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DETECTING ORDER AND CHAOS BY THE LINEAR DEPENDENCE INDEX (LDI) METHOD

Chris Antonopoulos, Tassos Bountis
Univ. of Patras, Greece
antonop@math.upatras.gr, bountis@math.upatras.gr

Abstract We introduce a new methodology for a fast and reliable discrimination between ordered and chaotic orbits in multidimensional Hamiltonian systems which we call the Linear Dependence Index (LDI). The new method is based on the recently introduced theory of the Generalized Alignment Indices (GALI). LDI takes advantage of the linear dependence (or independence) of deviation vectors of a given orbit, using the method of Singular Value Decomposition at every time step of the integration. We show that the LDI produces estimates which numerically coincide with those of the GALI method for the same number of m deviation vectors, while its main advantage is that it requires considerable less CPU time than GALI especially in Hamiltonian systems of many degrees of freedom.

1. INTRODUCTION

The fast and reliable discrimination of chaotic and ordered orbits of conservative dynamical systems is of crucial interest in many problems of nonlinear science. By the term conservative we characterize here systems which preserve phase space volume (or some other positive function of the phase space variables) during time evolution. Important examples in this class are $N-$degree of freedom (dof) Hamiltonian systems and $2N-$dimensional symplectic maps. As is well known, in such systems chaotic and regular orbits are distributed in phase space in very complicated ways, which often makes it very difficult to distinguish between them. In recent years, several methods have been developed and applied to various problems of physical interest in an effort to distinguish efficiently between ordered and chaotic dynamics. Their discrimination abilities and overall performance, however, varies significantly, making some of them more preferable than others in certain situations.

One of the most common approaches is to extract information about the nature of a given orbit from the dynamics of small deviations, evaluating the maximal Lyapunov Characteristic Exponent (LCE) σ_1. If $\sigma_1 > 0$ the orbit is characterized as chaotic. The theory of Lyapunov exponents was first applied to characterize chaotic orbits by Oseledec [19], while the connection between Lyapunov exponents and exponential divergence of nearby orbits was given in

[8, 21]. Benettin et al. [6] studied the problem of the computation of all LCEs theoretically and proposed in [7] an algorithm for their efficient numerical computation. In particular, σ_1 is computed as the limit for $t \to \infty$ of the quantity

$$L_1(t) = \frac{1}{t} \ln \frac{\|\mathbf{w}(t)\|}{\|\mathbf{w}(0)\|}\,, \text{ i.e. } \sigma_1 = \lim_{t\to\infty} L_1(t)\,, \tag{1}$$

where $\mathbf{w}(0)$, $\mathbf{w}(t)$ are deviation vectors from a given orbit, at times $t = 0$ and $t > 0$ respectively. It has been shown that the above limit is finite, independent of the choice of the metric for the phase space and converges to σ_1 for almost all initial vectors $\mathbf{w}(0)$ [19, 6, 7]. Similarly, all other LCEs, σ_2, σ_3 etc. are computed as limits for $t \to \infty$ of some appropriate quantities, $L_2(t)$, $L_3(t)$ etc. (see for example [7]). We note here that throughout the paper, whenever we need to compute the values of the maximal LCE or of several LCEs we apply respectively the algorithms proposed by Benettin et al. [8, 7].

Over the years, several variants of this approach have been introduced to distinguish between order and chaos such as: The Fast Lyapunov Indicator (FLI) [15, 14, 12, 16, 5], the Mean Exponential Growth of Nearby Orbits (MEGNO) [11, 10], the Smaller Alignment Index (SALI) [24, 25, 26], the Relative Lyapunov Indicator (RLI) [23], as well as methods based on the study of power spectra of deviation vectors [29] and spectra of quantities related to these vectors [13, 17, 28].

Recently, the SALI method was generalized to yield a much more comprehensive approach to study chaos and order in $2N$ – dimensional conservative systems, called the GALI$_m$ indices [27, 2]. These indices represent the volume elements formed by m deviation vectors ($2 \leq m \leq 2N$) about any reference orbit and have been shown to: (a) Distinguish the regular or chaotic nature of the orbit faster than other methods, (b) identify the dimensionality of the space of regular motion and (c) predict the slow (chaotic) diffusion of orbits, long before it is observed in the actual oscillations.

In the present paper, we improve the GALI method by introducing the Linear Dependence Indices (LDI$_m$). The new indices retain the advantages of the GALI$_m$ and display the same values as GALI, in regular as well as chaotic cases. More importantly, however, the computation of the LDI$_m$ is much faster in CPU time, especially if the dimensionality of phase space becomes large ($N \gg 10$). The main purpose of this paper, therefore, is to strongly advocate the use of LDI, for the most rapid and efficient study of the dynamics of multi – dimensional conservative systems.

For the computation of the LDI$_m$ we use information from the evolution of $m \geq 2$ deviation vectors from the reference orbit, as GALI does. However, while GALI requires the computation of many $m \times m$ determinants at every time step [27, 2] in order to evaluate the norm of the corresponding wedge product, LDI achieves the same purpose simply by applying Singular Value

Decomposition (SVD) to the $2N \times m$ matrix formed by the deviation vectors. LDI is then computed as the product of the corresponding singular values of the above matrix. This not only provides the same numerical values as the corresponding GALI$_m$, it also requires much less CPU time.

The paper is organized as follows: In section 2 we introduce the new index, explain in detail its computation and justify its validity theoretically. In section 3, we demonstrate the usefulness of the LDI method, by applying it to the famous Fermi – Pasta – Ulam (FPU) lattice model of N dof, for small and large N. Finally, in section 4 we present our conclusions, highlighting especially the advantages of the new index.

2. DEFINITION OF THE LINEAR DEPENDENCE INDEX (LDI)

Let us consider the $2N$ – dimensional phase space of a Hamiltonian system

$$H \equiv H(q_1(t), \ldots, q_N(t), p_1(t), \ldots, p_N(t)) = E \tag{2}$$

where $q_i(t),\ i = 1, \ldots, N$ are the canonical coordinates, $p_i(t),\ i = 1, \ldots, N$ are the corresponding conjugate momenta and E is the total energy. The time evolution of an orbit $\mathbf{x}(t)$ of (1) associated with the initial condition

$$\mathbf{x}(t_0) = (q_1(t_0), \ldots, q_N(t_0), p_1(t_0), \ldots, p_N(t_0))$$

at initial time t_0 is defined as the solution of the system of $2N$ first order differential equations (ODE)

$$\frac{dq_i(t)}{dt} = \frac{\partial H}{\partial p_i(t)}, \quad \frac{dp_i(t)}{dt} = -\frac{\partial H}{\partial q_i(t)},\ i = 1, \ldots, N. \tag{3}$$

Eqs. (13) are known as Hamilton's equations of motion and the reference orbit under study is the solution $\mathbf{x}(t)$ which passes by the initial condition $\mathbf{x}(t_0)$.

In order to define the Linear Dependence Index (LDI) we need to introduce the variational equations. These are the corresponding linearized equations of the ODE (13), about the reference orbit $\mathbf{x}(t)$ defined by the relation

$$\frac{d\mathbf{v}_i(t)}{dt} = \mathcal{J}(\mathbf{x}(t)) \cdot \mathbf{v}_i(t),\ i = 1, \ldots, 2N \tag{4}$$

where $\mathcal{J}(\mathbf{x}(t))$ is the Jacobian of the right hand side of the system of ODEs (13) calculated about the orbit $\mathbf{x}(t)$. Vectors $\mathbf{v}_i(t) = (\upsilon_{i,1}(t), \ldots, \upsilon_{i,2N}(t)),\ i = 1, \ldots, 2N$ are known as deviation vectors and belong to the tangent space of the reference orbit at every time t.

We then choose $m \in [2, 2N]$ initially linearly independent deviation vectors $\mathbf{v}_m(0)$ and integrate equation (3) together with the equations of motion (13).

These vectors form the columns of a $2N \times m$ matrix $\mathcal{A}(t)$ and are taken to lie along the orthogonal axes of a unit ball in the tangent space of the orbit $\mathbf{x}(t)$ so that $\mathbf{v}_m(0)$ are orthonormal. Thus, at every time step, we check the linear dependence of the deviation vectors by performing Singular Value Decomposition on $\mathcal{A}(t)$ decomposing it as follows

$$\mathcal{A}(t) = U(t) \cdot W(t) \cdot V(t)^{\top}, \tag{5}$$

where $U(t)$ is a $2N \times m$ matrix, $V(t)$ is an $m \times m$ matrix whose columns are the $\mathbf{v}_m(t)$ deviation vectors and $W(t)$ is a diagonal $m \times m$ matrix, whose entries $w_1(t), \ldots, w_m(t)$ are zero or positive real numbers. They are called the *singular values* of $\mathcal{A}(t)$. Matrices $U(t)$ and $V(t)$ are orthogonal so that $U^{\top}(t) \cdot U(t) = V^{\top}(t) \cdot V(t) = I$, where I is the rectangular $2N \times 2N$ unit matrix.

We next define the generalized Linear Dependence Index of order m or LDI_m as the function

$$\mathrm{LDI}_m(t) = \prod_{j=1}^{m} w_j(t) \tag{6}$$

with $m = 2, 3, \ldots, 2N$, where N is the number of dof of (1).

The reason for defining LDI through relation (6) is the following: According to [27] it is possible to determine whether an orbit is chaotic or lies on a d – dimensional torus by choosing m deviation vectors and computing the GALI_m index. If $\mathrm{GALI}_m \approx$ const. for $m = 2, 3, \ldots, d$ and for $m > d$ decay by a power law, the motion lies on a d – dimensional torus. If, on the other hand, all GALI_m indices decay exponentially the motion is chaotic. Thus, to characterize orbits we often have to compute GALI_m indices for m as high as N or higher.

A serious limitation appears, of course, in the case of Hamiltonian systems of large N, where $\mathrm{GALI}_N(t)$ involves the computation of $\binom{2N}{N} = \frac{(2N)!}{(N!)^2}$ determinants at every time step. For example, in a Hamiltonian system of $N = 15$ dof, $\mathrm{GALI}_{15}(t)$ requires, for a given orbit, the computation of 155117520 determinants at every time step while $\mathrm{LDI(t)} = \mathrm{LDI}_{15}(t)$ requires only the application of the SVD method for a 30×15 matrix $\mathcal{A}(t)$!

Clearly, at every point of the orbit $\mathbf{x}(t)$ the $2 \leq m \leq 2N$ deviation vectors span a subspace of the $2N$ – dimensional tangent space of the orbit, which is isomorphic to the Euclidean $2N$ – dimensional phase space of the Hamiltonian system (13). Thus, if k of the m singular values $w_k(t), k = 1, \ldots, m$ are equal to zero, then k columns of matrix $\mathcal{A}(t)$ of deviation vectors are linearly dependent with the remaining ones and the subspace spanned by the column vectors of matrix $\mathcal{A}(t)$ is $d(= m - k)$ – dimensional.

From a more geometrical point of view, let us note that the m variational equations (3) combined with the equations of motion (13) describe the evolution of an initial m – dimensional unit ball into an m (or less) dimensional ellipsoid in the tangent space of the Hamiltonian flow. Now, the deviation vectors $\mathbf{v}_i(t)$ forming the columns of $\mathcal{A}(t)$ do not necessarily coincide with the ellipsoid's principal axes. On the other hand, in the case of a chaotic orbit, every generically chosen initial deviation vector has a component in the direction of the maximum (positive) Lyapunov exponent, so that all initial tangent vectors in the long run, will be aligned with the longest principal axis of the ellipsoid. The key idea behind the LDI method is to take advantage of this fact to overcome the costly calculation of the many determinants arising in the GALI_m method and characterize a reference orbit as chaotic or not, via the trends of the stretching and shrinking of the m principal axes of the ellipsoid.

Thus, LDI solves the problem of orbit characterization by finding new orthogonal axes for the ellipsoid at every time step and taking advantage of the SVD method. Since the matrix V in (3) is orthogonal, we have $V^\top = V^{-1}$, so that equation (5) gives

$$\mathcal{A}_{2N\times m}\cdot V_{m\times m} = U_{2N\times m}\cdot W_{m\times m} \tag{7}$$

at every time step. Geometrically, Eq. (24.0) implies that the image formed by the column vectors of matrix V is equal to an ellipsoid whose i^{th} principal axis direction in the tangent space of the reference orbit is given by:

$$w_i\cdot u_i \tag{8}$$

where w_i are the singular values of matrix $\mathcal{A}(t)$ and u_i is the i^{th} column of matrix $U(t)$. This is, in fact, the content of a famous theorem stating that:

Theorem 2.1 *([1]) Let $\mathcal{A}$ be a $2N\times m$ matrix, and let U and W be matrices resulting from the SVD of $\mathcal{A}$. Then, the columns of $\mathcal{A}$ span an ellipsoid whose i^{th} principal axis is $w_i\cdot u_i$, where $W = diag(w_1, w_2, \dots, w_m)$ (singular values) and $\{u_i\}_{i=1}^m$ are the columns of U.*

According to this theorem, the principal axes of the ellipsoid created by the time evolution of equation (3) in the tangent space of the reference orbit $\mathbf{x}(t)$ at every time t, are stretched or shrunk, according to the singular values of $w_i > 1$ or $w_i < 1$ respectively for $i = 1, \dots, m$.

If it so happens that k of the singular values $w_i = 0$ as t grows, then the corresponding principal axes of the ellipsoid vanish and the ellipsoid is less than m – dimensional in the tangent space of the reference orbit because the corresponding deviation vectors of matrix $\mathcal{A}$ have become linearly dependent.

Thus, two distinct cases exist depending on whether the reference orbit $\mathbf{x}(t)$ is chaotic or ordered

1 If the orbit is chaotic, the m deviation vectors become linearly dependent so that $\mathrm{GALI_m}(t) \to 0$ exponentially [27]. Consequently, at least one of the singular values $w_i(t), i = 2, \ldots, m$ becomes zero and $\mathrm{LDI}_m(t) = \prod_{j=1}^{m} w_j(t) \to 0$ (also $\mathrm{LDI}(t) \to 0$) for all $m \geq i$.

2 If the orbit is ordered (i.e. quasiperiodic) lying on a d – dimensional torus, there is no reason [27, 25] for the m deviation vectors to become linearly dependent, as long as $m \leq d$. No principal axis of the ellipsoid is eliminated, since all singular values $w_i,\ i = 1, \ldots, m$ are nonzero and $\mathrm{LDI_m}(t)$ fluctuates around nonzero positive values. On the other hand, for $m \geq d$, the singular values $w_i,\ i = d+1, \ldots, m$ tend to zero following a power law [27], since $m - d$ deviations will eventually become linearly dependent with those spanning the d – dimensional tangent space of the torus [27, 25].

In the remainder of the paper, we apply the LDI indices and numerically demonstrate that

$$\mathrm{LDI}_m = \mathrm{GALI}_m, \quad m = 2, \ldots, 2N \tag{9}$$

for the same choice of m initially linearly independent deviation vectors $\mathbf{v}_i(0),\ i = 1, \ldots, m$. In particular, we present evidence that supports the validity of relation (9) and exploit it to identify rapidly and reliably ordered and chaotic orbits in a 1 – dimensional, N degree of freedom Fermi – Pasta – Ulam lattice under fixed and periodic boundary conditions [4, 3]. We propose that the validity of (9) is due to the fact that both quantities measure the volume of the same ellipsoid, the difference being that, in the case of the LDI, the principal axes of the ellipsoid are orthogonal. As we have not proved it, however, this is a point to which we intend to return in a future publication.

3. APPLICATION TO THE FPU HAMILTONIAN SYSTEM

In this section, we apply the LDI method to the case of a multidimensional Hamiltonian system. Our aim is the comparison of its performance and effectiveness in distinguishing between ordered and chaotic behavior compared with Lyapunov exponents as well as the SALI and GALI methods.

We shall use the N dof Hamiltonian system of the 1D lattice of the Fermi – Pasta – Ulam (FPU) β – model. The system is described by a Hamiltonian function containing quadratic and quartic nearest neighbor interactions

$$H_N = \frac{1}{2}\sum_{j=1}^{N} \dot{x}_j^2 + \sum_{j=0}^{N}\left(\frac{1}{2}(x_{j+1} - x_j)^2 + \frac{1}{4}\beta(x_{j+1} - x_j)^4\right) = E \tag{10}$$

where x_j is the displacement of the j^{th} particle from its equilibrium position, $\dot{x}_j$ is the corresponding conjugate momentum, β is a positive real constant and E is the constant energy of the system.

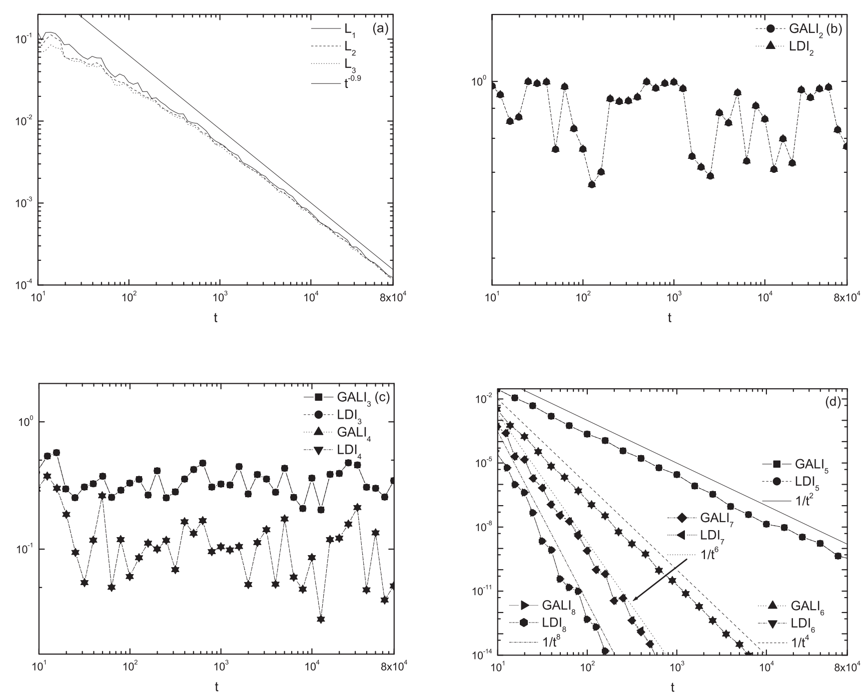

Fig. 1. The case of an ordered orbit:(a) The time evolution of the three maximal positive Lyapunov exponents. (b) The time evolution of GALI_2 and LDI_2. (c) The time evolution of GALI_3, LDI_3 and GALI_4, LDI_4. (d) The time evolution of GALI_m, LDI_m with $m = 5, \ldots, 8$ and the corresponding slopes of the fall to zero. In all panels we have chosen a neighboring orbit at a distance ≈ 2.1 from the OPM of the FPU Hamiltonian (10) with periodic boundary conditions for $N = 4$ and $E = 2$. All axes are logarithmic.

We start by focusing on an ordered case choosing a neighboring orbit of the stable out of phase mode (OPM) [9, 20, 4], which is a simple periodic orbit of the FPU Hamiltonian (10). This solution exists for every N, for fixed as well as periodic boundary condition (PBC)

$$x_{N+1}(t) = x_1(t), \ \forall t \tag{11}$$

and is given by

$$x_j(t) = -x_{j+1}(t), \ \dot{x}_j(t) = 0, \ j = 1, \ldots, N, \forall t. \tag{12}$$

In [9, 4] the stability properties of the OPM mode with periodic boundary conditions were determined using Floquet theory and monodromy matrix analysis and the energy range $0 \leq E(N) \leq E_c^{\mathrm{OPM}}(N)$ over which it is linearly stable was studied in detail.

It is known that for $N = 4$ and $\beta = 1$, the solution (12) with periodic boundary condition (12) is destabilized for the first time at the critical energy $E_c^{\mathrm{OPM}} \approx 4.51$. Below this critical energy, the OPM is linearly stable and is surrounded by a sizable island of stability. By contrast, for $E > E_c^{\mathrm{OPM}}$, the OPM is linearly unstable with no island of stability around it.

In Fig. 1(a), we have calculated the three maximal Lyapunov exponents of a neighboring orbit located at distance ≈ 2.1 away from the OPM at $E = 2 < E_c^{\mathrm{OPM}}$. At this energy, the OPM is linearly stable and thus all Lyapunov Exponents tend to zero following a simple power law. Next, in Fig. 1(b), we compute GALI_2 and LDI_2 for a final integration time $t = 8 \times 10^4$ and observe that GALI_2 and LDI_2 practically coincide fluctuating around non zero values indicating the ordered nature of the orbit. GALI_2 needs 558 seconds of computation time while LDI_2 takes about 912 seconds in a Pentium 4 3.2GHz computer.

In Fig. 1(c), we compute GALI_3, LDI_3 and GALI_4, LDI_4 for the same energy and initial condition. We see once more that GALI_3, LDI_3 and GALI_4, LDI_4 coincide fluctuating around non zero values. The GALI_3 computation now takes about 1044 seconds, LDI_3 about 838 seconds, GALI_4 needs 898 seconds and LDI_4 753 seconds.

Finally, in Fig. 1(d), we present GALI_m, LDI_m with $m = 5, \ldots, 8$ as a function of time for the same energy and initial condition. We observe again that GALI_m and LDI_m with $m = 5, \ldots, 8$, have the same values and tend to zero following a power law of the form $t^{-2(k-N)}$. All these results are in accordance with the formulae reported in [27] and suggest that the torus on which the orbit lies is 4 – dimensional, as expected from the fact that the number of dof of the system is $N = 4$.

In [4] we also studied the stability properties of a different simple periodic orbit of FPU called the SPO1 mode with fixed boundary conditions (FBC). Using monodromy matrix analysis we found that for $N = 5$ and $\beta = 1.04$, the SPO1 mode with FBC is destabilized for the first time at the critical energy $E_c^{\mathrm{SPO1}} \approx 6.4932$.

Thus, in order to study a chaotic case where things are different, we choose initial condition at distance of $\approx 1.27 \times 10^{-4}$ from the SPO1 orbit at the energy $E = 11$, where it is unstable.

In Fig. 2(a), we calculate Lyapunov exponents of the above mentioned orbit and find that the four maximal Lyapunov exponents tend to positive values. This is strong evidence that the nature of the orbit is chaotic. Next, in Fig. 2(b) we calculate GALI_2 and LDI_2 up to $t = 1200$. We see the indices again

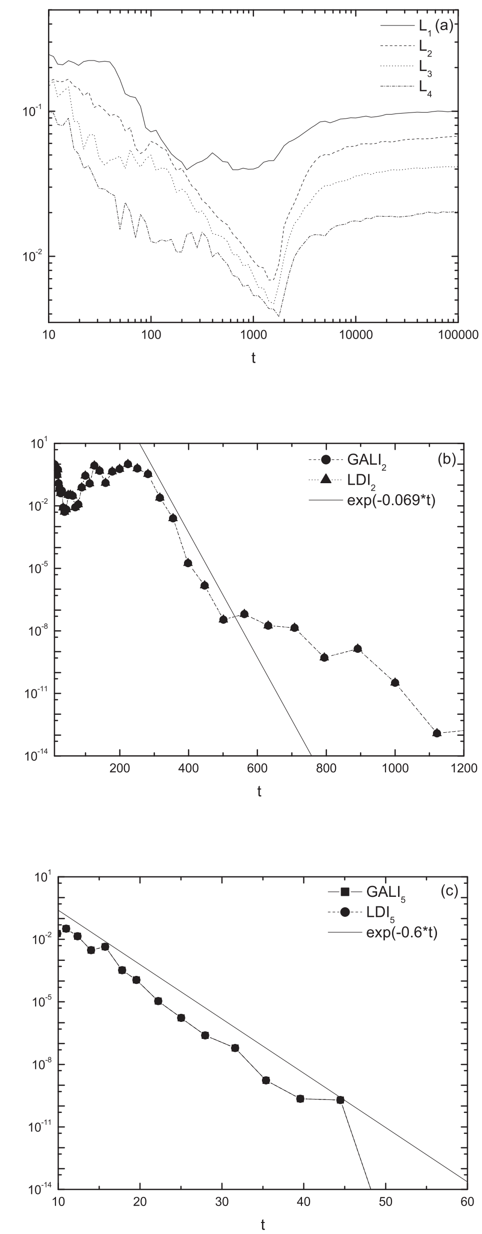

Fig. 2. The case of a chaotic orbit:(a) The time evolution of the four maximal Lyapunov exponents. (b) The time evolution of GALI_2, LDI_2 follows the approximate formula $e^{-(\sigma_1-\sigma_2)t}$ where $\sigma_1 \approx 0.124$ (solid straight line) and $\sigma_2 \approx 0.056$ for $t = 71$. (c) The time evolution of GALI_5, LDI_5 follows the approximate formula $\propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+(\sigma_1-\sigma_4)+(\sigma_1-\sigma_5)]t} \approx e^{-0.069t}$ (solid straight line) where $\sigma_1 \approx 0.197$, $\sigma_2 \approx 0.095$, $\sigma_3 \approx 0.047$, $\sigma_4 \approx 0.026$ and $\sigma_5 \approx 0.022$ for $t \approx 44$. We have used, in all figures, the same orbit of a distance of 1.27×10^{-4} from the SPO1 of the FPU Hamiltonian system (10) with fixed boundary conditions for $N = 5$ and $E = 11$.

coincide and tend to zero as $\propto e^{-(\sigma_1-\sigma_2)t}$ (solid straight line), as predicted by our theory [26, 27]. In this figure, we find $\sigma_1 \approx 0.124$ and $\sigma_2 \approx 0.056$ for time $t = 71$. The corresponding CPU time required for the calculation of all indices does not differ significantly, as they become quite small in magnitude, rather quickly.

Nevertheless, LDI_2 requires less CPU time than GALI_2. In Fig. 2(c), we calculate GALI_5 and LDI_5 for the same energy and initial condition as in the previous panels. We observe now that GALI_5 and LDI_5 coincide falling to zero as $\text{GALI}_5 \propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+(\sigma_1-\sigma_4)+(\sigma_1-\sigma_5)]t}$ (solid straight line) where $\sigma_1 \approx 0.197$, $\sigma_2 \approx 0.095$, $\sigma_3 \approx 0.047$, $\sigma_4 \approx 0.026$ and $\sigma_5 \approx 0.022$ for $t \approx 44$. Clearly, GALI_5 and LDI_5 distinguish the chaotic character of the orbit faster than GALI_2 or LDI_2. This is so, because GALI_2 or LDI_2 reaches the threshold 10^{-8} [25, 26, 27] for $t \approx 750$ while GALI_5 and LDI_5 for $t \approx 35$! The CPU times required for the calculation of GALI_5 and LDI_5 up to $t = 80$ are approximately 1.5 seconds each.

Thus, we conclude from these results that the LDI method performs at least as well as the GALI, predicting correctly the ordered or chaotic nature of orbits in Hamiltonian systems for low dimensions, i.e. at 2, 4 and 5 degrees of freedom. However, in higher dimensional cases, GALI indices become very impractical as they demand the computations of millions of determinants at every time step making the LDI method much more useful.

In order to show the advantages of the LDI method concerning the CPU time, we repeat the above analysis for the same Hamiltonian system (10), but now for $N = 15$ and energy $E = 2$, and for an initial condition very close to the unstable SPO1 [4].

In [4] it has also been shown that for $N = 15$ and $\beta = 1.04$, the SPO1 with fixed boundary conditions destabilizes at the critical energy $E_c \approx 1.55$. Thus, for energies smaller than E_c, SPO1 is linearly stable, while for $E > E_c$ it is unstable and is surrounded by a chaotic region.

In Fig. 3(a) we depict the time evolution of the five maximal Lyapunov exponents which converge to positive values for high enough t suggesting that the neighboring orbit is chaotic. In the second panel of the same figure we present the evolution of GALI_8 and LDI_8 together with the approximate exponential law. We remark once more that the values of the corresponding indices coincide until they become numerically zero. More interestingly, the CPU time required for the calculation of GALI_8 up to $t \approx 100$ is about 186 seconds while for the LDI_8 it takes only one second! This difference is very important, showing why LDI is preferable compared to the corresponding GALI index in Hamiltonian systems of many degrees of freedom.

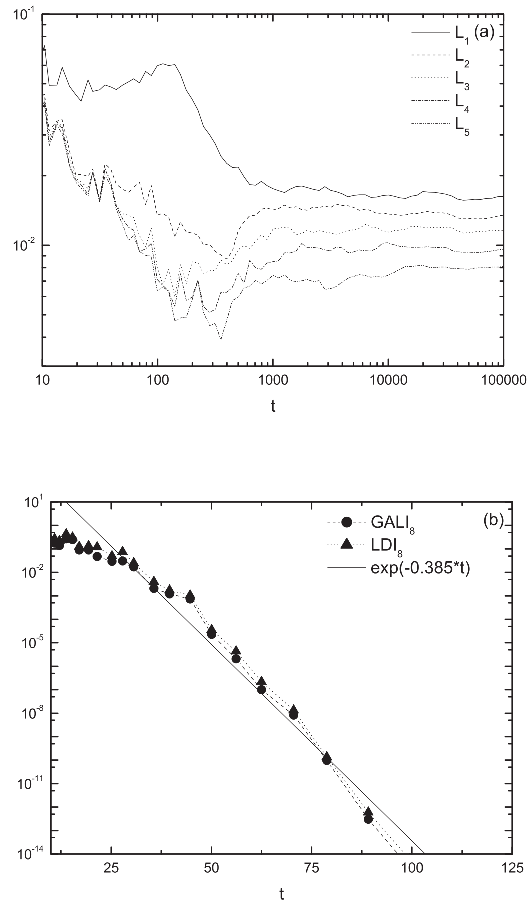

Fig. 3. (a) The time evolution of the five positive Lyapunov exponents. (b) The time evolution of the GALI_8, LDI_8 follows the approximate formula $\propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_8)]t} \approx e^{-0.385t}$ (solid straight line) where $\sigma_1 \approx 0.061$, $\sigma_2 \approx 0.011$, $\sigma_3 \approx 0.006$, $\sigma_4 \approx 0.005$, $\sigma_5 \approx 0.005$, $\sigma_6 \approx 0.004$, $\sigma_7 \approx 0.004$ and $\sigma_8 \approx 0.004$ for time $t \approx 141$. In all panels we have used initial conditions at a distance of 9×10^{-5} from the SPO1 orbit of Hamiltonian system (10) with fixed boundary conditions, $N = 15$ and $E = 2$.

4. CONCLUSIONS

In this paper we have introduced a new method for distinguishing quickly and reliably between ordered and chaotic orbits of multidimensional Hamiltonian systems and argued about its validity justifying it in the ordered and chaotic case. It is based on the recently introduced theory of the Generalized Alignment Indices (GALI). Following this theory, the key point in the distinction between order and chaos is the linear dependence (or independence) of deviation vectors from a reference orbit. Consequently, the method of LDI takes advantage of this property and analyzes m deviation vectors using Singular Value Decomposition to decide whether the reference orbit is chaotic or ordered. If the orbit under consideration is chaotic then the deviation vectors are aligned with the direction of the maximal Lyapunov exponent and thus become linearly dependent. On the other hand, if the reference orbit is ordered then there is no unstable direction and $m = 1, 2, ..., d \leq N$ deviation vectors are linearly independent. As a consequence, the LDI of order m (LDI_m) becomes either zero if the reference orbit is chaotic or it fluctuates around non zero values if the orbit is ordered if $m \leq d$.

After introducing the new method, we presented strong numerical evidence about its validity and efficiency in the interesting case of multidimensional Hamiltonian systems. One first main result is that GALI_m and LDI_m coincide numerically for the same m number of deviation vectors and for the same reference orbit. Moreover, it follows that it is preferable to use the LDI method rather than the equivalent GALI method especially in the multidimensional case of Hamiltonian systems, since the LDI needs considerably less CPU time than the corresponding GALI method for the same number of deviation vectors.

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