^1 \cdot \varphi_{[t_0,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus ... \in U$.

Theorem *Suppose that U is $\sigma-$closed and that f satisfies: a) it is non-anticipatory; b) it satisfies the next property of initialization with bounded initial time*

$$\forall u \in U, \exists w^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \forall t < t_0, x(t) = w^0,$$

where w^0 and t_0 depend on u only (thus $\exists w^0, \exists t_0$ commute); c) it is absolutely race-free stable with bounded final time, i.e.

$$\forall u \in U, \exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall x \in f(u), \forall t \geq t_f, x(t - 0) = w,$$

where w and t_f depend on u only (thus $\exists w, \exists t_f$ commute). Then for any sequence $u^k \in U, k \in \mathbf{N}$ of inputs, the unbounded sequence $t_0 < t_1 < t_2 < ...$ of real numbers exists so that the transitions $x(t_k - 0) \rightarrow x(t_{k+1} - 0)$, $k \in \mathbf{N}$, $x \in f(u)$ are synchronous-like, where $u \in U$ is given by

$$u = u^0 \cdot \varphi_{(-\infty,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus u^3 \cdot \varphi_{[t_2,t_3)} \oplus ...$$

Remark The previous theorem has two variants when 'f is absolutely race-free stable' is replaced by 'f is relatively race-free stable' and by 'f is $F-$relatively race-free stable' respectively.

Theorem *Let the system $f : U \rightarrow P^*(S^{(n)})$, with U $\sigma-$closed and make the next assumptions: a) f is non-anticipatory; b) it is initialized with fix initial time, i.e. $\exists w^0 \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \forall t < t_0, x(t) = w^0$, where w^0 and t_0 are constant (thus $\exists w^0, \exists t_0$ commute); c) the next controllability properties hold:*

$$\forall w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \exists u \in U, \forall x \in f(u), \forall t \geq t_f, x(t - 0) = w$$

$$\exists w \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \exists u \in U, \forall x \in f(u), \forall t \geq t_f, x(t - 0) = w \Longrightarrow$$

$$\Longrightarrow \forall w' \in \mathbf{B}^n, \exists t'_f \in \mathbf{R}, \exists v \in U, \forall x \in f(u \cdot \varphi_{(-\infty,t_f)} \oplus v \cdot \varphi_{[t_f,\infty)}),$$

$$\forall t \geq t'_f, x(t - 0) = w'.$$

Then for any sequence $w^k \in \mathbf{B}^n, k \geq 1$ *of binary vectors, an unbounded sequence* $t_0 < t_1 < t_2 < ...$ *of real numbers and a sequence of inputs* $u^k \in U, k \in \mathbf{N}$ *exist so that the input* $u \in U$ *defined by*

$$u = u^0 \cdot \varphi_{(-\infty,t_1)} \oplus u^2 \cdot \varphi_{[t_1,t_2)} \oplus u^3 \cdot \varphi_{[t_2,t_3)} \oplus ...$$

satisfies the property $x(t_k - 0) = w^k$, $x(t_k - 0) \to x(t_{k+1} - 0)$ *are synchronous−like for all* $k \in \mathbf{N}$ *and all* $x \in f(u)$.

Definition The transition $x(t') \to x(t")$ is called *monotonous*, if all the coordinate functions $x_i, i = \overline{1,n}$ restricted to the interval $[t', t"]$ are monotonous, i.e. they have on $[t', t"]$ at most one discontinuity point. The transition $x(t' - 0) \to x(t" - 0)$ is monotonous if all the coordinate functions $x_i, i = \overline{1,n}$ restricted to all the intervals $[t' - \varepsilon, t" - \varepsilon]$ with $\varepsilon > 0$ chosen sufficiently small are monotonous.

Definition If for $u \in U$ and $t_f < t'_f$ the transfer $x(t_f - 0) \to x(t'_f - 0)$ is synchronous-like and monotonous, $x \in f(u)$ then it is called *hazard-free*.

9. CONCLUSIONS

The asynchronous systems are a mathematical concept that is inspired by the modeling of the asynchronous circuits and the purpose of this paper is that of stating the stability problem for them. Furthermore, we can connect this topic with the notions of controllability and accessibility (by analogy we can adopt from [1] about eight definitions of controllability and four definitions of accessibility, but there exist also different points of view in the literature) we can change / replace the non-anticipation condition with other similar or dual conditions, we can suppose that f is generated by a generator function $\Phi : \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{B}^n$, that it satisfies supplementary inertial properties etc.

References

[1] M. Megan, Proprietés qualitatives des systemes linéaires controlés dans les espaces de dimension infinie, Monographies mathématiques, Université de Timisoara, Departement de mathématique, Timisoara, 1988.

[2] Ş E. Vlad, Topics in asynchronous systems, Analele Universitatii Oradea, Fasc Matematica, **10** (2003), 115-170.

ON ROOTS OF HYPERGEOMETRIC FUNCTIONS

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1. INTRODUCTION

It is known [9] that the normed conjugate product of gamma functions such as

$$\frac{2}{\pi}\Gamma(1-ix)\Gamma(1+ix) = \frac{2}{\pi}\frac{1}{\Pi_{n=1}^{\infty}(1+x^2/n^2)}, \tag{1}$$

is an infinitely divisible density. From the infinite divisibility of the above probability distribution and from numerical analysis of roots of the hypergeometric functions the author could guess that a probability distribution with the following density function consisting of conjugate product of gamma functions, that is

$$c\left|\frac{\Gamma(m+ix)}{\Gamma(m)}\right|^2 = \frac{c}{\Pi_{n=0}^{\infty}(1+x^2/(m+n)^2)}, \tag{2}$$

is infinitely divisible (cf. [1. 6.1.25]). In this case the hypergeometric function $F(-n,\, 2m; 2m+n+1; z)$ appears and it is necessary to show that the hypergeometric function $F(-n,\, 2m; 2m+n+1; z)$ has roots outside the unit disk. In this paper, by making use of Watson's formula, it is shown that all the roots of $F(-n,\, 2m; 2m+n+1; z)$ are situated outside the unit disk for any positive constant m and for each natural number n.

2. THE HYPERGEOMETRIC SERIES

In what follows, suppose that m is a positive number and $a_1 = m$, $a_2 = m+1, \ldots, a_{n+1} = m+n$. Let us consider the following density function instead of (2)

$$f(x) = \frac{c}{\Pi_{j=1}^{n+1}(x^2+a_j^2)}, \tag{3}$$

where c is a constant which satisfies the following equality

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The probability density function $f(x)$ is an approximation of the above right hand side of (2) in the sense of weak limit. Let us consider a characteristic function of the density fucntion (3). It holds that

$$\begin{aligned} \phi(t) &= \int_{-\infty}^{\infty} e^{itx} \frac{c}{\Pi_{j=1}^{n+1}(x^2+a_j^2)} dx \\ &= \pi c \sum_{j=1}^{n+1} \frac{\exp(-a_j|t|)}{a_j \Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}, \ -\infty < t < \infty. \end{aligned} \tag{4}$$

If we set $x = \exp(-|t|)$ we obtain a polynomial of the following form

$$\phi(t) = \pi c \sum_{j=1}^{n+1} \frac{x^{a_j}}{a_j \Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}, \quad 0 \le x \le 1,$$

and we have a function

$$\begin{aligned} & F(-n, 2m; 2m+n+1; z) \\ = \ & a_1 \Pi_{l=2}^{n+1}(-a_1^2+a_l^2) \sum_{j=1}^{n+1} \frac{z^{a_j - m}}{a_j \Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}. \end{aligned}$$

They are concretely the following formulae:

$$F(0, 2m; 2m+1; z) = 1,$$

$$F(-1, 2m; 2m+2; z) = 1 + \frac{(-1)(2m)}{2m+2} z, \tag{5}$$

$$F(-2, 2m; 2m+3; z) = 1 + \frac{(-2)(2m)}{2m+3} z + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)} \frac{z^2}{2!}, \tag{6}$$

$$\begin{aligned} F(-3, 2m; 2m+4; z) &= 1 + \frac{(-3)(2m)}{2m+4} z + \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)} \frac{z^2}{2!} \\ &+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+4)(2m+5)(2m+6)} \frac{z^3}{3!}, \end{aligned} \tag{7}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$
\begin{aligned}
& F(-n, 2m; 2m+n+1; z) \\
= \; & 1 + \frac{(-n)(2m)}{2m+n+1} z + \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)} \frac{z^2}{2!} \\
+ \; & \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)} \frac{z^3}{3!} + \cdots \\
+ \; & \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)(2m+2)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+n+k)} \\
\cdot \; & \frac{z^k}{k!} + \cdots \\
+ \; & \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)(2m+2)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)(2m+n+3)\cdots(2m+2n)} \frac{z^n}{n!} \,.
\end{aligned} \tag{8}
$$

We write the above function in the following form,

$$F(-n, 2m; 2m+n+1; z) = \Sigma_{k=0}^{n} \frac{(-n)_k (2m)_k z^k}{(2m+n+1)_k k!}.$$

The hypergeometric series $F(-n, b; c; z)$ is defined by

$$F(-n, b; c; z) = \Sigma_{k=0}^{n} \frac{(-n)_k (b)_k z^k}{(c)_k k!}.$$

Here $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ denotes the Pochhammer symbol.

3. THE HYPERGEOMETRIC SERIES HAS ROOTS OUTSIDE THE UNIT DISK

If $m = 1$ it is known [8] that the roots of $F(-n, 2; n+3; z)$ are situated outside the unit disk. If $n = 1$ the root of $F(-1, 2m; 2m+2; z)$ is $z_1 = (m+1)/m$ and if $n = 2$ the roots of $F(-2, 2m; 2m+3; z)$ are

$$z_1 = \frac{2m+4}{2m+1} + i\frac{1}{2m+1}\sqrt{\frac{3(m+2)}{m}}, \quad z_2 = \frac{2m+4}{2m+1} - i\frac{1}{2m+1}\sqrt{\frac{3(m+2)}{m}}$$

for any positive number m. These roots are outside the unit disk. Concerning the roots of the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ for n larger than 2 we obtain the following result:

Theorem 3.1. *If m is a positive number and n is a natural number, the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ has roots outside the unit disk.*

Proof. Since we have $F(-n, 2m; 2m+n+1; 0) = 1$, take and fix a positive number σ in the interval (0,1]. The curve of the function $F(-n, 2m; 2m+$

$n+1;\sigma e^{it})$ of the variable t is symmetric with respect to the real axis in the complex plane. In order to show that all the roots of the Gauss hypergeometric function $F(-n,2m;2m+n+1;z)$ are outside the unit disk, it suffices to show that $|F(-n,2m;2m+n+1;\sigma e^{it})|^2$ is positive for all t in the interval $[0,\pi]$. We make use of the Watson's formula

$$\begin{aligned}&F(-n,b;c;z)F(-n,b;c;Z)\\ =\ &\frac{(c-b)_n}{(c)_n}\, F_4[-n,b;c,1-n+b-c;zZ,(1-z)(1-Z)]\end{aligned}$$

in Slater's book (cf. [6. (8.4.2)]). Let $b=2m,\ c=2m+n+1$. Then

$$\begin{aligned}&F_4[-n,2m;2m+n+1,-2n;zZ,(1-z)(1-Z)]\\ =\ &\Sigma_{r=0}^{n}\Sigma_{s=0}^{n}\frac{(-n)_{r+s}(2m)_{r+s}(zZ)^r[(1-z)(1-Z)]^s}{(2m+n+1)_r(-2n)_s r!s!}.\end{aligned}$$

Let $z=\sigma e^{it},\ Z=\sigma e^{-it}$. Then $zZ=\sigma^2$ and $(1-z)(1-Z)=1+\sigma^2-2\sigma\cos t$. Let $x=\sigma^2$ and $y=1+\sigma^2-2\sigma\cos t$. From the above F_4 we obtain

$$\begin{aligned}&|F(-n,2m;2m+n+1;\sigma e^{it})|^2\\ =\ &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{r=0}^{n}\Sigma_{s=0}^{n}\frac{(-n)_{r+s}(2m)_{r+s}x^r y^s}{(2m+n+1)_r(-2n)_s r!s!}.\end{aligned}\tag{9}$$

It holds that

$$\begin{aligned}&|F(-n,2m;2m+n+1;\sigma e^{it})|^2\\ =\ &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(2m)_s y^s}{(-2n)_s s!}\left\{\Sigma_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!}\right\}.\end{aligned}$$

We see that the right hand side of the above equality is positive for all t in the interval $[0,\pi]$ since it holds that

$$\begin{aligned}&\Sigma_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!}\\ =\ &\frac{\Gamma(2m+n+1)}{\Gamma(2m+s)\Gamma(n-s+1)}\int_0^1 \tau^{2m+s-1}(1-\tau)^{n-s}(1-x\tau)^{n-s}d\tau.\end{aligned}$$

When $\sigma=1$, by Vandermonde's formula we see that

$$\begin{aligned}&|F(-n,2m;2m+n+1;e^{it})|^2\\ =\ &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s}s!}\\ =\ &\Sigma_{s=0}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!y^s,\end{aligned}$$

where $y = 2(1 - \cos t)$. $\square$

4. THE WRONSKIAN

It is often convenient for us to treat the polynomial $z^m F(-n, 2m; 2m+n+1; z)$ instead of $F(-n, 2m; 2m+n+1; z)$. Let us take a branch such that z^m is positive if z is positive. Consider the unit circle $C:\ z = e^{it}\ \ (0 \le t \le 2\pi)$. Let

$$u(m,n;t) = \text{ Re } e^{imt} F(-n, 2m; 2m+n+1; e^{it}),$$

$$v(m,n;t) = \text{ Im } e^{imt} F(-n, 2m; 2m+n+1; e^{it}).$$

We have

$$u(m,n;t) = \Sigma_{k=0}^{n} \frac{(-n)_k (2m)_k}{(2m+n+1)_k} \frac{\cos(m+k)t}{k!} \tag{10}$$

and

$$v(m,n;t) = \Sigma_{k=0}^{n} \frac{(-n)_k (2m)_k}{(2m+n+1)_k} \frac{\sin(m+k)t}{k!}. \tag{11}$$

We note that the curve of $F(-n, 2m; 2m+n+1; e^{it})$ in the complex plane does not always make a Jordan curve when t moves on the interval $[0, 2\pi]$. It is known [1] that the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ is a solution of the hypergeometric equation, namely

$$\begin{aligned} & z(1-z)\frac{d^2}{dz^2} F(-n, 2m; 2m+n+1; z) \\ + \quad & (2m+n+1-(2m-n+1)z)\frac{d}{dz} F(-n, 2m; 2m+n+1; z) \\ + \quad & 2mn\, F(-n, 2m; 2m+n+1; z) = 0. \end{aligned} \tag{12}$$

Lemma 1.. *If m is a positive number and n is a natural number, then the functions $u(m,n;t)$ and $v(m,n;t)$ are solutions of the following differential equation*

$$\sin\frac{t}{2}\, x''(t) - n\cos\frac{t}{2}\, x'(t) + m(m+n)\sin\frac{t}{2}\, x(t) = 0. \tag{13}$$

Proof. Let $h(z) = z^m F(-n, 2m; 2m+n+1; z)$. Then we obtain the following differential equation

$$z^2(1-z)h''(z) + (n+1+(n-1)z)zh'(z) - m(m+n)(1-z)h(z) = 0. \tag{14}$$

If $z = e^{it}$ we obtain the following equation

$$(1-e^{it})\frac{d^2 h(e^{it})}{dt^2} + n(1+e^{it})i\frac{dh(e^{it})}{dt} + m(m+n)(1-e^{it})h(e^{it}) = 0 \tag{15}$$

and from the real part and imaginary part of the above equation we have two equations

$$\begin{aligned}
& (1-\cos t)u''(m,n;t) - n\sin t\cdot u'(m,n;t) \\
& +m(m+n)(1-\cos t)u(m,n;t) \\
= \;& -\sin t\cdot v''(m,n;t) + n(1+\cos t)v'(m,n;t) \\
& -m(m+n)\sin t\cdot v(m,n;t) \qquad (16)
\end{aligned}$$

and

$$\begin{aligned}
& \sin t\cdot u''(m,n;t) - n(1+\cos t)u'(m,n;t) \\
& +m(m+n)\sin t\cdot u(m,n;t) \\
= \;& (1-\cos t)v''(m,n;t) - n\sin t\cdot v'(m,n;t) \\
& +m(m+n)(1-\cos t)v(m,n;t). \qquad (17)
\end{aligned}$$

Then we have

$$\begin{aligned}
& \sin\frac{t}{2}\Big\{\sin\frac{t}{2}\cdot u''(m,n;t) - n\cos\frac{t}{2}\cdot u'(m,n;t) \\
& +m(m+n)\sin\frac{t}{2}\cdot u(m,n;t)\Big\} \\
= \;& -\cos\frac{t}{2}\Big\{\sin\frac{t}{2}\cdot v''(m,n;t) - n\cos\frac{t}{2}\cdot v'(m,n;t) \\
& +m(m+n)\sin\frac{t}{2}\cdot v(m,n;t)\Big\} \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
& \cos\frac{t}{2}\Big\{\sin\frac{t}{2}\cdot u''(m,n;t) - n\cos\frac{t}{2}\cdot u'(m,n;t) \\
& +m(m+n)\sin\frac{t}{2}\cdot u(m,n;t)\Big\} \\
= \;& \sin\frac{t}{2}\Big\{\sin\frac{t}{2}\cdot v''(m,n;t) - n\cos\frac{t}{2}\cdot v'(m,n;t) \\
& +m(m+n)\sin\frac{t}{2}\cdot v(m,n;t)\Big\}. \qquad (19)
\end{aligned}$$

From the above equations we see that the functions $u(m,n;t)$ and $v(m,n;t)$ are solutions of the differential equation (13). $\square$

By using the Lemma 1 we can obtain the following result:

Theorem 4.1. *If m is a positive number and n is a natural number, then the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ does not have roots on the unit circle.*

Proof. In order to show that the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ does not have roots on the unit circle we will show that the following relation

$$W(t) = u(m,n;t)v'(m,n;t) - u'(m,n;t)v(m,n;t) = c(1-\cos t)^n \tag{20}$$

holds, where c is a positive constant not depending on the variable t. If $t_0 = 0$ or 2π, then $W(t_0) = 0$, while we have

$$\begin{aligned} u(m,n;t_0) &= \frac{(n+1)_n}{(2m+n+1)_n}\cos mt_0, \\ v(m,n;t_0) &= \frac{(n+1)_n}{(2m+n+1)_n}\sin mt_0, \end{aligned}$$

by Vandermonde's formula, and

$$u(m,n;t_0)^2 + v(m,n;t_0)^2 \neq 0.$$

Let

$$\alpha(t) = 2^{-2n}\sin^{-2n}(\frac{t}{2})$$

and

$$\beta(t) = 2^{-2n}m(m+n)\sin^{-2n}(\frac{t}{2}).$$

Then the differential equation (13) can be written in the following form,

$$\{\alpha(t)x'(t)\}' + \beta(t)x(t) = 0$$

and the wronskian $W(t)$ can be expressed in the following form

$$\alpha(t)W(t) = c_1 \ (const).$$

Therefore we obtain

$$W(t) = c_1\ 2^{2n}\sin^{2n}(\frac{t}{2}).$$

To determine c_1 we take $t = \pi$. Then it implies

$$W(\pi) = u(m,n;\pi)v'(m,n;\pi) - u'(m,n;\pi)v(m,n;\pi) = c_1\ 2^{2n}.$$

We can see that

$$\begin{aligned} & W(\pi) \\ = \ & \{\Sigma_{k=0}^{n}\frac{(-n)_k(2m)_k(-1)^k}{(2m+n+1)_k k!}\}\{\Sigma_{k=0}^{n}\frac{(-n)_k(2m)_k(m+k)(-1)^k}{(2m+n+1)_k k!}\} \\ = \ & \frac{(2m)_{n+1}2^{2n-1}}{(2m+n+1)_n}. \end{aligned} \tag{21}$$

Therefore we obtain $c = (2m)_{n+1}2^{n-1}/(2m+n+1)_n$ and c is a positive constant. $\square$

References

[1] M. Abramowitz, I. A. Stegun, Handbook of mathematical functions, Dover, New York, 1970.

[2] L. Bondesson, On the infinite divisibility of the half-Cauchy and other decreasing densities and probability functions on the nonnegative line, *Scand. Acturial J.,* (1985), 225-247.

[3] E. Koelink, W. Van Assche, Orthogonal polynomials and special functions, *Lecture Notes in Mathematics,* **1817** Leuven 2002, Springer 2003.

[4] M. J. Goovaerts, L. D'Hooge, N. De Pril, On the infinite divisibility of the product of two Γ-distributed stochastical variables, *Applied Mathematics and Computation,* **3** (1977), 127-135.

[5] K. Sato, Class L of multivariate distributions and its subclasses, *J. Multivariate Anal.,* **10** (1980), 207-232.

[6] L. J. Slater, Generalized hypergeometric functions, Cambridge University Press, 1966.

[7] F. W. Steutel, Preservation of infinite divisibility under mixing and related topics, *Math. Centre Tracts, Math. Centre, Amsterdam,* **33**, 1970.

[8] K. Takano, On a family of polynomials with zeros outside the unit disk, *International J. Comput. Numer. Anal. Appl.*, **1**, 1 (2002), 369-382.

[9] K. Takano, On infinite divisibility of normed product of Cauchy densities, *J. Comput. Applied Math.*, **150**(2003), 253-263.

[10] K. Takano, H. Okazaki, On the Gauss hypergeometric series with roots outside the unit disk, to appear.

[11] O. Thorin, On the infinite divisibility of the Pareto distribution, *Scand. Acturial. J.,* (1977), 31–40.

METHODOLOGY FOR BOND GRAPH REPRESENTATION OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

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Abstract This paper deals with the methodology for bond graph representation of nonlinear distributed parameter systems, which uses frequency response function as a basis in describing the required behavior. This methodology aids in the determination of control schemes, which are necessary to design the active element, such that the required frequency domain behavior is met.

Keywords: bond graph, frequency response function.

1. INTRODUCTION

Bond graphs were selected as the modeling tools from a number of reasons: the bond graph methodology yields a clear mapping of the topology of a system; there exists a straightforward method by which state equations can be extracted from the bond graphs; it allows for causality information to be contained in the system model; it easily handles the modeling of systems that involve multiple energy domains, and it easily allows for the modeling of system elements that exhibit nonlinear behavior.

2. NONLINEAR MODEL WITH DISTRIBUTED PARAMETERS

Such model is obtained by using the equations of continuity, momentum and energy. These partial differential equations (PDEs) correspond to the physical principles of mass conservation, Newton's second law and energy conservation. Under the assumptions that the fluid is compressible, viscous, isentropic, homogenous and one – dimensional they lead to a coupled nonlinear set of PDEs. They are linearized and written in a form using common notation. We perform a complex – plan curve fitting procedure in order to obtain the corresponding transfer function in analytical form, namely as a ratio of two polynomials. The results of the curve fitting procedure yields what we refer to in this case as the "data – based" transfer function.

3. TRANSFER FUNCTION MODEL

The derivation of a transfer function model for PDE follows the same steps as for the scalar case: apply the Laplace transformation with respect to time. This removes the time derivatives and turns the initial boundary value problem into a boundary value problem for the space variable; construct a suitable transformation for the space variable which removes the spatial derivatives and turns the boundary value problem into an algebraic equation; in order to obtain a multidimensional function, solve the algebraic equation for the transfer of the solution of the PDE;

As accurate as possible curve fit is desired, however the designer must exercise caution, since selecting too high a degree of polynomials can cause the curve - fitting algorithm to yield unstable transfer functions.

The bond graph structure that represents the system is obtained. A signal flow diagram was also developed, which provided information about the nature of the control system needed such that the system exhibited the desired frequency response behavior.

4. ALGORITHM FOR BOND GRAPH REPRESENTATION METHODOLOGY OF PDE

Step 1: obtain desired transfer function, given in analytical form, or determine from curve fit of experimental data; step 2: Determine theoretical transfer function; step 3: cast into bond graph framework – using primitive positive and negative elements bond graph; step 4: incorporate into system bond graph model; step 5: determine state equations from bond graph model, identify virtual state variables; step 6: identify separable bond graph elements, select hybrid or fully active system; step 7: select active devices physical realization, add idealized bond graph representation to model; step 8: determine control signal diagram, graphical representation of the differential equations of virtual state variable; step 9: perform simulations.

5. CONCLUSIONS

The use of the bond graph in representation distributed parameters systems (DPS) was shown. This methodology easily allows for simulations to be run for various physical systems and design parameters. The simulations results allow a systems designer to perform comparisons and make decisions regarding the system. Thus, the bond graph model of the DPS produces the desired response characteristics.

References

[1] Tanasescu, N. – *Identification of distributed parameter systems, applied at mass and heat exchanger processes*. Thesis, Univ. "Politehnica" Bucharest, 1995.

[2] Paynter, H. M., *Analysis and design of engineering systems bond graph methodology*, 1961.

[3] Karnopp, D. C., Rosenberg, R. C., *System dynamics,* New York, 1974.

[4] N, Tanasescu, A, Filipescu, O. Tanasescu, Mathematical modeling of the technological processes treated as distributed parameter dynamic systems, Vienna, 3^{rd} Mathmod, 2000.

SIMULATION MODELS OF HYDRO ENERGETIC EQUIPMENTS DYNAMIC BEHAVIOUR

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Abstract The paper deals with the theoretical issues of the models of simulation of the dynamic behavior of the hydro energetic equipments in order to identify the nature and causes of malfunction. The construction of our model is based on the physics laws of the work process, expressed by non-linear, linear and ordinary differential equations, experimental measurements followed by the identification and adjusting procedure. We describe the dynamic models used in the identification process and different ways to describe these models.

1. INTRODUCTION

In order to model any process we need to have knowledge regarding: structure, expressed by means of mathematical relations, flowcharts and graphs; values of the parameters (structural atributes); values of dependent variables (variables of state). The structure is essential in building the model and its selection may be decisive on the outcome of an identification experiment.

Fig. 1. Representation of the observability of the signals.

As the observability of input-output signals and of the noise contamination of these signals is concerned, in the identification problems there exist the situations described in fig. 1. The values $u(t)$ and $y(t)$ are the input values and output values of the process and $v(t)$ is a noise that contaminates, usually, the output. This criteria is expressed as a quadratic error function, expressed by $J(y, y_M) = \int_0^T \varepsilon^2(t)dt$. In this relation y and y_M are the input values for the process and for the model, defined on the time interval $[0, T]$.

The error is expressed by $\varepsilon = y - y_M = y - M(u)$, where $M(u)$ represents the output value of the model, if its input is u.

2. CONSTRUCTION OF THE MODEL

The elaboration of the model may be achieved in two ways: applying the laws of physics which govern the base of the process; using the results of observations, materialized through measurements, on the functional attributes (input, output, state) of the process. The details are presented in fig. 2.

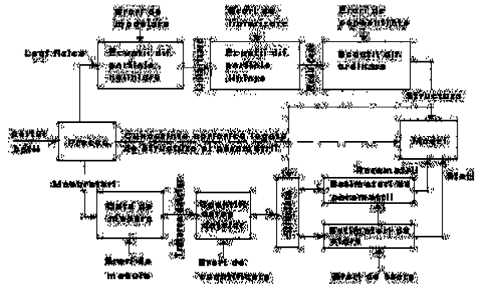

Fig. 2. Ways of elaboration of the model of a process.

The data obtained by measurements made with a frequency adequated to the dynamics of the process, are processed by some procedures (filtering, numbering, disposal of wrong data) before being used for the estimation of the parameters. The adjustment of the model is considered in the sense of adapting the parameters to a model with a fixed structure so that the transfer characteristics of the model to be as close as possible to the characteristics of the process in a way established apriori. The adjustment principle is imposed by the parity of the criterion with respect to the error (fig. 3).

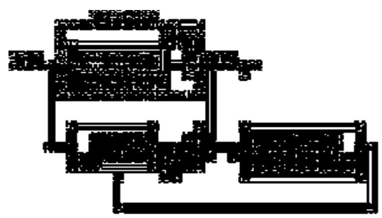

Fig. 3. Adjustment scheme of the model parameters.

Generally, we adopt even functions of the form $J = \int_0^T \varepsilon^2(t,\theta,\hat{\theta})dt = \int_0^T [y(t,\theta) - y_M(t,\hat{\theta})]^2 dt$, where $\hat{\theta}$ is the vector of parameters of the process; $\hat{\theta}$ is the vector of parameters of the adjustable model. The direction of optimal adjusting is showed by the first derivative by θ, that is $\frac{\partial J}{\partial \hat{\theta}} = \int_0^T \varepsilon(t,\theta,\hat{\theta}) \frac{\partial \varepsilon(t,\theta,\hat{\theta})}{\partial \hat{\theta}} dt$ and the rate of change of adjustment variation may be described as $\frac{d\hat{\theta}}{dt} = \hat{\theta} = -K\frac{\partial J}{\partial \hat{\theta}} = -2K \int_0^T \varepsilon(t,\theta,\hat{\theta}) \frac{\partial \varepsilon(t,\theta,\hat{\theta})}{\partial \hat{\theta}} dt = K' \int_0^T \varepsilon(t,\theta,\hat{\theta}) \frac{\partial y_M(t,\hat{\theta})}{\partial \hat{\theta}}$ where K and K' are proportionality factors. The problem arises due to the fact that neither $\frac{\partial J}{\partial \hat{\theta}}$ nor $\frac{\partial y_M(t,\hat{\theta})}{\partial \hat{\theta}}$ are measurable without additional instruments.

The correctness of the adjustment does not necessitate by all means the use of the size of these gradients but only their direction, so that the problem can be solved. If the rate of change of the process parameters is much slower than the speed of adjustment then the last relation offers a satisfactory description for the dynamics of the adjustment. The base relation used for this is the linear equation with finite differences $y(t)+a_1y(t-T)+a_2y(t-2T)=b_1u(t-2T)+b_2u(t-3T)$, or $y(t)=-a_1y(t-T)-a_2y(t-2T)+b_1u(t-2T)+b_2u(t-3T)$. These relationships allow us to express the output values $[y(t)]$ if the input values are known $[u(t)]$. The output at the moment t is computed as a linear combination of the previous inputs and outputs. The problem that was solved in the process of identification consists in using the measured inputs and outputs to determine the coefficients of the equation: a_1, a_2, b_1, b_2, to determine the number of delays in the output $(y(t-T), y(t-2T))$ and the number of delays in the input $u(t-2T)$ and $u(t-3T)$. The number of delayed inputs and outputs used define de order of the model.

3. WAYS OF DESCRIBING THE MODEL

A linear model defined on the space of states may be described by the equation $y=Gu+He$, where y is the output value; G is transfer function that describes the effect of inputs on the outputs; u is the value of the input; H is the transfer function that describes the effect of perturbations on the outputs; e is the value of the perturbations.

The representation in the state space is given by the following relations

$$x(t+1)=Ax(t)+Bu(t)+Ke(t),\ y(t)=Cx(t)+Du(t)+e(t)$$

where $x(t)$ is the vector of state variables; A is the matrix that defines the effect of previous states on the current state; B is the matrix that defines the effect of current inputs on the current state; K is the matrix that defines the effect of perturbations in the current state; C is the matrix that defines the effect of current states on the output; D is the matrix that defines the effect of inputs on the outputs.

The order of the model corresponds to the number of state variables.

4. DETERMINING MODEL CHARACTERISTICS

For the characterization of the models, in practice we use a series of methods that can simplify the measurements and offer sufficient information on the model. The most widely used methods are: the response to a signal of type unitar impulse of a dynamic model is the output that follows after applying an input of type impulse, whose value is 0 for any moment t, except for the moment $t=0$, when the value of the input $u(0)=1$; the response to a step signal of a dynamic model is the output that follows after applying an input of type step, with $u(t) = 1$ for all positive values of t; the response in frequency describes the system behavior when it is excited with a sinusoidal signal. If the input is sinusoidal with a given frequency, then the output will be also sinusoidal with the same frequency, but the amplitude and phase modified. The resulted phase and amplitude are represented by two diagrams, both of a function of the frequency, known as Bode diagrams; poles and zeros. The determination of „poles" and „zeros" is an equivalent mode of finding the coefficients of the last equation for an ARX model. Poles are connected to the output side and the zeros are connected to the input side. The number of poles and zeros is identical to the number of intervals between the largest and smallest delay in the system. By rearranging the poles of an equation we can obtain command structures that lead to the stability of the system.

5. CONCLUSION

For the simulation of the dynamic behavior of the hydroenergetic equipments it is necessary to identify them, so that we can establish an exact connection between the model and the process. In this paper, for the simulation of the dynamic behavior of the hydroenergetic equipment, we prove the necessity for a building o dynamic model based on measurements on the values of the process, this being the case of experimental identification. This way, the existence of apriori knowledge on the structure of the process or the values of some parameters, may be used for narrowing the area of search in a class of models or in the space of parameters. The data obtained from the measurements, made with a frequency adequated to the dynamics of the process, is processed by means of some procedures (filtering, numbering, disposal of wrong data) before being used for estimation of the parameters.

References

[1] Alban, L. E., Systematic analysis of gear failures, ASM Inst Metals Park, 1985.

[2] Bara, A., Identificarea sistemelor, Editura UT Press, Cluj-Napoca, 2001.

[3] Bathe, K.J., Finite element procedures in engineering analysis, Prentice Hall, Englewood Cliffs, N. J., 1982, p 735.

[4] Kafer, R., et.al., Latest development and verification tests of diagnostic methods for hydro-generators, Session GIGRE-Paris, 2000.

[5] Vereş, M., Diagnoza tehnică a echipamentelor hidroenergetice, Ed. Univ. Oradea, 2004.

A NOTE ON THE REACHABILITY SET OF PETRI NETS

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Abstract This paper treats the notion of reachability in Petri nets. More precisely, a method to compute the residue of the reachability set of a Petri net is present. Moreover, as an application, it is shown how this residue set can be used to compute the concurrency-degrees of Petri nets.

Keywords: parallel and distributed systems, Petri nets, reachability, computability.

1. INTRODUCTION

A Petri net is a mathematical model used for the specification and the analysis of parallel and distributed systems. An introduction about Petri nets can be found in [6].

Petri nets are a powerful language for system modelling and validation. They are now in widespread use for many different practical and theoretical purposes in various fields of software and hardware development.

The reachability set of a Petri net is the set of all the states which are reachable from the initial state of the system. In this paper we present a method to compute the residue (i.e. the set of minimal elements) of the reachability set of a Petri net, by applying a more general algorithm from [8].

It appears to be useful to have a measure of concurrency for parallel and distributed systems. What is the meaning of the fact that in the system S_1 the concurrency is greater than in the system S_2? The number of transitions which can fire simultaneously in a Petri net, which models a given real system, can be used as an intuitive measure of the concurrency of that system.

As an application of this result, we show how we can compute the inferior concurrency-degree of a Petri net, using the residue of its reachability set.

The remainder of this paper is organized as follows. Section 2 presents the basic terminology, notation and results concerning Petri nets. In Section 3 we present a method to compute the residue of the reachability set of a marked P/T-net, and, in Section 4, we show how we can compute the inferior concurrency-degree of a marked P/T-net. Section 5 concludes this paper and formulates some open problems.

2. PRELIMINARIES

Assume as known the basic terminology and notation about sets, relations and functions, vectors, multisets and formal languages.

This section establishes the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites to understand this paper (for details the reader is referred to [1], [3], [6]). Mainly, it follows [3].

2.1. MULTISETS AND INTEGER VECTOR SETS

The set of integers is denoted by $\mathbb{Z}$, and the set of nonnegative integers by $\mathbb{N}$.

First, let us just briefly remind that a *multiset* m, over a non-empty set S, is a function $m : S \to \mathbb{N}$, usually represented as a formal sum: $\sum_{s\in S} m(s)`s$. Sometimes it will be identified with a $|S|$-dimensional vector. The operations and relations on multisets are defined component-wise. $S_{MS}$ denotes the set of all multisets over $S$. The empty multiset $\sum_{s\in S} 0`s$ is denoted by $\emptyset$. The *size* of the multiset m is defined as $|m| = \sum_{s\in S} m(s)$. The multiset m is called *infinite* iff $|m| = \infty$.

In order to represent the infinity ("$+\infty$"), a new symbol, denoted usually by ω, is added to the set of non-negative integers $\mathbb{N}$, giving $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$. The symbol ω plays the role of "$+\infty$", the binary operations $+$, $-$, $\cdot$, min and max, and the relation $<$ been extended to $\mathbb{N}_\omega$ in the obvious way:
a) $n+\omega = \omega+n = \omega+\omega = \omega$,
b) $\omega - n = \omega$,
c) $(n+1)\cdot\omega = \omega\cdot(n+1) = \omega$ and $0\cdot\omega = \omega\cdot 0 = 0$,
d) $\min(n,\omega) = \min(\omega,n) = n$ and $\max(n,\omega) = \max(\omega,n) = \omega$,
e) $n < \omega$, for all $n \in \mathbb{N}$.

$\mathbb{N}^k$ (with $k \geq 1$) denotes the set of k-dimensional nonnegative integer vectors, and $\mathbb{N}^k_\omega$ denotes the set of k-dimensional vectors with components from the set $\mathbb{N}_\omega$. The relations $=, \geq, \leq$ for vectors (from $\mathbb{N}^k$ or $\mathbb{N}^k_\omega$) are understood componentwise and $x < y$ is a shorthand for ($x \leq y$ and $x \neq y$). The binary operations $+$, $-$, min, and max are evaluated componentwise too. This means, for instance, that

$$\min(\,(x_1,\ldots,x_k),(y_1,\ldots,y_k)\,) = (\,\min(x_1,y_1),\ldots,\min(x_k,y_k)\,)\,.$$

Definition 2.1. *The* residue *of a set* $X \subseteq \mathbb{N}^k$*, abbreviated* $res(X)$*, is the set of minimal elements of* X *(w.r.t. the partial order* $\leq$ *defined on* $\mathbb{N}^k$*)*

$$res(X) = minimal(X) = \{x \in X \mid \forall y \in X - \{x\} : y \not\leq x\}\,.$$

Remark 2.1. *By Dickson's lemma [2], any subset of* $\mathbb{N}^k$ *contains only finitely many incomparable vectors. Since, by the above definition, the elements of the residue of any subset of* $\mathbb{N}^k$ *are incomparable (w.r.t. the partial order* $\leq$*), it follows that the residue of any subset of* $\mathbb{N}^k$ *is a finite set.*

Petri nets

A *Place/Transition net*, shortly *P/T-net*, (finite, with infinite capacities), abbreviated *PTN*, is a 4-tuple $\Sigma = (S, T, F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbb{N}$ is the *weight function* of Σ satisfying $W(x, y) = 0$ iff $(x, y) \notin F$.

A *marking* of a *PTN* Σ is a function $m : S \to \mathbb{N}$, i.e. a multiset over S; sometimes it will be identified with a $|S|$-dimensional vector. The operations and relations on vectors are defined component-wise. $\mathbb{N}^S$ denotes the set of all markings of Σ.

A *marked PTN*, abbreviated *mPTN*, is a pair $\gamma = (\Sigma, m_0)$, where Σ is a *PTN* and m_0, called the *initial marking* of γ, is a marking of Σ.

In the sequel we often use the term "Petri net"(*PN*) or "net" whenever we refer to a *PTN* (*mPTN*) and it is not necessary to specify its type (i.e. marked or unmarked).

Let Σ be a Petri net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbb{N}$ and $\Delta t, \Delta w : S \to \mathbb{Z}$ are defined by: $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$, and

$$\Delta w(s) = \begin{cases} 0 & \text{, if } w = \lambda \\ \sum\limits_{i=1}^{n} \Delta t_i(s) & \text{, if } w = t_1 t_2 \ldots t_n \ (n \geq 1) \end{cases} \quad , \forall s \in S.$$

The sequential behaviour of a net Σ is given by the *firing rule*, which consists of

- the *enabling rule*: a transition t is *enabled* at a marking m in Σ (or t is *fireable* from m), abbreviated $m[t\rangle_\Sigma$, iff $t^- \leq m$;
- the *computing rule*: if $m[t\rangle_\Sigma$, then t may *occur* yielding a new marking m', abbreviated $m[t\rangle_\Sigma\, m'$, defined by $m' = m + \Delta t$.

In fact, for any transition t of Σ we have a binary relation on $\mathbb{N}^S$, denoted by $[t\rangle_\Sigma$ and given by: $m[t\rangle_\Sigma\, m'$ iff $t^- \leq m$ and $m' = m + \Delta t$. If $t_1, t_2, \ldots, t_n$, $n \geq 1$, are transitions of Σ, $[t_1 t_2 \ldots t_n\rangle_\Sigma$ will denote the classical product of the relations $[t_1\rangle_\Sigma, \ldots, [t_n\rangle_\Sigma$. Moreover, we also consider the relation $[\lambda\rangle_\Sigma$ given by $[\lambda\rangle_\Sigma = \{(m,m) \mid m \in \mathbb{N}^S\}$.

Let Σ be a *PTN*, and $m \in \mathbb{N}^S$. The word $w \in T^*$ is called a *transition sequence* from m in Σ if there exists a marking m' of Σ such that $m[w\rangle_\Sigma\, m'$. Moreover, m' is called *reachable* from m in Σ. We denote by $TS(\Sigma,m) = \{w \in T^* \mid m[w\rangle_\Sigma\}$ the set of all transition sequences from m in Σ, and by $[m\rangle_\Sigma = RS(\Sigma,m) = \{m' \in \mathbb{N}^S \mid \exists\, w \in TS(\Sigma,m) : m[w\rangle_\Sigma\, m'\}$ the set of all reachable markings from m in Σ.

If $\gamma = (\Sigma, m_0)$ is a *mPTN*, for the case $m = m_0$, the set $TS(\Sigma,m_0)$ is abbreviated by $TS(\gamma)$ and sometimes it is called the *language* of γ, and the set $RS(\Sigma,m_0)$ is abbreviated by $RS(\gamma)$ (or $[m_0\rangle_\gamma$) and it is called the *reachability set* of the net γ.

Functions $m : S \to \mathbb{N}_\omega$ are called *pseudo-markings*; sometimes they are identified with $|S|$-dimensional vectors. $\mathbb{N}_\omega^S$ denotes the set of all pseudo-markings. If $m(s) = \omega$, then the component s of m is called an ω-component; $\Omega(m)$ denotes the set of all ω-components of m. , i.e. $\Omega(m) = \{s \in S \mid m(s) = \omega\}$. Obviously, any marking is a pseudo-marking. The firing rule is extended to pseudo-markings in the straightforward way: (ER) $m[t\rangle_\Sigma$ iff $t^- \leq m$; (CR) $m[t\rangle_\Sigma\, m'$ iff $m[t\rangle_\Sigma$ and $m' = m + \Delta t$. The other notions from markings (i.e. transition sequence, reachable marking etc.) are extended similarly to pseudo-markings.

3. THE RESIDUE OF THE REACHABILITY SET

The residue of the reachability set has some practical importance for P/T-nets, for instance it can be used to compute the concurrency-degree of a Petri net (as we see in the next section).

As we already know, for any marked P/T-net $\gamma = (\Sigma, M_0)$, the residue of its reachability set, i.e. $res([m_0\rangle_\gamma)$, is a finite set (see Remark 2.1), but the problem is how to compute it.

In this section we show how we can compute the residue of the reachability set of a P/T-net, using an algorithm from [8].

Definition 3.1. *For each $m \in \mathbb{N}_\omega^k$, let $reg(m) = \{x \in \mathbb{N}^k \mid x \leq m\}$ be the* region *specified by m.*

Definition 3.2. *For each set of integer vectors $X \subseteq \mathbb{N}^k$ we define the predicate $p_X : \mathbb{N}_\omega^k \to \{true, false\}$ by $p_X(m) = (reg(m) \cap X \neq \emptyset)$, for all $m \in \mathbb{N}_\omega^k$.*

A set $X \subseteq \mathbb{N}^k$ is said to have property RES *iff the predicate $p_X(m)$ is decidable for each $m \in \mathbb{N}_\omega^k$.*

Definition 3.3. *A set $X \subseteq \mathbb{N}^k$ is called* right-closed *iff* $\forall\, x \in X, \forall\, y \in \mathbb{N}^k : x \leq y \Rightarrow y \in X$.

Valk & Jantzen [8] proved the following result about the computability of the residue of vector sets:

Theorem 3.1. *Let $X \subseteq \mathbb{N}^k$ be a right-closed set. Then $res(X)$ can be effectively constructed iff X has property RES.*

More exactly, the inverse implication was proved by giving an algorithm which compute the residue of a set $X \subseteq \mathbb{N}^k$ which has property RES (remark: the hypothesis X being a right-closed set was not used in the proof of the inverse implication).

The algorithm and its correctness proof can be found in [8]. Also, that article presents some applications to decidability problems in Petri nets of this result.

Thus, to compute the residue of the reachability set of a P/T-net, it suffices to show that the reachability set of any P/T-net has the property RES.

Definition 3.4. *The* Reachability Problem *(RP) : Given γ a mPTN and m a marking of γ, is m reachable in γ? The predicate associated with this problem is: $RP(\gamma, m) = true$ iff $m \in [m_0\rangle_\gamma$.*

Remark 3.1. *It is well-known [5], [4] that the problem (RP) is decidable (i.e. the predicate RP is recursively decidable).*

Propoziţia 3.1. *Let $\gamma = (\Sigma, m_0)$ be a mPTN. Then its reachability set $[m_0\rangle_\gamma$ has the property RES.*

Proof Let $\gamma = (\Sigma, m_0)$ be a marked P/T-net, with $\Sigma = (S, T, F, W)$, and $[m_0\rangle_\gamma$ its reachability set. We have to prove that the question $reg(m) \cap [m_0\rangle_\gamma \neq \emptyset$? is decidable for any pseudo-marking $m \in \mathbb{N}_\omega^S$.

Let $m \in \mathbb{N}_\omega^S$. We distinguish two cases:

i) $\Omega(m) = \emptyset$, i.e. $m \in \mathbb{N}^S$. In this case, the set $reg(m)$ is finite and we have that

$$p_{\,[m_0\rangle_\gamma}(m) = true \text{ iff } \exists\, m' \in reg(m) : RP(m') = true\,. \tag{1}$$

Therefore, in this case the predicate $p_{\,[m_0\rangle_\gamma}(m)$ is decidable;

ii) $\Omega(m) \neq \emptyset$, i.e. $m \in \mathbb{N}_\omega^S - \mathbb{N}^S$. In this case relation (1) still holds, but the set $reg(m)$ is infinite.

If $\Omega(m) = S$, then $m = (\omega, \ldots, \omega)$ and $reg(m) = \mathbb{N}^S$. Thus, the predicate $p_{\,[m_0\rangle_\gamma}(m)$ in this case is decidable, because $m_0 \in \mathbb{N}^S \cap [m_0\rangle_\gamma$.

Therefore, let us consider that $\emptyset \subset \Omega(m) \subset S$. Without loss of generality, we may assume that the set of locations of the net γ, $S = \{s_1, s_2, \ldots, s_k\}$, is indexed in such an order that $\Omega(m) = \{s_{j+1}, \ldots, s_k\}$, with $1 \leq j < k$. Thus, $\{s_1, s_2, \ldots, s_j\}$ is the set of finite components of the marking m, i.e.

$$m(s_1) = n_1 \in \mathbb{N}, \ldots, m(s_j) = n_j \in \mathbb{N}\,,$$

$$m(s_{j+1}) = \omega, \ldots, m(s_k) = \omega \,.$$

Therefore, $reg(m) = \{x \in \mathbb{N}^k \mid x_i \leq n_i, \forall\, 1 \leq i \leq j\}$ and in this case the question $reg(m) \cap [m_0\rangle_\gamma \neq \emptyset$? becomes the following decision problem:

(*) Given any *mPTN* γ (with k locations) and the integers $n_1, \ldots, n_j \in \mathbb{N}$, with $1 \leq j < k$, there exists a reachable marking $m' \in [m_0\rangle_\gamma$ such that $m'(s_i) \leq n_i$, for all $1 \leq i \leq j$?

But the decision problem $(*)$ is reducible to the following decision problem:

(**) Given any *mPTN* γ (with k locations) and the integers $n_1, \ldots, n_j \in \mathbb{N}$, with $1 \leq j < k$, there exists a reachable marking $m' \in [m_0\rangle_\gamma$ such that $m'(s_i) = n_i$, for all $1 \leq i \leq j$?

Indeed, the problem $(*)$ is equivalent to a finite conjunction of problems $(**)$.

Now, we prove that the problem $(**)$ is decidable.

So, let $\gamma = (\Sigma, m_0)$ be a *mPTN*, with $\Sigma = (S, T, F, W)$, and the integers $n_1, \ldots, n_j \in \mathbb{N}$, with $1 \leq j < k$, where $k = |S|$ ($S = \{s_1, \ldots, s_k\}$).

We construct a new *mPTN* $\gamma' = (\Sigma', m'_0)$, with $\Sigma' = (S', T', F', W')$, defined in the following way:

(i) $S' = S \cup \{\overline{s}, \tilde{s}\}$, with two new locations $\overline{s}$ and $\tilde{s}$;

(ii) $T' = T \cup \{t'_{j+1}, \ldots, t'_k\} \cup \{\overline{t}, \tilde{t}\}$, with $k-j+2$ new transitions $t'_{j+1}, \ldots, t'_k$, $\overline{t}$ and $\tilde{t}$;

(iii) the flow relation $F' \subseteq (S' \times T') \cup (T' \times S')$ is given by

$$\begin{aligned} F' = F \quad & \cup \{(\overline{s}, t), (t, \overline{s}) \mid t \in T\} \quad \cup \{(\overline{s}, \overline{t}), (\overline{t}, \tilde{s}), (\tilde{s}, \tilde{t})\} \cup \\ & \cup \{(s_i, \overline{t}) \mid 1 \leq i \leq j\} \quad \cup \{(s_i, t'_i), (\tilde{s}, t'_i), (t'_i, \tilde{s}) \mid j < i \leq k\}\,; \end{aligned} \tag{2}$$

(iv) the weight function $W' : (S' \times T') \cup (T' \times S') \to \mathbb{N}$ is given by

$$W'(x,y) = \begin{cases} W(x,y) & \text{, if } (x,y) \in F \\ 1 & \text{, if } (x,y) \in \{(\overline{s}, t), (t, \overline{s}) \mid t \in T\} \\ 1 & \text{, if } (x,y) \in \{(\overline{s}, \overline{t}), (\overline{t}, \tilde{s}), (\tilde{s}, \tilde{t})\} \\ n_i & \text{, if } (x,y) = (s_i, \overline{t})\,, 1 \leq i \leq j \\ 1 & \text{, if } (x,y) \in \{(s_i, t'_i), (\tilde{s}, t'_i), (t'_i, \tilde{s}) \mid j < i \leq k\} \\ 0 & \text{, if } (x,y) \notin F' \end{cases} \tag{3}$$

(v) the initial marking $m'_0 \in \mathbb{N}^{S'}$ is given by

$$m'_0(s) = \begin{cases} m_0(s) & \text{, if } s \in S \\ 1 & \text{, if } s = \overline{s} \\ 0 & \text{, if } s = \tilde{s} \end{cases} \tag{4}$$

By the way γ' is constructed, it is easy to see that to each transition sequence in γ of the form

$$m_0\,[w\rangle_\Sigma\, m\,, \tag{5}$$

such that

$$w \in T^*, m \in [m_0\rangle_\gamma \text{ and } m(s_i) = n_i\,, \forall\, 1 \le i \le j \tag{6}$$

it corresponds a transition sequence in γ' of the form

$$m_0'\,[w\rangle_{\Sigma'}\, m'\,[\,\bar{t}\,\rangle_{\Sigma'}\, m''\,[w'\rangle_{\Sigma'}\, \mathbf{0}1\,[\,\tilde{t}\,\rangle_{\Sigma'}\, \mathbf{0}\,, \tag{7}$$

where the markings $m', m'', \mathbf{0}1, \mathbf{0} \in \mathbb{N}^{S'}$ are defined by

$$m'(s) = \begin{cases} m(s) & \text{, if } s \in S \\ 1 & \text{, if } s = \bar{s} \\ 0 & \text{, if } s = \tilde{s} \end{cases} \quad , \quad m''(s) = \begin{cases} m(s) & \text{, if } s \in \{s_{j+1}, \ldots, s_k\} \\ 0 & \text{, if } s \in \{\bar{s}\} \cup \{s_1, \ldots, s_j\} \\ 1 & \text{, if } s = \tilde{s} \end{cases} \quad , \tag{8}$$

$$\text{and } \ \mathbf{0}(s) = 0\,, \forall\, s \in S'\,, \qquad \mathbf{0}1(s) = 0\,, \forall\, s \in S' - \{\tilde{s}\} \ \text{ and } \mathbf{0}1(\tilde{s}) = 1\,,$$

and the word $w' \in T'^*$ is given by

$$w' = {t'_{j+1}}^{m(s_{j+1})} \cdot \ldots \cdot {t'_k}^{m(s_k)} \tag{9}$$

(the order of the transitions $t'_{j+1}, \ldots, t'_k$ in the sequence w' is not important, i.e. more generally w' is an arbitrary sequence $w' \in \{t'_{j+1}, \ldots, t'_k\}^*$ such that t'_i occurs exactly $m(s_i)$ times in the word w', for each $j < i \le k$).

Moreover, the marking $\mathbf{0}$ is reachable in γ' only by transition sequences of the form (7).

From these facts, it follows easily that there exists a reachable marking $m \in [m_0\rangle_\gamma$ such that $m(s_i) = n_i$, for all $1 \le i \le j$, iff the marking $\mathbf{0}$ is reachable in the net γ'. Thus, we can conclude that the problem $(**)$ is decidable.

Therefore, the predicate $p_{[m_0\rangle_\gamma}(m)$ is decidable in this case, too. ■

As a consequence, we have:

Corollary 3.1. *The residue of the reachability set of any mPTN is computable.*

Proof This afirmation follows from Theorem 3.1 and Proposition 3.1 (the residue set $res([m_0\rangle_\gamma)$ is constructed by the algorithm given in [8]). ■

4. APPLICATION: COMPUTING THE CONCURRENCY-DEGREES FOR P/T-NETS

We recall the notion of concurrency-degree for Petri nets, which was defined in [9] in a more general way than the original definition introduced in [7], by taking into consideration also the transitions concurrently enabled with themselves.

Let us briefly remind those definitions from [9].

Definition 4.1. *A* step Y *of a P/T-net* Σ *is any non-empty and finite multiset over the set of transitions of* Σ*. The set of all steps of* Σ *is denoted by* $\mathbb{Y}(\Sigma)$*.*

Definition 4.2. *Let* Σ *be a P/T-net and* m *an arbitrary marking of* Σ*.*
i) The step-type concurrent behaviour of the net Σ *is given by the* step firing rule*, which consist of*

- *the* step enabling rule*: a step* Y *is* enabled *at the marking* m *in* Σ *(or* Y *is* fireable *from* m*), and we say also that* Y *is a multiset of transitions* concurrently enabled *at* m*, abbreviated* $m[Y\rangle_\Sigma$ *, iff* $\sum_{t\in T} Y(t)\cdot t^- \leq m$ *;*
- *the* step computing rule*: if the step* Y *is enabled at* m *in* Σ*, then* Y *may* occur *at* m *yielding a new marking* m'*, abbreviated* $m[Y\rangle_\Sigma m'$*, defined by* $m' = \left(m - \sum_{t\in T} Y(t)\cdot t^-\right) + \sum_{t\in T} Y(t)\cdot t^+$*.*

ii) A step Y *is called a* maximal step enabled *at the marking* m *in* Σ*, if* Y *is enabled at* m *in* Σ *and there exists no step* Y' *enabled at* m *in* Σ *with* $Y' > Y$*.*

Definition 4.3. *Let* Σ *be a P/T-net and* m *an arbitrary marking of* Σ*. The* concurrency-degree at the marking m *of the net* Σ *is defined by*

$$d(\Sigma, m) = \sup\{\ |Y|\ ;\ Y \in \mathbb{Y}(\Sigma) \wedge m[Y\rangle_\Sigma\}\ . \tag{10}$$

Remark 4.1. *Intuitively, the notion of concurrency-degree at a marking* m *of a Petri net* Σ *represents the maximum (i.e. the supremum) number of transitions concurrently enabled at the marking* m*.*

Definition 4.4. *Let* $\gamma = (\Sigma, m_0)$ *be a marked P/T-net.*
i) The inferior concurrency-degree *of the net* γ *is defined by*

$$d^-(\gamma) = \min\{\ d(\Sigma, m)\ |\ m \in [m_0\rangle_\gamma\}\ . \tag{11}$$

ii) The superior concurrency-degree *of the net* γ *is defined by*

$$d^+(\gamma) = \sup\{\ d(\Sigma, m)\ |\ m \in [m_0\rangle_\gamma\}\ . \tag{12}$$

iii) If $d^-(\gamma) = d^+(\gamma)$*, then this number is called the* concurrency-degree *of* γ *and it is denoted by* $d(\gamma)$*.*

Remark 4.2. *Directly from definitions we have*
i) $0 \leq d^-(\gamma) \leq d^+(\gamma) \leq \infty$ *.*
ii) The inferior concurrency-degree of the net γ*,* $d^-(\gamma)$*, represents the minimum number of transitions maximal concurrently enabled at any reachable marking of* γ*. In other words, at any reachable marking* m *of* γ *there exist*

$d^-(\gamma)$ transitions concurrently enabled at m.
iii) The superior concurrency-degree of the net γ, $d^+(\gamma)$, represents the supremum number of transitions maximal concurrently enabled at any reachable marking of γ. In other words, at any reachable marking m of γ there exist at most $d^+(\gamma)$ transitions concurrently enabled at m.
iv) The concurrency-degree of γ means that at any reachable marking m of γ there exist $d(\gamma)$ transitions concurrently enabled at m, and there is no reachable marking m' of γ with more than $d(\gamma)$ transitions concurrently enabled at m'.

What about the computability of these concurrency-degrees for P/T-nets?
In [9], we proved that the concurrency-degree at a marking, $d(\Sigma, m)$, is computable for any *PTN* Σ, and for any marking m, by showing that the computation of the degree $d(\Sigma, m)$ is reducible to solving the following integer linear programming problem

$$(P_{\Sigma,m}) \quad \begin{cases} \max \sum_{1\le i\le n} x_i \\ \\ \sum_{1\le i\le n} W(s_j, t_i)\cdot x_i \le m(s_j) \quad , \forall\ 1\le j\le k \\ \\ x_1, x_2, \ldots, x_n \in \mathbb{N} \end{cases} \tag{13}$$

where x_i are the variables of the problem and the non-negative integers $W(s_j, t_i)$ $\Big(= t_i^-(s_j)\Big)$, representing the weights of the net Σ, are the coefficients of the variables in the linear constraints (where $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_n\}$).

In [9], we also proved that the superior concurrency-degree, $d^+(\gamma)$, is computable for any $mPTN\ \gamma = (\Sigma, m_0)$

$$d^+(\gamma) = \max\{\ d(\Sigma, m) \mid m \in MCS(\gamma)\}\ , \tag{14}$$

where $MCS(\gamma)$ is the minimal coverability set of γ.

Now, we have the following result about the inferior concurrency-degree:

Theorem 4.1. *The inferior concurrency-degree, $d^-(\gamma)$, is computable for any $mPTN\ \gamma = (\Sigma, m_0)$:*

$$d^-(\gamma)\ =\ \min\ \{\ d(\Sigma, m) \mid m \in res([m_0\rangle_\gamma)\}\ . \tag{15}$$

Proof Relation (15) follows easily from Definition 4.4 and the fact that the degree $d(\Sigma, m)$ is a monotone increasing function in the argument m. Proceeding from Corollary 3.1, using this relation we conclude that the inferior concurrency-degree $d^-(\gamma)$ is computable. ■

5. CONCLUSION

In this paper we have presented a method to compute the residue (i.e. the set of minimal elements) of the reachability set of a marked P/T-net, by applying a more general algorithm from [8].

As an application of this result, we have showed how we can compute the inferior concurrency-degree of a marked P/T-net, using the residue of its reachability set.

Since Petri nets are used as suitable models for real-world parallel or distributed systems, the concurrency-degrees defined for Petri nets are an intuitive measure of the concurrency of the modelled systems, and, therefore, they have some practical importance. For instance, they are useful for the evaluation of the models in the process of designing such a system: after making a model of that system as a Petri net, the study of the concurrency-degree of the model will give information to the designers about the concurrency of that system, allowing them to notice the inefficient components of the system, and to make improvements of the model by remodelling those components.

Therefore, it is important to be able to compute the concurrency-degrees for Petri nets.

Some problems remain to be studied, for example: finding better algorithms to compute the residue of the reachability set for P/T-nets; finding better algorithms to compute the concurrency-degrees of P/T-nets; making some case studies on models of real-world systems.

References

[1] Best, E., Fernandez, C., *Notations and terminology on Petri Net Theory*, Arbeitspapiere der GMD **195** (1986)

[2] Dickson, L.E., *Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors,* American Journal of Mathematics, **35** (1913), 413–422.

[3] Jucan, T., Ţiplea, F.L., *Reţele Petri. Teorie şi practică*, The Romanian Academy Publishing House, Bucharest, 1999.

[4] Kosaraju, S.R., *Decidability of reachability in vector addition systems*, Proc. 14^{th} Annual ACM STOC (1982), 267–281.

[5] Mayr, E.W., *An algorithm for the general Petri net reachability problem*, Proc. 13^{th} Annual ACM STOC (1981), 238–246.

[6] Reisig, W., *Petri nets. An introduction*, EATCS Monographs on Theoretical Computer Science, Springer, Berlin, 1985.

[7] Ţiplea, F.L., Jucan, T., Dumbravă, Şt., *Modelling systems by Petri nets with different degrees of concurrency*, in: Proc. 4^{th} International Symposium on Automatic Control and Computer Science SACCS'93, Iaşi (1993), 48–54.

[8] Valk, R., Jantzen, M., *The residue of vector sets with applications to decidability problems in Petri nets,* Acta Informatica, **21** (1985), 643–674.

[9] Vidraşcu, C., Jucan, T., *Concurrency-degrees for P/T-nets,* Scientific Annals of the "Al. I. Cuza" University of Iaşi, Computer Science Section, **XIII** (2003), 91–103.

ABOUT FUZZY DATABASE QUERY LANGUAGES AND THEIR RELATIONAL COMPLETENESS THEOREM

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Two fuzzy database query languages are proposed. They are used to query fuzzy databases that are enhanced from relational databases in such a way that fuzzy sets are allowed in both attribute values and truth values. A fuzzy calculus query language is constructed based on the relational calculus, and a fuzzy algebra query language is also constructed based on the relational algebra. In addition, a fuzzy relational completeness theorem such that the languages have equivalent expressive power is proved.

Keywords: fuzzy database query languages; relational completeness theorem; fuzzy sets; attribute values; truth values; fuzzy calculus query language; relational calculus; fuzzy algebra query language; fuzzy set theory; information retrieval; query languages; relational databases.

1. INTRODUCTION

Database technology has been advanced up to the relational database stage with the purpose that user interfaces with databases may approach a level of human interfaces. It is recognized that the fuzzy theory is suitably applied to some human-oriented engineering fields, one of which is information processing, in particular database retrievals. In fact, fuzzy database models that allow fuzzy attribute values and fuzzy truth values in enhanced relational databases have been studied in [3] and [4]. However, these studies are restricted to just some particular applications and not grounded on theories of fuzzy database query languages. The fuzzy database systems would not be systematically developed on the basis of these studies; it is due to Codd's relational database theory that relational database systems have been systematically developed. It is desirable that theoretical foundations of fuzzy databases be established in order to systematically develop fuzzy database systems.

There was an excelent work done in the field of the fuzzy database theory; it develops a theoretical foundation for the fuzzy functional dependencies of fuzzy databases [1]. The work encourages further research for the rest of theoretical foundation of fuzzy databases. This paper aims to present a theoretical foundation of query languages to fuzzy databases. It proposes two fuzzy database query languages: a fuzzy calculus query language and a fuzzy algebra query language. In adition, it proves a relationalcompletness theorem such that both the languages are equivalent in expressive power to each other. With these theoretical foundations, fuzzy database query systems will be developed systematically.

2. A FUZZY DATABASE MODEL

A fuzzy database is defined as an enhanced relational database that allows fuzzy attribute values and fuzzy truth values; both of these are expressed as fuzzy sets.

Fuzzy data model. A fuzzy database consists of relations: a relation is a relation $R(t_1, \ldots, t_n)$ in a Cartesian product $P_1 \times \cdots \times P_n$ of domains P_i; each P_i is a set of fuzzy sets t_i over an attribute domain $D_i (1 \leq i \leq n)$. It is assumed that key attributes take ordinary nonfuzzy values. For the notational convenience, fuzzy sets are identified with their representative membership functions; for example, t_i also denotes a membership function.

Fuzzy attributes. Attribute values such as age have nonfuzzy values such as 20 in the relational database; attribute values are defined as fuzzy predicates such as „young" and „about forty" in the fuzzy database. For example, a fuzzy attribute value of „age of Dr. X is young" is expressed as a possibility function $P(age\ of\ X) =$YOUNG; here YOUNG denotes a fuzzy set that represents the fuzzy predicate „young". Thus attribute values are identified with fuzzy sets such as YOUNG.

Fuzzy truth values. Truth values of any tuples are either 1 (=true) or 0 (=false) in the relational database; truth values of any tuples are defined as fuzzy predicates such as „0.7" and „completely true" in the fuzzy database. Consider, for example, a tuple t that asserts a fuzzy proposition: „It is completely true that Dr. X is very much older than twenty". The truth value of t is expressed as a possibility distribution $P[T(t)] = N$; $T(t)$ denotes truth value of t and N denotes a fuzzy set that represents the fuzzy predicate „completely true". Thus the true values $T(t)$ are identified with fuzzy sets such as N over $z \in [0, 1]$; the value $z \in [0, 1]$ has the following meaning:

1) $z = 0$ means that the tuple t is completely false;

2) $0 < z < 1$ means that the tuple t is true to the degree expressed by the real number z;

3) $z = 1$ means that the tuple t is completely false.

In particular, each tuple t of the relation $R(t_1, \ldots, t_n)$ is given a unique truth value $T(t)$ by system designers at system generation time. In this case,

$T(t)$ determines a mapping $T : P_1 \times \cdots \times P_n \rightarrow P([0,1])$ where $P([0,1])$ is a set of fuzzy sets over $z \in [0,1]$.

3. QUERY BY TUPLE FUZZY CALCULUS

Tuple fuzzy calculus. A tuple fuzzy calculus (query language) is constructed as an enhancement of the tuple relational calculus. Formulas in the tuple fuzzy calculus are of the form $(t \mid f(t))$ where t is a fuzzy tuple variable; each ith component t_i is a fuzzy variable in P_i; f is a tuple fuzzy well-formed formula (WFF).

Tuple fuzzy WFF's are enhanced from those of the tuple relational calculus as follows.

1) **Atomic Tuple Fuzzy WFF's:** An atomic tuple fuzzy WFF consists of fuzzy sets and a fuzzy comparison operator $*$. The fuzzy comparison operator $*$ is one of the operators: equal; not equal; proper inclusion; inclusion. The fuzzy comparison operator $*$ is an enhancement from the arithmetic comparison operator $(=, \neq, <, >, \leq, \geq)$ in the relational calculus. Then the atomic tuple fuzzy WFF's are either of the following two types:

a) $(t_i) * (s_j)$; here, it is assumed that t and s are fuzzy tuple variables such that $D_i = D_j (1 \leq i, j \leq n)$.

b) $(t_i) * (c), (c) * (t_i)$; here, it is assumed that c is a fuzzy set over D_i.

2) **Logical Connectives and Quantifiers:** The logical connectives („AND", „OR", and „NOT") are used for tuple fuzzy WFF's.

Also, quantifiers („for all" and „there exists") are used for tuple fuzzy WFF's.

3) **Others:** Other definitions concerning tuple fuzzy WFF's are the same as in the tuple relational calculus.

Thus tuples in any relation $R(t_1, \ldots, t_n)$ that satisfy the formula $(t \mid f(t))$ form a set of Cartesian products of fuzzy sets.

It should be considered further whether or not to include fuzzy comparison operators $*$ expressed by fuzzy relations such as „much greater than", „is close to", „is similar to", and „is relevant to".

Query evaluation. Queries expressed in the tuple fuzzy calculus are evaluated by two steps as follows.

(Step1) Selecting resultant tuples: Consider that the query $(t \mid f(t))$ is issued to a relation $R(t_1, \ldots, t_n)$. Resultant tuples are those $r \in R(t_1, \ldots, t_n)$ each of which satisfies the formula $f(r)$.

(Step2) Calculating truth values of resultant tuples: Let any resultant tuple r be a projection of $t \in R(t_1, \ldots, t_n)$ onto the components $k_1, \ldots, k_j, \ldots, k_m$ where $1 \leq m \leq n$, $1 \leq k_1, \ldots, k_j, \ldots, k_m \leq n$. Then the truth value $T(r)$ is defined as a projection of $T(t)$ onto the components $k_1, \ldots, k_j, \ldots,$

k_m: $T(r) = Max.T(t)$, where the maximum is taken over those components $t_k (1 \leq k \leq n)$, such that $t_k \neq t_{k_j}$.

4. QUERY BY FUZZY ALGEBRA

Fuzzy algebra. A fuzzy algebra (query language) is constructed as an enhancement of the relational algebra. Fundamental fuzzy algebraic operations are union, set difference, Cartesian product, projection, and selection, which are defined as follows.

1) **Union:** Let R and S denote any relations in the fuzzy database. The union of R and S is a set of tuples that belongs to R or S. The union is equal to that in set theory.

Any resultant tuple t by the union of R and S inherits the truth value $T(t)$ from its original tuple in R or S.

2) **Set difference:** The difference $R - S$ of R from S is a set of tuples, each of which belongs to R and does not belong to S. The difference is equal to that in set theory.

Any resultant tuple t by the set difference $R - S$ inherits the truth value $T(t)$ from its original tuple in R.

3) **Cartesian product:** The Cartesian product $R \times S$ of R and S is a set of tuples, $\{r, s) \mid r : \text{tuple in } R,\ s : \text{tuple in } S\}$. The Cartesian product is equal to that in set theory.

The truth value $T(t)$ of the resultant type $t = (r, s)$ by the Cartesian product $R \times S$ is the minimum of $T(r)$ and $T(s)$, where $T(r)$ and $T(s)$ are truth values of r and s, respectively.

4) **Projection:** The projection $\mathrm{Proj}(k_1, \ldots, k_j, \ldots, k_m)(R)$ of R onto the k_jth attributes is a set of tuples of the k_jth attribute values. The projection is equal to that in set theory.

Let r denote any resultant tuple of the projection $\mathrm{Proj}(i_1, \ldots, i_j, \ldots, i_m)(R)$ of $t \in R$. Then the truth value $T(r)$ is the maximum of $T(t)$ taken over those components T_k, such that $t_k \neq t_{k_j}$.

5) **Selection:** Let G denote a fuzzy WFF involving the following constituents:

i) operands that are constant fuzzy sets and attribute item numbers of the relation R;

ii) the fuzzy set comparison operators $*$ (equal, not equal, proper inclusion, inclusion);

iii) logical conectives „AND", „OR", and „NOT".

The selection $\mathrm{Sel}_G(R)$ of the relation R is a set of tuples t in R each of which satisfies the fuzzy WFF G when any occurrences of the number i in G are replaced by the ith component of r in R.

When any resultant tuple r is made by the selection $\mathrm{Sel}_G(R)$, $t \in R$ inherits the truth value $T(t)$ from the original tuple t in R: $T(r) = T(t)$.

Some additional fuzzy algebraic operations such as intersection, quotient, θ-join, and natural join are defined as combinations of the fundamental fuzzy algebraic operations defined previously in the same way as in the relational algebra. For example, the θ-join and the natural join are defined as follows.

6) **θ-join:** The θ-join of R and S is defined as a combination of two fundamental fuzzy algebraic operations: the Cartesian product and the selection where θ is enhanced to a fuzzy comparison operator $*$ (equal, not equal, proper inclusion, inclusion). Truth values of resultant tuples by the θ-join are calculated as those of combinations of the two fundamental operations.

7) **Natural join:** The natural join of R and S is defined as a combination of three fundamental fuzzy algebraic operations: the Cartesian product, the selection, and the projection. Truth values of resultant tuples by the natural join are calculated as those of combinations of the three fundamental fuzzy algebraic operations.

Query evaluation. Any query by the fuzzy algebra is expressed as a combination of the fundamental fuzzy algebraic operations. Thus the resultant tuples r and their truth values $T(r)$ by this query are obtained as combinations of its constituent fundamental fuzzy algebraic operations.

Duplicate removal schemes and return methods of resultant tuples to users are the same as described in the fuzzy calculus.

5. RELATIONAL COMPLETENESS THEOREM FOR FUZZY DATABASE QUERY LANGUAGES

The relational database theory establishes the relational completeness theorem such that the relational calculus is equivalent in expressive power to the relational algebra [2]. A similar theorem in the fuzzy database is given.

Theorem: *The following three fuzzy database query languages have the same expressive power*:

1) *tuple fuzzy calculus;*
2) *domain fuzzy calculus;*
3) *fuzzy algebra.*

Proof: The fundamental idea of the proof of this theorem is given by Ullman [**?**, pp. 114-122]; it presents the proof of the relational completeness theorem for the relational database query languages. Ullman's proof techniques consist of the following three reduction techniques:

i) reduction of the relational algebra to the tuple relational calculus;

ii) reduction of the tuple relational calculus to the domain relational calculus;

iii) reduction of the domain relational calculus to the relational algebra.

The reduction technique ii) is just the transformation between variable expressions, and thus is not influenced by the enhancements of the fuzzy database query languages. Therefore, it should be proved here that the reduction techniques i) and iii) can also be extended to cover the enhancements of the fuzzy database query languages.

There are two essential enhancements in the fuzzy database query languages from the relational database.

1) The fuzzy database allows fuzzy sets as attribute values; the fuzzy comparison operators $*$ (equal, not equal, proper inclusion, inclusion) are used in the fuzzy database query languages instead of the arithmetic comparison operators $(=, \neq, <, >, \leq, \geq)$ used in the relational database query languages.

2) The fuzzy database allows fuzzy sets as truth values $T(t), t \in R$; truth values $T(r)$ of resultant tuples r are inherited from $T(t)$ of original tuples $t \in R$, or calculated as combinations of Cartesian products or projection of $T(t)$ of original tuples $T \in R$.

The enhancements 1) is easily incorporated into the reduction techniques i) and iii) by replacing the arithmetic comparison operators with the fuzzy comparison operators.

Next, consider the enhancement 2). Remember that the truth value $T(t)$ of the tuple t is defined just depending on the fuzzy set and fuzzy set operations that have been established in the fuzzy theory; calculations of $T(r)$ are made independently of any of the definitions of the fuzzy database query languages. Thus the reduction techniques i) and iii) can be extended to incorporate the enhancement 2). This completes the proof.

6. CONCLUDING REMARKS

This paper presents two fuzzy database query languages (fuzzy relational calculus and fuzzy relational algebra) based on the relational database query languages. In addition, it proves the relational completeness theorem such that both the languages are equivalent in expressive power to each other. As in the case of the relational database, this relational completeness theorem in the fuzzy database is expected to provide a criterion for the minimum fuzzy database query capability that must be implemented in any reasonable real fuzzy database query languages.

Further interesting studies are induced for extending the existing real relational database query languages, such as the international standard database language SQL. Such an example is the fuzzy query language developed by Rasmussen and Yager in 1997 [5], named *SummarySQL*. The purpose of the language is to let linguistic summaries be a part of a fuzzy query. In SummarySQL, we can evaluate a linguistic summary to find the measure of validity, but a linguistic summary can also be used as a predicate in a fuzzy query. When a predicate is evaluated in a query it will, like a linguistic summary, take a value in the unit interval; for this reason, SummarySQL will treat a linguistic summary as a fuzzy predicate.

Linguistic summaries are related to the class of operators called aggregate functions in the SQL, such as the average (AVG) and count (COUNT) functions. But, different from the usual aggregate functions, linguistic summaries always take truth values in the unit interval. The prototype interpreter built for the SummarySQL can be used to evaluate the following query examples.

A SummarySQL statement has the form: select attributelist from tablelist where conditions

The result from a query is a table where every tuple is associated with a truth value. The *from* clause defines the join-table we access through the query and is the result of joining the tables in the tablelist. A table in the tablelist can also be a subquery. The *select* clause defines a projection on the join-table by an attributelist. The conditions in the *where* clause represent a fuzzy expression over the attributes from the join-table. The predicates in the conditions can be „summaries" or have the form „attribute IS fuzzyterm" and can be conjuncted (AND), disjuncted (OR) or negated (NOT). The fuzzy expression is evaluated for each tuple in the join-table and the result is assigned to the associated truth value (μ).

A statement for summary has the form: summary quantifier from tablelist where conditions where the quantifier is a fuzzy quantifier and the *from* clause and the *where* clause are the same as in the fuzzy query. Compared to a summary of the form „Q objects in FDB are S", Q is equal to the quantifier in the *summary* clause and the summarizer S is equal to the fuzzy conditions defined in the *where* clause. The join-table defined in the *from* clause is equal to the fuzzy table FDB. As we mentioned earlier, a table in the tablelist can be a subquery, and the result of a subquery could be a fuzzy subset FDB. If we look at the fuzzy subset FDB as the subpopulation „the R objects in DB", we have the summary „Q R objects in DB are S".

References

[1] Raju, K. V. S. V. N., Majumdar, A. K., Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems, ACM Trans. Database Syst., **13**, 2 (1988), 129-166.

[2] Ullman, J. D., Principles of database systems, MD: Computer Science, Rockville, 1980.

[3] Umano, M., Relational algebra in fuzzy database, IEICE Tech. Rep. DE86-4, **86**, 192 (1986), 1-8.

[4] Zemankova-Leech, M., Kandel, A., Fuzzy relational databases - A key to expert systems, Verlag TUV Rheiland Gmbh, 1980.

[5] Rasmussen, D. and Yager, R. R., SummarySQL - A fuzzy tool for data mining, Intelligent Data Analysis, 1997.

THEORETICAL APPROACH FOR MEDICAL IMAGE ENHANCEMENT IN LOGARITHMIC REPRESENTATION

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Abstract

The logarithmic image processing (LIP) theory is a mathematical framework that provides a set of specific algebraic and functional operations and structures that are well adapted to the representation and processing of non-linear images, and more generally of non-linear signals, valued in a bounded intensity range. The purpose of this paper is to introduce a new mathematical LIP model focused on theoretical and practical aspects concerning the enhancement of the transmitted medical images and the physical absorption/transmission laws expressed within LIP mathematical framework. First of all the bounded interval (-1,1) is considered as the set of gray levels and we define two operations: addition$\langle + \rangle$ and real scalar multiplication $\langle \times \rangle$. With these operations, the set of gray levels becomes a real vector space. Then, defining the scalar product $(.|.)$ and the norm $||\ .||$, we obtain an Euclidean space of the gray levels. Secondly, we extend these operations and functions for color images. Finally, the experimental results, shown as enhanced medical images, reveal that this method has wide potential areas of impact which may include: Digital X-ray, Digital Mammography, Computer Tomography Scans, Nuclear Magnetic Resonance Imagery and Telemedicine Applications.

Keywords: Medical imagery, image enhancement, logarithmic image processing, real vector space, Euclidean space.

1. INTRODUCTION

Medical images represent, in most cases, a noisy image environment due, primarily, to the limitations placed on X-ray dose, for example. In a large number of situations the medical expert must inspect, visually segment and analyze various objects within a medical image in order to diagnose diseases. This is especially so in breast cancer screening and brain cancer tests. It is generally accepted that due to human factors there are varying degrees of inconsistencies in the diagnostic conclusions, reached by medical experts. In some cases, a false positive diagnostic conclusion is indicated, while in others it is a false negative. Obviously, the only means of reducing the amount of inconsistencies is the use of computer enhanced images. The user interface should allow for proper adjustments so the medical expert (the radiologists) can achieve the preferred image perception in order to come up with a final and correct decision.

For telemedicine applications, particularly, there is a data channel with a limited bandwidth between doctor and patient which may pose a potential bottleneck. This mathematical model compacts the high input dynamic range, potentially reducing the high bandwidth requirement.

Original digital X-ray pelvis image

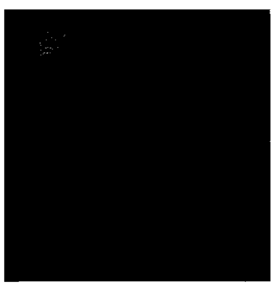
Contrast enhanced pelvis image

Original digital X-ray femur image

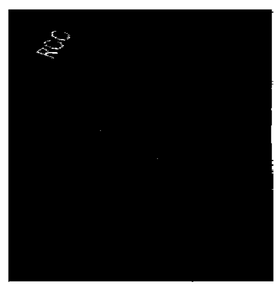
Contrast enhanced femur image

Original digital X-ray lungs image

Contrast enhanced lungs image



Original digital mammography image for breast cancer diagnose



Contrast enhanced mammography image revealing calcium formation

The following mathematical model proposed is useful to any medical imaging application where automatic contrast enhancement and sharpening is needed. Potential areas of impact may include: digital X-ray, digital Mammography, computer tomography scans, nuclear magnetic resonance imagery, telemedicine applications.

The most important performances of this mathematical model are: *dynamic range compression,* i.e. the ability to represent large input dynamic range into relatively small output dynamic range; *sharpening,* i.e. compensation for the blurring introduced into the image by the image formation process. This allows fine details to be seen more easily than before. *color constancy,* i.e. the ability to remove the effects of the illumination from the subject. This allows consistency of output as illumination changes.

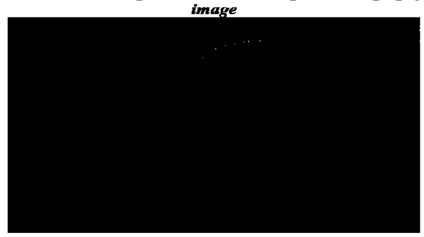
Original digital mammography image for breast cancer diagnose

Contrast enhanced mammography image revealing fibrillar formation

Original computer tomography scan

Contrast and sharpness enhanced computer tomography image

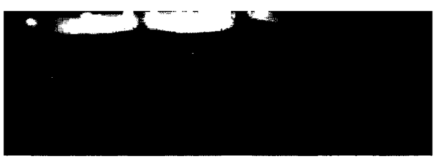
Original digital eye image



Contrast and sharpness enhanced digital eye image

A medical image is represented as a function defined on a two-dimensional spatial domain. Medical images can be classified by the space in which these functions take their values in two groups: scalar images and vector images. Scalar images are defined as functions with real (bounded) values. The most frequent case in practice is when the value at a point (x,y) is the measure of the luminosity at that point. Vector images are defined by real (bounded) vector functions.

Medical color images are the most common particular case of vector images: R(red), G(green) and B(blue) components are considered vector real components. Scalar image values are called gray levels and the functions that define scalar images are called gray level functions. In order to use these gray levels as some algebra elements, the most frequently mathematical model used is the classical one, based on the real numbers algebra (the case of linear processing [1], [2]). This leads to an implicit acceptance of the set of the gray level values as the whole real axis. The result is in contradiction with the fact that gray level functions are bounded. The practical solution to overcome this situation consists in truncating the values as soon as they go out of the true gray level set, be this truncation at the end of chain of operations or at each step. The problem is that in such a way we do not control in the right way the effect of the truncations. Within the mathematical model developed in this paper, the set of gray levels will be the bounded interval (-1,1) put in one-to-one correspondence with the physical interval of e.g. luminosities $(0,M)$, by a linear transformation $y=2(x-M/2)/M$. The problem is to organize the set (-1,1) as a real vector space i.e. closed with respect to an addition (inner group operation) and to a scalar multiplication with real numbers.



Original digital color image of a mouth infection

Contrast and sharpness enhanced digital mouth color image

Original digital color image of eye retina

Contrast and sharpness enhanced color image while preserving color constancy

The key of this approach is to use an adequate isomorphism between $\mathbb{R}$ and (-1,1), respectively between $\mathbb{R}$ and $(0,M)$ and the solution is of a logarithmic nature.

The paper is organized as follows: Section 2 and 3 introduces the addition and real scalar multiplication, the scalar product and the norm for the gray levels space respectively for the gray level images space. Section 4 and 5 introduces the addition and real scalar multiplication, the scalar product and the norm for the color space respectively for the color images space. Section 6 presents some experimental results and Section 7 outlines the conclusions.

2. THE REAL VECTOR SPACE OF GRAY LEVELS

We consider as the space of gray levels, the set E= (-1,1). In the set of gray levels E we define the addition $\langle + \rangle$ and the real scalar multiplication $\langle \times \rangle$.
Addition. $\forall u, v \in E, u\langle + \rangle v = \frac{u+v}{1+\frac{u \cdot v}{M^2}}$, where the operations in the right-hand side are meant in $\mathbb{R}$. The neutral element for addition is θ=0. Each element $v \in E$ has as its opposite the element $v\langle + \rangle w = -v$ and this verifies the following equation: $v\langle + \rangle w = \theta$. The addition $\langle + \rangle$ is stable,associative, commutative, has a neutral element and each element has an opposite. It means that this operation establishes on E a commutative group structure. We can also define the subtraction operation $\langle - \rangle$ by $\forall u, v \in E, u\langle - \rangle v = \frac{u-v}{1-\frac{u \cdot v}{M^2}}$. Using subtraction $\langle - \rangle$, we denote the opposite of v by $\langle - \rangle v$.
Scalar multiplication. For $\forall\ \lambda \in \mathbb{R}$, $\forall u \in E$, we define the product between λ and u by $\forall \lambda \in \mathbb{R}, \forall u \in E, \lambda\langle \times \rangle u = M \cdot \frac{(M+u)^\lambda - (M-u)^\lambda}{(M+u)^\lambda + (M-u)^\lambda}$, where again the operations in the right-hand side of the equality are meant in $\mathbb{R}$. The two operations, addition $\langle + \rangle$ and scalar multiplication $\langle \times \rangle$ establish on E a real vector space structure.
The Euclidean space of the gray levels. We define the scalar product $(.|.)_E$:$E \times E \rightarrow \mathbb{R}$ by $\forall u, v \in E, (u|v)_E = \varphi(u)\varphi(v)$ where $\varphi :$ (-1,1) $\rightarrow \mathbb{R}$ and $\varphi\ (x)$ = arcth(x).With the scalar product $(.|.)_E$ the gray levels space becomes an Euclidean space. The norm $||.||_E$:$E \rightarrow [0,\infty)$ is defined via the scalar product $\forall v \in E, ||v||_E = ((v|v)_E)^{1/2} = |\varphi(v)|$.

3. THE VECTOR SPACE OF THE GRAY LEVEL IMAGES

A gray level image is a function defined on a two-dimensional compact D from $\mathbb{R}^2$ taking the values in the gray level space E. We denote by $F(D, E)$ the set of gray level images defined on D. We can extend the operations and the functions from gray level space E to gray level images $F(D, E)$ in a very natural way:
Addition. $\forall f_1, f_2 \in F(D, E), \forall (x, y) \in D, (f_1\langle + \rangle f_2)(x, y) = f_1(x, y)\langle + \rangle f_2(x, y)$ The neutral element is the identically null function. The addition $\langle + \rangle$ is stable, associative, commutative, has a neutral element and each element has an opposite. As a conclusion, on the set $F(D, E)$ this operation establishes a

commutative group structure.
Scalar multiplication. $\forall\lambda \in \mathbb{R}, \forall f \in F(D,E), (x,y) \in D, (\lambda\langle\times\rangle f)(x,y) = \lambda\langle\times\rangle f(x,y)$ The two operations, addition $\langle+\rangle$ and scalar multiplication $\langle\times\rangle$, establish on $F(D,E)$ a real vector space structure.
The Hilbert space of the gray level images. Let f_1 and f_2 be two integrable functions from $F(D,E)$. We define the scalar product by
$\forall f_1, f_2 \in F(D,E), (f_1|f_2)_{L^2(E)} = \int_D (f_1(x,y)|f_2(x,y))_E dxdy$ With the scalar product the gray level images space $F(D,E)$ becomes a Hilbert space. Further on, we define the norm $\forall f \in F(D,E), ||f||_{L^2(E)}(x,y) = (\int_D ||f(x,y)||^2_E dxdy)^{1/2}$

4. THE REAL VECTOR SPACE OF THE COLORS

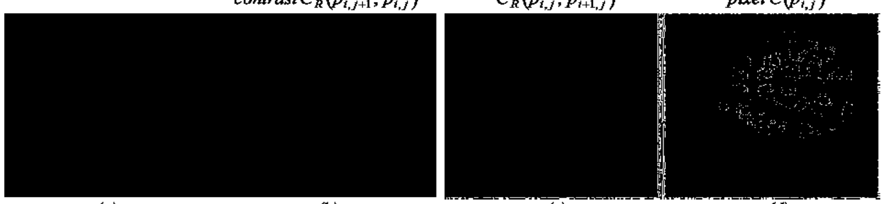
Next we consider as the space of colors, the cube $E_3(-1{,}1)^3$. Denote by r, g and b (red, green, blue) the three components of a vector $v\in E_3$. In the cube E_3 we define the addition $\langle+\rangle$ and the real scalar multiplication $\langle\times\rangle$.
Addition. $\forall v_1, v_2 \in E_3$ with $v_1 = (r_1, g_1, b_1)$, $v_2 = (r_2, g_2, b_2)$, the sum $v_1\langle+\rangle v_2$ is defined by: $v_1\langle+\rangle v_2 = (r_1\langle+\rangle r_2, g_1\langle+\rangle g_2, b_1\langle+\rangle b_2)$ The neutral element for addition is $\theta = (0{,}0{,}0)$. Each element $v = (r,g,b) \in E_3$ has its opposite element $w=(-r,-g,-b)$ and obviously $v\langle+\rangle w = \theta$. The addition $\langle+\rangle$ is stable, associative, commutative, has a neutral element and each element has an opposite. It follows that this operation establishes on E_3 a commutative group structure. We can also define the subtraction operation $\langle-\rangle$ by $v_1\langle-\rangle v_2 = (r_1\langle-\rangle r_2, g_1\langle-\rangle g_2, b_1\langle-\rangle b_2)$. Using subtraction operation $\langle-\rangle$, we denote the opposite of v, with $\langle-\rangle v$.
Scalar multiplication. For $\forall\ \lambda \in \mathbb{R}$, $\forall\ v = (r,\ g,\ b)\in E_3$ we define the product of λ by v by $\lambda\langle\mathrm{x}\rangle v = (\lambda\langle\mathrm{x}\rangle r, \lambda\langle\mathrm{x}\rangle g, \lambda\langle\mathrm{x}\rangle b)$. The two operations, addition $\langle+\rangle$ and scalar multiplication $\langle\times\rangle$ establish on E_3 a real vector space structure.
The Euclidean space of the colors. We define the scalar product $(.\ |\ .\ )_E : E_3 \times E_3 \to \mathbb{R}$ by $\forall v_1, v_2 \in E_3$ with $v_1 = (r_1, g_1, b_1), v_2 = (r_2, g_2, b_2)$, $(v_1|v_2)_{E_3} = \varphi(r_1)\cdot\varphi(r_2) + \varphi(g_1)\cdot\varphi(g_2) + \varphi(b_1)\cdot\varphi(b_2)$, where φ: $(-1{,}1) \to \mathbb{R}$ and $\varphi\ (x) = \mathrm{arcth}(x)$. With the scalar product $(.\ |\ .\ )_{E3}$ the color space becomes a three-dimensional Euclidean space. We define the norm $||.||_{E_3} : E_3 \to [0,\infty)$ by $\forall v = (r,g,b) \in E_3 \quad ||v||_{E_3} = (\varphi^2(r) + \varphi^2(g) + \varphi^2(b))^{1/2}$.

5. THE VECTOR SPACE OF THE COLOR IMAGES

A color image is a function defined on a two-dimensional compact D from $\mathbb{R}^2$ taking the values in the colors space E_3. We note by $F(D,E_3)$ the set of color images defined on D. We extend the operations and the functions from colors space E_3 to color images $F(D,E_3)$ in a natural way as follows
Addition. $\forall f_1, f_2 \in F(D,E_3), \forall(x,y) \in D, (f_1\langle+\rangle f_2)(x,y) = f_1(x,y)\langle+\rangle f_2(x,y)$ The neutral element is the identically null function. The addition $\langle+\rangle$ is sta-

ble, associative, commutative, has a neutral element and each element has an opposite. As a conclusion, these operations establish on the set $F(D, E_3)$ a commutative group structure.
Scalar multiplication. $\forall \lambda \in \mathbb{R}, \forall f \in F(D, E_3), \forall (x, y) \in D, (\lambda \langle \times \rangle f)(x, y) = \lambda \langle \times \rangle f(x, y)$.
The two operations, addition $\langle + \rangle$ and scalar multiplication $\langle \times \rangle$ establish on $F(D, E_3)$ a real vector space structure.
The Hilbert space of the color images. Let $f_1$and $f_2$be two integrable functions from $F(D, E_3)$. We define the scalar product by
$\forall f_1, f_2 \in F(D, E_3), \quad (f_1|f_2)_{L^2(E_3)} = \int_D (f_1(x, y)|f_2(x, y))_{E_3} dxdy$ With the scalar product the gray level images space $F(D, E)$ becomes a Hilbert space. Further on, we define the norm
$\forall f \in F(D, E_3), \quad ||f||_{L^2(E_3)}(x, y) = (\int_D ||f(x, y)||^2_E dxdy)^{1/2}$.

(a) Original image
(b) Horizontal relative contrast $C_R(p_{i,j+1}, p_{i,j})$
(c) Vertical relative contrast $C_R(p_{i,j}, p_{i+1,j})$
(d) Contrast for a pixel $C(p_{i,j})$



(a) Original image
(b) Horizontal relative contrast $C_R(p_{i,j+1}, p_{i,j})$
(c) Vertical relative contrast $C_R(p_{i,j}, p_{i+1,j})$
(d) Contrast for a pixel $C(p_{i,j})$

6. THE CONTRAST OF GRAY LEVEL IMAGES

The structure of vector space defined on the set $F(D, E)$ allows us to translate the notion of contrast from classical framework. We denote by $p_i = (x_i, y_i)$ the pairs of coordinates that define the spatial position of a pixel in an image.
The relative contrast between two pixels. The relative contrast between two distinct pixels $p_1, p_2 \in D$, for an image $f : D \to E$, is a logarithmic gray level denoted by $C_R(p_1, p_2)$, and defined by the relation $\forall p_1, p_2 \in D, p_1 \neq p_2 \quad C_R(p_1, p_2) = \frac{1}{d(p_1, p_2)} \langle \times \rangle \frac{f(p_1) - f(p_2)}{1 - \frac{f(p_1) \cdot f(p_2)}{M^2}}$, where $d(p_1, p_2)$ is the Euclidean distance between p_1 and p_2 in the $\mathbb{R}^2$ plane.
The absolute contrast between two pixels. From the relative contrast C_R we define the absolute contrast by the formulas $\forall p_1, p_2 \in D, p_1 \neq p_2 \quad C_A(p_1, p_2) =$

$|C_R(p_1,p_2)| = \frac{1}{d(p_1,p_2)} \langle\times\rangle \frac{|f(p_1)-f(p_2)|}{1-\frac{f(p_1)\cdot f(p_2)}{M^2}}$.
The contrast for a pixel. The contrast for an arbitrary pixel $p \in D$, for an image $f \in F(D,E)$ is a positive logarithmic gray levels image, denoted by $C(p)$, and defined by the mean of absolute contrast between the pixel p and the pixels $(p_i)_{i=1,n}$ that belong to a neighborhood V. Thus we have the formula $\forall p \in D \quad C(p) = \frac{1}{n}\langle\times\rangle(\langle+\rangle_{i=1}^{n} C_A(p,p_i))$. Usually $n = 8$ and the neighborhood V is a window with 3×3 dimensions and having the pixel p in the center.
Experimental results for the contrast. In the next figures we show the contouring of images with the contrast formulae defined above. We define $C(p)$as the contour image for an initial image, which obtained for each pixel using the contrast formulae. In the nest figures we have represented in section (a) the original image, in section (b) the horizontal relative contrast $C_R(p_{ij+1}, p_{ij}$, in section (c) the vertical relative contrast $C_R(p_{ij}, p_{i+1,j}$ and in section (d) the contrast for a pixel $C(p_{ij})$.

7. CONCLUSIONS

In this paper we presented a new mathematical model for gray level images and also for color images. The main idea is to define an algebraical structure on a bounded interval. The above examples show that the operations (addition, scalar multiplication) and the functions (scalar product, norm) ground an important model for images processing. Using this mathematical model we obtain very simple algorithms for images processing, especially for medical purposes.

REFERENCES

[1] K.R. Castleman, Digital image processing, Prentice Hall, Englewood Cliffs, NJ, 1996.

[2] A. Jain, Fundamentals of digital image processing, Prentice Hall, Englewood Cliffs, 1989.

[3] M. Jourlin, J.C. Pinoli, Logarithmic image processing. The mathematical and physical framework for the representation and processing of transmitted images, Advances in Imaging and Electron Physics, **115** (2001), 130-196.

[4] A. V. Oppenheim, Generalized superposition. Information and control, **11** (1967), 528-536.

[5] V. Patrascu, A mathematical model for logarithmic image processing, PhD thesis, "Politehnica" University of Bucharest, May 2001.

[6] C. Vertan, M. Zamfir, E.Zaharescu, V.Buzuloiu, C. Fernandez-Maloigne, Nonlinear color image filtering by color to planar shape matching, ICIP 2003, Barcelona, Spania, 2003.

[7] E. Zaharescu, Image Processing Using Mathematical Morphology, International Conference-Panonian Applied Mathematics and Meetings, PAMM1998, published in PAMM Monographic Booklets Series, Budapest, 1998. (ISSN 0133-3526).

[8] E. Zaharescu, Image processing using superior order morphological primitives, International Conference-Panonian Applied Mathematics and Meetings, PAMM1998, published in PAMM Monographic Booklets Series, Budapest, 1998. (ISSN 0133-3526).

[9] E. Zaharescu, A high-level programming language for image processing using mathematical morphology, International Conference on Discrete Mathematics and Theoretical Computer Science, DMTCS01, Constanta, Romania, July 6-10, 2001.

[10] E. Zaharescu, M. Zamfir, C. Vertan: Color morphology-like operators based on color geometric shape characteristics, Proc. of International Symposium on Signals Circuits and Systems SCS 2003, Iasi, Romania, 2003.

[11] E. Zaharescu, Extending mathematical morphology for color images and for logarithmic representation of images, PhD thesis, "Politehnica" University of Bucharest, September 2003.

ECONOMETRIC AND GEOMETRIC ANALYSIS OF COBB-DOUGLAS AND CES PRODUCTION FUNCTIONS

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Abstract Due to the fact that the sign of mixed second order partial derivatives of the functions depends on the scale, in the theory of production functions the scale plays an important role. The laws of changes of scale can be analyzed mathematically. Known analytic holomorphic functions, so far they were studied for the elliptic type of space. However, since the production functions can be different, it is necessary to study the theory of analytic functions of hyperbolic and parabolic types too. This needed theory is developed herein.

I. Cobb- Douglas and CES production functions are
(Cobb-Douglas)$F(x,y) = Ax^s y^t$, $x \geq 0, y \geq 0$, $A > 0$, $0 < s,t < 1$,
(CES)$P(x,y) = a_0(a_1x^n + a_2y^n)^{\frac{1}{\rho}}$,$x \geq 0, y \geq 0$,$a_2 = 1 - a_1$,$-1 < n, \rho < 0$, $0 < a_1, a_2 < 1$, $a_0 > 0$, where x is the capital invested and y is the labor force involved in the production. A, s, t, a_0, a_1, a_2, n, ρ are constants, evaluated according to the real production data and statistical information. First the Cobb-Douglas production function was studied for $s+t = 1$, but later, Douglas allowed the cases $s+t > 1$ and $s+t < 1$ too. The CES production function first was studied for $n - \rho = 0$ and later for $n - \rho > 0$ and $n - \rho < 0$ too. Assume that the volume of capital and labor force increase by 10%. Then
$F(1.10x, 1.10y) = A(1.10x)^s(1.10y)^t = (1.10)^{s+t}Ax^sy^t = (1.10)^{s+t}F(x,y)$and

$$P(1.10x, 1.10y) = a_0[a_1(1.10)^n x^n + a_2(1.10)^n y^n]^{\frac{1}{\rho}} = (1.10)^{n/\rho}P(x,y).$$

In this case, the volume of production increases $(1.10)^{s+t}$(Cobb-Douglas) and $(1.10)^{n/\rho}$(CES) times. The answer to the question "will this growth be greater or less than 10%?" depends on $s+t$ and n/ρ. Summarizing, we have:
$s+t = 1$and $n - \rho = 0$ – constant result of scale;
$s+t > 1$and $n - \rho$¿0 – positive result of scale;
$s+t < 1$and $n - \rho$¡0 – negative result of scale.

Cobb-Douglas and CES production functions can correspond to any value of the result of scale. This is the cause of their popularity among econometricians. However, the feature, providing their popularity is that their elasticity of substitution in their domain is equal to one, feature invariant relatively to the allowance of the sum of $s$and t and the quotient of n and ρ.

II. Given the surface representing the graph of the function $z = f(x, y)$, $(x, y) \in D \to \Delta z, D \in R^2$, $z \in R$, then the theorema ergerium of the surface at the point (x, y) is evaluated by formula $K = \frac{f_{xx}f_{yy} - f_{xy}^2}{1+f_x^2+f_y^2}$, where $f_{xx}, f_{yy}, f_{xy}, f_x, f_y$are partial derivatives of the function $f$with respect to the corresponding arguments. Since the denominator is never negative and zero, it follows that the sign of theorema ergerium depends on the nominator. We have

$$F_{xx}F_{yy} + F_{xy}^2 = A^2 st x^{2s-2} y^{2t-2}[1 - (s+t)] = X[1 - (s+t)],$$

$$P_{xx}P_{yy} - P_{xy}^2 = \frac{a_0^2}{\rho^2} a_1 a_2 n^2 (xy)^{n-2}(1-n)(a_1 x^n + a_2 y^n)^{\frac{2}{\rho}-2}(1-n/\rho) = Y(1-n/\rho),$$

hence the sign of theorema ergerium depends on $[1 - (s+t)]$ and $(1 - n/\rho)$ respectively (because other factors (that is $X$and$Y$) are always positive). We conclude that:

$s+t = 1$ and $n-\rho = 0$-constant result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of parabolic type;

$s+t > 1$and $n-\rho > 0$–positive result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of hyperbolic type;

$s+t < 1$and $n-\rho < 0$–negative result of scale geometrically means that the surfaces corresponding to Cobb-Douglas and CES functions are of elliptic type.

The three types of surfaces are determined by three geometries: Euclidean, of Bolyai-Lobachevsky, and Riemann spherical geometry. Thus:

1) if the production gives constant result of scale, then the computations must be performed on the basis of the law of the dual numbers of Euclidean geometry with an imaginary unity $i_p$where $i_p^2 = 0$ but $i_p \neq 0$;

2) if the production gives positive result of scale, then the computations must be performed on the basis of the law of the double numbers of the Bolyai-Labochevsky geometry with an imaginary unity $i_h$where $i_h^2 = 1$ but $i_h \neq \pm 1$ and there is a great probability of avoiding every possible crises;

3) if the production gives negative result of scale, then the computations must be performed on the basis of the law of the complex numbers of the

Riemann geometry with an imaginary unity i_e where $i_e^2 = -1$ and there is possibility of transferring the production to a higher level.

In the three dimensional space, three geometries correspond to three types of production.

Basic goal of economic theory is to find the ways of transferring the production with negative result of scale to the production with constant and henceforth to the production with positive result of scale.

III. Denote by D the economic space limited to the capital x and labor y. Suppose that a geodetic line passes through$A(x_0, y_0) \in D$, i.e. with the given (x_0, y_0)(capital and labor) we associate an ideal production. In other words, the ideal production corresponds to the least expense. Let the point$M(x, y) \in D$ which does not correspond to the ideal production. Then:

1) if the production is in the routine of constant result of scale, then with the coordinates x, y of the point M, a single ideal production can correspond;

2) if the production is in the routine of positive result of scale, then with the coordinates x, y of the point M, at lest two ideal productions can correspond;

3) if the production is in the routine of negative result of scale, then with the coordinates x, y of the point M, cannot correspond any ideal productions.

Perhaps, these results are obvious from the economic point of view, but we figured them out as a mathematic investigation and this is the power of our investigation.

Since the systems of coordinates are based on the parallelism of geodetic lines, this means that in the case of positive and negative result of scales, the ideal production is impossible to correspond and so the effectiveness of other productions cannot be compared with. However, this problem can be solved by leading the local system of coordinates or by defining the surface of production functions in the three dimensional Euclidean space.

Leading the system of curvilinear coordinates, with respect to which (consequently with respect to the scale) all types of surfaces are invariant is the road (calibrating) invariance and the corresponding geometry is the Wale geometry W_2. The importance of this invariance is stated at the beginning of the article.

Thus, leading the isothermal system of coordinates is directly connected with MES (Mixed Equation System). Stereographic transformation of the sphere S on the tangent plane $T_pS$of the sphere on the southern pole P determines MES and the vector fields on the sphere, if this transformation satisfies the commutativity of automorphism $T_pS \to S \to T_pS$, $u = u[x(u, v), y(u, v)]$, $v = [x(u, v), y(u, v)]$, where $(u, v) \in S - T_pS$ are the local coordinates of the point MES. If the above-stated invariance (invariance of elasticity of substitution) is satisfied relatively to the system of coordinates(u, v), then we can obtain the system of equations MES relatively to (x, y)

$$\frac{\partial u}{\partial u} = \frac{\partial v}{\partial v}; \quad \frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} i_m^2,$$

where $i_m^2 = 0,\pm 1$.

Relatively to the isothermal system of coordinates (x, y) we conclude that $u_x = \beta v_x + \gamma v_y,$, $i_m^2 v_x = \beta u_x + \gamma u_y$ where $\beta = \beta(x, y), \gamma = \gamma(x, y)$ describe the metrical features of the surface.

For $i_m^2 = 0$, we can get parabolic, for $i_m^2 = 1$, hyperbolic and for $i_v^2 = -1$, elliptic system of equations MES. The coefficients β, γ are related to the functions $u = u(x, y)$ and $v = v(x, y)$ as follows

$$\beta = -\frac{u_x u_y - v_x v_y i_m^2}{u_x v_y + u_y v_x i_m^2}; \quad \gamma = \frac{u_x^2 - i_m^2 v_x^2}{u_x v_y + i_m^2 u_y v_x};$$

Holomorphic solutions of the above stated system of equations appear are

$$w = u + i_m v = f(\xi(x, y) + i_m \eta(x, y)) = u[\xi(x, y), \eta(x, y)] + i_m v[\xi(x, y) + i_m \eta(x, y)],$$

where f is a holomorphic function with respect to its argument that does not depend on the conjugate function $\xi(x, y) - i_m \eta(x, y)$. Relatively to ξ, η the following system of equations

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \quad ; \quad \frac{\partial u}{\partial \eta} = i_m^2 \frac{\partial v}{\partial \xi}$$

is satisfied.

This is called the Cauchy-Riemann system and it was studied only for the elliptic types of surfaces. It is easy to show its power for the hyperbolic and parabolic types too.

To this aim, we must demonstrate the connection of the holomorphic function with Pauli matrices, namely
$\begin{pmatrix} 0 & -i_e \\ i_e & 0 \end{pmatrix}$, which is the σ_x matrix of Pauli, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is the σ_y matrix of Pauli and the σ_z matrix of Pauli, obtained by multiplying the first two matrices. We can conclude that holomorphic functions must satisfy the equality
$\begin{pmatrix} 0 & 1 \\ i_m^2 & 0 \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix} = i_m \begin{pmatrix} w_x \\ w_y \end{pmatrix}$,where $w = u + i_m v$

Rather

1)$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_\xi + i_e v_\xi \\ u_\eta + i_e v_\eta \end{pmatrix} = i_e \begin{pmatrix} u_\xi + i_e v_\xi \\ u_\eta + i_e v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for elliptic types of system of equations MES;

2)$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_\xi + i_h v_\xi \\ u_\eta + i_h v_\eta \end{pmatrix} = i_h \begin{pmatrix} u_\xi + i_h v_\xi \\ u_\eta + i_h v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for hyperbolic types of system of equations MES;

3)$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_\xi + i_p v_\xi \\ u_\eta + i_p v_\eta \end{pmatrix} = i_p \begin{pmatrix} u_\xi + i_p v_\xi \\ u_\eta + i_p v_\eta \end{pmatrix}$, the datum demonstrates the validity of Cauchy-Riemann system for parabolic types of system of equations MES.

In fact, the Pauli matrices are the particular cases of a certain Qmatrix, the connection of which with the holomorphic functions gives us the above stated system of equations MES. More precisely

$$Q = \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2 - i_m^2}{\gamma} & -\beta \end{pmatrix},$$

and when $\beta = 0$and $\gamma = 1$, we obtain the Pauli matrices. The system of equations MES (quasi-conformal transformation) will be found by means of the following relation

$$\begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2 - i_m^2}{\gamma} & -\beta \end{pmatrix}\begin{pmatrix} w_x \\ \\ w_y \end{pmatrix} = i_m \begin{pmatrix} w_x \\ \\ w_y \end{pmatrix}.$$

Specifically, if we denote by A the Q matrix when the imaginary unity is $i_e(i_m^2 = i_e^2 = -1)$, then we can conclude that$A^4 = E$, which means that$i_e^4 = 1$. Thus, there are two solutions of this equation, $i_e^2 = -1$ and $i_e^2 = 1$. Most mathematicians have omitted the second solution (i.e., $i_e^2 = 1$) supposing it to have real solutions, but in fact, it is also the imaginary unity.

If we denote by B the Q matrix when its imaginary unity is $i_h(i_m^2 = i_h^2 = 1)$, then we can conclude that$B^2 = E$, which demonstrates that$i_h^2 = 1$.

If we denote by C the Q matrix when its imaginary unity is $i_p(i_m^2 = i_p^2 = 0)$, then we can conclude that$C^2 = 0$, which demonstrates that$i_p^2 = 0$.

An arbitrary solution $w = u(x, y) + i_m v(x, y)$of the system of equations MES reads

$$u(x, y) + i_m v(x, y) = f\left[\int_c M\gamma dx - M\beta dy + i_m M dy\right],$$

where c is an arbitrary Jordan curve, which connects the fixed point $(x_0, y_0) \in D$ with the arbitrary point $(x, y) \in D$, and compactly belonging to D. Here f is an arbitrary function holomorphic with respect to its argument, and $M$is the integrating multiplier of the form $\gamma dx - \beta dy$, i.e. $M = \sqrt{\frac{1}{\gamma} \cdot \frac{\partial(\xi,\eta)}{\partial(x,y)}}$ and $\frac{\partial(\xi,\eta)}{\partial(x,y)}$ is the Jacobian of the transformation, $(x, y) \to (\xi, \eta)$.

In case of production functions (rather Cobb-Douglas and CES) we have $\xi = F(x, y), \quad \eta = y$, and $\beta = -F_y(x, y), \quad \gamma = F_x(x, y)$. Then $dF = \gamma dx - \beta dy = F_x dx + F_y dy$, therefore $\int\limits_c dF = \int\limits_c F_x dx + F_y dy = 0$, if c is a simple closed curve. Moreover, $\frac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x \eta_y - \xi_y \eta_x = F_x$ and $M = \sqrt{\frac{1}{F_x} \cdot F_x} = 1$.

References

[1] M. Blaug, Economic idea in retrospect, Delo LTD, Moscow, 1994.

[2] A. D. Alexandrov, N. Y. Nesvetaev, Geometry, Nauka, Moscow, 1990.(Russian)

[3] M. Zakhirov, Quantum mechanical interpretation of mixed quasi-conformal transformations, DAA.AN. Republic of Uzbekistan, **8** (1990), 9-11.

[4] M. Zakhirov, Geometrodynamics of an example from two particles dynamics, DAN.AN. UzRep, **8** (1992), 9-11.

[5] M. Zakhirov, Quantum signatures and mixed transformation, Problems of Computer Science and Applied Math., Tashkent **99** (1995).

[6] M. Zakhirov, Quantum physics model of economics, Trudy TSUE, (1999), 115-123.