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MOMENTS OF FIRST-PASSAGE PLACES AND RELATED RESULTS FOR THE INTEGRATED BROWNIAN MOTION

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Abstract We consider first-passage problems for the two-dimensional diffusion process $(X(t), Y(t))$, where $Y(t)$ is a Wiener process and $X(t)$ is its integral. Let $T(x, y)$ be the first time $Y^3(t)/X(t)$ leaves a certain region of the second quadrant. With the help of the method of similarity solutions, we obtain an exact solution to the Kolmogorov backward equation, subject to the appropriate boundary conditions, satisfied by the moments of $Y(T(x, y))$. Similarly, the probability that the process $(X(t), Y(t))$ will hit a given part of the boundary is explicitly computed.

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1. INTRODUCTION

First-passage problems are important in many applications. For example, in mathematical finance, one is interested in computing the time it takes some stocks to reach a given price, at which point an *option* is exercised. In biology, people want to determine the time needed for a neuron to *fire a spike*, once it has attained a fixed threshold. Other fields in which first-passage problems appear are those of physics, chemistry and electrical engineering, in particular.

In most cases, first-passage problems are concerned with determining the *time* it takes a stochastic process to reach or cross some boundary. In one dimension, suppose that $B(t)$ is a Brownian motion (starting from 0) and that τ is the time it takes for $B(t)$ to become equal to $a > 0$. Then, one has obviously $B(\tau) = a$. In the case when τ is the time until $B(t)$ leaves the interval $[-b, a]$, where $b > 0$, one might be interested in computing the probability that $B(\tau) = a$. In two or more dimensions, the problem of obtaining the distribution, or at least the moments, of *first-passage places* is generally quite difficult, as it involves solving partial differential equations subject to the appropriate boundary conditions. The author has considered such problems for the most important diffusion processes [3]-[6].

Let $Y(t)$ be a Wiener process with drift coefficient μ and diffusion coefficient σ^2, and $X(t)$ be its integral, so that

$$dX(t) = Y(t)dt, \tag{1}$$

$$dY(t) = \mu\, dt + \sigma dW(t), \tag{2}$$

where $W(t)$ is a standard Brownian motion. In this paper, we will see that sometimes it is possible to obtain relatively simple solutions to certain first-passage problems for the two-dimensional diffusion process $(X(t), Y(t))$ by making use of the method of similarity solutions to solve the Kolmogorov backward equation satisfied by the function of interest.

We define

$$T(x, y) = \inf\{t > 0 : Y^3(t)/X(t) = k_1 \text{ or } k_2 | X(0) = x,\ Y(0) = y\}, \tag{3}$$

where $y^3/x \in (k_2, k_1)$ and $-\infty < k_2 < k_1 < 0$, with $x < 0$ and $y > 0$. The moment generating function of the random variable $T(x, y)$, namely

$$L(x, y; \alpha) := E[e^{-\alpha T(x,y)}], \tag{4}$$

where $\alpha > 0$, satisfies the Kolmogorov backward equation

$$\frac{1}{2}\sigma^2 L_{yy} + \mu L_y + y L_x = \alpha L, \tag{5}$$

where $L_{yy} \equiv \frac{\partial^2 L}{\partial y^2}$, etc. This equation is subject to the boundary conditions

$$L(x, y; \alpha) = 1 \quad \text{if } y^3/x = k_1 \text{ or } k_2. \tag{6}$$

Ideally, we would like to first find the function $L(x, y; \alpha)$ and then invert the Laplace transform to obtain the probability density function of $T(x, y)$. Unfortunately, this is rarely possible for this type of problem and most often we must content ourselves with finding the function $L(x, y; \alpha)$ only, or at least the moments of $T(x, y)$.

The method of similarity solutions consists in assuming that there is a certain relationship between the variables x and y. For instance, because the first-passage time $T(x, y)$ is defined in terms of the ratio y^3/x, we could try a solution of the form

$$L(x, y; \alpha) = N(z; \alpha), \tag{7}$$

where $z := y/x^{1/3}$. Unfortunately, we find that this particular instance of the method of similarity solutions fails in the case of the moment generating function. However, if $\mu = 0$, this method enables us to explicitly compute the moments of $Y(T(x, y))$ and $X(T(x, y))$, as well as the probability that $Y^3(T(x, y))/X(T(x, y)) = k_2$. We will also obtain the expected value of $\ln Y(T(x, y))$. These functions will be computed in Sections 2, 3 and 4, respectively. The paper will then end with some remarks in Section 5.

2. MOMENTS OF $Y(T(X,Y))$ AND $X(T(X,Y))$

As mentioned in Section 1, we would like to explicitly compute the function $L(x,y;\alpha)$. Assuming that (7) holds, then (5) is transformed into

$$\frac{1}{2}\sigma^2\frac{1}{x^{2/3}}N''(z;\alpha)+\mu\frac{1}{x^{1/3}}N'(z;\alpha)-y\frac{y}{3x^{4/3}}N'(z;\alpha)=\alpha N(z;\alpha) \quad (8)$$

and the boundary conditions would be

$$N(z;\alpha)=1 \quad \text{if } z^3=k_1 \text{ or } k_2. \quad (9)$$

For this transformation to be valid, we must be able to express the coefficients of N, N' and N'' in (8) in terms of z. We see that this is not possible, even if we set μ equal to 0.

However, this method works in the case of the moments of $Y(T(x,y))$ (and $X(T(x,y))$). Indeed, the function

$$m_k(x,y):=E[Y^k(T(x,y))] \quad (10)$$

is a solution of the partial differential equation (p.d.e.)

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}m_k(x,y)+\mu\frac{\partial}{\partial y}m_k(x,y)+y\frac{\partial}{\partial x}m_k(x,y)=0, \quad (11)$$

together with the boundary conditions

$$m_k(x,y)=y^k \quad \text{if } y^3/x=k_1 \text{ or } k_2. \quad (12)$$

We must set μ equal to 0. If we choose σ equal to 1 (for simplicity), then $Y(t)$ is a standard Brownian motion and we find that (11) becomes

$$\frac{1}{2}n_k''(z)-\frac{z^2}{3}n_k'(z)=0, \quad (13)$$

where

$$n_k(z):=m_k(x,y) \quad (14)$$

(with $z:=y/x^{1/3}$). Because the boundary conditions (12) must also be expressed in terms of the variable z, we next define

$$n_k(z)\ (=n_k(y/x^{1/3}))\ =y^k r_k(z). \quad (15)$$

The function $r_k(z)$ is such that

$$r_k(z)=1 \quad \text{if } z^3=k_1 \text{ or } k_2 \quad (16)$$

and is a solution of

$$\frac{z^2}{2}r_k''(z)+z\left(k-\frac{z^3}{3}\right)r_k'(z)+\frac{1}{2}k(k-1)\,r_k(z)=0. \quad (17)$$

The general solution of the second order linear ordinary differential equation (o.d.e.)

$$z^2 r_k''(z) + z(a + bz^3) r_k'(z) + c\, r_k(z) = 0 \tag{18}$$

can be written as

$$r_k(z) = \exp\left\{-bz^3/6\right\} z^{-(a/2)-1} \times \tag{19}$$

$$\left[c_1 M\left(-\frac{1}{3} - \frac{a}{6}, \frac{1}{6}\left(1 + a^2 - 2a - 4c\right)^{1/2}, \frac{b}{3}z^3\right) \right.$$
$$\left. + c_2 W\left(-\frac{1}{3} - \frac{a}{6}, \frac{1}{6}\left(1 + a^2 - 2a - 4c\right)^{1/2}, \frac{b}{3}z^3\right)\right],$$

where c_1 and c_2 are constants, and $M(\cdot,\cdot,\cdot)$ and $W(\cdot,\cdot,\cdot)$ are Whittaker functions (see [1, p. 505]). We may now state the following proposition.

Proposition 2.1 *The k^{th} order moment of the random variable $Y(T(x,y))$ is given by $y^k r_k(z)$, where $r_k(z)$ is given in (19), with $a = 2k$, $b = -2/3$ and $c = k(k-1)$, and in which the constants c_1 and c_2 are* uniquely *determined from the boundary conditions (16).*

Proof. . We obtained an explicit solution to our problem by making use of a particular case of the method of similarity solutions. However, we must make sure that the solution obtained is indeed the one we are looking for.

Now, from the fact that $X(t)$ increases when $Y(t)$ is positive (see (1)) we deduce that $P[T(x,y) < \infty] = 1$. It then follows, using the results in [4, section 9.1], that we can assert that the solution to (11), (12) is *unique*, which completes the proof. □

Corollary 2.1 *The expected value of $Y(T(x,y))$, denoted by $m_1(x,y) = y\, r_1(z)$, where $r_1(z)$ is a solution of*

$$z\, r_1''(z) = [(2/3)z^3 - 1] r_1'(z), \tag{20}$$

can be expressed as

$$E[Y(T(x,y))] = y \exp\left\{y^3/9x\right\} (x^{2/3}/y^2) \times \tag{21}$$

$$\left[c_1 M\left(-\frac{2}{3}, \frac{1}{6}, -\frac{2}{9}\frac{y^3}{x}\right) + c_2 W\left(-\frac{2}{3}, \frac{1}{6}, -\frac{2}{9}\frac{y^3}{x}\right)\right],$$

where c_1 and c_2 are such that $E[Y(T(x,y))] = y$ if $y^3/x = k_1$ or k_2.

Remarks. i) The solution may also be expressed in terms of the incomplete gamma function.

ii) The uniqueness of the solution will be true in the other problems considered in this paper as well.

To complete this section, we will find the moments of the random variable $X(T(x,y))$. Let

$$e_k(x,y) := E[X^k(T(x,y))] \tag{22}$$

We set

$$e_k(x,y) = x^k f_k(x,y) \tag{23}$$

and we assume that $f_k(x,y) = \lambda_k(z)$, with $z = y/x^{1/3}$ as previously. The function $\lambda_k(z)$ is such that $\lambda_k(k_i^{1/3}) = 1$, for $i = 1, 2$, and satisfies the ordinary differential equation

$$\frac{1}{2}\lambda_k''(z) - \frac{1}{3}z^2\lambda_k'(z) + kz\lambda_k(z) = 0. \tag{24}$$

Since the general solution of the o.d.e.

$$\lambda_k''(z) + az^2\lambda_k'(z) + bz\lambda_k(z) = 0 \tag{25}$$

can be written as

$$\lambda_k(z) = \frac{\exp\left\{-az^3/6\right\}}{z}\times \tag{26}$$

$$\left[c_1 M\left(-\frac{1}{3}+\frac{b}{3a},\frac{1}{6},\frac{a}{3}z^3\right) + c_2 W\left(-\frac{1}{3}+\frac{b}{3a},\frac{1}{6},\frac{a}{3}z^3\right)\right],$$

where c_1 and c_2 are constants chosen so that the boundary conditions are satisfied, we can write an explicit expression for $E[X^k(T(x,y))]$.

3. PROBABILITY THAT $Y^3(T(X,Y))/X(T(X,Y)) = K_2$

We know that the two-dimensional diffusion process $(X(t), Y(t))$, starting from a point located in the second quadrant, is certain to eventually leave the region C defined by

$$C = \{(x,y) \in mathbbR^2 : -\infty < k_2 < y^3/x < k_1 < 0\}. \tag{27}$$

We are now interested in computing the probability π that $(X(t), Y(t))$ will leave C through the boundary $y^3/x = k_2$. Note that since $P[T(x,y) < \infty] = 1$, the probability $P[Y^3(T(x,y))/X(T(x,y)) = k_1]$ is simply equal to $1 - \pi$.

The probability π is actually a function $\pi(x, y)$ of the starting point (x, y). It can be shown that, with $\mu = 0$ and $\sigma = 1$, it satisfies the Kolmogorov backward equation

$$\frac{1}{2}\pi_{yy}(x, y) + y\,\pi_x(x, y) = 0, \tag{28}$$

subject to the boundary conditions

$$\pi(x, y) = \begin{cases} 1 & \text{if } y^3/x = k_2, \\ 0 & \text{if } y^3/x = k_1. \end{cases} \tag{29}$$

As in the previous section, we try a solution of the form $\pi(x, y) = \nu(z)$, where $z = y/x^{1/3}$. (28) then simplifies to

$$\frac{1}{2}z^2\nu''(z) - \frac{z^4}{3}\nu'(z) = 0 \quad \overset{z\neq 0}{\Longleftrightarrow} \quad \nu''(z) - \frac{2z^2}{3}\nu'(z) = 0 \tag{30}$$

and the boundary conditions become $\nu(k_1^{1/3}) = 0$, $\nu(k_2^{1/3}) = 1$. We find that the general solution of this o.d.e. may be written as follows

$$\nu(z) = c_1 + c_2\left(2\sqrt{3}\pi - 3\,\Gamma(2/3)\,\Gamma\left(\frac{1}{3}, -\frac{2}{9}z^3\right)\right) \quad \text{for } k_2 < z < k_1, \tag{31}$$

where $\Gamma(\cdot, \cdot)$ is the *incomplete gamma function*, which is defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}\,dt \tag{32}$$

if $\mathrm{Re}(a) > 0$. It can be expressed in terms of *confluent hypergeometric functions*. Using the fact that the solution to (28), (29) is *unique*, we can state the proposition that follows.

Proposition 3.1 *The probability $\pi(x, y) := P[Y^3(T(x, y))/X(T(x, y)) = k_2]$ is given by the function in (31), where $z = y/x^{1/3}$ and the constants c_1 and c_2 are uniquely determined from the boundary conditions (29).*

Remark. Notice that we did not have to transform the function $\nu(z)$ to express the boundary conditions in terms of z since $\nu(z)$ is equal to either of two constants on the boundary.

4. EXPECTED VALUE OF $\ln Y(T(X, Y))$

Finally, we obtain an explicit formula for the mathematical expectation $E[\ln Y(T)]$. Let $h(x, y) := E[\ln Y(T(x, y))]$. The function $h(x, y)$ satisfies the same p.d.e. as $m_k(x, y)$ (with $\mu = 0$ and $\sigma = 1$; see (11)

$$\frac{1}{2}h_{yy}(x, y) + yh_x(x, y) = 0. \tag{33}$$

This time, the boundary conditions are

$$h(x,y) = \ln y \quad \text{if } y^3/x = k_1 \text{ or } k_2. \tag{34}$$

Let

$$h(x,y) = g(x,y) + \ln y. \tag{35}$$

We find that

$$\frac{1}{2} g_{yy} + y g_x = \frac{1}{2y^2} \tag{36}$$

(and $g(x,y) = 0$ if $y^3/x = k_1$ or k_2). Assuming that $g(x,y) = \phi(z)$, where $z = y/x^{1/3}$, we reduce the preceding p.d.e. to the o.d.e.

$$\phi''(z) - \frac{2}{3}\, z^2\, \phi'(z) = \frac{1}{2z^2}, \tag{37}$$

which has the general solution

$$\phi(z) = \int_{k_2^{1/3}}^{z} \left[c\, e^{2w^3/9} - \frac{6^{2/3}}{20} w e^{w^3/9} M\left(\frac{1}{3}, \frac{5}{6}, \frac{2w^3}{9}\right) - \frac{w^2}{6} - \frac{1}{2w} \right] dw, \tag{38}$$

where $M(\cdot,\cdot,\cdot)$ is a Whittaker function and the constant c can be found by making use of the boundary condition $\phi(k_1^{1/3}) = 0$. That is,

$$c = \int_{k_2^{1/3}}^{k_1^{1/3}} \left[\frac{6^{2/3}}{20} w e^{w^3/9} M\left(\frac{1}{3}, \frac{5}{6}, \frac{2w^3}{9}\right) + \frac{w^2}{6} + \frac{1}{2w} \right] dw \Bigg/ \int_{k_2^{1/3}}^{k_1^{1/3}} e^{2w^3/9}\, dw. \tag{39}$$

Summing up, we have the following proposition.

Proposition 4.1 *The mathematical expectation $E[\ln Y(T(x,y))]$ is given by the formulas (38) and (39), in which $z = y/x^{1/3}$, for $k_2 \le z \le k_1$ (< 0).*

Remark 4.1 *Proceeding as above, we can also compute the mathematical expectation $E[\ln(-X(T(x,y)))]$. This time, the o.d.e. that we have to solve is*

$$\psi''(z) - \frac{2}{3}\, z^2\, \psi'(z) = -2z, \tag{40}$$

where $\psi(z) = \psi(y/x^{1/3}) = E[\ln(-X(T(x,y)))] - \ln(-x)$. We find that its solution that satisfies the boundary conditions $\psi(z) = 0$ if $z^3 = k_1$ or k_2 is quite similar to the one above.

5. CONCLUSION

In this paper, we have explicitly computed the moments of a random variable denoting a first-passage place for an important two-dimensional diffusion process, namely the process $(X(t), Y(t))$, where $Y(t)$ is a Wiener process with zero drift and $X(t)$ is its integral. We also obtained exact solutions to other related problems.

We used the method of similarity solutions to solve the appropriate p.d.e.'s. Because the solutions we were looking for are unique, we could appeal to any method to get the required solutions. Notice, however, that this method (at least the particular case we considered) did not enable us to find the moment generating function of the first-passage time $T(x, y)$. It would surely be interesting to obtain an explicit expression for this function. Similarly, in addition to the moments of $Y(T(x, y))$ and $X(T(x, y))$, we could try to find the distribution of these random variables.

Next, from the two-boundary problems, we could consider the limiting one-boundary problems obtained by letting k_2 decrease to $-\infty$ or k_1 increase to 0.

Finally, because of the importance of the Wiener process, we could try to solve other first-passage time and/or place problems with the help of the method of similarity solutions. In general, such problems in two or more dimensions give rise to really complicated formulas. Here, the solutions obtained are relatively simple, since only in the last section the solution involved an integral difficult to evaluate explicitly.

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