ROMAI J., **1**, 2(2005),
23–68

EXTENSIONS AND MAPPINGS OF TOPOLOGICAL SPACES

Laurenţiu I. Calmuţchi, Mitrofan M. Choban
Department of Mathematics, Tiraspol State University,
str. Gh. Iablocichin 5, Chişinău, MD-2069, Republic of Moldova

Abstract In the present paper the class of extensions of topological spaces and the methods of constructing of special extensions are investigated. The notions of quasi-compactness, compactness and double compactness are considered. Various problems of the theory of extensions are stated.

1991 MSC primary 54C25, 54D35, 54D40, 54D60, 54E15, 54F15; secondary 54A05, 54C05, 54C20, 54D80, 54E18.

Key words: compactness, extension, compactification, remainder, uniform space, singular mapping, superperfect mappings.

Introduction

Compactness is one of the most important notions. A quasi-compactness is a class of spaces which is multiplicative, hereditary with respect to closed subspaces and contains an infinite T_0-space.

The concept of a compact space was introduced by L. Vietoris [117], P.S. Alexandroff and P.S. Urysohn [2] and is due to the works of E. Borel, H. Lebesgue, K. Kuratowski, W. Serpinski, S. Saks (see [34, 44, 90]).

The general notion of compactness is due to the works of P. S. Alexandroff and P. S. Urysohn [2], E. Hewitt [64], R. Arens and J. Dugundji [7], L. Nachbin [83], S. Mrowka and R. Engelking [43,81], H. Herrlich [62], H. Herrlich and J. Vander Slot [63], M. Hušek and J. de Vries [67], Z. Frolik [51], R. N. Bhanmik and D. N. Misra [19], G. Viglino [118], A. P. Shostak [140] (see [44]).

For every space E there exists the minimal quasi-compactness P such that $E \in P$ (see [43, 44, 81]).

Theory of compactifications is a wide and vast branch of topology and its applications.

One-point compactification of the plane was studied by G. Riemann and compactifications of open subsets of the plane were studied by C. Caratheodory in connection with some problems of analytic functions. The notion of the extension was used by R. Dedekind and G. Cantor in the theory of real numbers and by F. Hausdorff in the theory of metric spaces (see [30, 34, 44, 90, 121]).

Let P be a quasi-compactness. A generalized P-extension of a space X is a pair (eX, f), where $eX \in P$, $f : X \to eX$ is a continuous mappings and the set $f(X)$ is dense in eX. If f is an embedding, then eX is called a P-extension or a P-compactification of the space X.

The general problems of the theory of P-extensions are the following.

First General Problem: To find the methods to construct and study the P-extensions and special P-extensions of a given space X.

Second General Problem: To study the class $GE(X)$ of all generalized P-extensions of a given space X.

Third General Problem. Under which conditions the class $GE(X)$ is a complete lattice?

Fourth General Problem. Let GE(X) be a lattice and let $\beta_P X$ be the maximal element in $GE(X)$. To study the properties of spaces $\beta_P X$ and $\beta_P X \setminus X$.

Fifth General Problem. Let X and Y be spaces. Under which conditions there exists a P-extension eX of X such that Y and $eX \setminus X$ are homeomorphic?

Various important problems of the theory of extensions were formulated in [3, 12, 17, 34, 49, 59, 90, 103, 119, 121, 129].

The purpose of the present paper is to investigate the class of P-extensions of topological spaces and the methods of constructing of new P-extensions of topological spaces.

In Section 1 we discuss the general notions and problems. We introduce the notion of double compactness. In the final part of the section we give examples and concrete problems of the theory of extensions.

Section 2 is devoted to investigation of the methods of construction of extensions.

The method of perfect mappings was used by M. C. Raybom [89] in the constructions of Hausdorff compactifications for locally compact spaces. We introduce the method of superperfect mappings for arbitrary spaces. These methods are used for investigation of the lattice of compactifications (see [1, 5, 27, 32, 55, 58, 68, 71,
72, 74, 76, 82, 95, 106, 114, 116, 119, 124]).

The method of singular mappings was introduced in [32] for construction of the Hausdorff compactifications of locally compact spaces.

The Wallman-Shanin method was introduced by W. H. Wallman [122] and N. A. Shanin [96, 97, 98, 99]. The notion of the base-ring was introduced by O. Frink [50], E. F. Steiner [106, 109], V. I. Zaitsev [128]. In [50] O. Frink formulated the problem: Is every Hausdorff compactification of a completely regular space of the Wallman-Shanin type? The problem of O. Frink was studied by many authors (see [49, 55, 79, 85, 90, 105, 106, 109, 113]) and it was negatively solved by V. M. Uljanov [115].

The spectrum of rings (see [15, 29, 52, 53, 58, 65, 66, 84, 90, 110, 111, 119, 124]) was used by L. I. Calmutskii [24, 28, 131, 132, 133] to introduce the notion of spectral compactifications.

In Section 3 we study the uniform extensions of completely regular spaces. The construction of maximal uniform extension μX of a space X is due to J. Diendonné and to F. Hausdorff [44]. The concept of a uniform space and the notion of a complete uniform space were introduced by A. Weil (see [44]). The completions of separable metric spaces were studied by J. M. Aarts and P. V. van Emde Boas [1]. The completions of arbitrary metric spaces were studied by V. K. Bel'nov [20,127]. An important part of the methods of construction of extensions of a space is to present the "new points" of the extension as a space with concrete properties. We simplify and extend the "Bel'nov's gluing method" to theory of uniform completions of arbitrary completely regular spaces.

In this article we shall use the following notation:

We denote by $cl_X A$ or clA the closure of a set A in a space X.

We denote by $|A|$ the cardinality of a set A.

We denote by $w(X)$ the weight of a space X.

The interval [0, 1] is denoted by I.

On the set $N = \{1, 2, \ldots\}$ we consider only the discrete topology.

We use the terminology from [44,34,90].

General notions and problems

Let L be a partially ordered set. Fix a non-empty subset A of L. We consider that $a = \vee A$ if $a \geq x$ for every $x \in A$ and if $b \geq x$ for each $x \in A$, then $b \geq a$. We consider that $c = \wedge A$ if $c \leq x$ for every $x \in A$ and if $b \leq x$ for each $x \in A$, then $b \leq c$.

The set L is called:

- an upper semi-lattice if there exists the element $\vee L$ and for every two elements $x, y \in L$ there exists the element $x \vee y = \vee\{x, y\}$;
- a lower semi-lattice if there exists the element $\wedge L$ and for every two elements $x, y \in L$ there exists the element $x \wedge y = \wedge\{x, y\}$;
- a complete upper semi-lattice if for every non-empty subset $A \subseteq L$ there exists the element $\vee A$;
- a lattice if L is an upper semi-lattice and a lower semi-lattice;
- a complete lattice if L is a lower semi-lattice and a complete upper semi-lattice.

We mention that in the complete lattice L for every non-empty subset $A \subseteq L$ there exists the element $\wedge A$.

Let L be a complete upper semi-lattice and M be a non-empty subset of L. If for every two elements $x, y \in M$ we have $x \vee y \in M$, then M is called an upper subsemi-lattice of L. In the similar way there are defined the notions of a lower subsemi-lattice and of a sublattice.

1.1. Extensions of spaces

1.1.1. Definition. A g-extension of a space X is called a pair (Y, f), where Y is a non-empty T$_0$-space, $f : X \to Y$ is a continuous mapping and $\{cl_Y f(A) : A \subseteq X\}$ is a closed base of the space Y.

1.1.2. Definition. A g-extension (Y, f) of a space X is called an extension of X if f is an embedding of X in Y.

1.1.3. Remark. If (Y, f) is a g-extension of a space X, then the set $f(X)$ is dense in Y.

Denote by $E(X)$ the family of all extensions of a space X and by $GE(X)$ the family of all g-extensions of the space X. The family $GE(X)$ is partially ordered in the standard way: $(Y_1, f_1) \leq (Y_2, f_2)$ if there exists a continuous mapping $\varphi : Y_2 \to Y_1$ such that $f_1(x) = \varphi(f_2(x))$ for every $x \in X$, i. e. $f_1 = \varphi \circ f_2$.

If (Y_1, f_1), $(Y_2, f_2) \in GE(X)$, $\varphi : Y_2 \to Y_1$ and $\psi : Y_1 \to Y_2$ are continuous mappings, $f_1 = \varphi \circ f_2$ and $f_2 = \psi \circ f_1$, then $\psi = \varphi^{-1}$ and φ and ψ are homeomorphisms. Thus $(Y_1, f_1) = (Y_2, f_2)$ provided $(Y_1, f_1) \leq (Y_2, f_2)$ and $(Y_2, f_2) \leq (Y_1, f_1)$.

If $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$, then $GE(X) = \{(Y, f) \in GE(X) : Y$ is a T_i-space$\}$ and $E_i(X) = E(X) \cap GE_i(X) = \{(Y, f) \in E(X) : Y$ is a T_i-space$\}$.

1.1.4. Proposition. Let $f : X \to Y$ be a continuous mapping of a space X into a T_i-space Y, the set $f(X)$ is dense in Y and $i \geq 3$. Then $(Y, f) \in GE_i(X)$.

Proof. Let F be a closed non-empty subset of Y and $y \in Y \setminus F$. There exist two open subsets U and V of Y such that $F \subseteq U$, $y \in V$ and $U \cap V = \emptyset$. We put $\Phi = cl_Y(f(X) \cap U)$. Then $F \subseteq \Phi$ and $y \notin \Phi$. Hence $\{cl_Y A : A \subseteq f(X)\}$ is a closed base of the space Y. The proof is complete.

1.1.6. Definition. A pair (Y, f) is called a weak g-extension (wg-extension) of a space X if $f : X \to Y$ is a continuous mapping, Y is a T_0-space and the set $f(X)$ is dense in Y.

We denote by $WGE(X)$ the family of all wg-extensions of a space X,
$WE(X) = \{(Y, f) \in WGE(X) : f$ is an embedding$\}$,
$WGE_i(X) = \{(Y, f) \in WGE(X) : Y$ is a T_i-space$\}$ and
$WGE_i(X) = WE(X) \cap WGE_i(X)$.

1.1.7. Proposition. Let X be a non-empty T_0-space. Then:
1. $WE(X)$ is n*ot* a set.
2. $WGE(E)$ is not a set.
3. If $i \geq 2$, then $WGE_i(X)$ is a set.
4. $GE(X)$is a set.

Proof. Let Z be a non-empty T_0-space and $Z \cap X = \emptyset$. We put $Y = X \cup Z$, $f(x) = x$ for every $x \in X$,- $\mathrm{Im}_0 = \{H \subseteq X : H$ is open in $X\} \cup \{X \cup V : V \subseteq Z$ and V is open in $Z\}$. Then Im_0 is a T_0-topology on Y and

$(Y, f) \in WE(X)$. Thus $WE(X)$ is not a set. Hence $WGE(X)$ is not a set, too.

Let $m = |X|$ and $\tau = 2^{2^\tau}$. If $(Y, f) \in GE(X) \cup WGE_2(X)$, then $|Y| \leq \tau$. Therefore $WGE_2(X) \cup GE(X)$ is a set. The proof is complete.

1.1.8. Proposition. Let X be an infinite T_1-space. Then $WE_1(X)$ is not a set. In particular, $WE_1(X)$ is not a set.

Proof. Let Z be a non-empty T_1-space and $Z \cap X = \emptyset$. We put $Y = X \cup Z$, $f(x) = x$ for every $x \in X$ and $\Gamma_1 = \{H \subseteq X : H \text{ is open in } X\} \cup \{V \subseteq Y : V \cap Z \text{ is open in } Z \text{ and the set } X \setminus V \text{ is finite}\}$. Then Γ_1 is a T_1-topology on Y, $\Gamma_0 \subseteq \Gamma_1$ and $((Y, \Gamma_1), f) \in WE(X)$. The proof is complete.

1.1.9. Remark. If in the proof of Proposition 1.1.7 or of Proposition 1.1.8 the space Z is compact, then the space (Y, Γ_0) or (Y, Γ_1) is compact, too.

If (Y_1, f_1), $(Y_2, f_2) \in WGE(X)$, then $(Y_1, f_1) \leq (Y_2, f_2)$ if there exists a continuous mapping $\varphi : Y_2 \to Y_1$ such that $f_1 = \varphi \circ f_2$.

1.1.10. Proposition. The relation $\leq$ is an ordering on $WGE_2(X)$.

Proof. Is obvious.

1.1.11. Example. Let X be a non-empty space. Then $\leq$ is not an ordering on $WE(X)$.

Let Z be a non-empty T_0-space, $Z \cap X = \emptyset$ and $b \in Z$. Consider the space $Y_1 = Z \cup X$ with the topology $\Gamma_0 = \{U \subseteq X : U \text{ is open in } X\} \cup \{V \cup X : V \text{ is open in } Z\}$ and subspace $Y_2 = \{b\} \cup X$ of Y_1. Let $f(x) = x$ for each $x \in X$. Then (Y_1, f), $(Y_2, f) \in WE(X)$. We put $\varphi(y) = y$ for every $y \in Y_2$, $f = \psi \,|\, X$ and $\psi(y) = b$ for every $y \in Z$. Then the mappings $\varphi : Y_2 \to Y_1$ and $\psi : Y_1 \to Y_2$ are continuous and $\varphi(x) = \psi(x) = x$ for each $x \in X$. Thus $(Y_1, f) \leq (Y_2, f)$, $(Y_2, f) \leq (Y, f)$ and $(Y, f) \neq (Y_2, f)$ provided $|Z| \geq 2$.

1.1.12. Example. Let X be an infinite T_1-space. Then $\leq$ is not an ordering on $WE_1(X)$.

Let Z be a T_1-space, $|Z| \geq 2$, $b \in Z$, $Y_1 = Z \cup X$ be a space with the topology $\Gamma_1 = \{U \subseteq X : U \text{ is open in } X\} \cup \{V \subseteq Y_1 : V \cap Z \text{ is open in } Z \text{ and the set } X \setminus V \text{ infinite}\}$, $Y_2 = \{b\} \cup X$ be a subspace of Y_1 and $f(x) = x$ for each $x \in X$. Then (Y_1, f), $(Y_2, f) \in WE_1(X)$, $(Y_1, f) \leq (Y_2, f), (Y_2, f) \leq (Y_1, f)$ and $(Y_1, f) \neq (Y_2, f)$.

Let X be a space. On the class $WGE(X)$ we consider the relation $\sim$:$(Y_1, f_1) \sim (Y_2, f_2)$ iff $(Y_1, f_1) \leq (Y_2, f_2)$ and $(Y_2, f_2) \leq (Y_1, f_1)$. Obviously, $\sim$ is a relation of equivalence. Denote by $WGE^0(X)$ the classes of equivalence on $WGE(X)$ and by $WE^0(X)$ the classes of equivalence on $WE(X)$.

Obviously $\leq$ is an ordering on the a class $WGE^0(X)$.

1.1.13. Proposition. Let $H = \{(Y_\alpha, f_\alpha) \in WGE(X) : \alpha \in A\}$ be a set, $f(x) = (f_\alpha(x) : \alpha \in A)$ for every $x \in X$ and Y be the closure of the set $f(X)$ in the space $\Pi\{Y_\alpha : \alpha \in A\}$. Then:

1. $(Y, f) \in WGE(X)$ and we put $(Y, f) = \vee H$.
2. $(Y_\alpha, f_\alpha) \leq (Y, f)$ for each $\alpha \in A$.

3. If $(Z, g) \in WGE(X)$ and $(Y_\alpha, f_\alpha) \leq (Z, g)$ for each $\alpha \in A$, then $(Y, f) \leq (Z, g)$.

4. If $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$ and Y_α is a T_i-space, then Y is a T_i-space.

5. If $H \cap WE(X) \neq \emptyset$, then $(Y, f) \in WE(X)$.

Proof. For every $\beta \in A$ we consider the projection $\varphi_\beta : Y \to Y_\beta$, where $\varphi_\beta(y_\alpha : \alpha \in A) = y_\beta$ for any $(y_\alpha : \alpha \in A) \in Y$. Then $f_\alpha = \varphi_\alpha \circ f$ for each $\alpha \in A$. The assertions 1, 2, 4 and 5 are proved. Let $(Z, g) \in WGE(X)$ and $(Y_\alpha, f_\alpha) \leq (Z, g)$ for each $\alpha \in A$. For any $\alpha \in A$ we fix a continuous mapping $\psi_\alpha : Z \to Y_\alpha$ such that $f_\alpha = \psi_\alpha \circ g$. Consider the mapping $\psi : Z \to \Pi\{Y_\alpha : \alpha \in A\}$, where $\psi(z) = (\psi_\alpha(z) : \alpha \in A)$. The mapping ψ is continuous, $\psi(g(X)) = f(X)$ and $\psi(Z) \subseteq Y$. The assertion 3 and Proposition are proved.

1.1.14. Question. Is it true that $WE^0(X)$ is a set for each topological space X?

Obviously, $WE^0(X)$ is a set for every space X iff $WGE^0(X)$ is a set for every space X.

1.1.15. Remark. Let X be a non-empty space, D_0 be a singleton space, $f_m : X \to D_0$ be the unique mapping of X into D_0, $f_M(x) = x$ for each $x \in X$. Then (X, f_M) is the maximal element in $WGE(X)$ and (D_0, f_M) is the minimal element in $WGE(X)$. Obviously, (X, f_M), $(D_0, f_M) \in GE(X)$.

1.1.16. Question. Let X be a space and H be a non-empty subset of the set $GE(X)$. Is it true that $\vee H \in GE(X)$?

1.1.17. Corollary. Let $i \geq 2$. Then $WGE_i(X)$ is a complete lattice.

1.1.18. Corollary. Let $i \geq 2$. Then $WE_i(X)$ is a complete upper semi-lattice.

1.1.19. Corollary. Let $i \geq 3$. Then $GE_i(X)$ is a complete lattice.

1.1.20. Corollary. Let $i \geq 3$. Then $E_i(X)$ is a complete upper semi-lattice.

1.2. The canonical functor $m : WGE(X) \to GE(X)$

Consider a topological space X. Fix a *wg*-extension (Y, f) of the space X. Let Γ_Y be the topology of the space Y. On Y consider a new topology Γ_{Yf} generated by the closed base $\{cl_Y H : H \subseteq f(X)\}$. There exist a set Y_f and a mapping $P_Y : Y \to Y_f$ such that $P_Y^{-1}(P_Y(H)) = H$ for every $H \in \Gamma_{Yf}$ and $\Gamma^0_{Yf} = \{P_Y(H) : H \in \Gamma_{Yf}\}$ is a T_0-topology on a set Y_f.

If $y \in Y$, then$P_Y^{-1}(P_Y(y)) = (\cap\{Y \in \Gamma_{Yf} : y \in U\}) \cap (\cap\{Y \setminus U : U \in \Gamma_{Yf}, y \notin U\})$. Consider the mapping $Pf : X \to Y_f$, where $Pf = P_Y \circ f$. By construction, $(Y_f, Pf) \in GE(X)$. We put $(Y_f, Pf) = m(Y, f)$, $Y_f = m(Y)$ and *Pf*=*m*(f).

The canonical functor $m : WGE(X) \to GE(X)$ is constructed.

From the construction it follows.

1.2.1. Proposition. If $(Y, f) \in GE(X)$, then $m(Y, f)$=(Y, f).

1.2.2. Question. Is it true that the functor m is covariant?

1.3. The canonical functors $m_i : WGE(X) \to WGE_i(X)$

Fix $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$. For every space Y there exist a unique T_i-space Y/i and a unique projection $i_Y : Y \to Y/i$ with the properties:

1. i_Y is a continuous mapping onto Y/i;

2. for every continuous mapping $\varphi : Y \to Z$ in a T_i-space Z there exists a unique continuous mapping $\bar{\varphi} : Y/i \to Z$ such that $\varphi = \bar{\varphi} \circ i_Y$.

3. if $\psi : Y \to Z$ is a continuous mapping, then there exists a unique continuous mapping $\bar{\psi} : Y/i \to Z/i$ such that $\bar{\psi} \circ i_Y = i_Z \circ \psi$.

The space Y/i with the projection i_Y is called the i-replic of the space Y.

Fix a space X. If $(Y, f) \in WGE(X)$, then we put $f_i = i_Y \circ f$ and $m_i(Y, f) = (Y/i, f_i)$. From the construction it follows.

1.3.1. Proposition. $m_i : WGE(X) \to WGE_i(X)$ is a covariant functor. If $(Y, f) \leq (Z, g)$, then $m_i(Y, f) \leq m_i(Z, g)$. If $(Y, f) \in WGE(X)$, then $m_i(Y, f) = (Y, f)$.

1.4. Compactness

The notion of compactness is due to E. Mrowka [81,43], E. Hewit [64], R. Arens and S. Dugundji [7].

A class P of topological T_0-spaces is called a strongly compactness if the following conditions are fulfilled:

C_1. the class P is non-empty;

C_2. there exists a space $X \in P$ such that $|X| > 2$;

C_3. the class P is multiplicative, i. e. if $\{X_\alpha \in P : \alpha \in A\}$ is a non-empty set of spaces from P, then $\Pi\{X_\alpha : \alpha \in A\} \in P$;

C_4. the class P is closed hereditary, i. e. if Y is a closed subspace of a space $X \in P$, then $Y \in P$;

C_5. if Y is a dense subspace of a space $X \in P$, then $\{cl_X A : A \subseteq Y\}$ is a closed base of the space X.

A class of spaces P with properties $C_1 - C_4$ is called a quasi-compactness. A quasi-compactness P of Hausdorff spaces is called a compactness.

Fix a quasi-compactness P. For every space X we put $WPGE(X) = \{(Y, f) \in WGE(X) : Y \in P\}$, $WPE(X) = WPGE(X) \cap WE(X)$, $PGE(X) = WPGE(X) \cap GE(X)$ and $PE(X) = PGE(X) \cap E(X)$.

If P is a compactness, then $WPGE(X) = PGE(X)$ and $WPE(X) = PE(X)$. From Proposition 1.1.7. it follows that $PGE(X)$ and $PE(X)$ are the sets for each space X.

1.4.1. Theorem. Let P be a compactness and X be a space. Then $\vee H \in WPGE(X)$ for every non-empty set $H \subseteq WPGE(X)$.

Proof. Follows immediately from the conditions C_3, C_4 and properties of Hausdorff spaces.

1.4.2. Corollary. Let P be a compactness. Then $WPGE(X)$ is a complete lattice for every space X.

Denote by $(\beta_P X,\, \beta_P)$ the maximal element of the lattice $WPGE(X)$, where P is a compactness.

1.4.3. Corollary. Let P be a compactness, X be a space and $WPE(X) \neq \emptyset$. Then:

1. $WPE(X)$ is a complete upper semi-lattice;
2. $(\beta_P X,\, \beta_P) \in WPE(X)$.

1.4.4. Theorem. Let P be a compactness. For every continuous mapping $f : X \to Y$ of a space X into a space Y there exists a continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$ such that $\beta_P \circ f = \beta_P f \circ \beta_P$. If every space $Z \in P$ is a T_2-space, then the mapping $\beta_P f$ is unique.

Proof. We may consider that $f(X)$ is dense in Y. Then $g = \beta_P \circ f : X \to \beta_P Y$ is a continuous mapping, the set $g(X)$ is dense in $\beta_P Y$ and $(\beta_P Y,\, g) \in WPGE(X)$. Thus there exists a continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$ such that $g = \beta_P \circ f \circ \beta_P$. The proof is complete.

1.4.5a. Corollary. Let P be a compactness and $f : X \to Y$ be a continuous mapping of a space X into a space $Y \in P$. Then $Y = \beta_P Y$ and there exists a unique continuous mapping $\beta_P f : \beta_P X \to Y$ such that $f = \beta_P f \circ \beta_P$.

1.4.5b. Remark. Let P be a quasi-compactness. Then there exists $\beta_P X \in WPGE(X)$ such that $\beta_P X \in \vee PGE(X)$.

1.4.6. Proposition. Let P be a strongly compactness. Then every space $x \in P$ is a Hausdorff space, i. e. P is a compactness.

Proof. Let $d(X) = \min\{|H| : H \subseteq X, cl_X H = H\}$ be the density of a space X. Consider the space F=$\{0,1\}$ with the topology Im $= \{\emptyset, \{1\}, \{1, 0\}\}$. Suppose that $X \in P$ and X is not a T_1-space. Then the space F is embeddable in X. Suppose that $F \subseteq X$. Denote by b, c the cardinality larger than 2^c (see Proposition 1.1.7). For some cardinal m the space Y is embeddable in $F^m \subseteq X^m$([44],Theorem 2.3.26). Let Z be the closure of Y in X^m. Then the space Z is separable and $|Z| \geq |Y| > 2^c$. If $S \in P$, then $|S| \leq \exp(\exp(d(S)))$. Thus $|S| \leq 2^c$ for every separable space $S \in P$. Therefore every space $S \in P$ is a T_1-space.

Suppose that $X \in P$ and X is not a T_2-space. There exist two distinct points $a, b \in X$ such that $V \cap W \neq \emptyset$ provided V and W are open subsets of X, $a \in V$ and $b \in W$. Fix a cardinal number $\tau > \exp(\exp(|X|))$. We put $\Phi = \{a, b\}$. In X^τ we consider the diagonal $\Delta(X)$ (see [44], p.110). Let Y be the closure of the set $\Delta(X)$ in X^m. Then $\Phi^\tau \subseteq Y$, $|\Delta(X)| = |X|$, $d(Y) \leq |X|$, $|Y| \leq \exp(\exp(|X)) < \tau|$ and $|\Phi^\tau| = 2^\tau = \exp(\tau)$, a contradiction. The proof is complete.

1.5. Double compactness

A class P of topological T_0-spaces is called a double compactness if the following conditions are fulfiled:

D_1. the class P is non-empty;

D_2. there exists a space $X \in D$ such that $|X| \geq 2$;

D_3. if Γ is a topology of the space $X \in P$ then there is determined the completely regular topology $d\Gamma$ on X such that $(X, d\Gamma) \in P$, $\Gamma \subseteq d\Gamma$ and $dd\Gamma = d\Gamma$;

D_4. if $f : X \to Y$ is a continuous mapping of a space (X, Γ) into a space (Y, Γ') and $X, Y \in P$, then f is a continuous mapping of the space $(X, d\Gamma)$ into a the space $(Y, d\Gamma')$;

D_5. if $\{(X_\alpha, \Gamma_\alpha) : \alpha \in A\}$ is a non-empty set of spaces, $(X_\alpha, \Gamma_\alpha) \in P$ for each $\alpha \in A$, $X = \Pi\{X_\alpha : \alpha \in A\}$, Γ is the product of topologies Γ_α on X and Γ' is the product of topologies $d\Gamma_\alpha$ on X, then $\Gamma' \subseteq d\Gamma$;

D_6. if $(X, \Gamma) \in P$, $Y \subseteq X$ and Y is a closed subset of the space $(X, d\Gamma)$, then $(Y, \Gamma\,|\,Y) \in P$ and $d(\Gamma\,|\,Y) \supseteq d\Gamma\,|\,Y$, where $\Gamma\,|\,Y = \{U \cap Y : U \in \Gamma\}$ for the topology Γ on X.

1.5.1. Proposition. Let P be a class of spaces, X be a space, $\{Y_\alpha : \alpha \in A\}$ be a non-empty family of subspaces of the space X, $Y = \cap\{Y_\alpha : \alpha \in A\}$ and $Y_\alpha \in P$ for each $\alpha \in A$. Then:

1. if P is a double compactness, then $Y \in P$;
2. if P is a compactness, then $Y \in P$.

Proof. We may consider that $X = Y_\alpha$ for some $\alpha \in A$. If X is a T_2-space, then Y is a closed subspace of the space $\Pi\{Y_\alpha : \alpha \in A\}$. The assertion 2 is proved. If P is a double compactness, then Y is a closed subspace of the space $\Pi\{Y_\alpha : \alpha \in A\}$ in the topology $d\Gamma$. The assertion 1 and Proposition are proved.

Fix a double compactness P. For every space X we put $PGE(X) = \{(Y, f) \in WGE(X) : Y \in P$ and $f(X)$ is a dense subset of the space $(Y, d\Gamma)$ and $PE(X) = WE(X) \cap PGE(X)$.

From the condition D_6 it follows that $PGE(X)$ and $PE(X)$ are sets.

1.5.2. Theorem. Let P be a double compactness. Then $PGE(X)$ is a complete lattice for every space X.

Proof. Let $\{(Y_\alpha, f_\alpha) : \alpha \in A\}$ be a non-empty subset of the set $PGE(X)$. Denote by Γ_α the topology of the space Y_α and by Γ the topology of the space $\Pi\{Y_\alpha : \alpha \in A\}$. Consider the mapping $f : X \to \Pi\{Y_\alpha : \alpha \in A\}$, where $f(x) = (f_\alpha(x) : \alpha \in A)$ for each $x \in X$. Let Y be the closure of the set $f(X)$ in the space $(\Pi\{Y_\alpha : \alpha \in A\}, d\Gamma)$. Then $(Y, f) \geq (Y_2, f_2)$ for each $\alpha \in A$. From the condition D_4 it follows that if $(Z, g) \in PGE(X)$ and $(Z, g) \geq (Y_\alpha, f_\alpha)$ for each $\alpha \in A$, then $(Z, g) \geq (Y, f)$. Thus $(Y, f) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in A\}$. The proof is complete.

1.5.3. Corollary. Let P be a double compactness, let X be a space and $PE(X) \neq \emptyset$. Then $PE(X)$ is a complete upper semi-lattice.

1.5.4. Theorem. Let P be a double compactness. Then:

1. for every continuous mapping $f : X \to Y$ of a space X into a space Y there exists a unique continuous mapping $\beta_P f : \beta_P X \to \beta_P Y$, $\beta_P \circ f = \beta_P f \circ \beta_P$;

2. for every continuous mapping $f : X \to Y$ of a space X into a space $Y \in P$ there exists a unique continuous mapping $\beta_P f : \beta_P X \to Y$ such that $f = \beta_P f \circ \beta_\Gamma$.

Proof. Let Z be the closure of the set $\beta_P(f(X))$ in the space $(\beta_P Y, d\Gamma)$. Then $(Z, \beta_P \circ f) \in PGE(X)$ and the assertion 1 is proved. If $Y \in P$, then $\beta_P Y = Y$. The proof is complete.

1.6. Examples

1.6.1. Example. Let C be the class of compact Haussdorff spaces. Then C is a strongly compactness. If $(Y, f) \in CGE(X)$, then we say that (Y, f) is a g-compactification of X. If $(Y, f) \in CE(X)$, then (Y, f) is called a compactification of X. For every space X the g-compactification $\beta X = \beta_P X$ is the Stone-Čech g-compactification of X. If X is a completely regular space, then βX is the Stone-Čech compactification of X.

1.6.2. Example. Let C_0 be the class of zero-dimensional compact spaces. Then C_0 is a strongly compactness. If $indX > 0$, then $C_0 E(X)=\emptyset$. If $indX=0$, then $mfX = \beta_{C_0} X$ is the Morita-Freudenthal compactification of X. We put $mfX = \beta_{C_0} X$ and $(mfX, mf) = (\beta_{G_0} X, \beta_{G_0})$. The g-compactification mfX is called the maximal zero-dimensional g-compactification of the space X.

1.6.3. Example. Let X be a completely regular space. A subset L of X is called bounded in X if the set $f(L)$ is bounded in the space of reals R for every continuous function $f : X \to R$. A space X is called μ-complete if the closure clL of every bounded subset L is compact. Let C_μ be the class of all μ-complete spaces. Then C_μ is a strongly compactness. The g-extension $(\beta_{G_\mu} X, \beta_{C_\mu}) = (\mu^* X, \mu^*)$ is called the maximal μ-completion of the space X. If X is a completely regular space, then $(\mu^* X, \mu^*) \in E(X)$ and $\mu^* X$ is the μ-completion of X.

1.6.4. Example. Let R be the space of reals. A space Z is called a realcompact space if it is homeomorphic to a closed subspace of some space R^A. The class R of all realcompact spaces is a strongly compactness. The g-extension $(\nu X, \nu) = (\beta_R X, \beta_R)$ is the maximal g-realcompactification of the space X. If X is a completely regular space, then νX is the realcompactification of X and $(\nu X, \nu) \in E(X)$. Every realcompact space is μ-complete. Therefore $\nu X \leq \mu^* X$ and $\mu^* X \subseteq \nu X$.

1.6.5. Example. Let U be the class of all complete uniform spaces. If X is a completely regular space, then by U_X we denote the universal uniformity on X (see [44]). Every uniform space is considered and a topological space too. Thus for every space X in $UGE(X)$ the maximal element $(\mu X, \mu)$ is determined, where μX is a complete uniform space, $\mu : X \to \mu X$ is a continuous mapping and the set $\mu(X)$ is dense in μX. The space μX is called the Diendonné completion of the space X. If X is completely regular, then μX is the completion of the uniform space (X, U_X). If $X = \mu X$, then the space X is called a Deudonné complete space. Every Deudonné complete space is μ-complete. For every space X we may consider that $\mu^* X \subseteq \mu X \subseteq \nu X \subseteq \beta X$.

1.6.6. Example. Let P be a compactness such that every space $(Y, \Gamma) \in P$ be completely regular. For every space $(X, \Gamma) \in P$ we put $d\Gamma = \Gamma$. Then P is a double compactness. Therefore every compactness of completely regular spaces may be considered as a double compactness.

1.6.7. Example. For every space (X, Γ) we put $c\Gamma = \{U \in \Gamma : U$ is a compact subset$\}$ and $d\Gamma$ is the topology generated by the open base $\{U_1 \cap U_2 \cap ... \cap U_n : n \in N,\ U_1,\ U_2,\ ..., U_n \in \Gamma\} \cup \{X \setminus U : U \in c\Gamma\}$.

A space (X, Γ) is called a spectral space if $c\Gamma$ is an open base of the space X, $U \cap V \in c\Gamma$ is an open base of the space X, $U \cap V \in c\Gamma$ for all $U,\ V \in c\Gamma$ and $(X, d\Gamma)$ is a compact Hausdorff space. Let S be the class of all spectral spaces. Then S is a double compactness. For every T_0-space X we have $SE(X) \neq \emptyset$, i. e. $\beta_S : X \to \beta_S X$ is an embedding.

1.6.8. Proposition. If (X, Γ) is a spectral space, then:

1. (X, Γ) is a compact T_0-space;
2. $(X, d\Gamma)$ is a zero-dimensional compact space;
3. $d\Gamma = \Gamma$ if (X, Γ) is a T_1-space.

Proof. Is obvious (see [132]).

1.6.9. Remark. The class of spectral compactifications of a space X was studied in [24,28,131,132,133].

1.6.10. Example. Let E=[0,1], $F = \{2^{-n} : n \in N\}$ and Im be the topology generated by the base $\{\{t \in E : a < t < b\}$: a, b are real numbers$\} \cup \{V_n = \{t \in E : t < 2^{-n}\} \setminus F : n \in N\}$(see [2] or [44], Example 1.5.7). Then E is a T_2-space and E is not regular. If X is the subspace of irrational numbers of E or $X = E \setminus F$, then $\{cl_E H : H \subseteq X\}$ is not a closed base of E. Thus $E \notin P$ for every compactness P. There exists a minimal quasi-compactness P of Hausdorff spaces such that $E \in P$. Therefore P is a compactness and P is not a strongly compactness.

1.6.11. Example. Let P be the class of all compact T_0-spaces. Then P is a quasi-compactness and P is not a compactness. Obviously $OPE(X) \neq \emptyset$ for every T_0-space X.

1.6.12. Example. Let P be the class of all compact T_1-spaces. Then P is a quasi-compactness and P is not a compactness. It is well–known that $\omega X \in PE(X)$ for every T_1-space X.

1.7. Problems

1.7.1. Problem. Let P be a compactness or a double compactness.

1. Under which conditions the lattices $PGE(X)$ and $PGE(Y)$ are isomorphic?

2. Under which conditions the upper semi-lattices $PE(X)$ and $PE(Y)$ are isomorphic?

3. Which topological properties of a space X are characterized in terms of the objects $PGE(X)$ and $PE(X)$?

4. Which properties of the lattice $PGE(X)$ are characterized by the properties of the space X?

5. Let X be a space and $PE(X) \neq \emptyset$. Which properties of the upper semi-lattice $PE(X)$ are characterized be the properties of the space X?

The program of matching "interesting" topological properties of a completely regular space X with "interesting" properties of the complete upper semi-lattice $PE(X)$ is very important in the theory of extensions. N. Boboc and G. Siretchi [22] has proved that $CE(X)$ is a lattice iff the space X is locally compact. In [76] K. D. Magil has proved that for two locally compact spaces X and Y the semi-lattices $CE(X)$ and $CE(Y)$ are isomorphic iff the spaces $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic.

Another program of investigation is to find the "interesting" compactness and double compactness.

1.7.2. Problem. Let P be a compactness or a double compactness, let X be a space and $PE(X) \neq \emptyset$.

1. Find the methods of constructions the extension $\beta_P X$, some extensions from $PE(X)$ or all extensions $PE(X)$.

2. Let Z be a space. Under which conditions there exists an extension $(Y, f) \in PE(X)$ such that $Y \setminus f(X)$ and Z are homeomorphic?

3. Under which conditions there exists an extension $(Y, f) \in PE(X)$ such that $\dim(Y \setminus f(X)) \geq m$, where $m \in N$?

4. Let $Z \in P$. Under which conditions there exist an extension $(Y, f) \in PE(X)$ and a closed subspace $Z' \subseteq Y \setminus f(X)$ such that Z and Z' are homeomorphic?

2. Some methods of construction of extensions

A mapping $f : X \to Y$ of a space X into a space Y is called:

- a perfect mapping if $f(X) = Y, f$ is continuous, closed and the fibers $f^{-1}(y),\ y \in Y$, are compact;

- a superperfect mapping if $f(X) = Y, f$ is continuous, perfect and there exists a compact set $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for each $x \in X \setminus \Phi$;

- a singular mapping if f is continuous and the set $f^{-1}(V)$ is non-compact for any non-empty open subset V of Y;

- an almost perfect mapping if $f(X) = Y, f$ is continuous, closed and there exists a closed compact set $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for each $x \in X \setminus \Phi$.

2.1. Method of perfect mappings

Let X be an open dense subspace of a space eX, $Y{=}eX\backslash X$ and $h : Y \to Z$ be a perfect mapping onto a space Z. We put $e_hX = Z \cup X$ and consider the mapping $f : eX \to e_hX$, where $f(x) = x$ for every $x \in X$ and $h = f\,|\,Y$. On a space e_hX we consider the quotient topology $\{W \subseteq e_hX : f^{-1}(W)$ is open in $eX\}$.

2.1.1. Property. The mapping f is perfect.

Proof. By construction, the mapping f is continuous and the fibers $f^{-1}(y)$, $y \in e_hX$, are compact. Let F be a closed subset of the space eX. Then $F_1 = h^{-1}(h(F \cap Y))$ is a closed subset of Y, $\Phi = F_1 \cup F$ is closed subset of eX and $\Phi = f^{-1}(f(F))$. Thus $f(F)$ is closed in e_hX. The proof is complete.

2.1.2. Property. If $i \in \{1, 2, 3, 4\}$ and eX is a T_i-space, then e_hX is a T_i-space. Moreover, if eX is a normal space, then e_hX is a normal space.

Proof. The property to be a T_i-space, $i \in \{1, 2, 3, 4\}$, is preserved by the perfect mappings.

2.1.3. Property. If eX and Z are T_0-spaces, then e_hX is a T_0-space.

Proof. Obvious.

2.1.4. Property. X is an open dense subspace of the space e_hX.

Proof. Obvious.

We put $LC(X) = \cup\{U : U$ is an open subset of X and cl_XU is compact$\}$ – the set of locally compactness of a space X. Let $RC(X){=}X\backslash LC(X)$. A space X is almost locally compact if the set $LC(X)$ is dense in X. If $RC(X){=}\emptyset$, then the space X is locally compact.

2.1.5. Theorem. Let eX be an extension of the almost locally compact space X, the set $LC(X)$ is open in eX, $Y{=}eX\backslash LC(X)$, $h : Y \to Z$ is a perfect mapping onto a space Z and $h^{-1}(h(x)) = \{x\}$ for every $x \in RC(X)$. Then there exist an extension e_hX of a space X and a perfect mapping $f : eX \to e_hX$ such that the set $LC(X)$ is open in e_hX.

Proof. Let $X_1 = LC(X)$. Then eX is an extension of the space X_1 and the set X_1 is open in eX. Properties 2.1.1 – 2.1.4 complete the proof.

2.2. Method of superperfect mappings

2.2.1. Theorem. If $f : X \to Y$ is an almost perfect mapping onto a T_1-space Y, then f is superperfect.

Proof. There exists a closed compact subset $\Phi \subseteq X$ such that $f^{-1}(f(x)) = \{x\}$ for every $x \in X \setminus \Phi$. Let $F = f(\Phi)$. If $y \in Y\backslash F$, then $f^{-1}(y)$ is a singleton. If $y \in F$, then $f^{-1}(y)$ is a compact set as a closed subset of the subspace Φ. Thus the fibers $f^{-1}(y)$ are compact. The proof is complete.

We say that a subset H of a space X is compact in X if the set cl_XH is compact.

A set $N(f) = \{x \in X : f^{-1}(f(x)) \neq \{x\}\}$ is called the kernel of a mapping $f : X \to Y$.

A mapping $f : X \to Y$ is almost perfect iff $f(X) = Y$, f is closed, continuous and the kernel $N(f)$ is compact in X.

2.2.2. Theorem. Let X be a subspace of a space X_1, $Y = X_1 \setminus X$, $h : Y \to Z$ be an almost perfect mapping onto a space Z and the set $cl_Y N(h)$ is closed in X. Then there exist a unique space S and an unique almost perfect mapping $f : X_1 \to S$ such that $N(f) = N(h)$ and $h = f\,|\,Y$.

Proof. We put $S = X \cup Z$, $Y_1 = cl_Y N(h)$, $X_2 = X_1 \setminus cl_Y N(h)$, $f(x) = h(x)$ for every $x \in Y$ and $f(x) = x$ for every $x \in X$. The space X_2 is open in X_1 and $g = h\,|\,Y_1 : Y_1 \to Z_1 = h(Y_1)$ is a continuous closed mapping. On S we consider the quotient topology. Obviously, $N(f) = N(h)$. By construction, f is a closed continuous mapping. The proof is complete.

2.2.3. Corollary. Let eX be an extension of a space X, $Y{=}eX\backslash X$, $h : Y \to Z$ be an almost perfect mapping onto a space Z and the set $cl_Y N(f)$ be closed in eX. Then there exist an extension e_hX of the space X and an almost perfect mapping $f : eX \to e_hX$ such that:

1. Z is a subspace of the space e_hX and $Z = e_hX \setminus X$;
2. $h = f\,|\,Y$;
3. $N(f) = N(h)$.

2.3. Method of singular mappings

Let P be a quasi-compactness.

A space X is called locally P-compact if for every point $x \in X$ there exists an open subset $U \subseteq X$ such that $x \in U$ and $cl_X U \in P$.

We say that a mapping $f : X \to Y$ is a P-singular mapping if f is continuous and $cl_X f^{-1}(V) \notin P$ for every non-empty open subset $V \subseteq X$.

Consider that the compactness P fulfills the following conditions:

S_1. If Y and Z are closed subspaces of a space X and Y, $Z \in P$, then $Y \cup Z \in P$.

S_2. If Y is a closed subspace of the space X, $Y \in P$, $Z \in P$ provided $Z \subseteq X \setminus Y$ is a closed subset of X and $X\backslash Y{=}\cup\{V : V$ is open in X and $cl_X V \subseteq X \setminus V\}$, then $X \in P$.

In the class of regular spaces Condition S1 follows from Condition S2.

2.3.1. Construction. Let $f : X \to Y$ be a P-singular mapping of a locally P-space X into a compact space $Y \in P$. Obviously that the set $f(X)$ is dense in Y. We put $eX = X \cup Y$, with the topology generated by the open base $\{U \subseteq X : U$ is open in $X\} \cup \{V \cup (f^{-1}(V) \setminus U) : V$ is open in Y, U is open in X and $cl_X \in UP\}$.

Property 1. $eX \in P$.

By construction, $Y \in P$ and $X = eX \setminus Y = \cup\{U \subseteq X : U$ is open in eX and $cl_X U \in P\}$. If U is open in X and $cl_X U$, then $cl_{eX} U = cl_X U$. Let Z be a closed subspace of eX and $Z \cap Y = \emptyset$. For every point $y \in Y$ there exist an open subset V_y of Y and an open subset U_y of X such that $cl_X U_y \in P$, $y \in V_y$

and $Z \cap (f^{-1}V \setminus cl_X U_y) = \emptyset$. Since Y is compact, there exists a finite set F such that $Y = \cup\{V_y : y \in F\}$. Then $Z \subseteq \cup\{cl_X U_y : y \in F\}$. By virtue of Condition S_1, $\cup\{cl_X U_y : y \in F\} \in P$ and $Z \in P$. Condition S_2 completes the proof.

Property 2. X is an open dense subspace of the space eX.

Obviously, X is open in eX. Let $y \in Y$, V be an open subset of Y, U be an open subset of X, $y \in V$ and $cl_X U \in P$. Then the set $W = V \cup (f^{-1}(V) \setminus cl_X U)$ is open in eX and $y \in W$. Since $cl_X f^{-1}(V) \notin P$, then $W \cap X = f^{-1}(V) \setminus cl_X U \neq \emptyset$. Thus the set X is dense in eX.

Property 3. Let $i \in \{0, 1, 2\}$ and X, Y be T_i-spaces. Then eX is a T_i-space.

Let $x, y \in eX$ and $x \neq y$.

Case 1. $x, y \in Y$ and $i \leq 1$.

If V is open in Y, $x \in V$ and $y \notin V$, then $W = V \cup f^{-1}(V)$ is open in eX, $x \in W$ and $y \notin W$.

Case 2. $x, y \in Y$ and $i = 2$.

There exist two open subsets V_1 and V_2 of Y such that $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2 = \emptyset$. The sets $W_i = V_i \cup f^{-1}(V_i)$ are open in $eX, x \in W_1, y \in W_2$ and $W_1 \cap W_2 = \emptyset$.

Case 3. $x \in X$ and $y \in Y$.

There exists an open subset U of X such that $x \in U$ and $cl_X U \in P$. We put $W = eX \setminus cl_X U = Y \cup (f^{-1}(Y) \setminus cl_X U)$. The set W is open in eX, $y \in W$ and $U \cap W = \emptyset$.

Case 4. $x, y \in X$.

Since X is an open subspace of the space eX and X is a T_i-space, the proof is complete.

Property 4. If every closed subset Z of X is compact provided $Z \in P$, then eX is a compact space.

Proof. Obvious.

Property 5. Let $\varphi : eX \to Y$ be the mapping for which $f = \varphi \,|\, X$ and $\varphi(y) = y$ for all $y \in Y$. Then φ is a continuous mapping.

Proof. If V is open in Y, then $\varphi(V \cup (f^{-1}(V) \setminus clU)) = V$. The proof is complete.

Property 6. Let X be a T_2-space and for every open subset U of X with $cl_X U \in P$ there exists an open subset W of X such that $cl_X U \subseteq W$, $cl_X W \in P$ and $cl_X W$ is a normal subspace of X. Then eX is a normal space.

Proof. Let F and Φ be two closed subsets of eX and $F \cap \Phi = \emptyset$.

Case 1. $F \subseteq Y$ and $\Phi \subseteq Y$.

There exists a continuous function $h : Y \to [0, 1]$ such that $F \subseteq h^{-1}(0)$ and $\Phi \subseteq h^{-1}(1)$. We put $g(x) = h(\varphi(x))$ for every $x \in eX$.

The function $g : eX \to [0, 1]$ is continuous $F \subseteq g^{-1}(0)$ and $\Phi \subseteq g^{-1}(1)$.

Case 2. $\Phi \cap Y = \emptyset$.

There exist the open subsets U and W of X such that $\Phi \subseteq cl_X U \subseteq W$ and $cl_X W \in P$. Then $cl_X W$ is a normal subspace of X and the set $cl_X W$ is closed in eX. There exists a continuous function $h : X \to [0, 1]$ such that $\Phi \subseteq h^{-1}(1)$ and $(F \cap X) \cup (X \setminus W) \subseteq h^{-1}(0)$. We put $g(y)$=0 for every $y \in Y$ and $g(x) = h(x)$ for every $x \in X$. The function $g : eX \to [0, 1]$ is continuous, $F \subseteq Y \subseteq g^{-1}(0)$ and $\Phi \subseteq g^{-1}(1)$.

Case 3. $F \subseteq Y$.

Let $\Phi_1 = Y \cap \Phi \neq \emptyset$. There exists a continuous function $g_1 : eX \to [-1, 1]$ such that $F \subseteq g_1^{-1}(1)$ and $\Phi_1 \subseteq g_1^{-1}(-1)$. The set $U = \{x \in eX : g_1(x) < 0\}$ is open in eX. We put $g_2(x) = \sup\{g_1(x),\ 0\}$. The function $g_2 : eX \to [0, 1]$ is continuous, $F \subseteq g_2^{-1}(1)$ and $\Phi_1 \subseteq U \subseteq g_2^{-1}(0)$. The set $\Phi_2 = \Phi \setminus U$ is closed in eX and $\Phi_2 \cap Y = \emptyset$. There exists a continuous function $g_3 : eX \to [0, 1]$ such that $F \subseteq g_3^{-1}(1)$ and $\Phi_2 \subseteq g_3^{-1}(0)$. Now we put $g(x) = g_3(x) \cdot g_2(x)$ for every $x \in eX$. The function $g : eX \to [0, 1]$ is continuous, $F \subseteq g^{-1}(1)$ and $\Phi \subseteq g^{-1}(0)$.

Case 4. $F_1 = F \cap Y \neq \emptyset$ and $\Phi_1 = \Phi \cap Y \neq \emptyset$.

There exists a continuous function $g_1 : eX \to [0, 2]$ such that $\Phi \subseteq g_1^{-1}(0)$ and $F_1 \subseteq g_1^{-1}(2)$. The set $U = \{x \in eX : g_1(x) > 1\}$ is open in eX. Let $F_2 = F \setminus U$. The set F_2 is closed in eX and $F_2 \cap Y = \emptyset$. There exists a continuous function $g_2 : eX \to [0, 1]$ such that $\Phi \subseteq g_2^{-1}(0)$ and $F_2 \subseteq g^{-1}(1)$. Now we put $g(x) = \min\{1,\ g_1(x) + g_2(x)\}$ for every $x \in eX$. The function $g : eX \to [0, 1]$ is continuous, $\Phi \subseteq g^{-1}(0)$ and $F \subseteq g^{-1}(1)$. The proof is complete.

2.3.2. Remark. In [31,32] the method of singular mappings was applied for the construction of Hausdorff compactifications of locally compact spaces.

2.4. Wallman-Shanin method

A family L of subsets of a space X is called an l-base on a space X if L is a closed base and $F \cup H,\ F \cap H \in L$ for all $F, H \in L$.

Let L be an l-base on the space X. An L-filter in the space X is a non-empty family ξ of subsets of X which satisfies the following conditions:

F_1. $\xi \subseteq L$ and $\emptyset \notin \xi$.

F_2. If $F, H \in L$, $F \subseteq H$ and $F \in \xi$, then $H \in \xi$.

F_3. If $F, H \in \xi$, then $F \cap H \in \xi$.

A maximal L-filter is called an L-ultrafilter. A filter ξ is called a free L-filter if $\cap\xi = \emptyset$.

A family L of subsets of the space X is called a net in the space X at a point $x \in X$ if for every neighbourhood U of x there exists $H \in L$ such that $x \in H \subseteq U$. A family L of subsets of X is a net in the space X if L is a net of X at each point $x \in X$ (see [9,10,11]).

For every point $x \in X$ we put $\xi_L(x) = \{F \in L : x \in F\}$.

2.4.1. Lemma. Let L be an l-base and $x \in X$. The following assertions are equivalent:

1. L is a net of the space X at the point x;
2. $\xi_L(x)$ is an L-ultrafilter.

Proof. Suppose that $\xi_L(x)$ is an L-ultrafilter. If $H \in L$ and $x \notin H$, then $H \notin \xi_L(x)$. Then $H \cap F = \emptyset$ for some $F \in \xi_L(x)$. Thus L is a net at the point $x \in X$. Consider that L is a net at the point $x \in X$, $H \in L$ and $H \notin \xi_L(x)$. Then there exists $F \in L$ such that $x \in F \subseteq X \setminus H$. Thus $\xi_L(x)$ is an L-ultrafilter. The proof is complete.

Denote $\omega_L X = \{\xi_L(x) : x \in X\} \cup \{\xi : \xi \text{ is a free } L\text{-ultrafilter}\}$. We identify the point $x \in X$ with the filter $\xi_L(x)$ and obtain $X \subseteq \omega_L X$. For every $F \in L$ we put $\langle F \rangle = \{\xi \in \omega_L X : F \in \xi\}$. Let $< L >= \{< F >: F \in L\}$.

2.4.2. Lemma. For every $H, F \in L$ we have $< H \cup F >=< H > \cup < F >$ and $< H \cap F >=< H > \cap < F >$

Proof. If $H \cup F \in \xi \in \omega_L X$, then $\xi \cap \{H, F\} \neq \emptyset$. Thus $< H \cup F >=< H > \cup < F >$. If $H \cap F = \emptyset$, then $< H > \cap < F >=< \emptyset >= \emptyset$. Let $\Phi = H \cap F \neq \emptyset$. If $\Phi \in \xi \in \omega_L X$, then $H, F \in \xi$ and $\xi \in< H > \cap < F >$. If $\xi \in< H > \cap < F >$, then $H, F \in \xi$ and $H \cap F \in \xi$. The proof is complete.

On $\omega_L X$ we consider the topology generated by a closed base $< L >$.

We say that the extension Y of a space X is an end – T_1-extension if the set $\{y\}$ is closed in Y for every point $y \in Y \setminus X$.

2.4.3. Theorem. If L is an L-base of a space X, then:

1. $\omega_L X$ is a compactification of the space X.
2. $\omega_L X \in E(X)$.
3. $\omega_L X$ is an end – T_1-extension of X.

Proof. For every $F \in L$ we have $< F > \cap X = F$ and $< F >$ is the closure of F in $\omega_L X$. By construction, $\omega_L X$ is a compact space. If $\xi \in \omega_L X$ is an L-ultrafilter, then $\{\xi\}$ is a closed subset of $\omega_L X$. The proof is complete.

2.4.4. Corollary. $\omega_L X$ is a T_1-space iff X is a T_1-space and L is a net of the space X.

2.4.5. Definition. If L is the family of all closed subsets of a space X, then $\omega X = \omega_L X$ is called the Wallman compactification of the space X.

The compactification ωX is a T_1-space iff X is a T_1-space. The compactification ωX for a T_1-space X was constructed by H. Wallman (see [122]). The T_1-compactifications of the type $\omega_L X$ were constructed by N. A. Shanin [96,98].The general case was examined in [29,133].

A compactification bX of a space X is called the compactification of the Wallman-Shanin type if there exists an l-base of X such that $bX = \omega_L X$.

In [18,100,115] it was proved that there exists a Hausdorff compactification bX of some discrete space X which is not of the Wallman-Shanin type.The papers [15,18,24,29,49,50,70,79,85,100,109,113,132,133] contained sufficient conditions provided the compactification to be of the Wallman-Shanin type.

2.5. $\omega\alpha$–compactification

Fix a space X and an l-base L of X.

2.5.1. Definition. A compactification bX of a space X is called an $\omega\alpha_L$-compactification if there exists a continuous closed mapping $f : \omega_L X \to bX$ such that $f(x) = x$ for each $x \in X$.

If L is the family of all closed subsets X, then an $\omega\alpha_L$-compactification is called an $\omega\alpha$-compactification. The $\omega\alpha$-compactifications of T_1-space were introduced and examined by P. C. Osmatescu [87].

If bX is an $\omega\alpha_L$-compactification of a space X, then the mapping $f : \omega_L X \to bX$ is a natural projection if f is continuous closed and $f(x) = x$ for every $x \in X$.

2.5.2. Proposition. Let bX be an $\omega\alpha_L$-compactification of a space X and $f : \omega_L X \to bX$ be the natural projection. Then:

1. $f(\omega_L X) = bX$;
2. $f(\omega_L X \setminus X) = bX \setminus X$;
3. bX is an end-T_1-extension of the space X;
4. $f^{-1}(x) = \{x\}$ for each $x \in X$;
5. $bX \in E(X)$;
6. the natural projection $f : \omega_L X \to bX$ is unique.

Proof. Let $(Y_1, f_1) \in GE(X)$, $(Y_2, f_2) \in WGE(X)$, $\varphi : Y_1 \to Y_2$ be a closed mapping and $f_2 = \varphi \circ f_1$. Then $(Y_2, f_2) \in GE(X)$. Thus the assertion 5 is proved.

Since f is a closed mapping and the set $f(\omega_L X)$ is dense in bX, then $bX = f(\omega_L X)$. The assertion 1 is proved.

Obviously, $bX \setminus X \subseteq f(\omega_L X \setminus X)$.

If $x \in \omega_L X \setminus X$, then the set $\{x\}$ is closed in $\omega_L X$ and the set $\{f(x)\}$ is closed in bX. Therefore the assertion 3 is proved.

Let $x \in X$, $y \in \omega_L X \setminus X$ and $f(y) = x$. There exists an L-ultrafilter ξ such that $y = \xi$ and $y \in cl_{\omega_L X} F$ for every $F = \xi$. Since f is continuous, then $x \in cl_{bX} F$ for every $F \in \xi$. There exists $H \in \xi$ such that $x \notin H$. Then $f(< H >) = cl_{bX} H$ and $cl_{bX} H \cap bX = H$, a contradiction. The assertion 4 is proved.

Let $f, g : \omega_L X \to bX$ be two continuous mappings and $f(x) = g(x)$ for all $x \in X$. Then $f(< H >) = cl_{bX} H = g(< H >)$ for each $H \in L$, $f(y) = \cap\{cl_{bX} H : H \in L, y \in < H >\}$ and $g(y) = \cap\{cl_{bX} H : H \in L, y \in < H >\}$ for every $y \in \omega_L X \setminus X$. Thus f(y)=$g(y)$ for every $y \in \omega_L X$. The proof is complete.

2.5.3. Theorem. The set $\Omega L(X)$ of all $\omega\alpha_L$-compactifications of the space X is a complete upper semi-lattice and $\omega_L X$ is the maximal element in $\Omega L(X)$.

Proof. Let $\{Y_\alpha : \alpha \in A\}$ be a non-empty subset of the set $\Omega L(X)$ and $f_\alpha : \omega_L X \to Y_\alpha$ be the natural projection of $\omega_L X$ onto Y_α. Consider the mapping $f : \omega_L X \to \Pi\{Y_\alpha : \alpha \in A\}$, where $f(y) = (f_\alpha(y) : \alpha \in A)$ for every $y \in \omega_L X$. We put $Y = f(\omega_L X)$. Then f is a continuous mapping, $f\,|\,X$ is an embedding of X into Y, Y is a compactification of X, f(x)$= x$ for each $x \in X$. For every $\alpha \in A$ there exists a projection $g_\alpha : Y \to Y_\alpha$, where $f_\alpha = g_\alpha \circ f$. Since $g_\alpha(A) = f_\alpha(f^{-1}(A))$ for each $A \subseteq Y$, the mapping g_α is closed. If $F \subseteq \omega_L X$, then $f(F) = Y \cap \Pi\{f_\alpha(F) : \alpha \in A\}$ and the mapping f is closed. Therefore $Y = \vee\{Y_\alpha : \alpha \in A\}$ and $\Omega L(X)$ is a complete upper semi-lattice. The proof is complete.

We put $a_L X = X \cup \{a\}$, where $a \notin X$ and $\{U \subseteq X : U \text{ is open in } X\} \cup \{a_L X \setminus F : F \subseteq X,\ F \text{ is closed in } \omega_L X\}$ is the open base of the space $a_L X$. The mapping $p : \omega_L X \to a_L X$, where $p^{-1}(a) = \omega_L X \setminus X$ and $f(x) = x$ for each $x \in X$, is continuous. Thus $a_L X \in WE(X)$ and $a_L X$ is a compactification of X.

2.5.4. Theorem. The following assertions are equivalent:

1. $\Omega L(X)$ is a complete lattice;
2. $a_L X$ is a minimal element of the lattice $\Omega L(X)$;
3. the set X is open in $\omega_L X$;
4. $a_L X$ is an end-T_1-extension of X.

Proof. Let $Y \in \Omega L(X)$, $y_1, y_2 \in Y \setminus X$ and $y_1 \neq y_2$. We put $Z = Y \setminus \{y_2\}$, $\varphi(y) = y$ for every $y \in Z, \varphi(y_2) = \varphi(y_1) = y_1$ and on Z consider the quotient topology. Then $\varphi : Y \to Z$ is a closed mapping, Z is a compactification of X, $Z \in \Omega L(X)$ and $Z \leq Y$. Thus the compactification $Y \in \Omega L(X)$ is not a minimal element in $\Omega L(X)$ provided $|\,Y \setminus X\,| \geq 2$.

Let Y be the minimal element in $\Omega L(X)$ and $f : \omega_L X \to Y$ be the projection. Then $Y \setminus X$ is a singleton, X is open in Y, $X = f^{-1}(X)$ is open in $\omega_L X$ and $Y = a_L X$.

If X is open in $\omega_L X$, then the mapping $p : \omega_L X \to a_L X$ is closed. The proof is complete.

2.5.5. Theorem. Let X be a locally compact space and the l-base L be a net in the space X. Then:

1. X is an open subset of $\omega_L X$.
2. $a_L X$ is an $\omega\alpha_L$-compactification of X.
3. $a_L X$ is the minimal element of the complete lattice $\Omega L(X)$.

Proof. For every point $x \in X$ there exists an open subset U_x such that $x \in U_x$ and the set $\Phi_x = cl_X U_x$ is compact. Every filter $\xi \in \omega_L X$ is an L-ultrafilter. If F is a closed subset of X, then $cl_{\omega_L X} F = \cap\{< H >: H \in L,\ F \subseteq H\}$. Fix $x \in X$. There exists $H_x \in L$ such that $x \notin H_x$ and $X \setminus U_x \subseteq H_x$. Since L is a net of X, there exists $F_x \in L$ such that $x \in F_x \subseteq U_x \cap (X \setminus H_x)$. Thus $\xi(x) \notin < H_x >$. Therefore $x \in \omega_L X \setminus < H_x >$. If $\xi \in \omega_L X \setminus X$, then there exists $H \in \xi$ such that $H \cap \Phi_x = \emptyset$. Then $H \subseteq X \setminus U_x \subseteq H_x$ and $\xi \in < H_x >$. Therefore $V_x = \omega_L X \setminus < H_x >$ is open in $\omega_L X$ and $x \in V_x \subseteq X$. The assertion 1 is proved. The Theorem 2.5.4. completes the proof.

2.5.6. Example. Let X be a non compact T_1-space. Denote by L_1 the family of all closed subsets of X. Fix $\xi \in \omega X \setminus X$. We put $L = \{F \cup H : F \in \xi,\ H \in L_1\}$. Then L is an l-base of X and $\mid \omega_L X \setminus X \mid = 1$. Thus $\Omega L(X)$ is a complete lattice and a singleton set. In this case $\omega_L X = a_L X$.

2.5.7. Corollary. The set $\Omega(X)$ of all $\omega\alpha$-compactifications of the space X is a complete upper semi-lattice with the maximal element ωX.

2.5.8. Corollary. If the space X is locally compact, then $\Omega(X)$ is a complete lattice with the maximal element ωX and minimal element aX, where $aX = a_L X$ for the l-base L of all closed subsets of X.

2.5.9. Corollary. For a T_3-space X the following assertions are equivalent:

1. X is locally compact;
2. $\Omega(X)$ is a complete lattice.

If X is a complete regular space X, then we denote by $SC(X)$ the family of all Hausdorff compactifications. In this case the Stone-Čech compactification βX is the maximal element in $SC(X)$ and $SC(X)$ is a complete subsemi-lattice of the upper semi-lattice $\Omega(X)$.

2.5.10. Corollary (N. Boboc and G. Siretchi [22]**).** For a complete regular space X the following assertions are equivalent:

1. X is locally compact;
2. $SC(X)$ is a complete lattice and sublattice of $\Omega(X)$.

If is well–know that the Stone-Čech compactification βX of a completely regular space X is a $\omega\alpha$-compactification [3] of the Wallman-Shanin type (see [3, 50, 86, 96, 109]).

From Theorem 2.1.5 it follows.

2.5.11. Corollary. Let X be an almost locally compact space, L be an l-base of X and $f : \omega_L X \setminus LC(X) \to Y$ be a continuous perfect mapping onto a T_1-space X such that $f^{-1}(x) = x$ for every $x \in X \setminus LC(X)$. Then there exists a unique $\omega\alpha_L$-compactification bX of the space X such that $bX \setminus LC(X)$ is homeomorphic to Y.

2.5.12. Corollary. Let X be a locally compact space, L be an l-base of X and Y be a T_1-space.

1. If there exists a closed mapping $f : \omega_L X \setminus X \to Y$ onto Y, then there exists a unique $\omega\alpha_L$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

2. If there exists a closed mapping $f : \omega X \setminus X \to Y$ onto Y, then there exists an $\omega\alpha$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

From Theorem 2.2.2 it follows.

2.5.13. Corollary. Let L be an l-base of a space X and Y be a T_1-space.

1. If $f : \omega_L X \setminus X \to Y$ is an almost perfect mapping onto Y and $cl_{\omega_L X} N(f) \subseteq \omega_L X \setminus X$, then there exists a unique $\omega\alpha_L$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

2. If $f : \omega X \setminus X \to Y$ is an almost perfect mapping onto Y and $cl_{\omega_L X} N(f) \subseteq \omega_L X \setminus X$, then there exists an $\omega\alpha$-compactification bX of X such that the remainder $bX \setminus X$ is homeomorphic to Y.

2.6. Spectral compactifications

Let $SE(X)$ be the set of all spectral compactifications of a space X.

A mapping $g : X \to Y$ of a space X into a space Y is a spectral mapping if g is continuous and a set $g^{-1}(U)$ is compact provided the set U is open and compact in Y.

If $Y, Z \in SE(X)$, then we consider that $Z \leq Y$ if there exists a spectral mapping $g : Y \to Z$ such that $g(x) = x$ for every $x \in X$. In this conditions $SE(X)$ is a complete upper semi-lattice with the maximal element $\beta_S X$ (see Example 1.6.7).

Let L be an l-base of a space X. The filter $\xi \subseteq L$ is a simple L-filter if $\xi \cap \{F, H\} \neq \emptyset$ provided $F \cup H \in \xi$ and $F, H \in L$. Every maximal L-filter is simple. The filter $\xi(x)$ is simple for every $x \in X$.

Denote by $s_L X$ the set of all simple L-filters. For every $H \in L$ we put $<< H >> = \{\xi \in s_L X : H \in \xi\}$. Then $<< L >> = \{<< H >> : H \in L\}$ is a closed base of the space $s_L X$. We identify $x \in X$ with $\xi(x)$. Then X is a subspace of $s_L X$, X is dense in $s_L X$ and the set $s_L X \setminus << H >>$ is open and compact in $s_L X$ for every $H \in L$. Thus $s_L X$ is a spectral compactification of X. We mention that $\omega_L X \subseteq s_L X$.

If bX is a spectral compactification of X, then $L = \{X \setminus U : U$ is an open and compact subset of $bX\}$ is an l-base of X and $bX = \omega_L X$ (see [24,25]).

In the papers [24,131,132] the class of all spectral compactifications was constructed and studied using the functional rings.

We mention that the spectrum of the simple ideals of a ring in the Zariski topology is a spectral space (see [132]).

3. Uniform extensions of topology spaces

In the present chapter every space is assumed to be a completely regular T_1-space.

A uniform space (X, U) is a set X and a family U of entourages of the diagonal $\Delta(X) = \{(x, x) : x \in X\}$ of X in $X \times X$ which satisfies the following conditions:

U_1. If $V \in U$ and $V \subseteq W$, then $W^{-1} = \{(x, y) : (y, x) \in W\} \in U$.

U_2. If $V, W \in U$, then $V \cap W \in U$.

U_3. For every $V \in U$ there exists $W \in U$ such that $2W \subseteq V$, where $2W = \{(x, y) :$ there exists $z \in X$ such that $(x, z), (z, y) \in W\}$.

U_4. $\cap U = \Delta(X)$.

Denote by $u - w(X, U)$ the weight of a uniform space (X,U). On a uniform space (X, U) we consider the topology $T(U)$, generated by the uniformity U.

Let X be a space with the topology T. We put $u-w(X) = \min\{u-w(X, U) : T(U) = T\} + \aleph_0$.

If X is discrete or metrizable, then $u - w(X) = \aleph_0$.

A pseudometric on a space X is a function $\rho : X \times X \to R$ into the reals such that $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. The pseudometric ρ is continuous if the sets $B(\rho, x, r) = \{y \in X : \rho(x, y) < r\}$, $x \in X$ and $r > 0$, are open in X.

Every uniformity is generated by a family of pseudometrics [44].

3.1. Lattice $UE(X)$

A uniform extension of a space X is a complete uniform space (eX, U) that contains X as a dense subspace.

Denote by $UE(X)$ the family of all uniform extensions of a space X.

If $(eX, U), (bX, V) \in UE(X)$, then we consider that $(e(X), U) \ge (bX, V)$ if there exists a uniformly continuous mapping $g : eX \to bX$ such that $g(x) = x$ for each $x \in X$.

3.1.1. Proposition. The set $UE(X)$ is a complete upper semi-lattice for every non-empty space X.

Proof. See Example 1.6.5

In the present chapter we consider the following two problems.

Problem 1. Let P be a property, X be a space and (Y, V) be a complete uniform space with the property P. Under which conditions there exists a uniform extension (Z, U) of X such that:

1. (Z, U) is a uniform space with the property P;
2. the uniform space (Y, V) is uniformly isomorphic to the subspace $Z \setminus X$ of (Z, U)?

Problem 2. Let X be a space and (Y, V) be a complete uniform space. Under which conditions there exists a uniform extension (Z, U) of X such that (Y, V) is uniformly isomorphic to some subspace $H \subseteq Z \setminus X$ of the space (Z, U)?

Concrete results related to the solution of the problems of this type play an important role in the study of classes of spaces and complete uniform spaces.

3.2. Discrete subspaces and uniform extensions

A subset L of a space X is strongly discrete if there exists a discrete family $\{H_x : x \in L\}$ of open subsets of X such that $L \cap H_x = \{x\}$ for every $x \in L$. For every space X we put $DS(X) = \{|L| : L$ is a strongly discrete infinite subset of $X\}$ and $d(X) = \min\{|H| : H$ is a dense subset of the space $X\}$.

If Y is a subspace of a space X, then we denote $DS(X, Y) = \{|H| : H \subseteq X \setminus Y$ and H is a strongly discrete infinite subset of $X\}$.

3.2.1. Proposition. Let Y be a subset of a space X, ρ and d be continuous pseudometrics on the space X, $r > 0$, $X_1 = \{x \in X : d(x, y) < 2r$ for some

$y \in Y\}$ and $X_2 = \{x \in X : d(x, y) < r$ for some $y \in Y\}$. Then there exists a continuous pseudometric ρ_1 on X such that $\rho_1(x, y) = \rho(x, y)$ and $\rho_1(z, u) = 0$ for all $x, y \in X \setminus X_1$ and $z, u \in X_2$. We say that ρ_1 is the (d, r)-modification of the pseudometric ρ.

Proof. Let $\rho_2 = d+\rho$. There exist a set Z, a metric ρ_3 on Z and a mapping $p : X \to Z$ such that $\rho_3(p(x), p(y)) = \rho_2(x, y)$ for every $x, y \in X$. We have $p^{-1}(p(x)) = \{y \in X : \rho_2(x, y) = 0\}$. There exists a continuous pseudometric ρ_4 on Z such that $\rho_4(p(x), p(y)) = \rho(x, y)$ for all $x, y \in X$.

Moreover, on Z there exists a continuous pseudometric d_1 such that $d_1(p(x), p(y)) = d(x, y)$ for all $x, y \in X$.

Now we put $Z_2 = \{z \in Z : d_1(z, y) < r$ for some $y \in p(Y)\}$ and $Z_1 = \{z \in Z : d_1(z, y) < 2r$ for some $y \in p(Y)\}$. By construction, $cl_Z Z_2 \subseteq Z_1$, Z_1 and Z_2 are open subsets of Z and $Z \setminus Z_1$ is closed subset of Z.

Since Z is a metric space, there exists a continuous pseudometrics ρ_5 on Z such that $\rho_5(x, y) = \rho_4(x, y)$ for all $x, y \in Z \setminus Z_1$ and $\rho_5(x, y) = 0$ for all $x, y \in Z_2$ (see [36, 101]). Obviously, $X_i = p^{-1}(Z_i)$, $i \in \{1, 2\}$. Thus, $\rho_1(x, y) = \rho_5(p(x), p(y))$ is the desired pseudometric.

3.2.2. Proposition. Let d be a continuous pseudometric on a space X, $Z \subseteq X$, ρ be a continuous pseudometric on a space Y, $r > 0$, Y_1 be a subset of Y, $f : Y_1 \to Z$ be an one-to-one mapping of Y_1 onto Z, $d(x, y) \geq 3r$ provided $x, y \in Z$ and $x \neq y$. We put $H_z = \{x \in X : d(x, z) \leq r\}$ for every $z \in Z$. Let $X_1 = X \cup Y$. Then:

1. $\{H_z : z \in Z\}$ is a discrete family of closed subsets of the space X;
2. there exists a pseudometric ρ_1 on X_1 such that:

– $\rho_1(x, y) = \rho(x, y)$ for all $x, y \in Y$;
– $\rho_1(y, f(y)) = 0$ for each $y \in Y$;
– $\rho_1(y, x) = \rho(y, f^{-1}(z)) + d(z, x)$ if $y \in Y$, $z \in Z$ and $x \in H_z$;
– $B(\rho_1, y, r) \cap X \subseteq \cup\{H_z : z \in Z\}$ for each $y \in Y$;
– for every $x \in X_1$ and $\varepsilon > 0$ the set $B(\rho_1, x, \varepsilon) \cap Y$ is open in Y and the set $B(\rho_1, x, \varepsilon) \cap X$ is open in X.

We say that ρ_1 is the (d, r)-extension of the pseudometric ρ.

Proof. There exist a metric space (Y_2, ρ_2) and a mapping $p : Y \to Y_2$ such that $\rho_2(p(y), p(z)) = \rho(y, z)$ for all $y, z \in Y$. We put $Y_3 = p(Y_1)$.

There exist a metric space (X_2, d_1) and a mapping $q : X \to X_2$ such that $d_1(q(x), q(y)) = d(x, y)$ for all $x, y \in X$. We put $Z_1 = q(Z)$. The mapping $q_1 = q\,|\,Z : Z \to Z_1$ is one-to-one. Let $g(z) = p(f^{-1}(q_1^{-1}(z)))$ for every $z \in Z_1$. By construction, $d_1(y, z) \geq 3r$ if $y,\ z \in Z_1$ and $y \neq z$.

The discrete sum $X_3 = Y_2 \oplus X_2$ is a metric space. Let $P_z = \{x \in X_2 : d_1(z, x) \leq r\}$ for every $z \in Z_1$. Then $H_z = q^{-1}(P_{q(z)})$ and $\{P_z : z \in Z_1\}$ is a discrete family of closed subsets of the space X_2.

We put $V_z = \{x \in X_2 : d_1(z, x) < r\}$, $V = \cup\{V_z : z \in Z_1\}$ and $P = \cup\{P_z : z \in Z_1\}$. On $Q = P \cup Y_2$ we consider the pseudometric ρ_3, where:

– $\rho_3(x, y) = \rho_2(x, y)$ if $x, y \in Y_2$;
– $\rho_3(z, g(z)) = \rho_3(g(z), z) = 0$ if $z \in Z_1$;
– $\rho_3(y, x) = \rho_3(x, y) = \rho_2(y, g(z)) + d_1(z, x)$ if $y \in Y_2,\ z \in Z_1$ and $x \in P_z$;
– $\rho_3(x, y) = d_1(x, y)$ if $x, y \in P_z$; for some $z \in Z_1$;
– $\rho_3(x, y) = d(x, z_1)+\rho_2(g(z_1), g(z_2))+d_1(z_2, y)$ if $z_1, z_2 \in Z_1, z_1 \neq z_2, x \in P_{z_1}, y \in P_{z_2}$.

By construction, the pseudometric ρ_3 is continuous on the closed subset Q of the metric space X_3. Thus, there exists a continuous pseudometric ρ_4 on X_3 such that $\rho_4(x, y) = \rho_3(x, y)$ for all $x, y \in Q$. Consider the mapping $\varphi : X \oplus Y \to X_3$, where $\varphi\,|\,X = q$ and $\varphi\,|\,Y = p$. Let $\rho_1(x, y) = \rho_4(\varphi(x), \varphi(y))$ for all $x, y \in X \oplus Y$. The proof is complete.

A space X is called a space of pointwise countable type if for every point $x \in X$ there exists a compact subset $\Phi(x) \ni x$ of countable character in X (see [10]).

A space X is called a space of countable type if for every compact subset $F \subseteq X$ there exists a compact subset $\Phi(F) \supseteq F$ of countable character in X (see [10,61]). M. Henriksen and J. R. Isbell [61] has proved that X is a space of countable type iff the remainder $\beta X \setminus X$ is a Lindelöf space.

3.2.3. Proposition. Let $X = Y \cup Z$, where Y is a closed subspace of the space X and every compact subset $F \subseteq Y$of a countable character in Y has a countable character in X. Then:

1. X is a first countable space iff Y and Z are the first countable spaces;
2. X is a space of pointwise countable type iff Y and Z are spaces of pointwise countable type;
3. X is a space of countable type iff Y and Z are spaces of countable type.

Proof. The assertions 1 and 2 are obvious.

Let X be a space of countable type. Then Y is a space of countable type as a closed subspace of X and Z is a space of countable type as an open subspace of space of countable type. Suppose that Y and Z are spaces of countable type. We put $Y_1 = cl_{\beta X}Y \setminus Y$ and $Z_1 = cl_{\beta X}Z \setminus Z$. By virtue of Theorem of M. Henriksen and J. R. Isbell [61], Y_1 and Z_1 are Lindelöf spaces. We affirm that $X_1 = \beta X \setminus X$ is a Lindelöf spaces. Let $\{U_\alpha : \alpha \in A\}$ be a cover of the set X_1 and U_α is open in βX for every $\alpha \in A$. Since Y_1 is a Lindelöf space and $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$, there exists a countable subset $A_1 \subseteq A$ such that $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$. Then $F = Y \setminus \cup\{U_\alpha : \alpha \in A_1\}$ is a compact subset of Y and there exists a compact subset $\Phi \subseteq Y$ such that $F \subseteq \Phi$ and Φ has countable character in X. Thus Φ is a G_δ-subset of βX and $Z_2 = Z \setminus \Phi$ is an F_δ-subset of Z_1. Thus Z_2 is a Lindelöf subspace of βX. Moreover, $Z_3 = Z_2 \setminus \cup\{U_\alpha : \alpha \in A_1\}$ is a Lindelöf subspace of βX. By construction $Z_3 \subseteq X_1 \subseteq \cup\{U_\alpha : \alpha \in A\}$. Therefore there exists a countable subset $A_2 \subseteq A$ such that $A_1 \subseteq A_2$ and $Z_3 \subseteq \cup\{U_\alpha : \alpha \in A_2\}$. So $\{U_\alpha : \alpha \in A_2\}$ is a countable cover of X_1. The M. Heriksen's and J. R. Isbell's theorem [61] completes the proof.

3.2.4. Proposition. Let $X = Y \cup Z$, Y be a closed subspace of the space X, Y_1 be a subset of Y and $\{V_n(y) : n \in N, y \in Y_1\}$ be a family of open subsets of the space X such that:

- if $y, z \in Y_1$ and $y \neq z$, then $cl_X V_1(y) \cap cl_Y V_1(z) = \emptyset$;
- if $y \in Y_1$ and $n \in N$, then $cl_X V_{n+1}(y) \subseteq \{y\} \cup V_n(y)$;
- the family $\{V_1(y) : y \in Y_1\}$ is discrete in Z and $\cup\{V_1(y) : y \in Y_1\} \subseteq Z$;
- if U is an open subset of Y, the set $U \cup \cup\{V_n(y) : y \in U \cap Y_1\}$ is open in X for every $n \in N$;
- if U is open in X and $y \in U \cap Y$, then $V_n(y) \subseteq U$ for some $n \in N$;
- $Y \cap Z = \emptyset$.

A. The following assertions are true:

1. If $V_n = \cup\{Y \cup V_n(y) : y \in Y_1\}$, then V_n is open in X and $cl_X V_{n+1} \subseteq V_n$ for every $n \in N$.

2. $Y = \cap\{V_n : n \in N\} = \cap\{cl_X V_n : n \in N\}$.

3. If the spaces Y and Z are paracompact spaces, then X is paracompact.

B. If for every open subset U of X and every point $x \in U$ there exist $n \in N$ and an open subset $V$of X such that $x \in V \subseteq V \cup \cup\{V_n(y) : y \in V \cap Y_1\} \subseteq U$, then the following assertions are true:

4. If the subspaces Y and Z are normal, then X is a normal space.

5. If Y and Z are perfectly normal spaces, then X is a perfectly normal space.

6. If a compact subset $F \subseteq Y$ has a countable character in Y, then F has a countable character in X.

7. If Y and Z are the first countable spaces, then X is first countable.

8. If Y and Z are spaces of countable (pointwise countable) types, then X is a space of countable (pointwise countable) type.

9. If Y and Z are metrizable spaces, then X is a metrizable space.

10. If Y and Z are complete metrizable spaces, then X is a complete metrizable space.

11. If Y and Z are paracompact p-spaces, then X is a paracompact p-space.

12. If Y and Z are Čech complete paracompact spaces, then X is a Čech complete paracompact space.

Proof. The assertions 1 and 2 are obvious.

Let Y and Z be paracompact spaces and ω be an open cover of the space X. There exists a sequence $\{\xi'_n = \{V'_\alpha : \alpha \in A_n\} : n \in N\}$ of open discrete families of the space Y such that for every $n \in N$ and every $\alpha \in A_n$ there is $W_\alpha \in \omega$ such that $V'_\alpha \subseteq W_\alpha$ and $Y = \cup\{V'_\alpha : \alpha \in \cup\{A_n : n \in N\}\}$. We put $V_\alpha = V'_\alpha \cup \cup\{W_\alpha \cap V_2(y) : y \in Y_1 \cap V'_\alpha\}$. Then $\xi_n = \{V_\alpha : \alpha \in A_n\}$ is a discrete family of open subsets of X.

There exists a sequence $\{\lambda_n = \{U_\beta : \beta \in B_n\} : n \in N\}$ of open discrete families of the space Z for which $Z = \cup\{U_\beta : \beta \in \cup\{B_n : n \in N\}\}$ and for every $\beta \in \cup\{B_n : n \in N\}$ there exists $W_\beta \in \omega$ such that $U_\beta \subseteq W_\beta$. For every $n,m \in N$ we put $\lambda_{nm} = \{U_\beta \setminus cl_X V_m : \beta \in B_n\}$. Then $\{\xi_n, \lambda_{nm} : n, m \in N\}$ is a σ-discrete refinement of the cover ω. Thus X is a paracompact space (see [44], Theorem 5.1.11).

Suppose that for every open subset U of X and point $x \in U$ there exist a natural number $n = n(x,U)$ and an open subset $V = V(x,U)$ of X such that $x \in V \subseteq \cup\{V \cup V_n(y) : y \in V \cap Y_1\} \subseteq U$.

Let Y and Z be normal spaces, F and Φ be closed subsets of X and $\Phi \cap F = \emptyset$.

Let $F \subseteq Y$ and $\Phi \cap Y = \emptyset$. For every $y \in F$ there exists $n = n(y) = n(y, X \setminus \Phi)$ and an open subset V_y of V such that $W_y \cap \Phi = \emptyset$, where $W_y = \cup\{V_y \cup V_n(y) : y \in V_y \cap Y_1\}$. Let $F_m = \{y \in F : n(y) \leq m\}$. Then $F = \cup\{F_m : m \in N\}$ and $F_n \subseteq F_{n+1}$ for every $n \in N$. We put $W_n = \cup\{V_y : y \in F_n\}$, $W_n^I = \cup\{W_n \cup V_n(y) : y \in W_n \cap Y_1\}$, $W_n'' = \cup\{W_n \cup V_{n+1}(y) : y \in W_n \cap Y_1\}$, $H_1 = \cup\{W_n' : n \in N\}$ and $H_2 = \cup\{W_n'' : n \in N\}$. By construction, $H_2 \subseteq H_1, cl_X H_2 \subseteq H_1 \cup Y$ and $H_1 \cap \Phi = \emptyset$. Thus $F \subseteq H_2$ and $\Phi \subseteq X \setminus cl_X H_2$.

Let $F \subseteq Y$ and $\Phi \cap Y \neq \emptyset$. There exist two open subsets H_1 and H_2 of Y such that $F \subseteq H_1, Y \cap \Phi \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_1' = \cup\{H_1 \cup V_2(y) : y \in H_1 \cap Y_1\}$, $H_2' = \cup\{H_2 \cup V_2(y) : y \in H_2 \cap Y_1\}$ and $\Phi_1 = \Phi \setminus H_2'$. Then Φ_1 is closed in X and $\Phi_1 \cap Y = \emptyset$. Thus there exist two open subsets $H_1'' \subseteq H_1'$ and H_2'' of X such that $F \subseteq H_1'', \Phi_1 \subseteq H_2''$ and $H_1'' \cap H_2'' = \emptyset$. Let $H_2''' = H_2' \cup H_2''$.Then $\Phi \subseteq H_2'''$ and $H_1'' \cap H_2''' = \emptyset$.

Suppose now that $F \cap Y = \emptyset$. Then there exist the open subsets H_1', H_1'', H_2' and H_2'' of X such that $H_1' \cap H_2' = H_1'' \cap H_2'' = \emptyset$, $F \cap Y \subseteq H_1', \Phi \subseteq H_2', F \setminus H_1' \subseteq H_1''$ and $\Phi \subseteq H_2''$. Let $H_1 = H_1' \cup H_2''$ and $H_2 = H_2' \cap H_2''$. Then $F \subseteq H_1, \Phi \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. The assertion 4 is proved. The assertion 5 follows from the assertions 1,2 and 4.

Let F be a compact subset of Y, $\{H_n : n \in N\}$ be a sequence of open subsets of Y and for every open subset $U \supseteq F$ there exists $n \in N$ such that $F \subseteq H_n \subseteq U$. We put $H_{nm} = \cup\{H_n \cup V_m(y) : y \in H_n \cap Y_1\}$. The sets H_{nm} are open in X. Let U be an open subset of X and $F \subseteq U$. There exists a finite set F' of F such that $F \subseteq \cup\{V(x,U) : x \in F'\}$. There exists n such that $F \subseteq H_n \subseteq \cup\{V(x,U) : x \in F'\}$ and $\max\{n(x,U) : x \in F'\} \leq n$. Then $F \subseteq H_{nm} \subseteq U$. Thus F has a countable character in X. The assertion 6 is proved. The assertions 7 and 8 follow from the assertion 6 and Proposition 3.2.3.

Let Y and Z be metric spaces. Fix a metric ρ_1 on a space Y. For every $y \in Y_1$ on a space $Z_y = clV_1(y)$ fix a metric d_y such that $V_n(y) \subseteq \{z \in Z_y : d_y(y,z) < 2^{-n}\}$. Now on $\cup\{Y \cup Z_y : y \in Y_1\}$ we construct the metric ρ_2 such that:

– $\rho_2(x,z) = \rho_1(x,z)$ if $x, z \in Y$;

– $\rho_2(x,z) = d_y(x,z)$ if $y \in Y_1$ and $x, z \in Z_y$;

– $\rho_2(x,z) = d_{y_1}(x,y_1) + \rho_1(y_1,y_2) + d_{y_2}(y_2,z)$ if $y_1, y_2 \in Y, y_1 \neq y_2, x \in Z_{y_1}$ and $z \in Z_{y_2}$.

Obviously, ρ_2 is a metric on a subspace $X_1 = \cup\{Y \cup Z_y : y \in Y_1\}$. Thus the paracompact space X is a union of two open metrizable subspaces $X_2 = \cup\{Y \cup V_1(y) : y \in Y_1\}$ and $X_3 = Z \setminus Y$.Thus X is a metrizable spaces. Suppose that Y and Z be complete metric spaces, In this case we consider that the metrics ρ_1 and d_y are complete. Then there exists a metric ρ on X such that:

– ρ is complete on $Z \setminus \cup\{V_2(y) : y \in Y_1\}$;

– $\rho(x,y) = \rho_2(x,y)$ if $x,y \in X_1$.

We affirm that the metric ρ is complete. Let $\{x_n : n \in N\}$ be a sequence of points and $\rho(x_n, x_m) < 2^{-n}$ provided $n \leq m$. Obviously, the sequence $\{x_n\}$ is convergent in X in the following cases:

– the set $\{n \in N : x_n \in Y\}$ is infinite;

– there exists $y \in Y_1$ such that the set $\{n \in N : x_n \in Zy\}$ is infinite;

– there exists $m \in N$ such that the set $\{n \in N : x_n \in Z \setminus \{V_m(y) : y \in Y_1\}\}$ is infinite.

Suppose that for every $n \in N$ there exists $y_n \in Y_1$ such that:

– $x_n \in Zy_n$;

– $y_n \neq y_m$ for $n \neq m$.

– In this case $\rho(y_n, y_m) < \rho(x_n, y_n) + \rho(y_n, y_m) + \rho(y_m, x_m) = \rho(x_n, x_m)$ and there exists $y \in Y$ such that $y = \lim y_n$. By construction, $y = \lim x_n$. The assertions 9 and 10 are proved.

Let Y and Z be two paracompact p-spaces. Fix a perfect mapping $\varphi : Y \to Y'$ onto a metric space Y'. Since Z is a paracompact space, we may consider that $V_n(y)$ is a co-zero set of Z for all $y \in Y_1$ and $n \in N$. Because the family $\{V_1(y) : y \in Y_1\}$ is discrete in Z, then there exists a perfect mapping $\Psi : Z \to Z'$ onto a metric space Z' such that $\Psi^{-1}(\Psi(V_n(y))) = V_n(y)$ for all $y \in Y_1$ and $n \in N$.

Let $X' = Y' \cup Z'$. Consider the mapping $g : X \to X'$, where $\varphi = g\,|Y$ and $\Psi = g\,|Z$. On X' we consider the quotient topology $\{U \subseteq X' : g^{-1}(U)$ is open in $X\}$. By construction, the mapping g is perfect. We put $Y_1' = g(Y_1) = \varphi(Y_1)$ and $V_n'(z) = \cup\{\Psi(V_n(y)) : y \in \Psi^{-1}(z)\}$ for all $z \in Y_1'$ and $n \in N$. By virtue of the assertion 9 the space X' is metrizable. If the spaces Y and Z are complete metrizable, then the space $X'$is complete metrizable. The assertions 11 and 12 are proved. The proof is complete.

3.2.5. Theorem. Let (e_1X, U_1) be a uniform extension of a space X, Y be an infinite discrete space and $|Y| \in DS(e_1X)$.Then there exists a uniform extension (eX, U) of X such that:

1. (eX, U) is a uniform extension of the space e_1X;
2. $eX = e_1X \cup Y$ and Y is a strongly discrete subset of the space *eX;*

3.$u - w(eX, U)$=$u - w(e_1X, U_1)$;

4. if e_1X is a paracompact space, then eX is a paracompact space;

5. if e_1X is a normal space, then eX is a normal space;

6. if e_1X is a collectionwise normal space, then eX is a collectionwise normal space;

7. if $e_1X$is a metacompact space, then eX is a metacompact space;

8. if e_1X is a perfectly normal space, then eX is a perfectly normal space;

9. if e_1X is Čech complete, then eX is a Čech complete space;

10. if e_1X is a p-space, then eX is a p-space;

11. the space eX has a countable base at every point $y \in Y$;

12. if e_1X is a first countable space, then eX is a first countable space;

13. if e_1X is a space of pointwise countable type, then eX is a space of pointwise countable type.

Proof. In the space e_1X we fix a discrete family $\{H_\alpha : \alpha \in A\}$ of non-empty open subsets, where $|A| = |Y|$. For every $\alpha \in A$ fix a point $b_\alpha \in X \cap H_\alpha$ and a continuous function $h_\alpha : e_1X \to [0, 1]$, where $h_\alpha(b_\alpha) = 1$ and $e_1X \setminus H_\alpha \subseteq h_\alpha^{-1}(0)$. Let $h : A \to Y$ be a mapping such that the set $h^{-1}(y)$ is countable for each $y \in Y$. We consider that $h^{-1}(y) = \{\alpha(n, y) : n \in N\}$ and $b_{\alpha(n,y)} = x(n, y)$ for every $y \in Y$ and $n \in N$.

There exists a family P of continuous pseudometrics on e_1X such that P generates the uniformity U_1 on e_1X and $\rho_1 + \rho_2 \in P$ for all $\rho_1, \rho_2 \in P$. We consider that $\rho(b_\alpha, b\mu) \geq 1$ for all $\rho \in P, \alpha, \mu \in A$ and $\alpha \neq \mu$.

Let $U(n, y) = \cup\{H_{\alpha(m,y)} \cup \{y\} : m > n\}, g_{(n,y)}(y) = 1, g_{(n,y)}(z) = 0$ and $g_{(n,y)}(x) = \Sigma\{h_{\alpha(m,y)} : m > n\}$ for every $y, z \in Y, y \neq z, x \in e_1X$ and n∈N.

We put $d_n(x, z) = \sum\{| (g_{(n,y)}(x) - g_{(n,y)}(z) | : y \in Y\}$ and $d(x, z) = \sum\{2^{-n}d_n(x, z) : n \in N\}$ for all $x, z \in eX$ and $n \in N$.

By construction:

– $d(y, z)$=2 if $y, z \in Y$ and $y \neq z$;

– $d(y, x(n, y))$=2^{-n} for every $y \in Y$ and $n \in N$;

– $B(d, y, 2^{-n-1}) \subseteq U(n, y)$ for every $y \in Y$ and $n \in N$;

– $B(d, x, r) \cap e_1X$ is open in e_1X for every $x \in eX$ and r¿0.

On eX we consider the topology generated by the open base $\{B(d, y, 2^{-n})$:$y \in Y, n \in N\}$∪$\{U \subseteq e_1X$:U is open in $e_1X\}$. At every point $y \in Y$ the space eX has a countable base and the set Y is strongly discrete in eX. From these properties of a space eX it follows the assertions 4-10 and 2. Obviously, $cl_{eX}\{b_\alpha : \alpha \in A\} \supseteq Y$. Therefore the sets X and e_1X are dense in eX.

Fix a pseudometric $\rho \in P$ and $n \in N$. Let $X_n = \cup\{B(d, y, 2^{-n}) : y \in Y\}$ and $Y_n = Y \cup (\{\beta_\alpha : \alpha \in A\} \cap X_n)$. By virtue of Proposition 3.2.1, on eX , there exists a continuous pseudometric $e_n(\rho)$ such that $e_n(\rho)$(x,y)= ρ(x,y), if $x, y \in eX \setminus X_n$, and $e_n(\rho)$(x,y)=0, if $x, y \in X_{n+2}$. Then $P_1 = \{\alpha + e_n(\rho) : \rho \in P, n \in N\}$ is a family of continous pseudometrics on eX which generates the topology of the space eX and some uniformity U on eX.

We affirm that the uniform space (eX, U) is complete, i.e.(eX, U) is a uniform extension of the spaces e_1X and X.

Let ξ be a Cauchy filter of closed subsets of the space eX.

Case 1. $Y \in \xi$.

Fix $\rho_1 = d + e_1(\rho) \in P_1$. There exists $\Phi \in \xi$ such that $\rho(x,y) < 1$ for all $x, y \in \Phi$ and $\Phi \subseteq Y$. Let $z \in \Phi$. If y∈Y and $y \neq z$, then $\rho_1(z,y) \geq d(z,y) = 2$. Thus Φ is a singleton set and $\cap\xi = \Phi \neq \emptyset$.

Case 2. $Y \notin \xi$.

There exists $\Phi \in \xi$ such that $\Phi \cap Y = \emptyset$. We may consider that $d(x,y) < 2^{-4}$ for all $x, y \in \Phi$. Let $Y_0 = \{y \in Y : d(y, \Phi) < 2^{-4}\}$. If $Y_0 = \emptyset$, then $X_4 \cap \Phi = \emptyset$, $F = X \setminus X_4$ is a closed subset of eX, $\Phi \subseteq F$ and $F \in \xi$. In this case ξ is a Cauchy filter of the uniform space (e_1X, U_1) and $\cap\xi \neq \emptyset$.

Suppose that $Y_0 \neq \emptyset$. Then the set Y_0 is a singleton set. If $y_1, y_2 \in Y_0$, then there exists $x_1, x_2 \in \Phi$ such that $d(y_1, x_1) < 2^{-4}$ and $d(y_2, x_2) \in \Phi$. Then $d(y_1, y_2) \leq d(y_1, x_1) + d(x_1, x_2) + d(x_2, y_2) < 3 \cdot 2^{-4} < 2^{-2} < 2$ and $y_1 = y_2$. Suppose that $Y_0 = \{y_0\}$. Since the set Φ is closed in eX and $y_0 \neq \Phi$, then there exists $n > 4$ such that $B(d, y_0, 2^{-n}) \cap \Phi = \emptyset$. In this case $X_n \cap \Phi = \emptyset$, $eX \setminus X_n \in \xi$ and ξ is a Cauchy filter of the uniform space (e_1X, U_1). Therefore $\cap\xi \neq \emptyset$. The proof is complete.

3.2.6. Theorem. Let (e_1X, U_1) be a uniform extension of a space X, (Y, ν) be a complete uniform space and $d(Y, \nu) \in DS(e_1X, U_1)$. Then there exists a uniform extension (eX, U) of X such that:

1.(eX, U) is a uniform extension of the space $(e_1X, T(U_1))$; 2. (Y, ν) is uniformly isomorphic to the subspace $eX \setminus e_1X$ of (eX, U); 3. $u - w(eX, U) \leq u - w(e_1X, U_1) + u - w(Y, \nu)$; 4. if e_1X and Y are paracompact Čech complete spaces, then eX is a paracompact Čech complete space; 5. if e_1X and Y are paracompact p-spaces, then eX is a paracompact p-space; 6. if e_1X and Y are paracompact spaces, then eX is a paracompact space; 7. if e_1X and Y are normal spaces, then X is a normal space; 8.if e_1X and Y are perfectly normal spaces, then X is a perfectly normal space; 9. if e_1X and Y are first countable spaces, then X is a first countable space; 10. if e_1X and Y are spaces of pointwise countable type, then X is a space of pointwise countable type; 11. if e_1X and Y are spaces of countable type, then X is a space of countable type; 12. if e_1X and Y are metrizable spaces, then X is a metrizable space.

Proof. Let Y_1 be a dense subset of the space Y and $|Y_1| = d(Y)$. In the proof of Proposition 3.2.1 there were constructed a uniform extension (e_2X, U_2) of a space e_1X, a continuous pseudometric d on e_2X and a family P_1 of continuous pseudometrics on e_2X such that:

– $Z = e_2X \setminus e_1X$ is a strongly discrete subspace of the space e_2X ;

– $|Z| = |Y_1|$, i.e. there exists a one-to-one correspondence $h : Y_1 \to Z$;

– if $x, y \in Z$ and $x \neq y$, then $d(x,y) = 3$ and $\rho(x,y) = 0$ for every $\rho \in P_1$;

– the space e_2X has a countable base at every point $y \in Z$;

– $\{B(d,y,2^{-n}) : n \in N\}$ is a base of the space e_2X at a point $y \in Z$;

– if $\rho \in P_1$, then there exists n∈N such that $\rho(x,z) = 0$ for all $x, z \in \cup\{B(d,y,2^{-n}) : y \in Z\}$;

– the family of pseudometrics $P_2 = \{d+\rho : \rho \in P_1\}$ generates the uniformity U_2 on a space e_2X;

– for every $n \in N$ the family P_1 generates on $e_1X \setminus \cup\{B(d,y,2^{-n}) : y \in Y_1\}$ the uniformity U_1.

Suppose that the uniformity V of the space Y is generated by the family P_3 of continuous pseudometrics.

We put $eX = e_1X \cup Y$.

For every $\rho \in P_3$ let $e(\rho)$ be the $(d,1)$-extension of the pseudometrics ρ on eX. We identify the point $y \in Y_1$ with the point $h(y) \in Z$.

Fix $\rho \in P_1$ and $n \in N$. Let ρ' be the $(d,2^{-n})$-modification of the pseudometric ρ on e_2X. We fix $z_0 \in Z$ and put $e_n(\rho)(x,y) = \rho'(x,y)$ if $x, y \in e_1X, e_n(\rho) = 0$ if $x, y \in Y$ and $e_n(\rho)(x,y) = e_n(\rho)(y,x) = \rho'(z_0,x)$ if $y \in Y$ and $x \in e_1X$. Now we put $P = \{e(\rho_1)+e_n(\rho_2) : \rho_1 \in P_3, \rho_2 \in P_1$ and $n \in N\}$. The pseudometrics P generates the uniformity U on eX.

Obviously, (Y,V) is a uniform subspace of the space (eX,U), e_1X is a dense subspace of the space eX.

Let ξ be a Cauchy filter of a space (X,U). The filter ξ is convergent in X in the following cases:

– $\Phi \subseteq Y$ for some $\Phi \in \xi$;

– $\Phi \subseteq e_1X \setminus \cup\{B(d,y,2^{-n}) : y \in Y_1\}$ for some n∈N and $\Phi \in \xi$;

– there exist $n \in N$, $y \in Y$ and $\Phi \in \xi$ such that $\Phi \subseteq B(d,y,2^{-n})$.

Suppose that for every $n \in N$ and $\Phi \in \xi$ the set $n(\Phi) = cl_X\{y \in Y_1 : \Phi \cap B(d,y,2^{-n}) \neq \emptyset\}$ is non-empty. Then $\eta = \{n(\Phi) : \Phi \in \xi, n \in N\}$ is a Cauchy filter of the space (Y,V). If $y \in \cap\eta$, then $y \in \cap\xi$. Therefore (X,U) is a complete space and a complete extension of Y and e_1X. Proposition 3.2.4 completes the proof.

3.2.7. Problems. Let P be a topological property and Y and e_1X be two spaces with the property P. Is it true that eX has the property P in the following cases:

a) P is the property to be a metacompact space;

b) P is the property to be a p-space;

c) P is the property to be a Čech complete space;

d) P is the property to be a space with a G_δ-diagonal;

e) P is the property to be a symmetrizable space.

3.3. The gluing operation and σ-discretness

For every point x of a space X we put $DS(x,\,X) = \cap\{DS(cl_XH) : H$ is open in X and $x \in H\}$, $\tau - ds(x,\,X) = \{\sup A : A \subseteq DS(x,\,X)$ and $|A| \leq \tau\}$ and $ds(x,\,X) = \aleph_0 - ds(x,\,X)$.

If X is a metric space or a space with a σ-discrete net, then $\sup DS(x, X) \in ds(x, X)$.

For every subset A of a space X, by A^d we denote the derived set of A, i. e. the set of accumulation points of A.

3.3.1. Definition ((see [20]) for metric spaces**).** Let (X_1, U_1) be a uniform space, let A be a non-empty subset of the set X_1^d and $\{(Y_x, V_x) : x \in A\}$ be a family of complete uniform spaces. A uniform extension (Y, V) of the space $X = X_1 \setminus A$ is obtained by gluing the space (Y_x, V_x) at the point x for every $x \in A$ if the following conditions are satisfied:

G_1. $Y = X \cup \cup\{Y_x : x \in A\}$;

G_2. (Y_x, V_x) is a uniform subspace of the space (Y, V) for each $x \in A$;

G_3. the subspace X is dense in Y;

G_4. the natural mapping $f : Y \to X_1$, where $f(x) = x$ for every $x \in X$ and $f^{-1}(x) = Y_x$ for each $x \in A$, is continuous;

G_5. if $x, y \in A$ and $x \neq y$, then $Y_x \cap Y_y = \emptyset$.

3.3.2. Definition. Let (X_1, U_1) be a uniform space, A be a non-empty subset of X_1^d and a uniform extension (Y,V) the space $X = X_1 \setminus A$ is obtained by gluing of the space (Y_x, V_x) at the point x for every $x \in A$. The gluing is strongly at the point $x_0 \in A$ if for every open subset H of Y , that contains Y_{x_0}, there exists an open subset U of X_1 such that $x_0 \in U$ and $Y_x \subseteq H$ for each $x \in A \cap U$.

A mapping $f : X \to Y$ is called:

- a closed mapping at a point $y \in Y$ if $f^{-1}(y) \neq \emptyset$ and for every open subset $U \subseteq X$, that contains $f^{-1}(y)$, there exists an open subset V of Y such that $y \in V$ and $f^{-1}(V) \subseteq U$;

- a perfect mapping of a point $y \in Y$ if $f^{-1}(y)$ is a compact subset and f is closed at y.

Gluing is strongly at a point $x \in A$ iff the natural mapping $p : Y \to X_1$ is closed at a point x.

From the E. Michal's theorem [78] it follows.

3.3.3. Corollary. Let (X_1, U_1) be a space of pointwise countable type, $A \subseteq X_1^d$, $X = X_1 \setminus A$ and the uniform extension (Y, V) of X is obtained by gluing the spaces $\{(Y_x, V_x) : x \in A\}$ at the points of A.

The following assertions are equivalent:

1. the natural mapping $p : Y \to X_1$ is perfect; 2. gluing is strongly at each point $x \in A$.

Let ρ be a continuous pseudometric on a space X. There exist a metric space $(X/\rho, \hat{\rho})$ and a natural projection $\pi_\rho : X \to X/\rho$, where $\pi_\rho^{-1}(\pi_\rho(x)) = \{y \in X : \rho(x, y) = 0\}$ and $\rho(x, y) = \widehat{\rho}(\pi_\rho(x), \pi_\rho(y))$ for all $x, y \in X$. The natural projection is continuous. Denote by $\overline{X/\rho}$ the completion of a metric space $(X/\rho, \hat{\rho})$.

3.3.4. Definition. Let ρ be a continuous pseudometric on a space X. The pseudometric ρ is a metric on a subset $A \subseteq X$ if for every point $x \in A$ and every open subset $U \subseteq X$ that contains x there exists $\varepsilon > 0$ such that $B(x, \rho, \varepsilon) \subseteq U$.

3.3.5. Proposition. Let (X_1, U_1) be a complete uniform space, $A \subseteq X_1^d$, $X = X_1 \setminus A$, a continuous pseudometric d on X_1 is a metric on the set A, $\{(Y_x, V_x) : x \in A\}$ is a family of complete uniform spaces and the uniform extension (Z, W) of the space $Z_1 = \overline{X_1/d} \setminus \pi_d(A)$ is obtained by gluing the space (Y_x, V_x) at a point $\pi_\rho(x)$ for every $x \in A$. Then:

1.there exists a uniform extension (Y, V) of a space X which is a gluing of the spaces (Y_x, V_x) at the points $x \in A$; 2. if $x \in A$ and the gluing of (Y_x, V_x) is strongly in Z, then the gluing of (Y_x, V_x) is strongly in X_1, too; 3. (Y, V) is uniformly isomorphic to some closed subspace of the Cartesian product of the spaces (X_1, U_1) and (Z, W).

Proof. Consider the projection $\pi_d : X_1 \to X_1/d$ and the set $A_1 = \pi_d(A)$. Let (Z, W) be the uniform extension of the space $Z' = \overline{X_1/d} \setminus A_1$ obtained by gluing each space (Y_x, V_x) at a point $\pi_d(x)$, $x \in A$. By definition, $Z = Z' \cup \cup\{Y_x : x \in A\}$.

We put $Y = X \cup \cup\{Y_x : x \in A\}$. Consider the mappings $p : Y \to X_1$ and $q : Y \to Z$, where:

- $p(x) = x$ and $q(x) = \pi_d(x)$ for each $x \in X$;
- $p^{-1}(x) = Y_x$ for each $x \in A$;
- $q(y) = y$ for every $y \in Y_x$ and $x \in A$.

Now we consider the mapping $\varphi : Y \to X_1 \times Z$, where $\varphi(y) = (p(y), q(y))$ for every $y \in Y$. By construction, $\varphi(y) \neq \varphi(z)$ provided $y, z \in Y$ and $y \neq z$. We identify $y \in Y$ with $\varphi(y)$ and consider $Y = \varphi(Y)$ as a uniform subspace of the uniform space $X_1 \times Z$. Since $\varphi(Y_x) = Y_x$, for every $x \in A$, (Y_x, V_x) is a uniform subspace of the space Y.

Since $p\,|\,X : X \to X_1$ is an embedding and the mapping $q\,|\,X \to Z$ is continuous, the space $X = \varphi(X)$ is a subspace of the space Y.

Let $(x, z) \in X_1 \times Z$ and $(x, z) \notin Y = \varphi(Y)$.

<u>Case 1</u>. $x \in X$.

In this case $x \in Y$ and $\pi_d(x) = \varphi(x) \neq z$. There exist two open subsets H_1 and H_2 of Z such that $\pi_d(x) \in H_1,\ z \in H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_3 = \pi_d^{-1}(H_1)$ and $H = H_3 \times H_2$. Then $(x,\ z) \in H$ and $H \cap \varphi(Y) = \emptyset$.

<u>Case 2</u>. $x \in A$.

In this case $z \notin Y_x$ and $z \neq \pi_d(x)$. If $r : Z \to \overline{X_1/d}$ is the natural projection, then $r(z) \neq \pi_d(x)$ and there exist two open subsets H_1 and H_2 of Z such that $\pi_d(x) \in H_1,\ r(z) \in H_2$ and $H_1 \cap H_2 = \emptyset$. Let $H_3 = \pi_d^{-1}(H_1)$, $H_4 = r^{-1}(H_2)$ and $H = H_3 \times H_4$. Then $(x, z) \in H$ and $H \cap \varphi(Y) = \emptyset$. Therefore $\varphi(Y)$ is a closed subset of the space $X_1 \times Z$.

Obviously, that the set X is dense in Y. The proof is complete.

3.3.6. Proposition. Let $(X_1,\, d)$ be a complete metric space, X be a dense subset of X, $X_1 \setminus X = L$, let $\{(Y_x,\, V_x) : x \in L\}$ be a family of complete uniform spaces, $L = \cup\{L_n : n \in N\}$, where L_n is a closed discrete subset of X_1 for each $n \in N$, $L_n \cap L_m = \emptyset$ for $n \neq m$, $\{H_x : x \in L\}$ be a family of open subsets of X_1, $\{L_n x : x \in L,\, n \in N\}$ be a family of closed discrete subsets of X_1 such that:

A$_1$. for every $n \in N$ the set $M_n = L_n \cup \cup\{L_n x : x \in L_n\}$ is closed in X_1;

A$_2$. for every $n \in N$ the family $\{H_x : x \in L\}$ is discrete in X_1;

A$_3$. if $m,\, n \in N$, $m \neq n$ and $x \in L$, then $x \in H_x \setminus L_m x$, $L_m x \cap L_n x = \emptyset$ and $L_m x \subseteq H_x$;

A$_4$. if $x \in L$ and $x_n \in L_n x$ for $n \in N$, then $d(x,\, x_n) < 2^{-n}$;

A$_5$. if $m,\, n \in N$, $n < m$ and $H_m = \cup\{H_x : x \in L_m\}$, then $M_n \cap clH_m = \emptyset$;

A$_6$. If $x \in L$, then $|L_n x| \leq |L_{n+1} x|$ for each $n \in N$, $\tau(x) = \sup\{L_m x : m \in N\}$ is an infinite cardinal and $d(Y_x) = \tau(x)$.

Then there exists a uniform extension $(Y,\, V)$ of the space X such that:

1. $Y = X \cup \cup\{Y_x : x \in L\}$ and $(Y,\, V)$ is a gluing of the spaces $(Y_x,\, V_x)$ at the points $x \in L$; 2. $u - w(Y,\, V) = \sup\{u - w(Y_x,\, V_x) : x \in L\}$; 3. if $x \in L$ and $x_n \in L_n x$, then the sequence $\{x_n : n \in N\}$ is convergent to some point of Y_x; 4. $Y_x \subseteq cl_Y(\cup\{L_n x : n \in N\})$ for every $x \in L$; 5. if $x \in L$ and $y \in Y_x$, then $\chi(y,\, Y) = \chi(y,\, Y_x)$; 6. if $\{Y_x : x \in L\}$ are paracompact p-spaces, then Y is a paracompact p-space; 7. if $\{Y_x : x \in L\}$ are Čech complete paracompact spaces, then Y is a Čech complete paracompact space.

Proof. Fix $x \in L$. There exists a set A_x of cardinality $\tau(x)$. Assume that $A_x = \cup\{A_n x : n \in N\}$, where:

- if $n \leq m$, then $A_n x \subseteq A_m x$;
- $|A_n x| = |L_n x|$ for each $n \in N$.

We may suppose that $L_n x = \{x_{n\alpha} : \alpha \in A_n x\}$ and $A_x \cap A_y = \emptyset$ for $x \neq y$. Let $\{y_\alpha : \alpha \in Ax\}$ be a dense subset of the space Y_x.

Let $Y = X \cup \cup\{Y_x : x \in L\}$, $L'_n = \cup\{L_i : i \leq n\}$ and $Y_n = (X_1 \setminus L'_n) \cup \cup\{Y_x : x \in L'_n\}$ for every $n \in N$. If $n,\, m \in N$ and $n < m$, then consider the natural projection $p_{(m,\,n)} : Y_m \to Y_n$, where:

– $p_{(m,\,n)}(x) = x$ if $x \in X$;

– if $x \in L'_n$ and $y \in Y_x$, then $p_{(n,m)}(y) = y$;

– $p^{-1}_{(m,\,n)}(x) = Y_x$ if $x \in L'_m \setminus L'_n$.

For every $n \in N$ there exist the projections $p_{(\omega,\,n)} : Y \to Y_n,\ p : Y \to X_1$ and $p_n : Y_n \to X_1$ such that:

– $p_{(\omega,\,n)}(x) = p(x) = p_n(x) = x$ if $x \in X$;

– if $x \in L'_n$ and $y \in Y_x$, then $p_n(y) = p(y) = x$ and $p_{(\omega,\,n)}(y) = y$;

– if $x \in L \setminus L'_n$, then $p_n(x) = x$ and $p^{-1}_{(\omega,\,n)}(x) = Y_x$.

We assume that $L_1 = \emptyset$ and $Y_1 = X_1$.

Fix $n \in N$, where $n \geq 2$. Let $X'_n = X_1 \setminus L'_n \setminus$. On a set $X_n = X'_n \cup \cup\{A_x : x \in L'_n\}$ there exists a complete metric d_n such that:

– the metrizable space (X'_n, d) is a subspace of the space (X_n, d_n);

– if $x \in L'_n$ and $\alpha \in A_x$, then $\lim d_n(d, x_{m\alpha}) = 0$ and $d_n(d, x_{m\alpha}) < 1$ for every $m \in N$;

– if $\alpha, \beta \in \cup\{A_x : x \in L'_n\}$ and $\alpha \neq \beta$, then $d_n(\alpha, \beta) \geq 3$.

By virtue of Theorem 3.2.5 there exists a uniform structure V_n on Y_n such that:

1. (Y_n, V_n) is a uniform extension of the space X'_n and $Y_n \setminus X'_n$ is uniformly isomorphic to the discrete sum of the uniform spaces $\{(Y_x, V_x) : x \in L'_n\}$; 2. $u - w(Y_n, V_n) = \sup\{u - w(Y_x, V_x) : x \in L'_n\}$; 3. if $x \in L'_n$ and $\alpha \in A_x$, then $y_\alpha = \lim x_{m\alpha}$; 4. if $\{(Y_x, V_x) : x \in L\}$ are paracompact spaces, then Y_n is a paracompact space; 5. if $\{Y_x : x \in L\}$ are p-spaces, then Y_n is a p-space; 6. if $\{Y_x : x \in L\}$ are Čech complete spaces, then Y_n is a Čech complete space.

Consider the mapping $\varphi : Y \to \Pi\{Y_n : n \in N\}$, where $\varphi(y) = (p_{(\omega, n)}(y) : n \in N)$ for every $y \in Y$. The set $\varphi(Y)$ is closed in $\Pi\{Y_n : n \in N\}$. We identify Y with $\varphi(Y)$ and consider (Y, V) as a closed subspace of $\Pi\{(Y_n, V_n) : n \in N\}$. The proof is complete.

3.3.7. Theorem. Let (e_1X, U_1) be a first-countable uniform extension of a space X, let $L_n \subset eX \setminus X$ be a strongly discrete subset of the space e_1X, let $L = \cup\{L_n : n \in N\}$, $\{(Y_x, V_x) : x \in L\}$ be a family of complete uniform spaces and let $d\{(Y_x, V_x) \in ds(x, X)$ for every $x \in L$. If (e_1X, U_1) is a Baire space, then there exists a uniform extension (eX, U) and a uniformly continuous mapping $g : eX \to e_1X$ such that:

1. $g(x) = x$ for every $x \in e_1X \setminus L$; 2. for every $x \in L$ the space (Y_x, V_x) is uniformly isomorphic to the subspace $g^{-1}(x)$ of (eX, U); 3. $u - w(eX, U) \leq u - w(e_1X, U_1) + \sup\{u - w(Y_x, V_x) : x \in L\}$; 4. if $\{e_1X_x : x \in L\}$ are Čech complete spaces, then eX is a paracompact Čech complete space; 5. if $(e_1X_x, Y_x) : x \in L$ are paracompact p-spaces, then eX is a paracompact p-space; 6. $\chi(y, eX) = \chi(y, Y_x)$ for every $y \in Y_x$ and $x \in L$.

Proof. For every $x \in L$ we fix a sequence $\{\tau_n(x) \in DS(x, X) : n \in N\}$ such that:

– $\tau_n(x) \leq \tau_{n+1}(x)$ for every $n \in N$;

– $d(Y_x, V_x) = \sup\{\tau_n(x) : n \in N\}$.

Let $\tau(x) = \sup\{\tau_n(x) : n \in N\}$, A_x be a set of cardinality $\tau(x)$, A_nx be a subset of A_x of cardinality $\tau_n(x)$ and $A_nx \subseteq A_{n+1}x$ for every $n \in N$.

Since $\{L_n : n \in N\}$ are strongly discrete sets of the first countable Baire space e_1X, then there exist a family $\{H_x : x \in L\}$ of open subsets of e_1X and a family $\{L_nx : x \in L, n \in N\}$ of strongly discrete sets of the space e_1X such that:

– $\{H_x : x \in L_n\}$ is a discrete family of e_1X for every $n \in N$;

– $\{x\} \cup \cup\{L_nx : n \in N\} \subseteq H_x$ for every $x \in L$;

– the set $M_n = \cup\{\{x\} \cup L_m x : x \in \cup\{L_i : i \leq n\},\, m \in N\}$ is closed in e_1X for every $n \in N$;

– if $n, m \in N,\ n < m$ and $H_m = \cup\{H_x : x \in L_m\}$, then $M_n \cap clH_m = \emptyset$;

– if $n, m \in N,\ n < m$ and $x \in L$, then $L_n x \cap L_m x = \emptyset$;

– if $x \in L$ and $x_n \in L_n x$ for every $n \in N$, then $x = \lim x_n$;

– $|L_n x| = \tau_n(x)$ for every $x \in L$ and $n \in N$.

For every $n, m \in N$ we put $L_{nm} = \cup\{L_m x : x \in L_n\}$. Then L_{nm} is a strongly discrete subset of the space e_1X.

Since e_1X is a first countable space and $\{L_n, L_{nm} : n, m \in N\}$ are strongly discrete subsets of e_1X, then there exists a continuous pseudometric d on e_1X such that d is a metric on the set $A = \cup\{L_n \cup L_{nm} : n, m \in N\}$. Consider the projection $\pi_d : e_1X \to e_1X$. Let (X_1, d_1) be the completion of the metric space $(e_1X/d, \hat{d})$. We put $L_n = \pi_d(L_n)$, $x = \pi_d(x)$ and $L_n x = \pi_d(L_n x)$ for all $n \in N$ and $x \in L$. The Propositions 3.3.6 and 3.3.5 complete the proof.

3.4. Ultrauniform spaces

An entourage U of the diagonal $\Delta(X)$ of a space X is called discrete if $U = U^{-1} = 2U$.

3.4.1. Definition. A uniform space (X, U) is said to be ultrauniform if there exists a base B of uniformity U such that:

– every entourage $U \in B$ is discrete;

– the base B is linearly ordered, i. e. if $U, V \in B$, then $U \subseteq V$ or $V \subseteq U$.

The completion of an ultrauniform space is ultrauniform.

3.4.2. Proposition (S. Nedev and M. Choban [137]**).** Every ultrauniform space is hereditarily paracompact.

Proof. Let B be a linearly ordered base of the uniformity U on a space X and every entourage from B be discrete. For every $x \in X$ and $V \in B$ we put $a(x, V) = \{y \in X : (x, y) \in V\}$. Since V is a discrete entourage, the family $\xi(V) = \{a(x, V) : x \in X\}$ is a discrete cover of the space X. If $U, V \in B$ and $U \subseteq V$, then $a(x, U) \subseteq a(X, V)$ for every $x \in X$. Thus $\{a(x, V) : V \in B, x \in X\}$ is a base of rank one of the space X. Every space with a base of rank one is hereditarily paracompact [8]. A family L of subsets of X has rank one if for every two sets $A, B \in L$ we have $A \subseteq B$, or $B \subseteq A$, or $A \cap B = \emptyset$. The proof is complete.

3.4.3. Lemma. Let (X, U) be an ultrauniform space and $\tau = u{-}w(X, U)$. Then:

1. $\dim X = 0$; 2. if $\{H_\alpha : \alpha \in A\}$ is a family of open subsets of X and $|A| < \tau$, then $\cap\{H_\alpha : \alpha \in A\}$ is open in X, i. e. X is a P_τ-space; 3. τ is a regular cardinal.

Proof is obvious.

3.4.4. Theorem. Let (e_1X, U_1) be an ultrauniform extension of a space X, (Y, V) be a complete ultrauniform space and τ be an infinite regular cardinal which satisfies the following conditions:

– $d(Y, V) \in DS(e_1X, U)$ and $\tau \in DS(e_1X, U)$;

– Y and e_1X are P_τ-spaces;

– the uniform space (Y, V) is discrete or $u - w(Y, V) = \tau$.

Then there exists a uniform extension (eX, U) of the spaces X and e_1X such that:

1. (eX, U) is an ultrauniform space; 2. the space (Y, V) is uniformly isomorphic to the subspace $eX \setminus e_1X$ of (eX, U); 3. $u - w(eX, U) = \tau$.

Proof. Since $\tau + d(Y, V) \in DS(e_1X, U)$, then there exists a family $\{M_\alpha : \alpha \in A\}$ of subsets of the space X such that:

– $|A| = d(Y, V)$ and $|M_\alpha| = \tau$ for each $a \in A$;

– if $\alpha, \beta \in A$ and $\alpha \neq \beta$, then $M_\alpha \cap M_\beta = \emptyset$;

– the set $M = \cup\{M_\alpha : \alpha \in A\}$ is strongly discrete in e_1X.

We may assume that $M_\alpha = \{x_{\alpha\beta} : \beta < \tau\}$.

Since e_1X is a P_τ-space, then either e_1X is a discrete space, or $u-w(e_1X, U_1) = \tau$. Therefore there exists a family $\{\gamma_\beta = \{H_{\beta\lambda} : \lambda \in \Gamma_\beta\} : \beta < \tau\}$ of open discrete covers of the space e_1X such that:

– $\{H_\beta = \cup\{H_{\beta\lambda} \times H_{\beta\lambda} : \lambda \in \Gamma_\beta\} : \beta < \tau\}$ is a base of some complete uniformity U_2 on eX and $U_1 \subseteq U_2$;

– if $\beta < \xi < \tau$, then γ_ξ is a refinement of γ_β, i. e. $H_\xi \subseteq H_\beta$;

– (e_1X, U_2) is a complete ultrauniform space;

– if $\beta < \tau$ and $\lambda \in \Gamma_\beta$, then $|H_{\beta\lambda} \cap M| \leq 1$.

Since $|A| = d(Y, V)$, we may fix a dense subset $Y_1 = \{y_\alpha : \alpha \in A\}$ of the space Y. The uniform space (Y, V) is either discrete or $u - w(Y, V) = \tau$. Therefore there exists a family $\{\omega_\beta = \{V_{\beta\mu} : \mu \in Q_\beta\} : \beta < \tau\}$ of open discrete covers of the space Y such that:

– $B = \{V_\beta = \cup\{V_{\beta\mu} \times V_{\beta\mu} : \mu \in Q_\beta\} : \beta < \tau\}$ is a base of the uniformity V on Y;

– if $\beta < \xi < \tau$, then ω_ξ is a refinement of ω_β, i. e. $V_\xi \subseteq H_\beta$.

For every $\alpha \in A$ and $\beta < \tau$ we put $M_{\alpha\beta} = \{x_{\alpha\xi} : \beta \leq \xi < \tau\}$.

For every $\beta < \tau$ and $\mu \in Q_\beta$ we put $W_{\beta\mu} = V_{\beta\mu} \cup \cup\{H_{\beta\lambda} \in \gamma_\beta : y_\alpha \in V_{\beta\mu}$ and $M_{\alpha\beta} \cap H_{\beta\lambda} \neq \emptyset$ for some $\alpha \in A\}$, $\bar{\gamma}_\beta = \{H_{\beta\lambda} : \lambda \in \Gamma_\beta\} \cup \{W_{\beta\mu} : \mu \in Q_\beta\}$ and $W_\beta = \cup\{H \times H : H \in \bar{\gamma}_\beta\}$. Let U be the uniformity on $eX = e_1X \cup Y$ generated by the base $\{W_\beta : \beta < \tau\}$. The uniform extension (eX, U) is desired. The proof is complete.

3.5. Extensions of locally compact spaces

In this section every space is assumed to be a completely regular T_2-space.

3.5.1. Proposition. Let bY be a Hausdorff compactification of a non-empty space Y. Then there exists a pseudocompact space X such that:

1. $Y = \beta X \setminus X$ and $bY = cl_{bX}Y$; 2. the space $X_1 = \beta X\setminus\ bY{=}X\ \setminus\ bY$ is a countable compact locally compact dense subspace of the space X and $\beta X_1 = \beta X$; 3. $\dim X = \dim bY$; 4. if $g : bY \to Z$ is a continuous mapping onto a compact space Z and $g^{-1}(g(y)) = y$ for every $y \in bY \setminus Y$, then there exists a compactification bX of X such that $g(Y){=}bX\ \setminus X$ and $Z = cl_{bX}g(Y)$.

Proof. Let ω_1 be the first uncountable ordinal number and W be the space of all ordinal numbers $\leq \omega_1$ in the topology generated by the linear order on W. We put $X = (W \times bY) \setminus (\{\omega_1\} \times Y)$ and $X_1 = (W \setminus \{\omega_1\}) \times bY$. Then $\beta X_1 = \beta X = W \times bY$.

Let $g : bY \to Z$ be a continuous mapping onto a compact space and $g^{-1}(g(y)) = y$ for every $y \in bY \setminus Y$. We put $bX = X_1 \cup Z$. Consider the mapping $\varphi : \beta X \to bX$, where $g = \varphi\,|\,bY$ and $\varphi(x) = x$ for every $x \in X_1$. Then $\varphi(x) = x$ for every $x \in X$. On bX consider the quotient topology. The proof is complete.

A space is called a continuum if it is a connected compact Hausdorff space. A space X is an arcwise connected or pathwise connected space if for every pair of points $a, b \in X$ there exists a continuous mapping $f : I \to X$ of the interval $I =[0, 1]$ into X such that $f(0){=}\ a$ and $f(1){=}\ b$. A space$X$ is locally arcwise connected if the family $\{U \subseteq X : U$ is an open arcwise connected subspace of $X\}$ is an open base of X (see [44]).

A space is called a Peano continuum if it is a locally arcwise connected continuum.

3.5.2. Definition. A space X is said to be a marginal arcwise connected space if there exist a cardinal τ, an embedding of X into I^m and a sequence of arcwise connected subspaces $\{X_n : n \in N\}$ of I^τ such that:

– $X = \cap\{X_n : n \in N\}$;

– for every open subset U of I^τ, which contains the closure clX of X in I^τ, there exists $n \in N$ such that $\cup\{X_i : i \geq n\} \subseteq U$.

3.5.3. Examples.

1. The Tychonoff cube I^m is a Peano continuum. 2. If a continuum X is a G_δ-subset of a Peano continuum Y, then X is a marginal arcwise connected space. 3. Every metrizable continuum is a marginal arcwise connected space.

The set $S(f) = Y \setminus \cup\{U : U$ is open in Y and the set $cl_X f^{-1}U$ is compact$\}$ is called the singularity set of the mapping $f : X \to Y$ of a space X into a space Y. If X is a locally compact space, then $S(\varphi) = \cap\{cl_Y f(X \setminus F) : F$ is a compact subset of $X\}$ (see [35]).

3.5.4. Proposition. Let $f : X \to Y$ be a continuous mapping of a locally compact non-compact space X into a compact space Y. Then:

1. $S(f) \neq \setminus$; 2. there exists a compactification bX of the space X such that the spaces $bX\ \setminus X$ and $S(f)$ are homeomorphic.

Proof. As in Section 2.3 we consider the compact space $Z = X \cup Y$ with the topology generated by the open base $\{U \subseteq X : U$ is open in $X\} \cup \{f^{-1}(V) \setminus F :$

V is open in Y and F is a compact subset of $X\}$. Then $S(f) = cl_Z X \setminus X$ and $bX = cl_Z X$. The proof is complete.

3.5.5.Theorem. Let X be a locally compact non-pseudocompact space and Y be a separable marginal arcwise connected space X. Then:

1. there exists a Hausdorff compactification bY of Y such that bY is a remainder of some Hausdorff compactification bX of X.

2. there exist a compact space Z, an embedding of Y in Z and a continuous mapping $\varphi : X \to Z$ such that $S(\varphi) = cl_Z Y$.

Proof. There exist a cardinal τ, an embedding of Y into I^τ and a sequence of arcwise connected subspaces $\{Y_n : n \in N\}$ of I^τ such that:

– $Y = \cap\{Y_n : n \in N\}$;

– for every open subset U of I^τ, which contains the closure clY of Y in I^τ, there exists $n \in N$ such that $\cup\{Y_i : i \geq n\} \subseteq U$.

We put $Z = I^\tau$ and $bY{=}cl_Z Y$. Fix a countable dense subset $B=\{b_n : n \in N\}$ of the space Y. Fix a point $b_0 \in Y$. For every $n \in N$ we fix a continuous mapping $g_n : I \to Y_n$ such that $g_n(0) = b_0$ and $g_n(1){=}b_n$.

Since X is non-pseudocompact, there exists a subset $A=\{a_n \in X{:}n \in N\}$ and a continuous function $f : X \to R$ such that $f(a_{n+1}) \geq 3 + f(a_n)$ for every $n \in N$. For every $n \in N$ we fix an open subset U_n of X and a continuous function $f_n : X \to I$ such that $a_n \in U_n, f(a_n) = 1, X \setminus U_n \subseteq f_n^{-1}$ and the set $cl_X U_n$ is compact.

Now we construct the mapping $\varphi : X \to Z$, where:

– $\varphi(x) = b_0$ if $x \in X \setminus \cup\{U_n : n \in N\}$;

– if $n \in N$ and $x \in U_n$, then $\varphi(x) = g_n(f_n(x))$.

Since the family $\{U_n : n \in N\}$ is discrete and $\varphi(x) = g_n(f_n(x))$ for every $n \in N$ and $x \in cl_X U_n$, the mapping φ is continuous.

Let $H_n = Z \setminus cl_Z Y_n$. For every $n \in N$ there exists $k = k(n) \in N$ such that $Y_i \cap H_n = \emptyset$ for every $i \geq k$.

Then $\varphi^{-1}(H_n) \subseteq \cup\{U_i : i \leq k\}$ and the set $cl_X \varphi^{-1}(H_n)$ is compact.

Since $bY = Z \setminus \cup\{H_n : n \in N\}$, we have $S(\varphi) \subseteq bY$. If U is open in Z and $U \cap bY \neq \emptyset$, then the set $N(U) = \{n \in N : b_n \in U\}$ is infinite. If $n \in N(U)$, then $a_n \in \varphi^{-1}(U)$. Therefore the set $\varphi^{-1}(U)$ is not compact and $bY = S(\varphi)$.

The assertion 2 is proved. The Construction 2.3.1 completes the proof.

3.5.6. Proposition. Let X be a locally compact non-pseudocompact space and bY be a compactification of a separable arcwise connected space Y. Then there exists a continuous mapping $\varphi : X \to bY$ such that $S(\varphi) = bY$, i.e. the mapping φ is singular.

Proof. As in the proof of Theorem 3.5.5. we consider that $Y \subseteq bY \subseteq I^\tau$ for some cardinal τ and put $Y_n = Y$ for each $n \in N$. The proof is complete.

3.5.7. Corollary. Let X be a locally compact non-pseudocompact space and K be a marginal arcwise connected compact space. Then K is a remainder of some Hausdorff compactification bX of X.

3.5.8. Corollary. (see [74], Theorem 3, when X is connected). Let X be a locally compact non-pseudocompact space and Y be a space which contains a dense separable and arcwise connected subspace. Then every compactification bY of $Y$is a remainder of X.

3.5.9. Corollary. (see [73] for Y metrizable). Let X be a locally compact non-pseudocompact space and Y be a separable Peano continuum. Then Y is a remainder of X.

3.5.10. Corollary. ([94], J.V.Rogers and [1], J.M.Aarts and P.van Emde Boas, for metrizable separable X). Let Y be metrizable continuum and let X be a locally compact non-pseudocompact space. Then Y is a remainder of X.

3.5.11. Theorem. Let X be a paracompact locally compact space. If the space X is not compact, then:

1. if $\tau \in DS(X)$, the cardinal τ is uncountable, Y is a compact space and $d(Y) \leq \tau$, then Y is a remainder of some compactification bX of the space X;
2. if *dimX=0* and Y is a remainder of some compactification of the discrete space D_m of the cardinality $m \leq \tau$,then Y is a remainder of some compactifion bX of the space X.

Proof. The space X can be represented as the union of a family $\{X_\alpha : \alpha \in A\}$ of disjoint closed-and-open subspaces of X each of which has the Lindelöf property ([44], Theorem 5.1.27).

There exist a locally compact metric space Z and a perfect mapping $\varphi : X \to Z$ of X onto Z such that $X_\alpha = \varphi^{-1}(\varphi(X_\alpha))$ for each $\alpha \in A$. Then $Z_\alpha = \varphi(X_\alpha)$ is an open-and-closed subset of Z for every $\alpha \in A$. If the set A is infinite and $\tau \in DS(X)$, then $\tau \leq |A|$. If the set A is countable, then the space X is Lindelöf and every closed discrete subspace of X is finite or countable .

<u>Case 1</u>. $\alpha_0 \in A$ and the subspace X_{α_0} is not compact.

In this case Z_{α_0} is a locally compact non compact space with a countable base.

Let Y be a metrizable connected compact space. By virtue of Aarts and Emde Boas theorem [1] (see Theorem 3.5.5) there exists a compactification bZ_{α_0} of the space Z_{α_0} such that $Y = bZ_{\alpha_0} \setminus Z_{\alpha_0}$. Then there exists a compactification $bX_{\alpha_0}$of the space X_{α_0} such that $Y = bX_{\alpha_0} \setminus X_{\alpha_0}$. Fix $y_0 \in Y$ and put $bX = X \cup Y$. On bX consider the following topology:

– the space X is an open subspace of the space bX;

– $bX_{\alpha_0} \setminus \{y_0\}$ is an open subspace of the space bX;

– if U is an open subset of bX and $y_0 \in U$, then $F = X \setminus (U \cup X_{\alpha_0})$ is a compact subset of X ;

– if V is an open subset of bX_{α_0}, $y_0 \in V$ and F is a compact subset of X, then $V \cup \cup\{X_\alpha \setminus F : \alpha \in A \setminus \{\alpha_0\}\}$ is open in bX.

In this conditions bX is a Hausdorff compactification of X and Y=$bX\setminus X$.

Case 2. $\tau = |A|$ is an infinite cardinal. For every $a \in A$ fix a point $a_\alpha \in X_\alpha$.

Let Y be a compact space and $d(Y) \leq \tau$. Let Y_1 be a dense subset of Y and $|Y_1| \leq \tau$. There exists a continuous mapping $\varphi : X \to Y$ into Y such that for every $y \in Y_1$ the set $\{\alpha \in A : \varphi(a_\alpha) = y\}$ is infinite. The continuous mapping φ is singular, i.e. the set $cl_X \varphi^{-1}(V)$ is not compact provided the set V is open and non-empty. By virtue of Construction 2.3.1 there exists a compactification bX of X such that $Y{=}bX \setminus X$.

Case 3. dimX=0.

We may assume that the set A is infinite and X_α is a compact subset of X for each $\alpha \in A$. Let Y be a compact space, D_m be a infinite discrete space, $m \leq \tau$, bD_m be a compactification of D_m and $Y{=}bD_m \setminus D_m$. We may assume that $m = \tau$. Consider that $D_m = D_\tau = \{d_\alpha : \alpha \in A\}$. Then there exists a mapping $\Psi : X \to D_\tau$ such that $\Psi^{-1}(d_\alpha) = X_\alpha$ for every $\alpha \in A$. The mapping Ψ is open and perfect. There exists a continuous extension $g : \beta X \to bD_\tau$ of the mapping Ψ. By construction, $g(\beta X \setminus X) = Y$. By virtue of Theorem 2.1.5 there exists a compactification bX of X such that $Y = bX \setminus X$. The proof is complete.

References

[1] Aarts J. M., Van Emde Boas P., Continua as remainders in compact extensions, New Arch. Wiskunde. Ser. 3, **15**,1 (1967), 34-37.

[2] Alexandroff P. S., Urysohn P. S., Mémoire sur les expaces topologiques compacts, Vern. Akad. Wetensch. Amsterdam **14**, 1929.

[3] Alexandroff P. S., On bicompact extensions of topological spaces, Matem. Sb. **5** (1939), 403-423.

[4] Alexandroff P. S., Ponomarev V. I., On bicompact extensions of topological spaces, Vestnik Moscow. Univ., **5** (1959), 93-108.

[5] Alo R., Shapiro H., Normal bases and compactifications, Math. Ann. **175** (1968), 337-340.

[6] Arens R., Extension of functions on fully normal spaces, Pacific Journ. Math. **2** (1952), 11-22.

[7] Arens R., Dugundji, Remark on the concept of compactness, Portugalial Math. **9** (1950), 141-143.

[8] Arhangelskii A. V., Filippov V. V., Spaces with bases of finite rank, Math. USSR Sb. **16** (1972), 147-158.

[9] Arhangelskii A. V., An addition theorem for the weight of sets lying in bicompacta, Dokl. Akad. Nauk SSSR, **126** (1959), 239-241.

[10] Arhangelskii A. V., Bicompact sets and the topology of spaces, Trans. Moscow Math. Soc. **13** (1965), 1-62.

[11] Arhangelskii A. V., Mappings and spaces, Russian Math. Surveys, **21**,4 (1966), 115-162.

[12] Arhangelskii A. V., On monolitic compactifications, Quet. and Ansuer in General Topology, **14** (1996), 251-258.

[13] Attia H., Choban M. M., On the Baire-category method in dimension theory, Serdica, **17** (1991), 255-263.

[14] Attia H., Choban M. M., On the dimension of remainder of extensions of normal spaces, Topology and its Appl. **36** (1990), 97-109.

[15] Baayen P. C., Van Mill J., Compactifications of locally compact spaces with zero-dimensional remainder, General Topology and Appl. **9** (1978), 125-129.

[16] Banaschewski B., Extensions of topological spaces, Canad. Math. Bull., **7**, 1(1964), 1-22.

[17] Banaschewski B., On Wallman's method of compactification, Math. Nachr. **27** (1963), 105-114.

[18] Bandt C., On Wallman-Shanin compactifications, Math. Naehr. **77** (1977), 332-351.

[19] Bhaumic R. N., Misra D. N., A generalization of K-compact spaces, Czechoslovak. Math. Journ. U:4 (1971), 625-632.

[20] V. Bel'nov, Some theorems on metrical extensions, Trudy Mosk. Matem. Ob-va, **30** (1974),115-138.

[21] Biles Ch. M., Wallman-type compactifications, Proc. Amer. Math. Soc., **25**, 2 (1970), 363-369.

[22] N. Boboc, Gh. Sireţchi, Sur la compactification d'un espace topologique, Bull. Math. Soc. Sci. R. P. Roumaine, **5** (53), 1961, 23-31.

[23] Bludova I. V., Nordo G., Pasynkov B. A., On the homeomorphism of spaces and Magil-type theorem, Question and Answers in General Topology, **19**, 1 (2001), 95-105.

[24] Calmuţchi L., On Zariski compactifications, Simpozionul al VI-lea Tiraspolean de Topologie generală şi aplicaţiile ei, Chişinău, 1991, 43-44.

[25] Calmuţchi L., Stone algebras and extensions of topological spaces, Izv. Acad. Nauk R.M. Matematika **3** (2000), 53-66.

[26] Calmuţchi L., On coreflections of category of T_0-spaces, Simpozionul al VI-lea Tiraspolean de Topologie generală şi aplicaţiile ei, Chişinău, 1991, 44-45.

[27] Calmuţchi L., Remainders of some spaces, Buletinul A.S.M, Matematica, **2** (2001), 92-96.

[28] Calmuţchi L., Stonean algebras and extension of topological spaces, Învăţământul universitar din Moldova la 70 ani, Materialele conferinţei ştiinţifico-metodice, Chişinău, 9-10 octombrie (2000), p. 21-23.

[29] Calmuţchi L. L., Cioban M. M., On Wallman-Shanin compatification of T_0-spaces, Anal. Univ., Oradea Fasc. Matem. **6** (1997/98), 130-141.

[30] Caratheodory C., Über die Begrenzung einfach zuzammenhander Gebiete, Math. Ann. **73** (1913), 323-370.

[31] Caterino A., Faulkner G. D., Vipera M. C., Construction of compactifications using essential semilattice homomorphisms, Proc. Amer. Math. Soc. **116** (1992), 85-860.

[32] Caterino A., Vipera M. C., Singular compactifications and compactification lattices, Rend. Circ. Matem. Palermo (2), Suppl. **24** (1990), 299-309.

[33] Čech E., On bicompact spaces, Ann. of Math. **38** (1937), 823-844.

[34] Chandler R. E., Hausdorff compactifications, Dekker, New York, 1976.

[35] Chandler R. R., Tzuny Fu-Chien, Remainders in Hausdorff compactifications, Proc. Amer. Math. Soc., **70**, 2 (1978), 196-202.

[36] Choban M., Nedev S. "Paracompact extensions and orderability of spaces, Buletinul Acad. de Ştiinţe a Republicii Moldova, Matematica, **3** (1996), 43-51.

[37] Choban M., Closed maps, Encyclopedia of General Topology, Elsevier Science Publ., 2004,89-91.

[38] Choban M. M., Nedev S., Strong P-embeddings, C. R. Acad. Bulg. **49**,1 (1966), 9-12.

[39] Choban M. M., Pavel D., Almost periodicity and compactifications of topological spaces, Acta et Commentationes, Analele Univ. de Stat din Tiraspol, **3** (2003), 118-135.

[40] Choban M. M., Perfect mappings defined on spaces of countable type, Vestnik Moscow, Univ., **6** (1967), 87-93.

[41] Cioban M. M., Attia H., On the dimension of remainders of extensions of normal spaces, Topology and its Applications **36** (1990), 97-109.

[42] Daw A., Watson S., Universal T_0-spaces, Dept. of Math., York University, Canada, Report no **89-24**, 1989.

[43] Engelking R., Mrowka S., On E-compact spaces, Bull. Acad. Polon. Sci. Ser. Math. **6** (1958), 429-436.

[44] Engelking R., General topology, Warszawa, 1977.

[45] Engelking R., Sur la compactification des espaces metriques, Fund. Math. **48** (1960), 321-324.

[46] Fedorchuk V. V., On ordered spaces, Soviet Math. Dokl, **7** (1966), 1011-1014.

[47] Fine N. S., Gillman L., Extensions of continuous functions on βN, Bull. Amer. Math. Soc, **86** (1960), 276-301.

[48] Fine N. S., Gillman L., Remote points in βE, Proc. AMS **13** (1962), 29-36.

[49] Frendenthal H., Kompaktifizierungen and Bikompaktifizierungen, Indag. Math. **13** (1951), 184-192.

[50] Frink O., Compactifications and semi-normal spaces, Amer. Journ. Math. **86** (1964), 602-607.

[51] Frolik Z., Generalizations of compact and Lindelöf spaces, Czechoslovak Math. Journ. **9** (1959),172-217.

[52] Gelfand I. M., Kolmogoroff A. N., On rings of continuous functions on topological spaces, Doklady Acad. Nauk SSSR, **22** (1939),11-15.

[53] Gelfand I. M., Raikov D., Shilov G., Commutative normed rings, Fizmatgiz, Moskva, 1960.

[54] Gilman L., Gerison M., Rings of continuous functions, Princeton, New York, 1960, (Springer, Berlin, 1997).

[55] Gryzlov A. A., On compactifications of discrete spaces, Fundam. Priklad. Matem. **2**,3 (1996),803-848.

[56] Hager A., Jonson D., A note on certain subalgebras of $C(X)$, Canad. Journ. Math. **20** (1968),389-393.

[57] Hager A., On inverse-closed subalgebras of $C(X)$, Proc. London Math. Soc. (3), **19** (1969), 233-257.

[58] Harris D., Semirings and T_1-compactifications, Trans. AMS, **188** (1974), 241-258.

[59] Hart K. P., The Stone-Čech compactifications of the real line, Recent Progress in General Topology, Elsevier Science Publ., North-Holland, Amsterdam, 1992, 317-352.

[60] Hausdorff F., Erweiterung einer stetigen Abbildung, Fund. Math. **30** (1938), 40-47.

[61] Henriksen M., Isbell S. R., Some properties of compactifications, Duke Math. Journ. **25** (1958), 83-106.

[62] Herrlich H., Fortsetzbarkeit stetiger Abbildungen und Kompaktheitsgrad topologischer Räume, Math. Z. , **96** (1967), 64-72.

[63] Herrlich H., Van der Slot J., Properties which are closely related to compactness, Proc. Koninul. Nederl. Acad. Wet. A **70**, 5 (1967), 524-529.

[64] Hewitf E., Rings of real-valued continuous functions, I, Trans. AMS **64**, 1948, 45-99.

[65] Hindmann N., Strauss D., Algebra in the Stone-Čech compactification, Berlin, 1948.

[66] Hofmann K. H., Lawson J. D., The spectral theory of distributive continuous lattices, Trans. AMS **246** (1978), 285-310.

[67] Hušek M., J. de Vries, Preservation of products by functors closed by reflections, Topology Appl. **27** (1987), 171-189.

[68] Kaufman R., Ordered sets and compact spaces, Colloq. Math. **17** (1967), 35-39.

[69] Kunen K., On the compactifications of the integers, Notices AMS **17** (1970), 299, Abstract 70T-G7.

[70] Lowen L., A Wallman-Shanin-type compactifications for approach spaces, Rocky Mountain J. Math. **30**, 4 (2000), 1381-1419.

[71] Magill K. D., Countable compactifications, Canad. J. Math. **18** (1966), 616-620.

[72] Magiil K. D., On the remainders of certain metric spaces, Trans. AMS **160** (1971), 411-417.

[73] Magill K. D., A note on compactifications, Math. Zeitschr, **94** (1966), 322-325.

[74] Magill K. D., More on remainders of spaces in compactifications, Bull. Acad. Polon. Sci., Ser. Math. **18** (1970),449-451.

[75] Magill K. D., N-point compactifications, Amer. Math. Monthly, **72** (1965), 1075-1081.

[76] Magill K. D., The lattice of compactifications of a locally compact space, Proc. London Math. Soc. **18** (1968),231-244.

[77] Matyushichev K., On e-compactifications and e-compactifiable spaces, Siberian Math. Journ. **41**, 4 (2000), 737-741.

[78] Michael E., A note on closed maps and compact sets, Israel J. Math. **2** (1964), 173-176.

[79] Morita K., On bicompactification of semibicompact spaces, Sci. Rep. Tokyo Bunrika Daigaku, Sect A, **4** (1952), 200-207.

[80] Morita K., On bicompactifications of semicompact spaces, Sci. Rep. Tokyo, Bunrika Daigaku, **4** (1952), 222-229.

[81] Mrowka S., Compactness and product spaces, Colloq. Math. **7** (1959-60), 19-22.

[82] Myškis A. D., On the concept of boundary. The definition of a boundary by means of continuous mappings. On the equivalence of certain methods of definition of a boundary, Trans. AMS, Series 1, **8** (1962), 11-61.

[83] Nachbin L., On the continuity of positive linear transformations, Proc. Internat. Congress of Math., Cambridge, Mass, 1950, vol. **1**, Providence (1952), 464-465.

[84] Nielsen R., Sloyer C., Ideals of semicontinuous functions and compactifications of T_1-spaces, Math. Ann. 1970, **187**, 4, 329-331.

[85] Njastad O., Hereditary Wallman spaces, Math. Japonika, **27**, 3 (1982), 327-333.

[86] Njastad O., On Wallman-type compactifications, Math. Zeitschr. **91** (1966), 267-276.

[87] Osmatescu P.C.,, $\omega\alpha$-extensions, Vestnic Moscow. Univ., Ser.Matem., **6** (1963), 45-54.

[88] Osmatescu P.C.,, One-point T_1-bicompactifications , Bull. Pol. Acad. Sci., Math., **34**,11-12 (1986), 707-712.

[89] Parovicenco I.I., On supreme of families of H-clased extensions, Soviet Math. Dokl. **11** (1970),1114-1118.

[90] Porter J. R. and Woods R. G., Extensions and absolutes of Hausdorff Spaces, Springer, Berlin, 1987.

[91] Rayburn M. C., On Hausdorff compactifications, Pacific J. Math. **44** (1973), 707-714.

[92] Riemann B., Gesammelte mathematische Werke, Leipzig, 1892.

[93] Robinson S. M., Some properties of $\beta X \backslash X$ for complete spaces, Fund. Math. **64** (1968),335-340.

[94] Rogers J. W., On compactification with continua as remainders, Fund. Math. **70** (1971), 7-11.

[95] Rudin W., Homogeneity problems in the theory of Čech compactifications, Duke Math. Journ. **23** (1956), 404-419.

[96] Shanin N. A., On special extensions of topological spaces, Doklady Akad. Nauk SSSR, **38** (1943), 6-9.

[97] Shanin N. A., On the product of topological spaces, Trudy Matem. Inst. Steklova Akad. Nauk SSSR, **24**, 1948.

[98] Shanin N. A., On the theory of bicompact extensions of topological spaces, Doklady Akad. Nauk SSSR, **38** (1943), 154-156.

[99] Shanin N. A., On separation in topological spaces, DAN SSSR, **38** (1943), 110-113.

[100] Shapiro L. B., Reduction of the main problem on compactifications of Wallman type, Soviet Math. Dokl. **15** (1974),1020-1023.

[101] Shapiro L. B., Three examples in the theory of compactifications of topological spaces, Soviet. Math. Dokl. **15** (1974),1140-1143.

[102] Simon B., Some pictorial compactifications of the real line, Amer. Math. Mathly, **76** (1969), 536-538.

[103] Skljarenko E. G., Some questions in the theory of bicompactifications, Izv. AN SSSR **26** (1962), 427-452, AMS Transl. Ser. **58** (1966), 216-244.

[104] Smirnov Yu. M., On the dimension of the remainders of bicompact extensions of proximity spaces and topological spaces, AMS Trans. 2, **84** (1969), 197-251, I. Matem. Sb. **69** (1966), 141-160, II. Matem. Sb. **71** (1968),454-482.

[105] Steiner A. K., Steiner E. F., Compactifications as closure of graphs, Fund. Math. **63** (1968),221-223.

[106] Steiner A. K., Steiner E. F., Wallman and z-compactifications, Duke Math. Journ. **35** (1968),269-276.

[107] Steiner A. K., Steiner E. F., Relative types of points in $\beta N \backslash N$, Trans. AMS **160** (1971), 279-286.

[108] Steiner E. F., Normal families and completely regular spaces, Duke Math. Journ. **33**, 4 (1966), 743-745.

[109] Steiner E. F., Wallman spaces and compactifications, Fund. Math. **61** (1968),295-304.

[110] Stone M. H., Application of the theory of Boolean rings to general topology, Trans. AMS, **41** (1937), 375-481.

[111] Suciu I., Function algebras, Bucureşti, Ed. Acad. R.S. România, 1973.

[112] Sultan A., Measure, compactification and representation, Canad. Journ. Math. **30**, 1 (1978), 54-65.

[113] Thrivikraman T., A net characterization of the Wallman-type compactifications, Proc. National Acad. Sciences, India, **50** (A), II (1980), 113-118.

[114] Thrivikraman T., On the lattice of compactifications, J. London Math. Soc. **4** (1972), 711-717.

[115] Uljanov V. M., Solution of a basic problem on compactification of Wallman type, Soviet Math. Dokl. **18** (1977), 567-571.

[116] Verbeek A., Superextentions of topological spaces, Math. Centre Tracts, Amsterdam, **41**, IV (1972), 155p.

[117] Vietoris L., Stetige Mengen, Monatsh. für Math und Phys. **31** (1921), 173-204.

[118] Viglino G., C-compact spaces, Duke Math. Journ. **36**, 4 (1969), 761-764.

[119] Vries de H., Compact spaces and compactification an algebraic approach, Assen, the Netherlands, 1962.

[120] Wajch E., Compactifications with finite remainders, Comment. Math. Univer. Carolinae, **29**, 3 (1988), 559-565.

[121] Walker R. C., The Stone-Čech compactification, Springer, New York, 1974.

[122] Wallman W. H., Lattices and topological spaces, Ann. of Math. **39**, 2 (1938), 112-126.

[123] Wine J. D., More paracompactifications using filter bases, Glasnik Matem. **25** (1990), 371-380.

[124] Wyler O., Filter space, monads, regularity, completions, Lect. Notes Math., **378**, Springer, Berlin, (1974),591-637.

[125] Zariski O., Samuel P., Commutative algebra, vol. 1. Van Nostrand, Princeton 1958; vol. 2. Van Nostrand, Princeton, 1960.

[126] Attia, H., Choban, M.M., On the dimension of the Stone-Čech remainders, Buletinul Acad. de Ştiinţe a RSS Moldova, **2** (1990), 43-49. (In Russian).

[127] Belnov, V.K., Metrical extensions, I:Vestnik Moskovovskogo Universiteta, **4** (1970), 60-65; II:Vestnik Moskovovskogo Universiteta, **5** (1970), 7–14. (In Russian).

[128] Zaitsev, V.I., On the theory of Tychonov spaces. Vestnik Moskovovskogo Universiteta, **3** (1967), 48–57. (In Russian).

[129] Ivanov, A.A., Spatial structures: the theory and applications, Zapiski Nauchnyh Seminarov LOMI, **287** (2002), 228. (In Russian).

[130] Ivanov, A.A., Extension structures, Issledovania po topologii: Zapiski Nauchnyh Seminarov LOMI, **36** (1973), 50-125. (In Russian).

[131] Calmuţchi, L.I., Function semirings and the charachterization of the T_1-spaces, Doklady Akademii Nauk Belorusskoi SSR, **30**, 11 (1986), 972-974. (In Russian).

[132] Calmuţchi L.I. The spectra of semirings and bicompactifications of T_0-spaces, Bulletin Academy Sciences of the Georgian SSR, 134:1, 1989, 45-48. (In Russian).

[133] Calmuţchi, L.I., On the Wallman extensions of T_0-spaces, Issledovania po obshchei algebre, geometrii i ih prilogeniah, Kishinev, Ştiinţa, 1986, 77-85. (In Russian).

[134] Calmuţchi, L.I., Proximal relations and regular extensions of T_0-spaces, Issledovania po obshchei algebre, geometrii i ih prilogeniah, Kishinev, Ştiinţa, 1986, 85-95. (In Russian).

[135] Calmuţchi, L.I., Choban, M.M., On the Nagata theorem on lattice of semicontinuous functions, Sibirskii Matem. Journ. **30**, 2 (1989), 185-191. (In Russian).

[136] Karmazin, A.P., Quasiisometries, theory of preends and metric structures of spatial domains, Surgut, 2003, 210 p. (In Russian).

[137] Nedev, S.I., Choban, M.M., Non-Arhimedian metrics and the spaces of Kurepa-Fréchet, Annal. de l'Univer. de Sofia, Mathematica, **67** (1976), 234-247. (In Russian).

[138] Smirnov, Yu.M., On the dimension of remainders of the proximity and the topological spaces, Doklady AN SSSR, **168**,3 (1969), 528-531. (In Russian).

[139] Suvorov, G.D., Metric theory of preends and boundary properties of representations with bounded Dirichlet integrals, Naukova Dumka, Kiev, 1981. (In Russian).

[140] Shostak, A.P., The E-compact extensions of the topological spaces, Funct. Analiz i ego Prilojenia, **8**, 1 (1974), 62-68. (In Russian).