ROMAI J., **2**, 2(2006),
153–164

# COMPUTATIONAL METHODS FOR FIRST KIND INTEGRAL EQUATIONS

Elena Pelican
*Ovidius Univ. of Constanta*
epelican@univ-ovidius.ro

**Abstract** The first kind integral equations model many classes of real-world problems (e.g. backwards heat equation, inverse scattering problems, the hanging cable, geological prospecting, computerized tomography, electric potential problems, etc). That is why it is very important to know how to solve them. In this paper we realize a study on computational methods for solving this type of equations: collocation and projection methods, spline techniques and different types of regularization methods.

**Keywords:** first kind integral equation, linear least-squares problem, minimal norm solution, spline functions, regularization metods, projection, collocation.
**2000 MSC:** 45Q05, 45B05, 65F20, 65F22.

## 1. INTRODUCTION

Let $K : L^2([a, b]) \to L^2([a, b])$ be the (compact) integral operator $Kx(t) = \int_a^b k(t, s)x(s)ds$, and the equation

$$Kx(t) = y(t), \quad \forall\ t \in [0, 1]. \tag{1}$$

with square-integrable kernel $k : [a, b] \times [a, b] \to I\!R$, and $y \in L^2([a, b])$. Our task is to find $x(s)$ when the data (function $y(s)$ and the kernel) are known exactly, or only approximately. As in most cases $y \notin R(K)$ (where by $R(K)$ we denoted the range of $K$), the equation (1) has no longer solution. Thus, if in addition, we suppose that $y \in D(K^+)$, where by $D(K^+) = R(K) \oplus R(K)^\perp$ we denoted the domain for the Moore-Penrose pseudoinverse of the linear compact operator $K$ from (1), we can reformulate (1) as the least-squares problem: find $\bar{x} \in L^2([0, 1])$ such that

$$\| K\bar{x} - y \|_{L^2([0,1])} = \min! \tag{2}$$

where $\| f \|_{L^2([0,1])} = (\int_0^1 (f(t))^2 dt)^{\frac{1}{2}}$. It is well-known that, if $y \in D(K^+)$, then the problem (2) has a minimal norm solution, $x_{LS}$, given by $x_{LS} = K^+y$. This solution also satisfies (in classical sense) the associated normal equation $K^*Kx = K^*y$, where $K^*$ is the adjoint of $K$.

## 2. COMPUTATIONAL METHODS

In what follows, we describe the collocation method, the projection one, the spline technique, and Tikhonov regularization method.

### 2.1. COLLOCATION METHOD

In this case, it is required that the kernel $k$ is continuous. For $n \geq 2$ arbitrary fixed and $T_n = \{t_1, \ldots, t_n\}$ the set of (collocation) points in $[0,1]$ $(0 \leq t_1 < t_2 < \ldots < t_n \leq 1)$, we consider the collocation discretization of (1): find $x \in L^2([0,1])$ such that

$$Kx(t_i) = y(t_i), \ \forall \ i = 1, \ldots, n. \tag{3}$$

If $t_i \in T_n$ we define $k_{t_i} : [0,1] \longrightarrow I\!R$ and $\tilde{y_i}$ by $k_{t_i}(s) = k(t_i, s), \ \forall \ s \in [0,1], \ \tilde{y_i} = y(t_i), \ i = 1, \ldots, n$. Then, the equation (3) can be written (3) can be written as

$$C_n x = \tilde{y}, \tag{4}$$

where $\tilde{y} \in I\!R^n$ and $C_n : L^2 \longrightarrow I\!R^n$ are defined by $C_n z = (\langle k_{t_1}, z \rightarrow, \ldots, \langle k_{t_n}, z \rightarrow, \ldots, \langle k_{t_n}, z$ If

$$y \in R(K), \tag{5}$$

let $x^{LS}$ be the minimal norm least-squares solution of (1) and let $x_n^{LS}$ be the similar one for (3) (or (4)), given by

$$x^{LS} = K^+ y, \ \ x_n^{LS} = C_n^+ \tilde{y}. \tag{6}$$

**Assumption CW.** There exists a sequence of positive integers $0 < n_1 < n_2 < \ldots < n_p < n_{p+1} < \ldots$ such that $\dim(Y_{n_p}) \ < \ \dim(Y_{n_{p+1}}), \ \forall \ p \geq 1$, with $Y_n = \mathrm{span}\{k_t, \ t \in T_n\}$.

**Remark 2.1** *The above assumption* **CW** *tells us that the number of linearly independent functions $k_t$ in the subspaces $Y_n$ tends to infinity together with $n$, but not all the functions in each $Y_n$ are linearly independent, as in the original assumption* [10].

The following result is proved in [11] (Theorem 2.4).

**Theorem 2.1** *Under the assumption* **CW**, *if* (5) *holds, and* $\lim_{n \longrightarrow \infty} \Delta_n = 0$, *where by* $\Delta_n$ *we denoted* $\sup_{t \in [0,1]} \left( \inf_{t_i \in T_n} |t - t_i| \right)$, *then*

$$\lim_{n \longrightarrow \infty} \| x_n^{LS} - x^{LS} \| = 0. \tag{7}$$

In [11] it is proven that $x_n^{LS}$ can be computed as

$$x_n^{LS}(t) = \sum_{j=1}^{n} \alpha_j k(s_j, t), \; t \in [0, 1], \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the minimal norm solution of the system $A_n \alpha = b_n$, and the entries for matrix $A_n$ and vector $b_n$ are given by

$$(A_n)_{ij} = \int_0^1 k(s_i, t) k(s_j, t) dt, \quad (b_n)_i = y(s_i), \quad i, j = 1, \dots, n.$$

For the case $y \in R(K) \oplus R(K)^{\perp}$, instead of (1), it is considered normal equation

$$\tilde{Q} x = w, \tag{9}$$

where $\tilde{Q} = K^* K, \quad w = K^* y$. Because of the equality [2] $\tilde{Q}^+ w = K^+ y$, it follows that the equations (1) and (9) have the same minimal norm solution $x^{LS}$ given by (6). Then, we replace (2) by the problem: find $x \in L^2([0, 1])$ such that $\sum\limits_{i=1}^{n} \left( \tilde{Q} x(t_i) - w(t_i) \right)^2 = \min!$ In this case, under a assumption similar to as **CW**, Theorem 2.1 still holds [11], where $x_n^{LS}$ is the minimal norm solution for (9).

Also, $x_n^{LS}$ can be computed as $x_n^{LS}(t) = \sum\limits_{j=1}^{n} \alpha_j \tilde{Q}(s_j, t), \; t \in [0, 1]$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the minimal norm solution of the (consistent) system $Q_n \alpha = \tilde{w}$ and, in this case, the entries for matrix $Q_n$ and vector $\tilde{w}$ are given by

$$(Q_n)_{ij} = \int_0^1 \tilde{Q}(s_i, t) \tilde{Q}(s_j, t) dt, \quad \tilde{w} = (w(t_1), \dots, w(t_n)), \quad i, j = 1, \dots, n.$$

With well-posed problems, better results are obtained as we refine the discretization. However, for the first kind integral equations, refining the discretization causes the discrete problem to more mirror the ill-posed nature of continuous problem. All the above mentioned matrices are rank-defficient, very ill-conditioned, and symmetric. Thus, using a classical direct or iterative method to solve these systems is not a good idea. A class of iterative solvers for relatively dense symmetric linear systems are the Kovarik-like approximative orthogonalization algorithms (see [9]).

**Algorithm KOBS** Let $A_0 = A$ a symmetric matrix.
for $k = 0, 1, \dots$ do: $K_k = (I - A_k)(I + A_k)^{-1}, \; A_{k+1} = (I + K_k) A_k$.

**Theorem 2.2** *If none of the eigenvalues of $A$ is in the set $E = \left\{ -\frac{1}{\alpha_j}, j \in N, \; \alpha_0 = 1, \alpha_{j+1} = 2\alpha_j + 1 \right\}$ then the sequence $(A_k)_{k \geq 0}$ generated as above is well-defined, convergent, and $\lim\limits_{k \to \infty} A_k = A^+ A$.*

In order to avoid the computation of the inverse at each step of the previous algorithm, we shall use a modified version of that one. The inverse $(I + A_k)^{-1}$ will be approximated by ($q \geq 1$ arbitrary fixed) $S(A_k; q) = \sum_{i=0}^{q} a_i(-A_k)^i$, with $a_0 = 0, a_{j+1} = \frac{2j+1}{2j+2} \cdot a_j,\ j > 0$. [9]

**Algorithm MKOBS** Let $A_0 = A$ be a symmetric matrix with $\sigma(A) \subset [0,\ 1]$. We construct the sequence $(A_k)_{k\geq 0},\ (A_k)_{k\geq 0}$ via

$$K_k = (I - A_k)S(A_k; n_k); \quad A_{k+1} = (I + K_k)A_k. \tag{10}$$

In order to solve the linear least-squares problem of the form $\|Ax - b\| = min!$ the following right hand side (rhs, for short) version of algorithm **MKOBS** was proposed in [9].

**Algorithm MKOBS-rhs** Let $A_0 = A,\ b^0 = b$; for $k = 0, 1, \ldots$ do

$$K_k = (I - A_k)S(A_k; n_k), \quad A_{k+1} = (I + K_k)A_k, \quad b^{k+1} = (I + K_k)b^k \tag{11}$$

In [9] the following results are proved .

**Theorem 2.3** *(i) If the problem* (1) *is consistent, then the sequence* $(b^k)_{k\geq 0}$ *is convergent and*

$$\lim_{k\to\infty} b^k = A^+b = x_{LS} \tag{12}$$

*(ii) If the problem* (1) *is not consistent, then the sequence* $(A_k b^k)_{k\geq 0}$ *is convergent and*

$$\lim_{k\to\infty} A_k b^k = A^+b = x_{LS} \tag{13}$$

*In this case,* $\lim_{k\to\infty} \left\|b^k\right\| = \infty$

**Remark 2.2** *The last relationship can generate problems. That is why, in practice, it is used a modified version of MKOBS-rhs algorithm.*

**Algorithm MKOBS-rhs-1**

$$K_k = (I - K_k)(I - \frac{1}{2}A_k), \quad A_{k+1} = (I + K_k)A_k, \quad \alpha^{(k+1)} = (I + K_k)^2\alpha^{(k)} \tag{14}$$

**Remark 2.3** *The above algorithm* **MKOBS-rhs-1** *has the same convergence behaviour as described in Theorem* 2.3.

## 2.2. SPLINE TECHIQUES USING THE PROJECTION METHOD

We shall start by briefly presenting the projection method used to solve the equation (1). Let $n \geq 1$ be arbitrary fixed and $\{v_1, v_2, \ldots, v_n\} \subseteq \overline{R(K)}$ a set of vectors with $\|v_i\| = 1, \ \forall\, i \in N$. We consider the following discretization of the equation (1): find $x \in X_n$ such that

$$\langle Kx, v_i \rangle = \langle y, v_i \rangle, \ \forall\, i = 1, \dots n, \tag{15}$$

where $X_n = span\{K^*v_1, \dots, K^*v_n\}$. If, for any $n \geq 1$, the set $\{v_1, v_2, \dots, v_n\} \subseteq \overline{R(K)}$ is linearly independent, then the discrete problem (15) has a unique solution $x_n \in X_n$ given by [3]

$$x_n = (K^*v_1, K^*v_2, \dots, K^*v_n)Q_n^{-1}(\langle y, v_1 \rangle \langle y, v_2 \rangle, \dots, \langle y, v_n \rangle)^t \tag{16}$$

or, equivalently,

$$x_n = \sum \tag{16}$$

or, equivalently,

$$x_n = \sum_{j=1}^{n} \alpha_j K^* v_j, \tag{17}$$

where system

$$Q_n \alpha = b$$

whith $Q_n = (\langle K^*v_i, K^*v_j \rangle)_{i,j=\overline{1,n}}$, $b = (b_1, \dots, b_n)^t \in I\!R^n$, $b_i = \langle y, v_i \rangle$. Since $K^+y = K^+P_{\overline{RK}}y$, the solution $K^+y = K^+P_{\overline{RK}}y$, the solution of (15) $y \in R(K)$. The following result is proved in [3] (Theorem 2.6).

**Theorem 2.4** *Under the above condition of linearly independency, and if* $span\{v_1, \dots, v_n, \dots\}$ *is dense in* $\overline{R(K)}$*, then* $\lim\limits_{n \to \infty} x_n = x_{LS}$*, where* $x_{LS}$ *is the minimal norm solution of the least-square problem associated with (1).*

**Remark 2.4** *In [11] it is proved that the previous theorem still holds even if we do not have the linear independent functions, but under a milder condition, similarly to Assumption* **CW**.

The main idea presented in [4] is to use, in the projection, method using as $v_i$ the spline functions. For this, let $a = x_1 < x_2 < \ldots < x_n = b$ be a partition of $[a, b]$. In [4] it is required that $y$ is $b - a$- periodic function. This is not a restrictive condition since we can define the other (eventually needed) values as $y(s_j) = y(s_{j+n}),\ j \leq 0$, and $y(s_j) = y(s_{j-n}),\ j > n$, and the knots $x_j$ with $j < 0$ or $j > n$ are chosen according to the periodicity. We shall denote by $s_{i,2m-1}(x)$ the local polynomial spline of $2m - 1$ degree constructed on knots $x_i, \dots, x_{i+2m}, \quad i = -2m+1, \dots, n-1$. The formulas for the local spline and the algorithms of their stable calculation is given in [1]. In our example we shall use the cubic spline polynomials (so, $m = 2$). Thus, the approximated minimal norm solution will be a linear combination of such splines.

In [4] it is proved that (Theorem 1.2.1) any solution for the initial equation obtained by the computation method is also solution obtained by the projection method. As in most cases, the first method is more tractable to deal with, we shall use this one in numerical experiments.

**Remark 2.5** *Another way to approximate the minimal norm solution is using the trigonometric spline.*

## 2.3. REGULARIZATION METHODS

Even if we formulate (1) in the least-square sense, if $K$ is of infinite rank, we have difficulties with in solving it because the Moore-Penrose inverse $K^+ : D(K^+) = R(K) \oplus R(K)^\perp \to I\!R$ is unbounded, and, as we have a noise in the data, namely

$$\| y - y_\delta \| \leq \delta, \tag{18}$$

one cannot expect the solution of the perturbed least-squares equation to be a good approximation to the exact least-squares $x_{LS} = K^+ y$. This is due to the fact that by its very nature, the initial problem is ill-posed. In order to overcome this shortcoming, it is considered the regularized equation of the normal equation

$$K^* K x_\delta = K^* y_\delta, \tag{19}$$

where adjoint of the operator $K$. Such (regularized) equations are computationally more tractable, but, in this case, another difficulty arises: to find a good regularization parameter. This task can be an expensive procedure. For example, for the standard Landweber iteration, for an $n$-point discretization of (1) $2in^2$ operations are required , where $i$ is the number of iterations, which can be quite large; also, for the Tikhonov method the cost is $\frac{n^3}{2} + \frac{jn^3}{6}$.

In what follows, we shall briefly present multilevel schemes which reduce the above mentioned computational cost (for details see [8]).
**Auxiliary Results**. For the compact operator $K$, let $\{u_n,\ v_n,\ \mu_n\}$ be the singular system given by the singular value decomposition theorem (for short, the SVD theorem): $\{v_n\}$ is the orthonormal eigenvector system for $K^* K$ with the eigenvectors $\lambda_1^2 \geq \lambda_2^2 \geq \ldots,\ \mu_n = |\lambda_n|^{-1}$, and $u_n = \mu_n K v_n$. It is known that $\{v_n\}$ and $\{u_n\}$ form orthonormal bases in $\overline{R(K^*)}$ and $\overline{R(K)}$ respectively. Also, the Picard Criteria for solvability and stability of (1) states the following [5].

**Theorem 2.5** *Equation* (1) *has a solution if and only if*
(i) $y \in N(K^*)^\perp$, and
(ii) $\sum\limits_{n=1}^{\infty} \mu_n^2 \, |(y, u_n)|^2 < \infty$.

*Under these assumptions, the solution is*

$$x = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n. \tag{20}$$

**Remark 2.6** *Problems appear when $y$ is perturbed by $\delta y$, because, in this case, either for $y + \delta y$ the condition* (ii) *may not hold, or if it does, the series $\sum\limits_{n=1}^{\infty} \mu_n(\delta y, u_n)$ may be notable (as $\mu_n \to \infty$). This is due to the fact that $R(K)$ is not closed (or* $\dim R(K) = \infty$*).*

**Theorem 2.6** *If $y \in D(K^+)$, then the minimal norm solution (for exact data) is given by*

$$x_{LS} = K^+ = \sum_{n=1}^{\infty} \mu_n (Py, u_n) v_n = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n, \tag{21}$$

*where $P$ is the orthogonal projector onto $\overline{R(K)}$.*

**Remark 2.7** *As in the previous remark, if $R(K)$ is not closed, the perturbed least-squares has the same instability problem.*

**Landweber Iteration. Tikhonov Regularization.** The aforementioned problems can be solved using the regularization algorithms (the main results can be found in [6]). The Landweber iteration and the Tikhonov regularization methods are defined as

$$x_{n+1}^{\delta} = x_n^{\delta} + \mu(K^* y_\delta - K^* K x_n^{\delta}),\ x_0^{\delta} = 0,\ 0 < \mu < \frac{2}{\| K^* K \|} = \frac{2}{\| K \|^2}, \tag{22}$$

and

$$x_{\alpha(\delta)}^{\delta} = [K^* K + \alpha(\delta)]^{-1} K^* y_\delta, \tag{23}$$

respectively, where $x_\alpha,\ x_\alpha^\delta$ are the solutions of the regularized equation with exact, and perturbed data respectively. The following estimations hold.

**Theorem 2.7**

$$\| M, \quad \| x_\alpha - x_\alpha^\delta \| \leq \delta \sqrt{M r(\alpha)}. \tag{24}$$

**Remark 2.8** *For the Landweber-Fridman iteration, $M = 1$, $r(n) = \mu n$, and if $n(\delta)$ is chosen such that $\delta^2 \mu n(\delta) \to 0$, $\delta \to 0$, then $x_{n(\delta)}^{\delta} \to x_{LS}$; for the Tikhonov scheme, $M = 1$, $r(\alpha) = \frac{1}{\alpha}$, and if $\frac{\delta^2}{\alpha(\delta)} \to 0$, $\delta \to 0$, then $x_{\alpha(\delta)}^{\delta} \to x_{LS}$.*

The Morozov discrepancy principle chooses the unique $\alpha(\delta)$ with the property $\| Kx_{\alpha(\delta)} - y_\delta = \delta \|$ .

For the first kind integral equation, the Landweber iteration is

$$x_n^\delta(s) = x_{n-1}^\delta(s) + \int_a^b k(v,s)\left[y_\delta(t) - \int_a^b k(v,t)x_{n-1}^\delta(t)\ \mathrm{dt}\right]\ \mathrm{dv},$$

which is solved after being discretized as

$$\tilde{x}_n^\delta = \tilde{x}_{n-1}^\delta + hK_{hh}^t\left[\tilde{y}_{\delta h} - hK_{hh}\tilde{x}_{n-1,h}^\delta\right],$$

where $h$ is the step size of the discretization, and $K_{hh}$ is the discretized kernel with stepsize $h$. The theory assures us [6], [8] that both

$$\| x_n^\delta(s) - x_{LS} \| \to 0,\ \delta \to 0$$

and

$$\| \tilde{x}_{n,h}^\delta - x_{LS} \| \to 0,\ \delta \to 0,$$

and also, the quadrature error goes to 0. The idea of the multilevel schemes is to monotorize the residual; if the residual does not change much after a coarse-grid correction, then only additional Landweber iteration on the fine grid should be performed. In [8] it is said that $\alpha$ should be not too small to permit magnification of roundoff errors which can be obtained on a grid coarser than $H$. If this grid is $4h$, letting $H = 2h$, the number of operation is less than in the standard approach.

The standard form of the Tikhonov scheme is

$$(K^*K + \alpha(\delta)I)x_{\alpha(\delta)}^\delta = K^*y_\delta. \tag{25}$$

The zeroth order stabilizer is $f(x) = \| x \|_{L^2}^2$ which applied to a first kind integral equation produces an integro-differential equation with boundary conditions as follows

$$\int_a^b\int_a^b k(v,s)k(v,t)x_{\alpha(\delta)}^\delta(t)\ \mathrm{dv}\ \mathrm{dt} + \alpha(\delta) = \int_a^b k(v,s)y_\delta(v)\ \mathrm{dv},$$

with $x_{\alpha(\delta)}^\delta(a) = x_1,\ x_{\alpha(\delta)}^\delta(b) = x_2$. For the parameter choice the quasi-optimal method is used i.e. $\alpha_k = \mu\alpha_{k-1},\ 0 < \mu < 1$. Then the parameter that minimizes $\| x_{\alpha_n(\delta)}^\delta - x_{\alpha_{n-1}(\delta)}^\delta \|$ is chosed. The idea of the Tikhonov multilevel schemes consists in: using $n$ levels, the coarsest level is solved using the discrepancy principle with Choleschy decomposition, and then the higher levels are solved using the discrepancy stopping criterion with an iterative system solver. Thus, the operations number reduces significantly.

## 3. NUMERICAL EXPERIMENTS

**Problem 1.**(P1) Consider the the integral equation with the kernel

$$k(s,t) = \frac{1}{\sqrt{(1+(s-t)^2)^3}}, \quad s,t \in [0,1].$$

The problem is a simplified version of a problem arising in the field of electrical potential generated by a known electric field. It was specifically chosen as a model problem to test the algorithms presented, since this kernel is a symmetric function, thus being appropriate for applying the iterative solvers described in Section 2.1.

| Collocation points | KOBS iterations | MKOBS iterations |
|---|---|---|
| 10 | 18 | 18 |
| 50 | 16 | 16 |
| 100 | 16 | 16 |
| 200 | 15 | 15 |

*Table 1* P1: KOBS and MKOBS maximum admissible error $10^{-6}$.

For $y(s) = \sin(\arctan(1-s)) - \sin(\arctan(-s))$, a solution is $x(t) = 1$ ($y \in R(K)$). The solutions for (P1) with KOBS and MKOBS are very similar fig. 1 and, at the same time, very close to the known solution $x(t) = 1$.

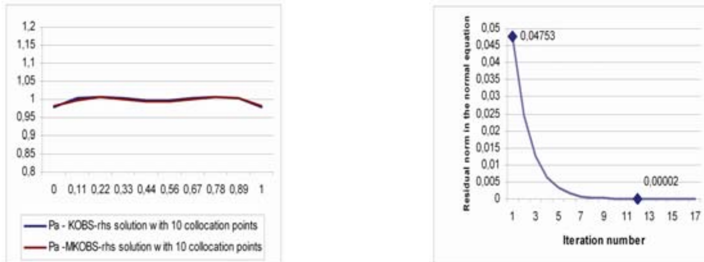
**Problem 2.** Let the equation (derived from antenna design theory)

$$\int_{-\pi}^{\pi} \cos(st)x(t)\ \mathrm{dt} = 2\pi\left[S((1+s)\pi) + S((1-s)\pi)\right],$$

where $S(s) = \int_0^s \frac{\sin(u)}{u}\ \mathrm{du}$. It has the solution $x(t) = 2\pi\frac{\sin(t)}{t}$. After we transformed this equations from $[-\pi,\pi]$ to $[0,1]$, discretize it, and using the values $h = \sqrt{\frac{12\delta}{\|K_{hh}\|}}$, $\alpha$ is chosen using the Morozov principle on the coarsest grid (stepsize $4h$). In addition, $\alpha \geq 0.00005$ in order to prevent propagation of roundoff errors in the interpolation procedure. The noise is $tr(K_{hh}^t K_{hh})\delta$ The data are presented in Tables 2, 3.

**Problem 3.** Let the Phillip's equation

$$\int_{-3}^{3} k(t-s)x(t)\ \mathrm{dt} = y(s),\ \ s \in [-6,6]$$




a) KOBS and MKOBS solutions b) Residual norm for MKOBS-rhs

*Fig. 1.* P1: KOBS and MKOBS with collocation discretization with 10 points.

| $\delta/h$ | iterations | $\parallel err \parallel_2$ |
|---|---|---|
| 0.0005/0.0078125 | 21 | 0.0324145 |
| 0.00025/0.0039063 | 95 | 0.0241030 |
| 0.0001/0.0019531 | 348 | 0.0103711 |

*Table 2* Results obtained with a standard Landweber iteration.

| $\delta/h$ | $\alpha$ | $\parallel err \parallel_2$ |
|---|---|---|
| 0.0005/0.0078125 | 0.02293 | 0.0254669 |
| 0.00025/0.0039063 | 0.00717 | 0.0158831 |
| 0.0001/0.0019531 | 0.00250 | 0.00739896 |

*Table 3* Results obtained with a multigrid Landweber iteration.

where $k(u) = \begin{cases} 1+\cos(\pi u/3), & |u| \leq 3 \\ 0, & |u| \geq 3, \end{cases}$ and
$y(s) = \begin{cases} (6-s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] + \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [0,6] \\ (6+s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] - \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [-6,0] \end{cases}$, with the exact solution $x(t) = \begin{cases} 1+\cos(\pi t/3), & |t| \leq 3 \\ 0, & |t| \geq 3. \end{cases}$
The initial value of $\alpha$ is 1, $\mu = 0.5$, a noise $y(s_j)\delta\theta_j$ where $\theta$ is a random number chosen from a uniform distribution on $[-1,1]$, and the generalized discrepancy principle. The results are those from Tables 4 and 5.

| $\delta/h$ | $\alpha$ | $\| err \|_2$ |
|---|---|---|
| 0.0002/0.015625 | 0.005722 | 0.0490729 |
| 0.0001/0.0078125 | 0.002576 | 0.0322052 |
| 0.00002/0.00390625 | 0.0009570 | 0.0233487 |

*Table 4* Results obtained by using a standard Tikhonov method with 0th order stabilizer.

| $\delta/tol$ | $\alpha$ | $\| err \|_2$ |
|---|---|---|
| $0.0002/10^{-6}$ | 0.003725 | 0.0391557 |
| $0.0001/10^{-6}$ | 0.001572 | 0.0260117 |
| $0.00002/10^{-8}$ | 0.0007053 | 0.0227495 |

*Table 5* Results obtained by using the multigrid Tikhonov method.

## 4. ACKNOWLEDGEMENTS

The paper was supported by the PNCDI INFOSOC Grant 131/2004.

## References

[1] de Boor, C., *Practical guides on spline*, Radio and Svjaz, Moscow, 1985.

[2] Boullion, T. L., P. L. Odell, *Generalized inverse matrices,* Willey-Interscience, New-York, 1971.

[3] Engl, H. W., *On the convergence of regularization methods for ill-posed linear operator equation,* Improperly Posed Problems and Their Numerical Treatment, Birkhauser, Basel-Boston-Stuttgart, 81-95, Surv. Math. Ind., **3** (1993), 71–143.

[4] Grebennikov, A.I., *Spline approximation method and its applications*, Max Press, Moscow, 2004.

[5] Groetsch, C.W., *Inverse problems in the mathematical sciences*, Vieweg, 1993.

[6] Groetsch, C.W., *The theory of Tikhonov regularization for Fredholm equation of the first kind*, Pitman, Boston, 1984.

[7] Kress. R., *Linear integral equations,* Springer, Berlin, 1989.

[8] Lee, B., *Multilevel approaches to ill-posed equations,* Master Thesis, Simon Frasier University, Burnaby, 1992.

[9] Mohr, M., C. Popa, U. Rude, *A Kovarik type algorithm without matrix inversion for the numerical solution of symmetric least-squares problems*, Lehrstulbericht 05-2, 2005

[10] Nashed, M. Z., G. Wahba, *Convergence rates of approximate least-squares solutions of linear integral and operator equation of the first kind.* Math. of Comp., **28(125)**, (1974).

[11] Pelican, E., *Extensions of projection and collocation methods for first kind integral equations,* Rev. Roumaine Math. Pures Appl., **3(51)**(2006), to appear.