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NUMERICAL BUILDING OF THE ORTHOGONAL MAPPING OF THE CURVILINEAR QUADRANGLE DOMAIN INTO THE RECTANGLE

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Abstract We propose and study the mathematical formulation of the problems of building the orthogonal mapping of the physical curvilinear quadrangle domain into the square or rectangular computation domain. The problem is formulated as a system of two coupled Laplace equations with non-linear boundary conditions, containing both unknown functions. The problem is solved by the finite difference method using the sweep matrix method and Seidel linearization. The program built in FORTRAN was tested on the exact analytical solution.

1. INTRODUCTION

The contemporary level of the computer technique development allows us to formulate and solve the problems of the dynamic prognosis for the conduct of the compound mechanical constructions of complex form. These problems are mathematicaly formulated as a system of partial differential equations of hyperbolic type. One of the effective methods for solving this kind of problems is the method of finite differences (MFD). This very method allows building in the rectangular-type domains the numerical scheme for solving the problem with minimal unwanted effects of physical nature, such as dispersion and dissipation. In a more complex domain of the MFD, encountering the problems of approximation of the boundary conditions, as well as the algorithms and the programs for the problem solving, becomes considerably more complex. That is why it is actual the elaboration of methods to map physical domain having a form of a curvilinear rectangle on the computation domain which has the form of a quadrangle or rectangle.

To the problem of numerical building of the arbitrary domain mapping on the quadrangle has been dedicated a large number of works, the survey of which is presented in [1]. However, algorithms used nowadays are considerably complex and mostly they are used to solve the hydrodynamics problems. In this paper we formulate the problem and propose an effective method for the numerical building of the orthogonal mapping of the curvilinear rectangle on the quadrangle.

After obtaining the orthogonal mapping the problem of calculation of the dynamic concentration of the stresses is solved in the orthogonal system of coordinates. If referred to this system, the elasticity theory equations have a more complex structure compared with the Cartesian system of coordinates. However, due to the rectangular form of the computation domain and the build by the author conservative difference scheme [2, 3], it is possible to effectively solve the problems of the stress concentration in domains of complex form.

2. BUILDING OF THE ORTHOGONAL MAPPING

Let us examine the dynamic problem of the elasticity theory in the domain presented in the left-hand side (fig.1). If we try to solve this problem numerically in the Cartesian system of coordinates Oxy by frontiers of the difference method, we encounter considerable difficulties of approximating the domain and the given boundary conditions.

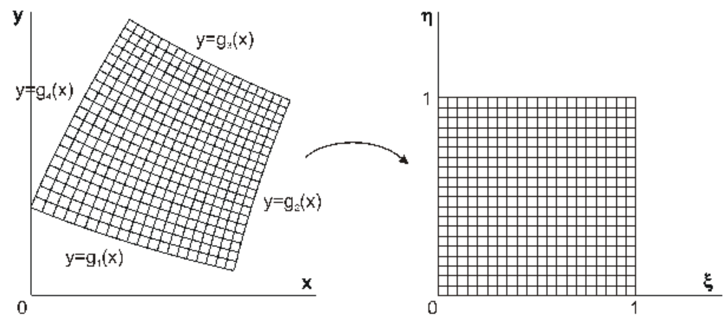

Fig. 1. Physical and computation domains.

Thus we propose the following method: first we perform the conformal or orthogonal mapping of the initial curvilinear domain written in the coordinates Oxy on the quadrangle or rectangular domain written in coordinates $O\xi\eta$. To this aim two problems involving the Laplace equations, which allow to determine the direct and inverse functions of the mapping: $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ and $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ are solved.

In the most general case in order to build the orthogonal mapping in the quadrangle domain written in the coordinates $O\xi\eta$ it is necessary to solve the following system of equations [1]

$$
\begin{aligned}
g_{22}\frac{\partial^2 x}{\partial \xi^2} + g_{11}\frac{\partial^2 x}{\partial \eta^2} + g_{11}g_{22}\left(\frac{\partial x}{\partial \xi}\Delta\xi + \frac{\partial x}{\partial \eta}\Delta\eta\right) = 0,\\
g_{22}\frac{\partial^2 y}{\partial \xi^2} + g_{11}\frac{\partial^2 y}{\partial \eta^2} + g_{11}g_{22}\left(\frac{\partial y}{\partial \xi}\Delta\xi + \frac{\partial y}{\partial \eta}\Delta\eta\right) = 0,
\end{aligned}
$$

$$g_{11} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2, \quad g_{22} = \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2.$$

This problem is a complex nonlinear problem of mathematical physics. Therefore, in order to elaborate an effective numerical method to solve it, we examine the case $g_{11} = g_{22} = 1$. Then the above system is transformed into the system of two Laplace equations $\Delta x = 0$, $\Delta y = 0$. We mention that the problem solution does not always ensure an injective mapping of the curvilinear domain on the quadrangle.

The definition of the unknown functions $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ in the domain $O\xi\eta$ is presented in the fig. 2. Here $\Delta x = 0$ and $\Delta y = 0$ are the Laplace equations and n_x^k, n_y^k stand for the cosines of the normal to the boundaries of the initial domain $y = g_k(x)$; $n_x^k / n_y^k = -g_k'(x(\xi, \eta))$, $g_k'(x) = dg_k(x)/dx$.

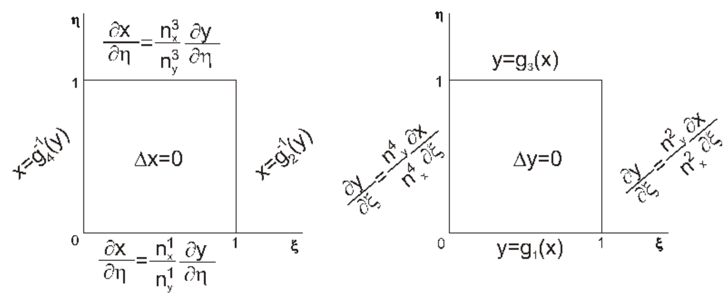

Fig. 2. The definition of the functions $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$.

Up to the first order the boundary conditions are those for the initial physical domain. The boundary conditions of the second order contain derivatives and are obtained by transforming the conditions asserting that on the corresponding boundary the normal derivative to the boundary is equal to zero: $\frac{\partial \xi}{\partial n} = \frac{\partial \xi}{\partial x} n_x^k + \frac{\partial \xi}{\partial y} n_y^k = 0$ on the boundaries with numbers $k = 1$ and 3; $\frac{\partial \eta}{\partial n} = \frac{\partial \eta}{\partial x} n_x^k + \frac{\partial \eta}{\partial y} n_y^k = 0$ on the boundaries with numbers $k = 2$ and 4.

These two problems are coupled, since the unknown functions are part of the nonlinear boundary conditions of both problems. Thus, even in the case when the Laplace equations are used, the problem of determining the unknown functions is a complex nonlinear problem of mathematical physics. But, due to the quadrangle form of the solving domains of these problems, the problems are solved well enough by the method of the finite differences using either the direct sweep method or rapidly converging method of alternating directions. There are some difficulties occurring during the process of the linearization of the boundary conditions, but they are overcome by using the Seidel iterative procedure.

The difference scheme for solving the problem is the following:

$$
\begin{gathered}
-\Delta_h x_{ij} = -x_{ij,\bar{\xi}\xi} - x_{ij,\bar{\eta}\eta} = 0, i = \overline{1, N-1}, j = \overline{1, M-1},\\
x_{0j} = g_4^{-1}(y_{0j}), x_{Nj} = g_2^{-1}(y_{Nj}), j = \overline{0, M},\\
-\frac{2}{h_2} x_{i0,\eta} - x_{i0,\bar{\xi}\xi} = -g_1'(x_{i0})\left[-\frac{2}{h_2} y_{i0,\eta} - y_{i0,\bar{\xi}\xi}\right], i = \overline{0, N},\\
\frac{2}{h_2} x_{iM,\bar{\eta}} - x_{iM,\bar{\xi}\xi} = -g_3'(x_{iM})\left[\frac{2}{h_2} y_{iM,\bar{\eta}} - y_{iM,\bar{\xi}\xi}\right], i = \overline{0, N};
\end{gathered}
$$

$$
\begin{gathered}
-\Delta_h y_{ij} = -y_{ij,\bar{\xi}\xi} - y_{ij,\bar{\eta}\eta} = 0, i = \overline{1, N-1}, j = \overline{1, M-1},\\
y_{i0} = g_1(x_{i0}), y_{iM} = g_3(x_{iM}), i = \overline{0, N},\\
-\frac{2}{h_1} y_{0j,\xi} - y_{0j,\bar{\eta}\eta} = -\frac{1}{g_4'(x_{0j})}\left[-\frac{2}{h_1} x_{0j,\xi} - x_{0j,\bar{\eta}\eta}\right], j = \overline{0, M},\\
\frac{2}{h_1} y_{Nj,\bar{\xi}} - y_{Nj,\bar{\eta}\eta} = -\frac{1}{g_2'(x_{Nj})}\left[\frac{2}{h_1} x_{Nj,\xi} - x_{Nj,\bar{\eta}\eta}\right], j = \overline{0, M}.
\end{gathered}
$$

The generally accepted notation for the left and right difference derivatives is used; $h_1 = 1/N$, $h_2 = 1/M$, $x_{ij} = x(\xi_i, \eta_j)$, $y_{ij} = y(\xi_i, \eta_j)$, $\xi_i = ih_1$, $i = \overline{0, N}$, $\eta_j = jh_2$, $j = \overline{0, M}$. The complex form of the second degree boundary conditions is due to the necessity of building difference approximations of the second oder at all points of the difference net, including the boundary nodes, where the values of the derivatives of the unknown functions are assigned .

The described numerical method for solving of the defined problem was implemented as a Fortran program. The test estimations carried out for solving the known orthogonal mapping [4] $x = \sqrt{\frac{\xi+\sqrt{\xi^2+\eta^2}}{2}}$, $y = \sqrt{\frac{-\xi+\sqrt{\xi^2+\eta^2}}{2}}$ revealed a quite good agreement of the obtained numerical results and the analytical solution.

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