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STABILITY SURFACES IN A CONVECTION PROBLEM FOR A MICROPOLAR FLUID. I. THEORETICAL RESULTS

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Abstract The simplest direct method and two direct and one variational methods based upon Fourier series expansions to solve eigenvalue problems involving ordinary differential equations of high order and the coefficients of which depend on several parameter are presented. The two main steps, namely the determination of the characteristic and secular equation, are shown. A concrete example is worked out.

1. INTRODUCTION

The linear stability theory problems involve mostly high order ordinary differential equations with constant coefficients and some homogeneous boundary conditions. To solve these problems several methods have been developed; the most known are the direct method and the methods based on Fourier series expansions [1]. An example of application of the Fourier series expansions methods is worked out. In Section 2 we describe the simplest direct method, in Section 3 we give details for the application of the methods based on series. For a concrete example, in Section 4 we give theoretical results obtained by applying three methods.

2. THE DIRECT METHOD

The direct method is one of the most frequently used method to solve eigenvalue problems and the simplest. The general solution of the equations of the problems is written as a linear combination (with unknown coefficients) of a basis consisting of the eigenvectors and, in the case of multiple characteristic values, generalized eigenvectors. The method consists in substituting the general solution in the boundary conditions and then imposing the condition that the Cramer determinant of the obtained linear algebraic system in the unknown coefficients to be zero. This condition generates the secular equation. If the determinant is zero, then the system has nontrivial solutions. The general solution of the equations depends on the multiplicity of the roots λ_i of the characteristic equation associated with the eigenvalue problem. Whence the importance of discussing the multiplicity of λ_i. If the λ_i's are distinct, then the general solution is written as a linear combination of hyperbolic sinus and cosinus functions. If λ_i are multiple, of m_{λ_i} multiplicity respectively, then the general solution is a product of a polynomial in z by $\cosh\lambda_i z$ and $\sinh\lambda_i z$, of degree $m_{\lambda_i} - 1$.

3. METHODS BASED ON FOURIER SERIES EXPANSION

Frequently, the linear problems in hydrodynamic stability theory are eigenvalue problems consisting of high order differential equations with constant coefficients depending on several parameters which make the discussion of multiplicity of the roots of the characteristic equation very difficult. In this way, the application of the direct method becomes very difficult and alternative methods must be used. Some of these methods are based on Fourier series: the direct methods based on series and the variational methods based on series too. The direct methods based on series are: of Chandrasekar - Galerkin type and of Budiansky- DiPrima type.

In the first one, the unknown functions are written in the form of expansions in Fourier series upon a complete set of function (in $L^2[a, b]$) which satisfy all boundary conditions. The characteristic equation is the equality to zero of an infinite determinant.

The second one is a direct expansion approach too, however the functions are chosen such that they satisfy only part of the boundary conditions of the problem. In this case the characteristic equation is the equality to zero of an infinite series.

The variational methods can also be of the Chandrasekar - Galerkin type or Budiansky-DiPrima type depending of whether all the boundary conditions are satisfied or only part of them. A functional J is derived whose stationary points are the eigenfunctions of the considered problem and conversely. To solve the eigenvalue problem, for example, $Lf = 0$, is equivalent to solving the associated variational problem $\delta J = 0$. We say that a variational principle is established. Then the unknown functions of $\delta J = 0$ are written in form of expansion in Fourier series upon complete sets that satisfies or not all the boundary conditions. Then, they are substituted in the variational problem.

4. THE SECULAR EQUATION OF A STABILITY PROBLEM

Consider a particular stability problem [2] and write its secular equation using the direct Chandrasekar - Galerkin method, the direct Budianski-DiPrima method and the variational Budiansky-DiPrima method. In [2] the problem was investigated only by the direct method.

If the exchange of stability principle is valid, the linear neutral stability in the presence of normal modes perturbations of a conduction state of a layer of a thermally conducting micropolar fluid, situated between two horizontal rigid walls maintained at constant temperature and subject to an external magnetic field, is governed by the following two-point problem

$$\begin{cases} (1+R)\Big[(D^2-a^2)^2 - QD^2\Big]W + R(D^2-a^2)Z - R_a a^2\Theta = 0, \\ \Big[A(D^2-a^2) - 2R\Big]Z - R(D^2-a^2)W = 0, \\ (D^2-a^2)\Theta + W - \overline{\delta}Z = 0, \end{cases} \tag{1}$$

$$W = DW = Z = \Theta = 0 \text{ at } z = \pm 0.5. \tag{2}$$

The number $R_a > 0$ stands for the Rayleigh number, $a > 0$ is the wave number, A, R, $\overline{\delta}$ are micropolar parameters, Q is the intensity of the magnetic field and the functions W, Θ, $Z : [-0.5, 0.5] \to \mathbb{R}$ characterize the amplitude of the perturbation of the vertical component of the velocity, temperature and the vertical component of the spin vorticity, respectively.

The Chandrasekar-Galerkin method

Let us apply the Chandrasekar method to (1)-(2). To this aim, we expand the unknown functions W, Θ, Z upon sets of orthonormal functions that satisfy all the boundary conditions. Assume that W, Θ and Z are even functions of z.

Taking into account (2) develop W upon the total set (in $L^2(-0.5;0.5)$) $\{C_n\}_{n\geq 1}$, $C_n(z) = \dfrac{\cosh(\lambda_n z)}{\cosh(\lambda_n/2)} - \dfrac{\cos(\lambda_n z)}{\cos(\lambda_n/2)}$ where λ_n are the roots of the equation $\tanh(\lambda/2) + \mathrm{tg}(\lambda/2) = 0$.

This means that $C_n = DC_n = 0$ at $z = \pm 0.5$. Choosing $W = \sum\limits_{n=1}^{\infty} W_{2n-1}C_n$, it follows that $Z = 0$ at $z = \pm 0.5$ too.

The unknown functions Θ, Z are developed upon the orthonormal set $\{E_{2n-1}\}_{n\geq 1}$, where $E_{2n-1}(z) = \sqrt{2}\cos(2n-1)\pi z$, i.e.

$$\Theta = \sqrt{2}\sum_{n=1}^{\infty} \Theta_{2n-1}\cos(2n-1)\pi z, Z = \sqrt{2}\sum_{n=1}^{\infty} Z_{2n-1}\cos(2n-1)\pi z.$$

It is immediate that $\Theta = Z = 0$ at $z = \pm 0.5$.

The Chandrasekar - Galerkin procedure imposes that the left-hand side of the equation obtained by introducing all these expansions in (1) is orthogonal to E_{2m-1}, $m=1,2,\ldots$. Denoting $l_{nm}=(C_n,E_{2m-1})$, $q_{nm}=(D^2C_n,E_{2m-1})$, $r_{nm}=(D^4C_n,E_{2m-1})$ we obtain

$$\begin{cases} (1+R)\sum\limits_{n=1}^{\infty}[r_{nm}-(2a^2+Q)q_{nm}+a^4l_{nm}]W_{2n-1}- \\ -R[(2m-1)^2\pi^2+a^2]Z_{2m-1}-R_a a^2\Theta_{2m-1}=0, \\ \\ -R\sum\limits_{n=1}^{\infty}(q_{nm}-a^2l_{nm})W_{2n-1}-[A((2m-1)^2\pi^2+a^2)+2R]Z_{2m-1}=0, \\ \\ \sum\limits_{n=1}^{\infty}l_{nm}W_{2n-1}-\overline{\delta}Z_{2m-1}-\Big[(2m-1)^2\pi^2+a^2\Big]\Theta_{2m-1}=0. \end{cases} \tag{3}$$

where $(.,.)$ is the inner product in $L^2(-0.5;0.5)$.

Eliminating Θ_{2m-1} and Z_{2m-1} between $(3)_{1,2,3}$ we obtain the following infinite system of linear algebraic equations in the constant coefficients W_1, W_3, W_5,...

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big\{(1+R)[r_{nm}-(2a^2+Q)q_{nm}+a^4l_{nm}]+\frac{R^2A_m(q_{nm}-a^2l_{nm})}{AA_m+2R}- \\ &-R_a a^2\Big[\frac{l_{nm}}{A_m}+\frac{\overline{\delta}R(q_{nm}-a^2l_{nm})}{A_m(A_m+2R)}\Big]\Big\}W_{2n-1}=0, \end{aligned} \tag{4}$$

where $A_m=(2m-1)^2\pi^2+a^2$.

This leads to the secular equation

$$\lim_{n\to\infty}\begin{vmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1} & d_{n2} & \ldots & d_{nn} \end{vmatrix}=0$$

where d_{nn} is the coefficient of W_{2n-1} in (4). From the secular equation we obtain a hypersurface S of equation $R_a=R_a(a,R,A,\overline{\delta},Q)$. It becomes a surface for three fixed parameters and a curve for four fixed parameters.

If we take some fixed values of three of the parameters a,R,A, then we can represent the secular surface in various two-dimensional planes: (a,R_a), (A,R_a), (R,R_a) or for two-fixed parameters, in the three-dimensional spaces: (A,a,R_a), (A,R,R_a), (a,R,R_a).

The Budiansky-DiPrima method

To apply the Budiansky - DiPrima method we expand the unknown functions W,Θ,Z upon total sets of orthonormal functions that do not satisfy all boundary conditions. We develop the functions W, Θ, Z upon the total set $\{E_{2n-1}\}_{n\in\mathbb{N}}$, where E_{2n-1} are defined before. We have

$$\begin{cases} W = \sum_{n=1}^{\infty} \sqrt{2} W_{2n-1} \cos(2n-1)\pi z, \\ Z = \sum_{n=1}^{\infty} \sqrt{2} Z_{2n-1} \cos(2n-1)\pi z, \\ \Theta = \sum_{n=1}^{\infty} \sqrt{2} \Theta_{2n-1} \cos(2n-1)\pi z. \end{cases} \tag{5}$$

Thus, the functions W, Z, Θ satisfy the conditions $W = Z = \Theta = 0$ at $z = \pm 0.5$, while the condition $DW = 0$ at $z = \pm 0.5$ introduces a constraint for the problem (1)-(2).

Using the backward integration technique, we obtain the expressions of all derivatives that appear in (1). We replace these expressions in (1), impose the condition that the obtained equations be orthogonal to E_{2m-1}, $m = 1, 2, ...$ and we have

$$\begin{cases} (1+R)\{2\sqrt{2}(-1)^{n+1} D^2 W(0.5)(2n-1)\pi + [A_n^2 + Q(2n-1)^2\pi^2] W_{2n-1}\} - \\ - R A_n Z_{2n-1} - R_a a^2 \Theta_{2n-1} = 0, \\ \\ R A_n W_{2n-1} - (A A_n + 2R) Z_{2n-1} = 0, \\ \\ W_{2n-1} - \overline{\delta} Z_{2n-1} - A_n \Theta_{2n-1} = 0. \end{cases} \tag{6}$$

We considered only the limit case $Q = 0, \overline{\delta} = 0$; the case Q, $\overline{\delta} \neq 0$ will be treated elsewhere.

In this case, the system (6) becomes

$$\begin{cases} (1+R) A_n^2 W_{2n-1} - R A_n Z_{2n-1} - R_a a^2 \Theta_{2n-1} = \\ = 2\sqrt{2}(-1)^n (1+R)(2n-1)\pi\alpha \\ \\ R A_n W_{2n-1} - (A A_n + 2R) Z_{2n-1} = 0, \\ \\ W_{2n-1} - A_n \Theta_{2n-1} = 0, \end{cases} \tag{7}$$

where $\alpha = D^2 W(0.5) \neq 0$.

The constraint has the form

$$\sum_{n=1}^{\infty} (-1)^n \sqrt{2}(2n-1)\pi W_{2n-1} = 0 \tag{8}$$

The expressions of the unknown constants W_{2n-1} are obtained by solving the system (7). Replacing these expressions in (8), we obtain the secular equation

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2 A_n (AA_n + 2R)}{(AA_n + 2R)[A_n^3 + RA_n^3 - R_a a^2] - R^2 A_n^3} = 0. \tag{9}$$

The variational method

The problem (1) can be written in the form $Lf = 0$,

$$\begin{cases} (1+R)(D^2 - a^2)^2 W + R(D^2 - a^2)Z - R_a a^2 \Theta = 0, \\ -R(D^2 - a^2)W + [A(D^2 - a^2) - 2R]Z = 0, \\ W + (D^2 - a^2)\Theta = 0. \end{cases} \tag{11}$$

where $L : \mathcal{A} \to [\mathcal{C}^0(-0.5, 0.5)]^3$ is the matricial linear differential operator

$$L = \begin{pmatrix} (1+R)(D^2 - a^2 I)^2 & R(D^2 - a^2 I) & -R_a a^2 I \\ -R(D^2 - a^2 I) & A(D^2 - a^2 I) - 2RI & \mathcal{O} \\ I & \mathcal{O} & D^2 - a^2 I \end{pmatrix},$$

I is the identity operator on $\mathcal{A}$, $\mathcal{O}$ is the null operator on $\mathcal{A}$, the set $\mathcal{A}$ consists of all vector functions $f = (W, Z, \Theta)$ satisfying (2) and

$$f \in [\mathcal{C}^\infty(-0.5, 0.5)]^4 \cap \mathcal{C}^0(-0.5, 0.5) \times \mathcal{C}^0(-0.5, 0.5) \times \mathcal{C}^0(-0.5, 0.5).$$

The domain of definition $\mathcal{A}$ is embedded in $[L^2(-0.5, 0.5)]^3$.

We say that an operator L is symmetric if the equality

$$(Lf, g) = (f, Lg)$$

holds for every $f, g \in \mathcal{A}$. It is selfadjoint if it is symmetric and $L = L^*$. In our case direct (and quite easy) computations show that L is not selfadjoint.

Remark. Sometimes [6] a nonselfadjoint matricial differential linear operator can become selfadjoint by multiplying one or more equations in its corresponding system of equations by some constants.

If we multiplying the equation $(10)_1$ by (-1) and the equation $(10)_3$ by $(R_a a^2)$, the equation (10) becomes $L_1 f = 0$, where L_1 is the selfadjoint operator

$$L_1 = \begin{pmatrix} -(1+R)(D^2 - a^2 I)^2 & -R(D^2 - a^2 I) & R_a a^2 I \\ -R(D^2 - a^2 I) & A(D^2 - a^2 I) - 2RI & \mathcal{O} \\ R_a a^2 I & \mathcal{O} & R_a a^2 (D^2 - a^2 I) \end{pmatrix}.$$

The inner product $(L_1 f, g)$ is obtained by multiplying the equations $(10)_{1,2,3}$ by $g = (W^*, Z^*, \Theta^*)$, respectively, by adding the results and then integrating the obtained sum over $[-0.5, 0.5]$. Integrating by parts in the expression $(L_1 f, g)$ and taking into account the boundary conditions (2) we are lead to

$$(L_1 f, g) = (f, L_1^* g).$$

Since the operator L_1 is selfadjoint, we check that $L_1 = L_1^*$.

The eigenvalue problem for the selfadjoint operator consist in the system of ordinary differential equations

$$\begin{cases} -(1+R)(D^2-a^2)^2 W^* - R(D^2-a^2)Z^* + R_a a^2 \Theta^* = 0, \\ -R(D^* - a^2)W^* + [A(D^2-a^2) - 2R]Z^* = 0, \\ R_a a^2 W^* + R_a a^2 (D^2-a^2)\Theta^* = 0, \end{cases} \tag{11}$$

and the boundary conditions

$$W^* = DW^* = Z^* = \Theta^* = 0 \text{ at } z = \pm 0.5.$$

Define the functional $J : \mathcal{A}_1 \to \mathbb{R}$ by $J(f) = (L_1 f, f)$, where the set $\mathcal{A}_1$ is defined as follows

$$\mathcal{A}_1 = \{f = (W, Z, \Theta) \in [\mathcal{C}^\infty(-0.5, 0.5)]^3 | W, Z, \Theta \text{ satisfy } W = Z = \Theta = 0\}.$$

We have

$$J(f) = (1+R)D^2 W DW\Big|_{-0.5}^{0.5} + \int_{-0.5}^{0.5} \Big\{ -(1+R)[(D^2W)^2 + 2a^2(DW)^2 + a^4 W^2] +$$
$$+2R[DZDW + a^2 WZ] + 2Ra^2 W\Theta - [A(DZ)^2 + Aa^2 Z^2 + 2RZ^2] -$$
$$-R_a a^2 (D\Theta)^2 - R_a a^4 \Theta^2 \Big\} \tag{12}$$

Theorem 1. $L_1(f) = 0$ *iff* $\delta J(f) = 0$.

If $\overline{f} \in \mathcal{A}_1$ is a solution of the equation $L_1 \overline{f} = 0$ it follows that the variation of the functional J at f in the $\mathcal{A}_1$ vanishes, i.e. $\delta J(\overline{f}) = 0$, in other words f makes the functional J stationary. We have (for simplicity we removed the bar)

$$\delta J(f) = 2 \int_{-0.5}^{0.5} \Big\{ -(1+R)(D^4 W - 2a^2 D^2 W + a^4 W) + R(D^2 Z - a^2 Z -$$
$$-R_a a^2)\Theta \Big\} \delta W + \Big\{ -R(D^2 W - a^2 W) + [A(D^2 Z - a^2 Z) - 2RZ] \Big\} \delta Z +$$
$$+ \Big\{ W + (D^2\Theta - a^2\Theta) \Big\} \delta\Theta = 0. \tag{13}$$

Since δW, δZ and $\delta\Theta$ are arbitrary,$\delta J(f) = 0$ implies that (11) hold. Conversely, if (11) hold, it follows that $\delta J(f) = 0$, whence the Theorem 1.

Replacing in (12) the series expansions for W, DW, D^2W, Z, DZ, Θ and $D\Theta$ we obtain a function in the coefficients W_{2n-1}, Z_{2n-1} and Θ_{2n-1}. Imposing to this function to be stationary we obtain an infinite linear system in these coefficients. Eliminating Z_{2n-1} and Θ_{2n-1} between the equations of this algebraic system it is obtained W_{2n-1}. Substituting the obtained expression for W_{2n-1} in the constraint (8) we are lead to (9). Therefore, as it was expected, we obtained the same secular equation.

The use of the functional J diminishes the computations to the half because J contains only two derivatives of W and one derivative for Z and Θ, i.e. half of the derivatives occurring in the system (1).

5. CONCLUSIONS

After describing the simplest direct method, and three direct and variational methods based on Fourier series, we treated a concrete example by the direct and variational Budiansky and DiPrima method, leading to the secular equation defining the manifolds separating the stability domain from the instability domain in some parameter space. The numerical study of these manifolds will be done elsewhere.

The method that we choose to apply depends on the form of the eigenvalue problem and must avoid complicate Fourier coefficients.

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