ROMAI J., **2**, 1(2006),
65–70

PERTURBED BIFURCATION IN A DEMAND-OFFER MODEL

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Abstract For a demand-offer model with cubic demand function the universal unfoldings and the transient manifolds are determined. The sections in the static bifurcation diagram and the universal unfoldings of the manifold **S** are characterized. The cases are enumerated where these universal unfoldings cannot be found, the respective germs having a greater than three or infinite codimension.

The mathematic model associated with the economical demand-offer model with a cubic demand function is the Cauchy problem for the system of ordinary differential equations(s.e.d.o.)

$$\begin{cases} \dot{x} = \alpha x^3 + cx^2 - y, \\ \dot{y} = \mu x - y - \gamma, \end{cases} \tag{1}$$

with α, c, μ, γ- the four parameters and x, y- the state functions. In the following we use the notation, definitions from [GS] (for perturbed bifurcation) and [GMO] (for general theory of bifurcation and nonlinear dynamics). The method of normal form is that presented in [AP] and [Kus].

1. DETERMINATION OF THE EQUILIBRIUM MANIFOLD AND S MANIFOLDS

Proposition 1.1 *The bifurcation equation for (1) is*

$$\alpha x^3 + cx^2 - \mu x + \gamma = 0. \tag{2}$$

It is the geometric locus of equilibria (we considered only x because $(0.1)_2$ shows that y is a bijective function of x for $\mu \neq 0$). Sections in the static bifurcation diagram for the s.e.d.o. (1) in the (x,α,c) space, for $\mu = \gamma = 1$ are represented in fig. 1a; in the (x,c,μ) space, for $\alpha = \gamma = 1$, in fig. 1b, in the (x, c, γ) space, for $\alpha = \mu = -1$, in fig.1c.

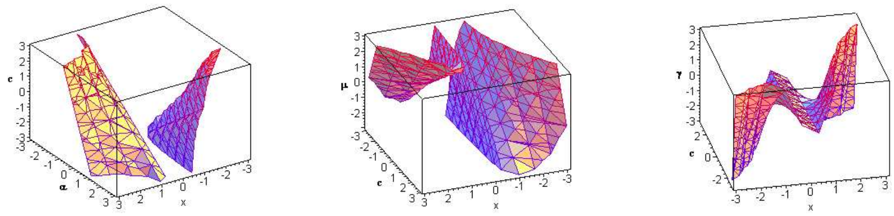

Fig. 1. Sections in the static bifurcation diagram (2) for: a) $\mu = \gamma = 1$; b) $\alpha = \gamma = 1$; c) $\alpha = \mu = -1$.

Proposition 1.2 *For the equilibrium points* $P_1\left(\frac{9\alpha\gamma+\mu c}{6\mu\alpha+2c^2}, \mu\left(\frac{9\alpha\gamma+\mu c}{6\mu\alpha+2c^2}\right)-\gamma\right)$ *of s.e.d.o. (1), the linearized s.e.d.o. has at the origin a saddle-node bifurcation, and the* $\mathbf{S}$ *manifold is*

$$\mathbf{S} = \left\{(\alpha, c, \mu, \gamma) \mid \; 4c^3\gamma - \mu^2c^2 + 18\alpha\mu\gamma c + 27\alpha^2\gamma^2 - 4\alpha\mu^3 = 0 \;\right\}. \tag{3}$$

The normal form for the linearized s.e.d.o. around the origin, in the phase space, for a parameter $(\alpha, c, \mu, \gamma) \in \mathbf{S}$ *fixed, is*

$$\begin{pmatrix} \dot{n}_2 \\ \dot{n}_1 \end{pmatrix} = \begin{pmatrix} \mu - 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} + \begin{pmatrix} -2a_1\mu n_1 n_2 \\ -\mu a_1 n_1^2 \end{pmatrix} + \mathcal{O}(\mathbf{n}^3), \tag{4}$$

with $a_1 = \dfrac{3\alpha x_0 + c}{\mu - 1}$*. The graphical representation of* $\mathbf{S}$*, for a fixed value of the parameter* μ*, is drawn in* $fig.2a$*.*

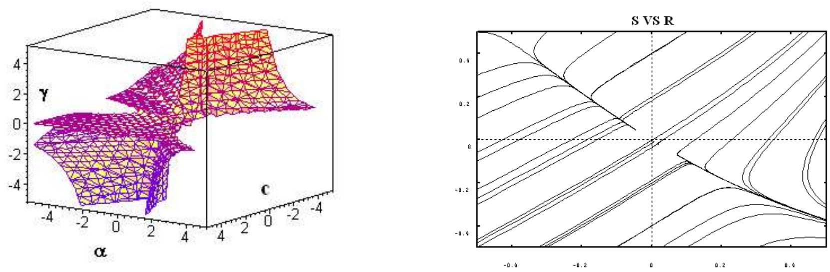

Fig. 2. a) The graphical representation of $\mathbf{S}$ in the (c, α, γ) space for $\mu = -1$; b)The phase portrait for the linearized system around the origin, for $(\alpha, c, \mu, \gamma) = (1, 2, -1, \frac{4}{27})$, when $x_0 = -\frac{1}{3}$.

Expresion (1.2) follows by eliminating x_0 between (1.1) and the condition that the equilibrium point (x_0, y_0) has a null eigenvalue.

$Fig.2b$, reveals the presence of a saddle-node.

2. PERTURBED BIFURCATION IN THE STATIC BIFURCATION DIAGRAM

Case $\alpha \neq 0$.

Proposition 2.1 *The static bifurcation diagram is an equation $G(x) = 0$, where the G function is*

$$G(x, a, b, d) = x^3 + ax^2 + bx + d. \tag{5}$$

The germ G, for x selected as a state function, b as a control parameter, a and d as small parameters, is the universal unfolding for the fork at the origin.

Proposition 2.2 *For d selected as control parameter, a and b as small parameters, then G in (5) is a two parameter unfolding of a hysteresis point situated at the origin.*

Let $\mathcal{E}_{x,\lambda}$ be the set of germs in x and λ variables.

Proposition 2.3 *The germ $g_1(x, a) = x^3 + ax^2$ has an infinite codimension, and, generally, any germ $g(x, \lambda) = x^2 q(x, \lambda)$, with $q(x, \lambda) \in \mathcal{E}_{x,\lambda}$ has an infinite codimension.*

Case $\alpha = 0$, $\mathbf{c} \neq 0$. For this case, the unfolding of the static bifurcation diagram for the s.e.d.o. (1) is

$$G(x, \gamma, \mu, c) = cx^2 - \mu x + \gamma, \text{ i.e. } \dot{x} = cx^2 - \mu x + \gamma. \tag{6}$$

Proposition 2.4 *The unfolding (6) factorizes with the universal unfolding of the limit point.*

2.1. DETERMINATION OF THE UNIVERSAL UNFOLDINGS FOR S, CORRESPONDING TO THE NON-HYPERBOLIC EQUILIBRIA P_1

Case $\boldsymbol{\gamma} \neq 0$.

Proposition 2.5 *For the bifurcation equation (3), assuming c as a state function, α as a control parameter and $\gamma \neq 0$ as fixed, then the origin $(c, \alpha) = (0, 0)$ is a bifurcation winged-cusp point, if $\mu = 0$. In addition, the function from (3) represents a map strongly equivalent with the normal form of the winged cusp.*

In fig. 3 we give sections in the transient manifolds of the universal unfoldings $H(x, \lambda, \alpha, \beta, \gamma) = x^3 + \lambda^2 + \alpha + \beta\lambda + \gamma x\lambda$ of the winged cusp, with plane $\gamma = const.$, together with all the persistent bifurcations.

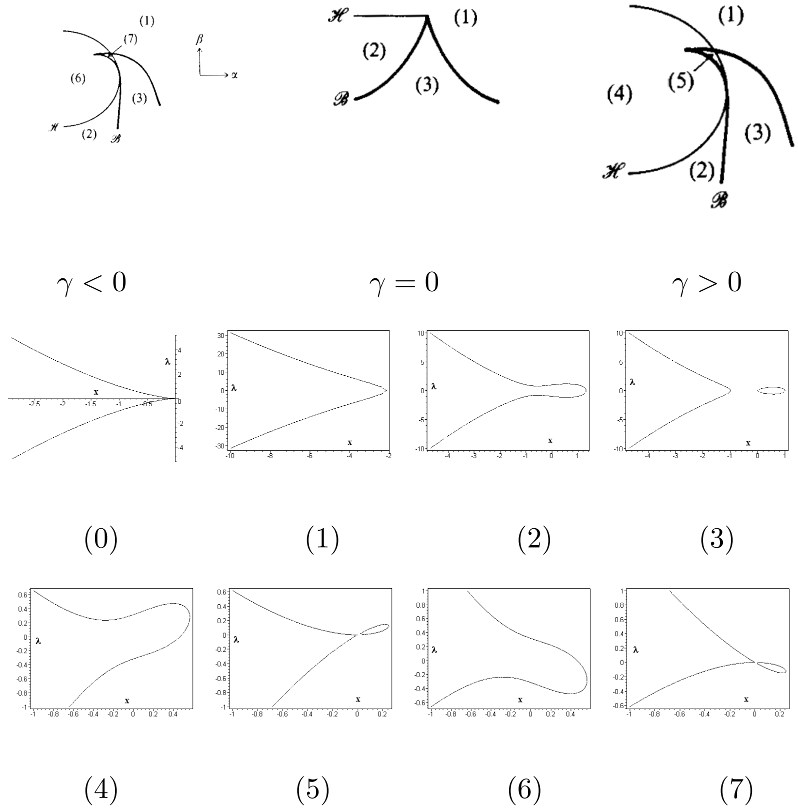

Fig. 3 Sections in the transient manifolds of the universal unfoldings $H(x, \lambda, \alpha, \beta, \gamma) = x^3 + \lambda^2 + \alpha + \beta\lambda + \gamma x\lambda$ of the winged cusp, with plans $\gamma = const.$ and the persistent unfoldings.

Proposition 2.6 *For the bifurcation equation (3), taking $\mu = 0$, c as state function, $\gamma \neq 0$ as a control parameter, i.e. for the bifurcation problem $G_2(c, \alpha, \gamma) = G_1(c, \alpha, \gamma, 0)/\gamma = 0$, it follows that the origin $(c, \gamma) = (0, 0)$ is a bifurcation hysteresis point.*

Proposition 2.7 *For the bifurcation equation (3), with $\mu = 0$, choosing c as a state function, γ as control parameter, and $\alpha \neq 0$ as fixed, then $G_1(c, \alpha, \gamma, 0)$ is an infinite codimension bifurcation germ.*

case $\gamma = 0$. The bifurcation problem (3) becomes

$$G_1(c, \alpha, 0, \mu) = -4\alpha\mu^3 - \mu^2 c^2 = 0. \tag{7}$$

Proposition 2.8 *In (7), for μ considered as a state function, $\alpha \neq 0$ as fixed, c as a control parameter, we have that at the origin an infinite codimension bifurcation occurs.*

Proposition 2.9 *The bifurcation problem (7), for μ taken as a state function, $\alpha \neq 0$ as fixed, c as a control parameter, has at the origin a fork bifurcation.*

Proposition 2.10 *In (7), for c taken as a state function, μ as a control parameter, $\alpha \neq 0$ as fixed, then $G_1(c, \alpha, 0, \mu)$ has an infinite codimension.*

Proposition 2.11 *For (7) with $\mu \neq 0$ fixed, c chosen as a state function, α as a control parameter, we have at the origin $(c, \alpha) = (0, 0)$ a zero codimension limit point.*

Case $\boldsymbol{\alpha} \neq 0$ fixed.

Proposition 2.12 *The bifurcation problem (3), with μ chosen as a state function, γ as a control parameter, $\alpha \neq 0$ as fixed, has at the origin $(\mu, \gamma) = (0, 0)$, a cusp bifurcation, for $c = 0$.*

Indeed for $c = 0$, from (3) we obtain the unfolding

$$G_1(\mu, \gamma, \alpha, 0) = -4\alpha\mu^3 + 27\alpha^2\gamma^2. \tag{8}$$

Proposition 2.13 *i) If in (8) we take μ as a state function, α as a control parameter and γ as fixed, we obtain that $G_1(\mu, \alpha, \gamma, 0)$ is a germ in (μ, α) with an infinite codimension.*

ii) If in (8) we take γ as a state function, $\alpha \neq 0$ as fixed and μ as the control parameter , then we obtain that $G_1(\gamma, \mu, \alpha, 0)$ is a zero codimension germ, strongly equivalent with the limit point germ, $g(x, \lambda) = x^2 + \lambda$.

iii) If in (8) we take γ as a state function, we fixe $\mu \neq 0$, and α is the control parameter, then $G_1(\gamma, \alpha, \mu, 0)$ is a germ in $\mathcal{E}_{\gamma,\alpha}$ with an infinite codimension.

iv) Taking in (8), α as a state function, $\gamma \neq 0$ as fixed and μ as a control parameter, we obtain that $G_1(\alpha, \mu, \gamma, 0) = 0$ is equivalent with the bifurcation problem $g(x, \lambda) = x^2 - \lambda^2 = 0$, i.e. it is a simple bifurcation.

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