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CONTINUITY OF CHARACTERISTICS OF A THIN LAYER FLOW DRIVEN BY A SURFACE TENSION GRADIENT

Emilia Borsa, Adelina Georgescu
University of Oradea, University of Piteşti

Abstract The flow of a thin layer down an inclined rigid impermeable plane driven simultaneously by the gravity and a surface tension gradient is considered. Geometric and mechanical reasons suggested us the continuity of the depth of the layer, the pressure and the volume flux of the fluid. By using suitable asymptotic expansions it was shown that, indeed, this is the case.

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1. PROBLEM FORMULATION

Consider the flow of a thin liquid layer and use a lubrication (thin-layer) approximation. The liquid is a Newtonian viscous fluid, of constant density ρ and coefficient of dynamic viscosity μ. The thin layer has a steady flow, down a plane inclined at an angle α, $0 \leq \alpha \leq \pi$ to the horizontal. The flow is driven simultaneously by gravity and a surface tension gradient, $\partial\gamma/\partial x > 0$.

With respect to the Cartesian coordinates system Oxyz as indicated in fig.1,

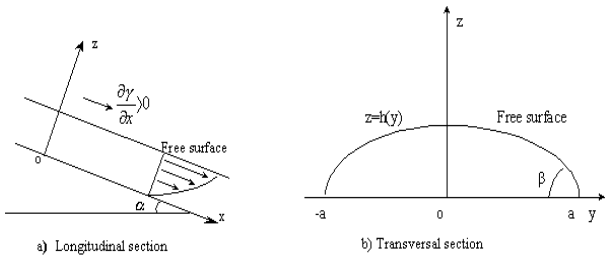

Fig.1. The geometry of the thin layer.

the fluid velocity has the form $\mathbf{u} = u(y,z)\mathbf{i}$. Introduce the notation: p is the pressure in the fluid , p_a is the atmospheric pressure, $\mathbf{g}$ is the gravitational acceleration, $\overline{\gamma}$ is the reference value of surface tension, β is the angle of the three-phase contact line (it is assumed constant, such that $\beta < \pi/2$), $2a$ is the

width of the layer, h_m is the maximum depth of the liquid ($h_m = h(0)$). A constant value of β means that any contact angle hysteresis is ignored. We neglected the viscosity of the air above the liquid and we took the surface curbature to be $h'' = d^2h/dy^2$, because $h^2 << h$. Denote by $z = h(y)$ the free-surface profile. Then, the Navier-Stokes equations are

$$0 = -p_x + \rho g \sin\alpha + \mu(u_{yy} + u_{zz}),\ 0 = -p_y,\ 0 = -p_z - \rho g \cos\alpha. \tag{1}$$

In the thin-layer theory, [1], [5] these equations reduce to

$$0 = -p_x + \rho g \sin\alpha + \mu u_{zz},\ 0 = -p_y,\ 0 = -p_z - \rho g \cos\alpha. \tag{2}$$

and are to be integrated subject to the boundary conditions

$$u = 0,\ \text{on } z = 0; \tag{3}$$

$$p - p_a = -\overline{\gamma}h'', \gamma u_z = \partial\gamma/\partial x \text{ on } z = h(y). \tag{4}$$

The origin of the stress τ could be very diverse. We are interested in the case when τ comes from a local variation of surface, i.e $\tau = \partial\gamma/\partial x$.

Impose the following contact conditions

$$h = 0,\ h = \pm tan\beta,\ \text{at } y = \pm a. \tag{5}$$

The angle of the layer and the horizontal straightline belongs to the following three cases : i) $\alpha \in (0, \pi/2)$, ii)$\alpha = \pi/2$, iii)$\alpha \in (\pi/2, \pi)$. Taking into account the closed form expressions (Section 2) for the velocity profile, free surface profile, pressure and nondimensional volum flux, in all these cases, in Section 3 their continuity at $\alpha = \pi/2$ is shown.

2. CLOSED FORM SOLUTIONS

In [2] we obtained the following results. The velocity of the fluid $u(y, z)$ is

$$u(y,z) = \frac{\rho g sin\alpha}{2\gamma}(-z^2 + 2zh) + \frac{1}{\mu} \cdot \frac{\partial\gamma}{\partial x} \cdot z. \tag{6}$$

The free surface profide $z = h(y)$ is given by

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot \frac{coshB - coshB\xi}{sinhB},\ \alpha \in (0, \pi/2) \tag{7}$$

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot B \cdot \frac{1 - \xi^2}{2},\ \alpha = \pi/2 \tag{8}$$

$$h(y) = \frac{a \cdot tan\beta}{B} \cdot \frac{cosB\xi - cosB}{sinB},\ \alpha \in (\pi/2, \pi) \tag{9}$$

where $B = a(\frac{\rho g|cos\alpha|}{\gamma})^{1/2}$, for $\alpha \neq \pi/2$; $B = 0$, for $\alpha = \pi/2$, $\xi = y/a \in [-1, 1]$.

The pressure of the fluid $p = p(z)$ has the expression

$$p(z) = p_a - \rho g z cos\alpha + tan\beta\sqrt{\rho g\gamma\,|cos\alpha|} \begin{cases} cothB, & \alpha \in (0, \pi/2) \\ B^{-1}, & \alpha = \pi/2 \\ cotB, & \alpha \in (\pi/2, \pi) \end{cases} \tag{10}$$

The nondimensional volume flux reads

$$\overline{Q} = F(B) + \frac{9}{2}\frac{\partial\gamma}{\partial x}\frac{1}{B}\frac{1}{tan\beta\,|tan\alpha|}G(B), \tag{11}$$

where

$$F(B) = 15Bcoth^3B - 15coth^2B - 9BcothB + 4,\ \alpha \in (0, \pi/2) \tag{12}$$

$$F(B) = \frac{12}{35}B^4,\ \alpha = \pi/2 \tag{13}$$

$$F(B) = -15Bcot^3B + 15cot^2B - 9BcotB + 4,\ \alpha \in (\pi/2, \pi) \tag{14}$$

and

$$G(B) = 3Bcoth^2B - 3cothB - B,\ \alpha \in (0, \pi/2) \tag{15}$$

$$G(B) = \frac{4}{15}B^3,\ \alpha = \pi/2 \tag{16}$$

$$G(B) = 3Bcot^2B - 3cotB + B,\ \alpha \in (\pi/2, \pi). \tag{17}$$

Remark 2.1. *Rigorously speaking, for the case $\alpha = \pi/2$, these formulae cotain nondeterminations of the form $0/0$. It is easy to see that the elimination of this nondetermination proceeds immediately. However, we kept the above forms for the sake of symmetry. On the other hand, this kind of expressions are currently used in the litterature on thin layers.*

3. CONTINUITY OF H, P AND Q AT $\alpha = \pi/2$

Some of the above expressions are sufficiently complicated to allow us to decide if h, p and Q are continuous or not at $\alpha = \pi/2$. The asymptotic study carried out in this section shows that they are continuous, indeed.

3.1. CONTINUITY OF H(Y)

For the sake of simplity, we consider the following functions $f_1(x) = \frac{coshx - coshx\xi}{xsinhx}$, $f_2(x) = \frac{cosx\xi - cosx}{xsinx}$, where B:=x. The asymptotic expansion of $f_1(x)$ and $f_2(x)$, as x→0, are

$$f_1(x) \sim \frac{\left(1 + \frac{x^2}{2!} + ...\right) - \left(1 + \frac{x^2\xi^2}{2!} + ...\right)}{x\left(\frac{x}{1!} + \frac{x^3}{3!} + ...\right)} = \frac{x^2\left(\frac{1}{2!} - \frac{\xi^2}{2!}\right) + ...}{x^2 + \frac{x^4}{3!} + ...} = \frac{\frac{1-\xi^2}{2!}}{1 + \frac{x^2}{3!} + ...} \sim \frac{1-\xi^2}{2};$$

$$f_2(x) \sim \frac{\left(1 - \frac{x^2\xi^2}{2!} + \frac{x^4\xi^4}{4!} + ...\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}...\right)}{x\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)} = \frac{x^2\left(\frac{1}{2!} - \frac{\xi^2}{2!}\right) + ...}{x^2 - \frac{x^4}{3!} + ...} \sim \frac{1-\xi^2}{2}.$$

Here we used [6] and the asymptotic expansions $cosx \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!}...$ and $sinx \sim \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$, as x→ 0. In fact, $\frac{1-\xi^2}{2}$ is the asymptotic representation of $f_1(x)$ and $f_2(x)$, as x→0 and it was obtained by performing the asymptotic expansion of the involved sines and cosines. The same results can by obtained by using the l$'$*Hospital* rule, namely

$$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} \frac{sinhx - \xi sinhx\xi}{sinhx + xcoshx} = \lim_{x\to 0} \frac{coshx - \xi^2 coshx\xi}{xsinhx + 2coshx} = \frac{1-\xi^2}{2};$$

$$\lim_{x\to 0} f_2(x) = \lim_{x\to 0} \frac{sinx - \xi sinx\xi}{sinx + xcosx} = \lim_{x\to 0} \frac{cosx - \xi^2 cosx\xi}{-xsinx + 2cosx} = \frac{1-\xi^2}{2}.$$

Therefore

$$\lim_{\alpha \nearrow \pi/2} h(y, \alpha) = h(y, \pi/2) \tag{18}$$

and

$$\lim_{\alpha \searrow \pi/2} h(y, \alpha) = h(y, \pi/2). \tag{19}$$

Here α is the asymptotic variable. The relations (18), (19) and (8) show that h(y,α) is continuous at $\alpha = \pi/2$.

3.2. CONTINUITY OF P

Consider the following functions $g_1(x) = xcothx$ and $g_2(x) = xcotx$, where B:=x. Using the asymptotic expansions [5] $cothx \sim \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...$ and $cotx \sim \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...$, as x→ 0, we have $g_1(x) \sim x(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...) = 1 + ...$, $g_2(x) \sim x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) = 1 + ...$, and, by means of the l$'$ Hospital rule, we get $\lim_{x\to 0} g_1(x) = \lim_{x\to 0} \frac{coshx + xsinhx}{coshx} = 1$;

$\lim\limits_{x\to 0} g_2(x) = \lim\limits_{x\to 0} \frac{x}{tanx} = \lim\limits_{x\to 0} cos^2 x = 1$, whence

$$\lim_{\alpha\nearrow \pi/2} p(y,\, \alpha) = p(y, \pi/2), \tag{20}$$

$$\lim_{\alpha\searrow \pi/2} p(y,\, \alpha) = p(y, \pi/2). \tag{21}$$

The relation (20), (21) and (10) show that p(y,α) is continuous at $\alpha = \pi/2$.

3.3. CONTINUITY OF *Q*

The following asymptotic representations $F(B) \sim \frac{12}{35}B^4$, $G(B) \sim \frac{4}{15}B^3$, as $B \to 0$ (i.e. as $\alpha \to \pi/2$) hold. Indeed, consider the functions $F_1(x) = \frac{15xcoth^3x - 15xcoth^2x - 9xcothx + 4}{x^4}$, $F_2(x) = \frac{-15xcot^3x + 15cot^2x - 9xcotx + 4}{x^4}$,
$G_1(x) = \frac{3xcoth^2x - 3cothx - x}{x^3}$, $G_2(x) = \frac{3cot^2x - 3cotx + x}{x^3}$, and take into account that $coth^2x \sim \frac{1}{x^2} + \frac{x^2}{9} + ... + \frac{2}{3} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...$, $cot^2x \sim \frac{1}{x^2} + \frac{x^2}{9} + ... - \frac{2}{3} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...$

Remark that the first nonvanishing term of the asymptotic expansion of F_1, F_2, G_1, G_2 is obtained if the first four terms in the asymptotic expansion of $coth^2x$ and cot^2x are considered. We have

$F_1(x) \sim 5 \cdot \frac{1}{x}(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) \cdot [\frac{3x(\frac{1}{x^2} + \frac{x^2}{9} + ... + \frac{2}{3} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...)}{x^3} - \frac{3(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - ...) - x}{x^3}] + \frac{-4x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) + 4}{x^4} =$

$= (\frac{5}{x^2} + \frac{5}{3} - \frac{5x^2}{45} + \frac{10x^4}{945} + ...)(\frac{4}{15} - \frac{36}{945}x^2 + ...) + \frac{4}{45} - \frac{8x^2}{945} + ... = \frac{4}{9} + \frac{4}{45} - \frac{36}{189} = \frac{12}{35}$,

$F_2(x) \sim 5 \cdot \frac{1}{x}(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) \cdot [\frac{-3x(\frac{1}{x^2} + \frac{x^2}{9} + ... - \frac{2}{3} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...)}{x^3} + \frac{3(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) - x}{x^3}] + \frac{-4x(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + ...) + 4}{x^4} =$

$= (\frac{5}{x^2} - \frac{5}{3} - \frac{5x^2}{45} - \frac{10x^4}{945} - ...)(-\frac{9}{45} - \frac{30}{945}x^2 - \frac{3}{45} - \frac{6}{945}x^2 ...) + \frac{4}{45} + \frac{8x^2}{945} + ... = \frac{12}{35}$,

$G_1(x) \sim \frac{3}{x^2}(\frac{x^2}{9} - \frac{2x^2}{45} + \frac{4x^4}{945} - \frac{2x^4}{135} + ...) - \frac{3}{x^3}(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + ...) - \frac{1}{x^2} = \frac{1}{3} - \frac{6}{45} + \frac{12x^2}{945} - \frac{6x^2}{135} + \frac{3}{45} - \frac{6x^2}{945} + ... = \frac{1}{3} - \frac{3}{45} = \frac{4}{15}$,

$G_2(x) \sim \frac{3}{x^2}(\frac{x^2}{9} - \frac{2x^2}{45} - \frac{4x^4}{945} + \frac{2x^4}{135} + ...) - \frac{3}{x^3}(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - ...) - \frac{1}{x^2} = \frac{1}{3} - \frac{6}{45} - \frac{12x^2}{945} - \frac{6x^2}{135} - \frac{3}{x^4} + \frac{1}{x^2} + \frac{3}{45} + \frac{6x^2}{945} + ... + \frac{1}{x^2} = \frac{1}{3} - \frac{3}{45} = \frac{4}{15}$

and , by the l$'$Hopital rules, we get $\lim\limits_{x\to 0} F_1(x) =$

$$\lim_{x\to 0} \frac{15xcoshx(1 + sinh^2x) - 15sinhx(1 + sinh^2x) - 9xcoshxsinh^2x + 4sinh^3x}{x^4 sinh^3x} =$$

$$= \lim_{x\to 0} \frac{15xsinhx - 27coshxsinh^2x + 6xsinh^3x + 12cosh^2xsinhx}{4x^3 sinh^3x + 3x^4 sinh^2xcoshx} =$$

$$= \lim_{x\to 0} \frac{-36sinh^2x + 36xcoshxsinhx}{12x^2 sinh^2x + 20x^3 sinhxcoshx + 3x^4 + 6x^4 sinh^2x} =$$

$$= \lim_{x\to 0} \frac{36(x+2xsh^2x-shxchx)}{24xsinh^2x+84x^2sinhxcoshx+32x^3+64x^3sinh^2x+12x^4sinhxcoshx} =$$

$$= \lim_{x\to 0} \frac{144xsinhxcoshx}{180x^2+24sinh^2x+216xsinhxcoshx+360x^2sinh^2x+12x^4+176x^3sinhxcoshx+24x^4cosh^2x} =$$

$$= \lim_{x\to 0} \frac{144xcoshx}{180\frac{x}{sinhx}+24\frac{sinhx}{x}+216coshx+360xsinhx+12x^3\frac{x}{sinhx}+176x^2sinhxcoshx+24x^3coshx} =$$

$= \frac{12}{35}$;

$\lim_{x\to 0} F_2(x) = \lim_{x\to 0} \frac{-15xcos^3x+15sinxcos^2x-9xcosxsin^2x+4sin^3x}{x^4sin^3x} = \lim_{x\to 0} \frac{27xcos^2x-27sinxcosx+9xsin^2x}{7x^5} =$
$\lim_{x\to 0} \frac{-36xcosx+36sinx}{35x^3} = \lim_{x\to 0} \frac{36xsinx}{35\cdot 3x^2} = \frac{12}{35}$;

$\lim_{x\to 0} G_1(x) = \lim_{x\to 0} \frac{4xsinhxcoshx-4sinh^2x}{3x^2sinh^2x+2x^3sinhxcoshx} = \lim_{x\to 0} \frac{4xsinhx}{6xsinhx+3xcosh^2x+6xcosh^2x+2x^3sinhx} =$
$\lim_{x\to 0} \frac{4}{6+3coshx\cdot\frac{x}{sinhx}+6coshx\cdot\frac{x}{sinhx}+2x^2} = \frac{4}{15}$,

$\lim_{x\to 0} G_2(x) = \lim_{x\to 0} \frac{3xcos^2x-3sinxcosx+xsin^2x}{x^3\cdot sin^2x} = \lim_{x\to 0} \frac{-4xcosx+4sinx}{5x^3} = \lim_{x\to 0} \frac{4xsinx}{15x^2} = \frac{4}{15}$;

whence

$$\lim_{\alpha\nearrow\pi/2} Q(y,\,\alpha) = Q(y,\pi/2) \tag{22}$$

and

$$\lim_{\alpha\searrow\pi/2} Q(y,\,\alpha) = Q(y,\pi/2). \tag{23}$$

The relations (22), (23) and (11)-(17) show that $Q(y,\,\alpha)$ is continuous at $\alpha = \pi/2$.

Consequently, by direct asymptotic expansions and by the l'Hopital rule, we proved that the following characteristics $h(y,\,\alpha)$, $p(y,\,\alpha)$ and $Q(y,\,\alpha)$ of the flow of the thin layer are continuous at $\alpha = \pi/2$.

References

[1] Acheson, D.J., *Elementary fluid dynamics*, Oxford University Press, 1990,238–259.

[2] Borsa, E., *Viscous fluid flows driven by surface tension gradients*, Series of Applied and Industrial Mathematics, **17** , Ed. University of Pitesti, 2004.(Romanian)

[3] Georgescu, A., *Asymptotic treatment of differential equations,* Chapman & Hall, London, 1995. (first published by Ed. Tehn., Bucharest, 1989).

[4] Chifu, E., Gheorghiu, C.I. and Stan, I., *Surface mobility of surfactant solution. Numerical analysis for the Marangoni and gravity flow in a thin liquid layer of triangular section*, Rev. Roumaine Chim., **29**, (1984), 31–42.

[5] Ockendon, H., Ockendon, J.R., *Viscous flows*, Cambridge University Press, 1995.

[6] D.J., RIjic, I.M., Grandstein, I.S., *Tables of integralls, sums, series and products,* Ed. Tehn., Bucharest, 1955. (Romanian)

[7] Tipei, N., *Theory of lubrication*, Stanford University Press, 1962.