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THEORETICAL PROPERTIES IN HYPERGRAPHS

Silviu Dumitrescu, Bogdana Pop, Vlad Monescu
"Transilvania" University of Brasov, Computer Science Department

Abstract Well-known relations in graphs are adapted to the hypergraphs. The correctness of these relations using the hyperedge replacement grammars is proved. The number of nodes and hyperedges of an arbitrary hypergraph, paths in string-graphs and their length and also cycles in string-graphs and their length are investigated. In the end, the compatibility between the theoretically properties of hypergraphs and the replacement process of hyperedges are defined, proving these compatibilities for the previous relations.

1. MATHEMATICAL RELATIONS IN HYPERGRAPHS

We show some of the most important properties of hypergraphs. A hypergraph [1, 5] is a special structure, that is why its properties are different from an ordinary graph.

1.1. NUMBER OF NODES AND HYPEREDGES

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be a hypergraph over a fixed and arbitrary set C, of labels. Denote by INT_H, the set of internal nodes of the hypergraph H, meaning the difference between V_H and ext_H.

Let $HRG = (N, T, P, S)$ be a hyperedge replacement grammar with $N \subseteq C$ and $T \subseteq C$, and a derivation in this grammar: $A^\bullet \Rightarrow R \Rightarrow^* H$, $A \in N$. For every $e \in E_R$ we have $lab_R(e)^\bullet \Rightarrow^* H(e)$, meaning that every labeled edge $lab_R(e)$ will have a support $lab_R(e)^\bullet$ and will be replaced by a hypergraph $H(e)$ by applying many productions from P. Therefore, the set of nodes of the hypergraph H consists of the set of nodes of R union with the set of internal nodes of components $H(e)$.

Proposition 1.1 *Let $H \in \mathcal{H}_T$ and $A^\bullet \Rightarrow R \Rightarrow^* H$ be a derivation of H. For every $e \in E_R$ we have that $l_R(e)^\bullet \Rightarrow^* H(e)$. Then*

$$|V_H| = |V_R| + \sum_{e \in E_R} |INT_{H(e)}|,$$
$$|INT_H| = |INT_R| + \sum_{e \in E_R} |INT_{H(e)}|,$$
$$|E_H| = \sum_{e \in E_R} |E_{H(e)}|.$$

Proof: We can prove these by using the induction related to the cardinality of the set E_R, such that $V_H = V_R + \sum\limits_{e \in E_R} INT_{H(e)}$. The verification is immediate if $|E_R| = 0$ then $H = R$ and $V_H = V_R$. The induction step will consider the context independence and from this the derivation definition. Therefore, the external nodes of the derived hypergraph are added to the hypergraph nodes to be derivate. The only eventual new node in the formula are internal nodes of the derived hypergraph.

For the second formula we have $V_H - [ext_H] = V_R - [ext_R] + \sum\limits_{e \in E_R} INT_{H(e)}$ because $[ext_H] = [ext_R]$. Therefore $INT_H = INT_R + \sum\limits_{e \in E_R} INT_{H(e)}$.

The third formula is obvious.

Remark 1.1 *We can define the following functions*

$$size(H) = |V_H| + |E_H|,\ intsize(H) = |INT_H| + |E_H|,$$

with the calculations formulae

$$\begin{aligned} size_H &= |V_R| + \sum_{e \in E_R} intsize(H(e)), \\ intsize(H) &= |INT_R| + \sum_{e \in E_R} intsize(H(e)). \end{aligned}$$

Also, we can define the density function: $dens(H) = |E_H|/|V_H|$ *if* $|V_H| > 0$ *given by the formula*

$$dens(H) = \frac{\sum\limits_{e \in E_R} |E_{H(e)}|}{|V_R| + \sum\limits_{e \in E_R} |INT_{H(e)}|}.$$

The proof of these formulae is immediate, if we consider the previous proposition.

1.2. PATHS IN GRAPHS, PATHS OF MINIMUM OR MAXIMUM LENGTH

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be a hypergraph over the set of labels C, arbitrary but fixed. A path that unite v_0 and v_n, two nodes from V_H, represents a sequence $p = v_0 e_1 v_1 e_2 \ldots e_n v_n$ of nodes and edges so that for every $i = 1, n$, v_i and v_{i-1} are incident nodes of the edge e_i. If in a path every node occurs only once we say that the path is simple. The length of a path, denoted by $length(p)$, represents the number of edges that this one contains.

Let 2-graf H. Denote by $PATH_H$ the set of simple paths uniting $begin_H$ and end_H, and by $numpath(H)$ the cardinal of $PATH_H$. Denote by $minpath(H)$ and $maxpath(H)$ the path of minimum or maximum length.

Proposition 1.2 *Let H be a graph-string and the derivation $A^\bullet \Rightarrow R \Rightarrow^* H$. We have the following relations*

$$\begin{aligned}
numpath(H) &= \sum_{p\in PATH_R} \prod_{e\in p} numpath(H(e)),\\
minpath(H) &= \min_{p\in PATH_R} \sum_{e\in p} minpath(H(e)),\\
maxpath(H) &= \max_{p\in PATH_R} \sum_{e\in p} maxpath(H(e)).
\end{aligned}$$

Proof: Let p be an arbitrary path in $PATH_R$. Assume that the path passes through the nodes $v_0, \ldots, v_n$. Because $A \in N$ and $A^\bullet$ is a 2-graph, then the hypergraph R will have the edges $e_1, \ldots, e_n$ and e_1 will be the edge that binds nodes $v_0 = begin_H$ and v_1, …, e_i will bind v_{i-1} and v_i, …, e_n will connect v_{n-1} and $v_n = end_H$. Every edge of these will be replaced by hypergraphs. Therefore, for every $e_i \in R$ we have $lab_R(e_i)^\bullet \Rightarrow^* H(e_i)$. Therefore, $numpath(H_p) = \prod_{e\in p} numpath(H(e))$. Because the process is repeating for every path from $PATH_R$ the first relation is demonstrated. The second and the third can be shown considering the global minimum and maximum from the local minimum and maximum on every possible path in R.

1.3. CYCLES IN GRAPHS, NUMBER OF SIMPLE CYCLES, MINIMUM LENGTH AND MAXIMUM LENGTH OF SIMPLE CYCLES

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be an hypergraph over a set of labels C, arbitrary but fixed. If in a path from H we have $v_0 = v_n$ then we say that the path is a cycle. A cycle is simple if the nodes are distinct and $n \geq 3$. The length of a cyle is defined as for the paths.

For a 2-graph, H, we say that $CYCLE_H$ is the set of simple cycles, $numcycle(H)$ the cardinal of this set, $mincycle(H)$ and $maxcycle(H)$ the minimum length and the maximum length of these cycles, respectively.

Proposition 1.3 *In a string-graph we have*

$$\begin{aligned}
&numcycle(H) = \sum_{c\in CYCLE_R} \prod_{e\in c} numpath(H(e)) + \sum_{e\in E_R} numcycle(H(e)),\\
&mincycle(H) = \min\{\min_{c\in CYCLE_R} \sum_{e\in c} minpath(H(e)), \min_{e\in E_R} mincycle(H(e))\},\\
&maxcycle(H) = \max\{\max_{c\in CYCLE_R} \sum_{e\in c} maxpath(H(e)),\\
&\max_{e\in E_R} maxcycle(H(e))\}.
\end{aligned}$$

Proof: Like in the previous demonstration we consider a cycle c from $CYCLE(R)$. This cycle is $v_0e_1v_1e_2v_2 \ldots e_nv_ne_{n+1}v_0$. For all $e_i, i = \overline{1, n+1}$ we have $lab(e_i)^\bullet \Rightarrow^* H(e_i)$. Any path in $H(e_i)$ that begins from v_{i-1} and ends at v_i forms a cycle with edges from c after the elimination of e_i. At these cycles we add the proper cycles (formed only with edges from $H(e_i)$). The algorithm is repeating for every edge from the cycles $c \in CYCLE(R)$ and after that for every cycle from R. The demonstration for the second and third relations is similar.

2. COMPATIBLE FUNCTIONS

We will define the compatible function like a generalization of the functions defined in the previous sections. Therefore, the f_0 function, defined on the hypergraph set and taking values in natural number set is compatible with the derivation process in a hyperedge replacement grammar, if for every hypergraph H and every derivation of it the value $f_0(H)$ can be calculated from the values assumed by the function on subgraphs $H(e)$ that compounds it and are results of the derivation process.

Definition 2.1 *[9] (Compatible function)*

1 *Let $\mathcal{HRG}$ be the class of hyperedges replacement grammars over the set of labels C, arbitrary but fixed, I a finit set of indices, VAL a set of values, $f : \mathcal{H}_C \times I \to VAL$ a function and f' a function defined on triples $(R, assign, i)$, where $R \in \mathcal{H}_C$, $assign : E_R \times I \to VAL$, $i \in I$, with values in VAL. We say then that f is $(\mathcal{HRG}, f')$-* compatible *if, for every HRG=(N,T,P,S)$\in \mathcal{HRG}$, every derivation $A^\bullet \Rightarrow R \Rightarrow^* H$, $A \in N$ and $H \in \mathcal{H}_T$, and for every $i \in I$ we have*

$$f(H, i) = f'(R, f(H(-), -), i)$$

2 *A function $f_0 : \mathcal{H}_C \to VAL$ is called $\mathcal{HRG}$-*compatible *if there exist the functions f, f' and an index i_0 such that $f_0 = f(-, i_0)$ and f is $(\mathcal{HRG}, f')$-compatible.*

We recall that $f(H(-), -)$ represents a notation for the function defined by $f(H(-), -)(e, j) = f(H(e), j)$.

Theorem 2.1 *If in the previous definition $I = \{all, int\}$, $VAL = \mathbf{N}$ and*

$$\begin{aligned}
&f(H, all) = |V_H|; \textit{total number of nodes},\\
&f(H, int) = |INT_H|; \textit{internal number of nodes},\\
&f'(R, assign, all) = |V_R| + \sum_{e \in E_R} assign(e, int),\\
&f'(R, assign, int) = |INT_R| + \sum_{e \in E_R} assign(e, int).
\end{aligned}$$

Then f is $(\mathcal{HRG}, f')$-compatible and if $f_0 = f(-, all)$, then the function number of nodes is $\mathcal{HRG}$-compatible.

Proof: We have $f(H, all) = |V_H| = |V_R| + \sum_{e \in E_R} |INT_{H(e)}| = |V_R| + \sum_{e \in E_R} f(H(e), int) = f'(R, f(H(-), -), all)$. Also, we choose $f_0 = f(-, all)$.

Corollary 2.1 *In a similar way we can assert that the function number of paths, minimum length and maximum length, number of cycles, minimum length and maximum length of simple cycles respectively is $\mathcal{ERG}$-compatible, where $\mathcal{ERG}$ is the 2-graph class.*

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