ROMAI J., **2**, 1(2006),
209–218

OPTIMAL CONTROL AND STOCHASTIC UNIFORM OBSERVABILITY

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Abstract G.Da Prato and I. Ichikawa solved in [3] the quadratic control problem (1), (2), under stabilizability and detectability conditions. We replace the detectability condition with the uniform observability property and we obtain an optimal control and the minimal value of the cost functional (2). So, we generalize the results obtained by T.Morozan in [7] for finite dimensional case and we also conclude that our result is distinct from the one of G.Da Prato and I. Ichikawa. We use the above results to give a method for the numerical computation of the optimal cost. Under the same hypotheses we solve the tracking problem (1), (3).

Keywords: quadratic control, tracking, Riccati equation, detectability, stabilizability and observability
2000 MSC: 93E20, 49K30, 93B07.

1. NOTATION AND STATEMENT OF THE PROBLEM

Let H, U, V be separable real Hilbert spaces. Denote by $L(H,V)$ the Banach space of all bounded linear operators from H into V (if $H = V$ we put $L(H,V) = L(H)$). Denote by $\mathcal{H}$ the subspace of $L(H)$ formed by all self-adjoint operators. Write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators . Then $S \in L(H)$ is called nonnegative ($S \geq 0$) if S is self-adjoint and $\langle Sx, x\rangle \geq 0$ for all $x \in H$. We set $L^+(H) = \{S \in L(H), S \geq 0\}$. For each interval $J \subset \mathbf{R}_+$, we denote by $C_s(J, L(H))$ the space of all strongly continuous mappings $G(t) : J \subset \mathbf{R}_+ \to L(H)$ and by $C_b(J, L(H))$ the subspace of $C_s(J, L(H))$, which consist of all mappings $G(t)$ such that $\sup_{t\in J} \|G(t)\| < \infty$. Let E is a Banach space; denote by $C(J, E)$ the space of all continuous mappings $G(t) : J \subset \mathbf{R}_+ \to E$. We need the following assumption:

P_1: a) $A(t)$, $t \in [0, \infty)$ is a closed linear operator on H with constant domain D dense in H;

b) there exist $M > 0$, $\eta \in (\frac{1}{2}\pi, \pi)$ and $\delta \in (-\infty, 0)$ such that $S_{\delta,\eta} = \{\lambda \in C; |\arg(\lambda - \delta)| < \eta\} \subset \rho(A(t))$, for all $t \geq 0$ and $\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda-\delta|}$ for all $\lambda \in S_{\delta,\eta}$;

c) there exist numbers $\alpha \in (0,1)$ and $\widetilde{N} > 0$ such that

$A(t)A^{-1}(s) - I \leq \widetilde{N}\,|t-s|^{\alpha}, t \geq s \geq 0$, where $\rho(A)$, $R(\lambda, A)$ are the resolvent set of A and the resolvent of A respectively.

It is known [4] that if P_1 holds, then the family $A(t), t \geq 0$ generates an evolution operator $U(t,s)$. For any $n \in \mathbf{N}$ we have $n \in \rho(A(t))$. The operators $A_n(t) = n^2 R(n, A(t)) - nI$ are called the Yosida approximations of $A(t)$. If we denote by $U_n(t,s)$ the evolution operator relative to $A_n(t)$, for each $x \in H$ one has $\lim\limits_{n\to\infty} U_n(t,s)x = U(t,s)x$ uniformly on any bounded subset of $\{(t,s);\ t \geq s \geq 0\}$.

Let $(\Omega, F, F_t, t \in [0,\infty), P)$ be a stochastic basis and assume that P_1 holds. Consider the following stochastic equation with control

$$dy(t) = A(t)y(t) + B(t)u(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t), \qquad y(s) = x \in H, \quad (1)$$

denoted by $\{A, B; G_i\}$, where $u \in U_{ad} = \{u \in L^2(\mathbf{R}_+ \times \Omega, U),\ u$ is F_t-adapted$\}$, w_i's are independent real Wiener processes relative to F_t. Suppose that the hypotheses

$P_2 : B \in C_b([0,\infty), L(U,H)), B^* \in C_b([0,\infty), L(H,U)), C \in C_b([0,\infty), L(H,V)), C^*C, G_i \in C_b([0,\infty), L(H)),\ K(t) \in C_b([0,\infty), L^+(U))$ and there exists $\delta_0 > 0$ such that $K(t) \geq \delta_0 I$ for all $t \in [0,\infty)$

are fulfiled. Denote $\widetilde{Z} = \sup\limits_{r\in[0,\infty]} \|Z(r)\|$ for $Z = B, C, G_i, K$.

Assume that P_1, P_2 hold. It is known that under uniform observability and stabilizability conditions the Riccati equation of stochastic control (3) has a unique, uniformly positive, bounded on $\mathbf{R}_+$ and stabilizing solution (see Theorem 3.2). We use this result to solve both a quadratic control problem and a tracking problem associated with (1).

Quadratic control problem. We look for an optimal control $u \in \mathfrak{U}_{ad}$, which minimize the following quadratic cost

$$I_s(u) = E\int_s^{\infty} \|C(t)y(t)\|^2 + < K(t)u(t), u(t) > dt, \quad (2)$$

where $\mathfrak{U}_{ad} = \{u \in U_{ad}$ such as $E\,\|y(t)\|^2 \to 0$ as $t \to \infty\}$.

Tracking problem. Given a signal $r \in C_b(\mathbf{R}_+, H)$ we want to minimize the cost

$$J(s,u) = \overline{\lim_{t\to\infty}} \frac{1}{t-s} E\int_s^t \|C(\sigma)\,(y(\sigma) - r(\sigma))\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle\, d\sigma \quad (3)$$

in a suitable class of controls u subject to the equation $\{A, B; G_i\}$.

2. BOUNDED SOLUTIONS OF LINEAR STOCHASTIC EQUATIONS

Assume that P_1, P_2 hold. With (1) associated the equation

$$dy(t) = A(t)y(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t), \qquad y(s) = x \in H, \tag{4}$$

denoted by $\{A; G_i\}$.

It is known [3] that $\{A; G_i\}$ has a unique mild solution in $C([s,T], L^2(\Omega; H))$ that is adapted to F_t; namely the solution of

$$y(t) = U(t,s)x + \sum_{i=1}^{m} \int_s^t U(t,r)G_i(r)y(r)dw_i(r). \tag{5}$$

Let $y(t,s;x)$ be the mild solution of $\{A; G_i\}$.

We have the following definition (see [5] for the autonomous case) :

Definition 2.1 *We say that $\{A; G_i\}$ is uniformly exponentially stable if there exist constants $M \geq 1, \omega > 0$ such that*

$$E\left\|y(t,s;x)\right\|^2 \leq Me^{-\omega(t-s)} \left\|x\right\|^2 \text{ for all } t \geq s \geq 0 \text{ and } x \in H.$$

Definition 2.2 [3] *We say that $\{A, B; G_i\}$ is stabilizable if there exists $F \in C_b([0,\infty), L(H,U))$ such that $\{A + BF; G_i\}$ is uniformly exponentially stable.*

Lemma 2.1 *Assume P_1 and $F \in C_b([0,\infty), L(H,U))$. If $h \in C_b(\mathbf{R}_+, H)$, then the equation*

$$g_n'(t) = -\left(A_n^* + F^*\right) g_n(t) - h(t), g(T) = x_0 \in H \tag{6}$$

where $A_n, n \in \mathbf{N}$ are the Yosida approximations of A and the weak differentiability is considered, has a unique solution. The functions $(t,x) \to \langle g_n'(t), x\rangle, n \in \mathbf{N}$ are continuous on $[0,\infty) \times H$. Moreover if $V_n(t,s)$ (resp. $V(t,s)$) is the evolution operator generated by $A_n + F$ (resp. $A + F$), then we have

$$\langle g_n(t), y\rangle \underset{n\to\infty}{\to} \left\langle V_{A+F}^*(T,t)x_0, y\right\rangle + \left\langle \int_t^T V_{A+F}^*(\sigma,t)h(\sigma)d\sigma, y\right\rangle, \tag{7}$$

uniformly with respect to $t \in [0,T]$.

Proof. Since $\frac{\partial V_{A_n+F}(t,s)x}{\partial s} = -V_{A_n+F}(t,s)(A_n(s) + F(s))x$, it is easy to see that $g_n(t) = V_{A_n+F}^*(T,t)x + \int_t^T V_{A_n+F}^*(\sigma,t)h(\sigma)d\sigma$ is the unique solution of the equation (6). Using Lemma 3 in [9] it follows that for each $y \in H$, $\lim_{n\to\infty} V_{A_n+F}(t,s)y = V_{A+F}(t,s)y$ uniformly with respect to $0 \le s \le t \le T$ and (7) holds. $\square$

Remark 2.1 *Assume that P_1 holds and $\{A, B; G_i\}$ is stabilizable with the stabilizing sequence $F \in C_b([0,\infty), L(H,U))$ and let $h \in C_b(\mathbf{R}_+, H)$. Since $\{A + BF, G_i\}$ is uniformly exponentially stable it follows that there exist constants $M \ge 1, \omega > 0$ such as $\|V_{A+BF}(t,s)\| \le Me^{-\omega(t-s)}$ for all $t \ge s \ge 0$. Hence, the integral*

$$g(s) = \int_s^\infty V^*_{A+BF}(\sigma,s)h(\sigma)d\sigma \tag{8}$$

is convergent in H and $g(s)$ is bounded on $\mathbf{R}_+$. Moreover $\left\langle \int_t^T V^*_{A+BF}(\sigma,t)h(\sigma)d\sigma, x \right\rangle \underset{T\to\infty}{\to} \left\langle \int_s^\infty V^*_{A+BF}(\sigma,s)h(\sigma)d\sigma, x \right\rangle$. If we consider the solution of (6) with the initial condition $g_n(T) = g(T)$ it is not difficult to see that $\langle g_n(s), y\rangle \underset{n\to\infty}{\to} \langle g(s), y\rangle$ for all $y \in H$.

3. THE RICCATI EQUATION OF STOCHASTIC CONTROL AND THE UNIFORM OBSERVABILITY

Consider the system $\{A; G_i; C\}$ formed by equation $\{A; G_i\}$ and the observation relation $z(t) = C(t)y(t,s;x)$.

Definition 3.1 *[7] The system $\{A; G_i; C\}$ is uniformly observable if there exist $\tau > 0$ and $\gamma > 0$ such that $E\int_s^{s+\tau} \|C(t)y(t,s;x)\|^2 dt \ge \gamma \|x\|^2$ for all $s \in \mathbf{R}_+$ and $x \in H$.*

In the deterministic case it is known (see [6] for the autonomous case) that uniform observability implies detectability. In [9] we proved that this assertion is not true in the stochastic case.

Let us consider the Riccati equation

$$\begin{aligned} P'(s) + A^*(s)P(s) + P(s)A(s) + \sum_{i=1}^m G_i^*(s)P(s)G_i(s) + \\ + C^*(s)C(s) - P(s)B(s)(K(s))^{-1}B^*(s)P(s) = 0. \end{aligned} \tag{9}$$

If $A_n(t), n \in \mathbf{N}$ are the Yosida approximations of $A(t)$, then we introduce the approximate equation

$$P_n'(s) + A_n^*(s)P_n(s) + P_n(s)A_n(s) + \sum_{i=1}^{m} G_i^*(s)P_n(s)G_i(s) + \\ + C^*(s)C(s) - P_n(s)B(s)(K(s))^{-1}B^*(s)P_n(s) = 0. \tag{10}$$

Lemma 3.1 *[3] Let $0 < T < \infty$ and let $R \in L^+(H)$. Then there exists a unique mild (resp. classical) solution P (resp. P_n) of (3) (resp. (3)) on $[0,T]$ such that $P(T) = R$ (resp. $P_n(T) = R$). They are given by*

$$P(s)x = U^*(t,s)RU(t,s)x + \int_s^t U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)Q(r)G_i(r) + \\ + C^*(r)C(r) - P(r)B(r)(K(r))^{-1}B^*(r)P(r)]U(r,s)xdr \tag{11}$$

$$P_n(s)x = U_n^*(T,s)RU_n(T,s)x + \int_s^T U_n^*(r,s)[\sum_{i=1}^{m} G_i^*(r)P_n(r)G_i(r) + \\ + C^*(r)C(r) - P_n(r)B(r)(K(r))^{-1}B^*(r)P_n(r)]]U_n(r,s)xdr \tag{12}$$

and for each $x \in H$, $P_n(s)x \to P(s)x$ uniformly on any bounded subset of $[0,T]$. Moreover, if we denote these solutions by $P(s,T;R)$ and $P_n(s,T;R)$ respectively, then they are monotone in the sense that $P(s,T;R_1) \leq P(s,T;R_2)$ if $R_1 \leq R_2$.

We say [3] that P is a mild solution on an interval J of (3, if $P \in C_s(J, L^+(H))$ and if $P(s,T;P(s))$ satisfies (11) for all $s \leq t, s,t \in J$. Moreover, if P is a mild solution on $\mathbf{R}_+$ of (3) and $\sup\limits_{s \in \mathbf{R}_+} \|P(s)\| < \infty$, then P is said to be a *bounded solution.*

Remark 3.1 *From the above lemma it is easy to deduce that for all $\alpha, \beta \in \mathbf{R}_+, \alpha \leq \beta$ we have*

$$P(s,\alpha,0) \leq P(s,\beta,0) \tag{13}$$

for all $s \in [0,\alpha]$.

Definition 3.2 *A mild solution of (3) is called stabilizing for $\{A, B; G_i\}$ if $\{L \overset{not}{=} A - BS; G_i\}$ is uniformly exponentially stable, where $S(t) = K^{-1}(t)B^*(t)P(t)$.*

We introduce the following hypothesis:

P_3 : The evolution operator $U(t,s)$ has an exponentially growth, i.e. there exist M_0 and ω positive constants such that $\|U(t,s)\| \leq M_0 e^{\omega(t-s)}$.

Theorem 3.1 *[9] Assume that $\{A, G_i; C\}$ is uniformly observable and P_3 holds. If $P(t)$ is a nonnegative bounded solution of (3) then*

a) there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0, \infty)$,

b) P is a stabilizing solution (for the stochastic system $\{A, B; G_i\}$).

Proposition 3.1 *Under the assumptions of Theorem 9, the Riccati equation (3) has at most one nonnegative and bounded solution.*

Proof. Since it is not very difficult to prove (see Corollary 3.2 in [3]) that any bounded and stabilizing solution of (3) is maximal (see [3]) in the class of bounded solutions, the conclusion follows from the above theorem . □

The following theorem is a consequence of the above results.

Theorem 3.2 *Assume that P_3 holds, $\{A, G_i; B\}$ is stabilizable and $\{A, G_i; C\}$is uniformly observable. Then the Riccati equation (3) has a unique nonnegative bounded on $\mathbf{R}_+$ solution $P(t)$, which is a stabilizing solution and there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0, \infty)$ (P is uniformly positive on $\mathbf{R}_+$).*

Proof. From Theorem 4.1 in [3] it follows that under stabilizability conditions the equation (3) has a bounded solution on $\mathbf{R}_+$. From Theorem 3.1 and Proposition 3.1 we deduce the conclusion. □

Remark 3.2 *Assume that the hypotheses of the above theorem hold. a) From Lemma 4.1 and the stochastic version of the Theorem 3.1 from [3], it is easy to see that the unique uniformly positive solution $P(.)$ of the Riccati equation (3) is the strong limit of the monotone sequence of nonnegative operators $P(., \alpha, 0), \alpha \in \mathbf{R}_+$ (see (13)). Consequently, if $M = \lim_{\alpha \to \infty} \sup_{t \in [0,\alpha]} \|P(., \alpha, 0)\|$, then $P(t) \leq MI$ for all $t \in [0, \infty)$.*

b) Using the same type of proof as in [9] we can see that if $\delta = (1/2)\gamma - r_1 \widetilde{C}^2$, where $r_1 = (m+1)(M_0^2 e^{2\omega\tau} \widetilde{B}^2 1/\delta_0) \exp(\tau M_0^2 e^{2\omega\tau} (m+1) \sum_{i=1}^{m} \widetilde{G}_i^2)$, τ, γ, M_0, ω and δ_0 are those introduced by the Definition 3.1, P_3 and P_2 hold, then $P(t) \geq \delta I$ for all $t \in [0, \infty)$.

c) From the proof of the statement b) of the Theorem 3.1 (see [9]) it follows that the mild solution $z(t, s; x)$ of $\{L = A - BS; G_i\}$ satisfies the inequality

$$E \|z(t, s; x)\|^2 \leq \beta e^{-\alpha(t-s)} \|x\|^2 \text{ for all } t \geq s \geq 0 \text{ and } x \in H,$$

where $\alpha = -\frac{\ln\left(1 - \frac{\delta}{M}\right)}{\tau}$ and β depends only on M_0, ω, τ (which are introduced above) and m. Hence the evolution operator U_L, associated with the operator L, satisfies the inequality $\|U_L(t,s)\|^2 \leq \beta e^{-\alpha(t-s)}$too.

Assume that the hypotheses of the Theorem 11 holds. Theorem 13 gives an estimate for the convergence rate of the sequence $P(.,T,0)$ to the unique solution of the Riccati equation (3).

Theorem 3.3 *Assume that* P_3 *holds,* $\{A, G_i; B\}$ *is stabilizable and* $\{A, G_i; C\}$*is uniformly observable. If* $P(.)$ *is the unique nonnegative and stabilizing solution of the Riccati equation (8), then*

$$\|P(t) - P(t,T,0)\| \leq \varepsilon$$

for all $t \leq \Im \leq T$ *and* $T \geq T_{\varepsilon,\Im} = -\alpha \ln \frac{\varepsilon}{\widehat{C} e^{\frac{\widehat{K}}{\alpha}}}$.

Proof. If $P(.)$ is the solution of the Riccati equation (8) such as the system $\{L = A - BS; G_i\}$ is uniformly exponentially stable, then [3] $P(.) - P(.,T,0)$ is a mild solution of the equation

$$Z' + L^* Z + ZL + \sum_{i=1}^{m} G_i^* Z G_i + ZB(K)^{-1} B^* Z = 0$$

with the final condition $Z(T) = P(T)$. Denote $\widehat{K} = \beta \left(2 \frac{M \widetilde{B}^2}{\delta_0} + \sum_{i=1}^{m} \widetilde{G}_i^2 \right) e^{\alpha \Im}$ and $\widehat{C} = M \beta e^{\alpha \Im}$, where $\Im > 0$ is fixed, we use the integral equation (see (11)) satisfied by Z, the statement a) of the above remark and the Gronwall's inequality to get

$$\|P(t) - P(t,T,0)\| \leq \widehat{C} e^{-\alpha T} + \int_t^T \widehat{K} e^{-\alpha u} \|P(u) - P(u,T,0)\| \, du \text{ and}$$

$$\|P(t) - P(t,T,0)\| \leq \widehat{C} e^{-\alpha T} e^{\frac{\widehat{K}}{\alpha}} \text{ for all } t \leq \Im \leq T. \tag{14}$$

Now, obviously, for all $t < \Im$, $\varepsilon > 0$ and $T \geq T_{\varepsilon,\Im} = -\alpha \ln \frac{\varepsilon}{\widehat{C} e^{\frac{\widehat{K}}{\alpha}}}$ we have $\|P(t) - P(t,T,0)\| \leq \varepsilon$. □

4. OPTIMAL QUADRATIC CONTROL

A consequence of Theorem 3.2, Theorem 4.2 and the stochastic version of Theorem 3.1 from [3] is the following theorem:

Theorem 4.1 *Assume that the hypotheses of the Theorem 11 are fulfilled and consider the control problem (1), (2). The optimal control is given by the feedback law*

$$\widetilde{u}(t) = -K(t)^{-1} B^*(t) P(t) y(t),$$

where P is the unique bounded positive solution of (3) ($y(t)$ is the corresponding solution of (1)) and the optimal cost is

$$I_s(\widetilde{u}) = \langle P(s)x, x\rangle . \tag{15}$$

If all operators in (1) and (2) are time invariant, then $P(.)$ is constant.

The following conclusion provides a numerical method for the computation of the optimal cost.

Conclusion 1 *Assume that the hypotheses of the above theorem hold and let $H = R^n$. From Remark 3.2 we deduce an algorithm which allows us to obtain the optimal cost (15) in the finite dimensional case. Let $\varepsilon > 0$ and $t \in [0, \mathcal{T}]$ be fixed.*

1. We can use the Runge Kutta method to obtain the solutions $P(t, \alpha, 0)$, $\alpha \in \mathbf{R}_+$ of the Riccati equation (3). We compute the constant M from the statement a) of Remark 12 (as a limit of an monotonously increasing sequence of real numbers). Consequently we can obtain all the constants from relation (14).

3. Using the constant $T_{\varepsilon,\mathcal{T}}$, introduced in Section 3, we see that $P(t, T_{\varepsilon,\mathcal{T}}, 0)$ is a good approximation (with the error ε) of the solution $P(t), t \in [0, \mathcal{T}]$ of the Riccati equation (3). Hence $\left\langle P(t, T_{\varepsilon,\mathcal{T}}, 0)x, x\right\rangle$ is an approximation of the optimal cost (15)

5. TRACKING PROBLEM

Consider the set of admissible controls $\mathcal{U}_{ad} = \{u$ is an U- valued random variable, F_s- measurable such as $\overline{\lim\limits_{t\to\infty}} \frac{1}{t-s} E \int\limits_s^t \|u(\sigma)\|^2 \, d\sigma < \infty$ and $\sup\limits_{t\geq s} E \|y(t)\|^2 < \infty$, where $y(t)$ is the solution of (1)$\}$.

Theorem 5.1 *Assume that the hypotheses of Theorem 3.2 hold. Let P be the unique and bounded on $\mathbf{R}_+$ solution of the Riccati equation (3) and $g(t)$ be given by (8), where $F(t) = -K^{-1}(t)B^*(t)P(t)$ and $h(t) = C^*(t)C(t)r(t)$. Then the optimal control is*

$$u(\sigma) = K^{-1}(\sigma)B^*(\sigma)[g(\sigma) - P(\sigma)y(\sigma)]$$

and the optimal cost is

$$J(s) = \inf_{u\in U_{ad}} J(s, u) =$$

$$\overline{\lim_{t\to\infty}} \frac{1}{t-s} [\int_s^t \|C(\sigma)r(\sigma)\|^2 \, d\sigma - \int_s^t ||K^{-1/2}B^*(\sigma)g(\sigma)||^2 d\sigma].$$

Proof. Let $P_n(s) = P_n(s, t_1; P(t_1))$ and let $g_n(s)$ be the solution of (6) with the final condition $g_n(t_1) = g(t_1)$, where $F_n(t) = -B(t)S_n(t)$, $S_n(t) = K^{-1}(t)B^*(t)P_n(t)$ and $h(t) = C^*(t)C(t)r(t)$. Consider the function $F_n(t,x) = \langle P_n(t)x, x\rangle - 2\langle g_n(t), x\rangle$, which is continuous together its partial derivatives F_t, F_x, F_{xx} on $[0,\infty) \times H$, according to Lemmas 3.1 and 2.1. Let $u \in \mathcal{U}_{ad}$ and let $y_n(t)$ be its response, where the approximate system associated with (1) is considered [3]. Using the Ito's formula for $F_n(t,x)$ and $y_n(t)$ we get

$$E\langle P_n(t_1)y_n(t_1), y_n(t_1)\rangle - 2E\langle g_n(t_1), y_n(t_1)\rangle - \langle P(s)x, x\rangle + 2\langle g_n(s), x\rangle =$$
$$-E\int_s^{t_1} \|C(\sigma)[y_n(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle\, d\sigma +$$
$$E\int_s^{t_1} \left\|K^{1/2}\{S_n(\sigma)y_n(\sigma) + u(\sigma) - K^{-1}B^*(\sigma)g_n(\sigma)\}\right\|^2 d\sigma$$
$$+\int_s^{t_1} \|C(\sigma)r(\sigma)\|^2\, d\sigma - \int_s^{t_1} ||K^{-1/2}B^*(\sigma)g_n(\sigma)||^2 d\sigma$$
$$+2E\int_s^{t_1} \langle (S_n(\sigma) - S(\sigma))\, y_n(\sigma), B^* g_n(\sigma)\rangle\, d\sigma.$$

Passing to the limit as $n \to \infty$ we obtain

$$E\langle P(t_1)y(t_1), y(t_1)\rangle - 2E\langle \widetilde{g}(t_1), y(t_1)\rangle - \langle P(s)x, x\rangle + 2\langle g(s), x\rangle =$$
$$-E\int_s^{t_1} \|C(\sigma)[y(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle\, d\sigma +$$
$$E\int_s^{t_1} \left\|K^{1/2}\{S(\sigma)y(\sigma) + u(\sigma) - K^{-1}B^*(\sigma)g(\sigma)\}\right\|^2 d\sigma$$
$$+\int_s^{t_1} \|C(\sigma)r(\sigma)\|^2\, d\sigma - \int_s^{t_1} ||K^{-1/2}B^*(\sigma)g(\sigma)||^2 d\sigma.$$

Since $P(t)$ and $g(t)$ are bounded on $\mathbf{R}_+$, we multiply the last relation by $\frac{1}{t_1 - s}$ and taking the limit as $t_1 \to \infty$ and, then, the infimum we get the conclusion. □

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