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SADDLE-NODE BIFURCATION IN AN EPIDEMIC MODEL

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Abstract A mathematical epidemic model consisting of a Cauchy problem for a system of two first order ordinary differential equations (ode's) is studied. For some particular parameters saddle-node singularities are found. The normal forms of the governing equations are derived using the method of Arrowsmith and Place [1].

Keywords: dynamical system, normal form, saddle-node, epidemic model.

1. MATHEMATICAL MODEL

From the epidemiological point of view, the individuals of a population can be in one of the following states: susceptibles (which were no infected till the considered moment, but can be contaminated in the future), contagious (source of contamination for the neighbor susceptible individuals), immunized (which make the virosis and gained the immunity against the respective virus). In the following sections we investigate the bifurcations of the mathematical epidemic model, described by the Cauchy problem

$$\begin{cases} x_t|_{t=0} = x_0, \\ y_t|_{t=0} = y_0, \end{cases} \tag{1}$$

for the ode's [2]

$$\begin{cases} \dot{x} = -\mu xy + \rho, \\ \dot{y} = \mu xy - \nu y, \end{cases} \tag{2}$$

where $x,\ y$ are the state functions, x are *susceptible individuals*, y are *infected individuals*; $\mu, \rho,\ \nu$ are nonnegative parameters.

2. THE EQUILIBRIUM SET

The equilibrium states are the stationary (in fact, constant) solutions of (1), (2), i.e. they satisfy the system of algebraic equations

$$\begin{cases} -\mu xy + \rho = 0, \\ \mu xy - \nu y = 0, \end{cases} \tag{3}$$

The number of the solutions of (3), i.e. the cardinal of the equilibrium set of (1), (2), depends on the three parameters μ, ρ and ν. More precisely, the following cases occur:

- $\mu = 0,\ \rho,\ \nu \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\rho = 0,\ \mu,\ \nu \neq 0 \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ 0),\ x_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = \mu x_0 - \nu$. We recall that $p_\pm$ are the eigenvalues of the matrix defining the system obtained by linearizing (2) about the equilibrium point;
- $\nu = 0,\ \mu,\ \rho \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\mu = \rho = 0,\ \nu \neq 0\ \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ 0),\ x_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = -\nu$;
- $\mu = \nu = 0,\ \rho \neq 0\ \Rightarrow$ (2) has no equilibria;
- $\rho = \nu = 0,\ \mu \neq 0\ \Rightarrow$ (2) has two equilibrium points $\mathbf{e}_1 = (x_0,\ 0)$ and $\mathbf{e}_2 = (0,\ y_0),\ x_0,\ y_0 \in \mathbb{R}$, possessing the eigenvalues $p_+ = 0,\ p_- = \mu x_0$ and $p_+ = 0,\ p_- = y_0$ respectively;
- $\mu = \rho = \nu = 0\ \Rightarrow$ (2) has an infinity of equilibria $\mathbf{e} = (x_0,\ y_0),\ x_0,\ y_0 \in \mathbb{R}$, possessing the eigenvalues $p_\pm = 0$;
- $\mu,\ \rho,\ \nu \neq 0\ \Rightarrow$ (2) has an unique equilibrium $\mathbf{e} = (\nu\mu^{-1},\ \rho\nu^{-1}),\ \mu,\ \rho,\ \nu \in \mathbb{R}$, possessing the eigenvalues $p_\pm = \dfrac{\mu\rho}{2\nu}\left(-1 \pm \sqrt{1 - \dfrac{4\nu^2}{\mu\rho}}\right)$.

Correspondingly, the cardinal of the equilibrium set is $0, \infty, 0, \infty, 0, 2, \infty$ and 1 respectively.

Moreover, for only one or two vanishing parameters the existing equilibria are saddle-nodes, for all vanishing parameters the equilibria are of the double zero type while for no vanishing parameters the unique equilibrium can be either a saddle, or a node, or a focus. Consequently the single nonhyperbolic equilibria are saddle-nodes, and this occurs for: 1) $\rho = 0,\ \mu,\ \nu \neq 0,\ \mu x_0 - \nu \neq 0$; 2) $\mu = \rho = 0,\ \nu \neq 0$; 3) $\rho = \nu = 0,\ \mu \neq 0,\ x_0,\ y_0 \neq 0$; 4)

$\rho = 0, \mu, \nu \neq 0,\ \mu x_0 - \nu = 0$; 5) $\rho = \nu = 0,\ \mu \neq 0,\ x_0 = 0$ and/or $y_0 = 0$ and a double zero equilibrium for 6) $\mu = \rho = \nu = 0$. Moreover, since in the case 2) the system (2), becomes linear and its phase portrait corresponds to the phase trajectories defined by the equations $x = x_0,\ y = y_0 e^{-\nu t}$, it follows that the only interesting cases are 1) and 3). In the following we study them, while the cases 4), 5) and 6) will be treated elsewhere.

3. NORMAL FORM FOR SADDLE-NODE SINGULARITY

As we saw, in the cases $\rho = 0$ and $\rho = \nu = 0$ the singularities have one zero eigenvalue, i.e. they are the saddle-nodes. In the following we deduce the particular normal forms of the governing equations at these singularities.

Theorem 3.1. *For $\rho = 0$ and for given nonvanishing values of μ and ν, namely $\mu = \mu_0,\ \nu = \nu_0,\ \mu_0, \nu_0 \neq 0$ such that $\mu_0 x_0 - \nu_0 \neq 0$, the Cauchy problem (1), (2) has the following normal form at the singularity $\mathbf{e} = (x_0, 0)$*

$$\begin{cases} \dot{s_1} = 0, \\ \dot{s_2} = (\mu_0 x_0 - \nu_0) s_2 + s_1 s_2. \end{cases} \tag{4}$$

In addition, the equilibrium point $\mathbf{e}$ is a degenerated saddle-node.

Proof. Consider two fixed values $\mu = \mu_0,\ \nu = \nu_0$ and let us carry the stationary point $(x_0,\ 0)$ at the origin of coordinates by the change of coordinates $q_1 = x - x_0,\ q_2 = y - 0$. Then (2) becomes

$$\begin{cases} \dot{q_1} = -\mu_0 q_1 q_2 - \mu_0 x_0 q_2, \\ \dot{q_2} = \mu_0 q_1 q_2 + \mu_0 x_0 q_2 - \nu_0 q_2. \end{cases} \tag{5}$$

Applying the transformation $\mathbf{r} \to P^{-1}\mathbf{q}$, with $P = (v_+,\ v_-)$, where v_+ and v_- are eigenvectors corresponding to the eigenvalues p_+ and p_- respectively, ode's (5) become

$$\begin{cases} \dot{r_1} = -\mu_0 \nu_0 r_2 (r_1 + \mu_0 x_0 r_2), \\ \dot{r_2} = \mu_0 r_2 (r_1 + \mu_0 x_0 r_2) + (\mu_0 x_0 - \nu_0) r_2. \end{cases} \tag{6}$$

In order to eliminate the nonresonant terms of order two from (6) we use the transformation (deduced by the normal form method [1])

$$\begin{cases} r_1 = s_1 - \dfrac{\mu_0 \nu_0}{\mu_0 x_0 - \nu_0}\, s_1 s_2 - \dfrac{\mu_0^2 x_0 \nu_0}{2(\mu_0 x_0 - \nu_0)}\, s_2^2, \\ r_2 = s_2 + \dfrac{\mu_0^2 x_0}{\mu_0 x_0 - \nu_0}\, s_2^2, \end{cases} \tag{7}$$

and we obtain the normal form (4). Relations (7) are valid for $\mu_0 x_0 - \nu_0 \neq 0$, i.e. case 2). Comparing (4) with the general normal form at a saddle-node singularity, it follows that in our case the singularity is degenerated. □

Theorem 3.2. *For $\rho = \nu = 0$, $\mu \neq 0$ the Cauchy problem (1), (2) has the following normal form at the singularities $\mathbf{e}_1 = (x_0, 0)$, $\mathbf{e}_2 = (0, y_0)$, with x_0, $y_0 \neq 0$:*

- *for $\mathbf{e}_1$ it is*

$$\begin{cases} \dot{m_1} = 0, \\ \dot{m_2} = \mu_0 x_0 m_2 + \mu_0 m_1 m_2; \end{cases} \tag{8}$$

- *for $\mathbf{e}_2$ it is*

$$\begin{cases} \dot{n_1} = -(\mu_0 y_0 n_1 + \mu_0 n_1 n_2), \\ \dot{n_2} = 0. \end{cases} \tag{9}$$

Both equilibria are degenerated saddle-nodes.

Proof. **Equilibrium $\mathbf{e_1}$.** The change of coordinates $k_1 = x - x_0$, $k_2 = y - 0$, $\mu = \mu_0$ leads to the ode's

$$\begin{cases} \dot{k_1} = -\mu_0 k_1 k_2 - \mu_0 x_0 k_2, \\ \dot{k_2} = \mu_0 k_1 k_2 + \mu_0 x_0 k_2. \end{cases} \tag{10}$$

Applying the transformation $\mathbf{l} \to P^{-1}\mathbf{k}$, with $P = (v_+,\ v_-)$, where v_+ and v_- are eigenvectors corresponding to the eigenvalues p_+ and p_- respectively, ode's (10) become

$$\begin{cases} \dot{l_1} = 0, \\ \dot{l_2} = \mu_0 (k_1 k_2 - k_2^2) + \mu_0 x_0 k_2. \end{cases} \tag{11}$$

Elimination of the nonresonant terms of order two from (11) leads immediately to the normal form (8).

Equilibrium $\mathbf{e_2}$. Similarly, we obtain the normal form (9). □

References

[1] Arrowsmith, D. K., Place, C. M., *An introduction to dynamical systems*, Cambridge University Press, 1990.

[2] Jones, D. S., Sleeman, B. D., *Differential equations and mathematical biology*, Chapman & Hall/ CRC Math. Biol. and Med. Series, 2003.