Contents

ROMAI J., **2**, 2(2006),
1–13

DETECTING ORDER AND CHAOS BY THE LINEAR DEPENDENCE INDEX (LDI) METHOD

Chris Antonopoulos, Tassos Bountis
Univ. of Patras, Greece
antonop@math.upatras.gr, bountis@math.upatras.gr

Abstract We introduce a new methodology for a fast and reliable discrimination between ordered and chaotic orbits in multidimensional Hamiltonian systems which we call the Linear Dependence Index (LDI). The new method is based on the recently introduced theory of the Generalized Alignment Indices (GALI). LDI takes advantage of the linear dependence (or independence) of deviation vectors of a given orbit, using the method of Singular Value Decomposition at every time step of the integration. We show that the LDI produces estimates which numerically coincide with those of the GALI method for the same number of m deviation vectors, while its main advantage is that it requires considerable less CPU time than GALI especially in Hamiltonian systems of many degrees of freedom.

1. INTRODUCTION

The fast and reliable discrimination of chaotic and ordered orbits of conservative dynamical systems is of crucial interest in many problems of nonlinear science. By the term conservative we characterize here systems which preserve phase space volume (or some other positive function of the phase space variables) during time evolution. Important examples in this class are $N-$ degree of freedom (dof) Hamiltonian systems and $2N-$ dimensional symplectic maps. As is well known, in such systems chaotic and regular orbits are distributed in phase space in very complicated ways, which often makes it very difficult to distinguish between them. In recent years, several methods have been developed and applied to various problems of physical interest in an effort to distinguish efficiently between ordered and chaotic dynamics. Their discrimination abilities and overall performance, however, varies significantly, making some of them more preferable than others in certain situations.

One of the most common approaches is to extract information about the nature of a given orbit from the dynamics of small deviations, evaluating the maximal Lyapunov Characteristic Exponent (LCE) σ_1. If $\sigma_1 > 0$ the orbit is characterized as chaotic. The theory of Lyapunov exponents was first applied to characterize chaotic orbits by Oseledec [19], while the connection between Lyapunov exponents and exponential divergence of nearby orbits was given in

[8, 21]. Benettin et al. [6] studied the problem of the computation of all LCEs theoretically and proposed in [7] an algorithm for their efficient numerical computation. In particular, σ_1 is computed as the limit for $t \to \infty$ of the quantity

$$L_1(t) = \frac{1}{t} \ln \frac{\|\mathbf{w}(t)\|}{\|\mathbf{w}(0)\|} \text{, i.e. } \sigma_1 = \lim_{t\to\infty} L_1(t) \,, \tag{1}$$

where $\mathbf{w}(0)$, $\mathbf{w}(t)$ are deviation vectors from a given orbit, at times $t = 0$ and $t > 0$ respectively. It has been shown that the above limit is finite, independent of the choice of the metric for the phase space and converges to σ_1 for almost all initial vectors $\mathbf{w}(0)$ [19, 6, 7]. Similarly, all other LCEs, σ_2, σ_3 etc. are computed as limits for $t \to \infty$ of some appropriate quantities, $L_2(t)$, $L_3(t)$ etc. (see for example [7]). We note here that throughout the paper, whenever we need to compute the values of the maximal LCE or of several LCEs we apply respectively the algorithms proposed by Benettin et al. [8, 7].

Over the years, several variants of this approach have been introduced to distinguish between order and chaos such as: The Fast Lyapunov Indicator (FLI) [15, 14, 12, 16, 5], the Mean Exponential Growth of Nearby Orbits (MEGNO) [11, 10], the Smaller Alignment Index (SALI) [24, 25, 26], the Relative Lyapunov Indicator (RLI) [23], as well as methods based on the study of power spectra of deviation vectors [29] and spectra of quantities related to these vectors [13, 17, 28].

Recently, the SALI method was generalized to yield a much more comprehensive approach to study chaos and order in $2N$ – dimensional conservative systems, called the GALI_m indices [27, 2]. These indices represent the volume elements formed by m deviation vectors ($2 \leq m \leq 2N$) about any reference orbit and have been shown to: (a) Distinguish the regular or chaotic nature of the orbit faster than other methods, (b) identify the dimensionality of the space of regular motion and (c) predict the slow (chaotic) diffusion of orbits, long before it is observed in the actual oscillations.

In the present paper, we improve the GALI method by introducing the Linear Dependence Indices (LDI_m). The new indices retain the advantages of the GALI_m and display the same values as GALI, in regular as well as chaotic cases. More importantly, however, the computation of the LDI_m is much faster in CPU time, especially if the dimensionality of phase space becomes large ($N \gg 10$). The main purpose of this paper, therefore, is to strongly advocate the use of LDI, for the most rapid and efficient study of the dynamics of multi – dimensional conservative systems.

For the computation of the LDI_m we use information from the evolution of $m \geq 2$ deviation vectors from the reference orbit, as GALI does. However, while GALI requires the computation of many $m \times m$ determinants at every time step [27, 2] in order to evaluate the norm of the corresponding wedge product, LDI achieves the same purpose simply by applying Singular Value

Decomposition (SVD) to the $2N \times m$ matrix formed by the deviation vectors. LDI is then computed as the product of the corresponding singular values of the above matrix. This not only provides the same numerical values as the corresponding GALI$_m$, it also requires much less CPU time.

The paper is organized as follows: In section 2 we introduce the new index, explain in detail its computation and justify its validity theoretically. In section 3, we demonstrate the usefulness of the LDI method, by applying it to the famous Fermi – Pasta – Ulam (FPU) lattice model of N dof, for small and large N. Finally, in section 4 we present our conclusions, highlighting especially the advantages of the new index.

2. DEFINITION OF THE LINEAR DEPENDENCE INDEX (LDI)

Let us consider the $2N$ – dimensional phase space of a Hamiltonian system

$$H \equiv H(q_1(t), \ldots, q_N(t), p_1(t), \ldots, p_N(t)) = E \tag{2}$$

where $q_i(t),\ i = 1, \ldots, N$ are the canonical coordinates, $p_i(t),\ i = 1, \ldots, N$ are the corresponding conjugate momenta and E is the total energy. The time evolution of an orbit $\mathbf{x}(t)$ of (1) associated with the initial condition

$$\mathbf{x}(t_0) = (q_1(t_0), \ldots, q_N(t_0), p_1(t_0), \ldots, p_N(t_0))$$

at initial time t_0 is defined as the solution of the system of $2N$ first order differential equations (ODE)

$$\frac{dq_i(t)}{dt} = \frac{\partial H}{\partial p_i(t)}, \quad \frac{dp_i(t)}{dt} = -\frac{\partial H}{\partial q_i(t)},\ i = 1, \ldots, N. \tag{3}$$

Eqs. (13) are known as Hamilton's equations of motion and the reference orbit under study is the solution $\mathbf{x}(t)$ which passes by the initial condition $\mathbf{x}(t_0)$.

In order to define the Linear Dependence Index (LDI) we need to introduce the variational equations. These are the corresponding linearized equations of the ODE (13), about the reference orbit $\mathbf{x}(t)$ defined by the relation

$$\frac{d\mathbf{v}_i(t)}{dt} = \mathcal{J}(\mathbf{x}(t)) \cdot \mathbf{v}_i(t),\ i = 1, \ldots, 2N \tag{4}$$

where $\mathcal{J}(\mathbf{x}(t))$ is the Jacobian of the right hand side of the system of ODEs (13) calculated about the orbit $\mathbf{x}(t)$. Vectors $\mathbf{v}_i(t) = (\upsilon_{i,1}(t), \ldots, \upsilon_{i,2N}(t)),\ i = 1, \ldots, 2N$ are known as deviation vectors and belong to the tangent space of the reference orbit at every time t.

We then choose $m \in [2, 2N]$ initially linearly independent deviation vectors $\mathbf{v}_m(0)$ and integrate equation (3) together with the equations of motion (13).

These vectors form the columns of a $2N \times m$ matrix $\mathcal{A}(t)$ and are taken to lie along the orthogonal axes of a unit ball in the tangent space of the orbit $\mathbf{x}(t)$ so that $\mathbf{v}_m(0)$ are orthonormal. Thus, at every time step, we check the linear dependence of the deviation vectors by performing Singular Value Decomposition on $\mathcal{A}(t)$ decomposing it as follows

$$\mathcal{A}(t) = U(t) \cdot W(t) \cdot V(t)^{\top}, \tag{5}$$

where $U(t)$ is a $2N \times m$ matrix, $V(t)$ is an $m \times m$ matrix whose columns are the $\mathbf{v}_m(t)$ deviation vectors and $W(t)$ is a diagonal $m \times m$ matrix, whose entries $w_1(t), \ldots, w_m(t)$ are zero or positive real numbers. They are called the *singular values* of $\mathcal{A}(t)$. Matrices $U(t)$ and $V(t)$ are orthogonal so that $U^{\top}(t) \cdot U(t) = V^{\top}(t) \cdot V(t) = I$, where I is the rectangular $2N \times 2N$ unit matrix.

We next define the generalized Linear Dependence Index of order m or LDI_m as the function

$$\mathrm{LDI}_m(t) = \prod_{j=1}^{m} w_j(t) \tag{6}$$

with $m = 2, 3, \ldots, 2N$, where N is the number of dof of (1).

The reason for defining LDI through relation (6) is the following: According to [27] it is possible to determine whether an orbit is chaotic or lies on a d – dimensional torus by choosing m deviation vectors and computing the GALI_m index. If $\mathrm{GALI}_m \approx$ const. for $m = 2, 3, \ldots, d$ and for $m > d$ decay by a power law, the motion lies on a d – dimensional torus. If, on the other hand, all GALI_m indices decay exponentially the motion is chaotic. Thus, to characterize orbits we often have to compute GALI_m indices for m as high as N or higher.

A serious limitation appears, of course, in the case of Hamiltonian systems of large N, where $\mathrm{GALI}_N(t)$ involves the computation of $\binom{2N}{N} = \frac{(2N)!}{(N!)^2}$ determinants at every time step. For example, in a Hamiltonian system of $N = 15$ dof, $\mathrm{GALI}_{15}(t)$ requires, for a given orbit, the computation of 155117520 determinants at every time step while LDI(t) $= \mathrm{LDI}_{15}(t)$ requires only the application of the SVD method for a 30×15 matrix $\mathcal{A}(t)$!

Clearly, at every point of the orbit $\mathbf{x}(t)$ the $2 \leq m \leq 2N$ deviation vectors span a subspace of the $2N$ – dimensional tangent space of the orbit, which is isomorphic to the Euclidean $2N$ – dimensional phase space of the Hamiltonian system (13). Thus, if k of the m singular values $w_k(t), k = 1, \ldots, m$ are equal to zero, then k columns of matrix $\mathcal{A}(t)$ of deviation vectors are linearly dependent with the remaining ones and the subspace spanned by the column vectors of matrix $\mathcal{A}(t)$ is $d(= m - k)$ – dimensional.

From a more geometrical point of view, let us note that the m variational equations (3) combined with the equations of motion (13) describe the evolution of an initial m – dimensional unit ball into an m (or less) dimensional ellipsoid in the tangent space of the Hamiltonian flow. Now, the deviation vectors $\mathbf{v}_i(t)$ forming the columns of $\mathcal{A}(t)$ do not necessarily coincide with the ellipsoid's principal axes. On the other hand, in the case of a chaotic orbit, every generically chosen initial deviation vector has a component in the direction of the maximum (positive) Lyapunov exponent, so that all initial tangent vectors in the long run, will be aligned with the longest principal axis of the ellipsoid. The key idea behind the LDI method is to take advantage of this fact to overcome the costly calculation of the many determinants arising in the GALI$_m$ method and characterize a reference orbit as chaotic or not, via the trends of the stretching and shrinking of the m principal axes of the ellipsoid.

Thus, LDI solves the problem of orbit characterization by finding new orthogonal axes for the ellipsoid at every time step and taking advantage of the SVD method. Since the matrix V in (3) is orthogonal, we have $V^\top = V^{-1}$, so that equation (5) gives

$$\mathcal{A}_{2N\times m} \cdot V_{m\times m} = U_{2N\times m} \cdot W_{m\times m} \tag{7}$$

at every time step. Geometrically, Eq. (24.0) implies that the image formed by the column vectors of matrix V is equal to an ellipsoid whose i^{th} principal axis direction in the tangent space of the reference orbit is given by:

$$w_i \cdot u_i \tag{8}$$

where w_i are the singular values of matrix $\mathcal{A}(t)$ and u_i is the i^{th} column of matrix $U(t)$. This is, in fact, the content of a famous theorem stating that:

Theorem 2.1 *([1]) Let $\mathcal{A}$ be a $2N \times m$ matrix, and let U and W be matrices resulting from the SVD of $\mathcal{A}$. Then, the columns of $\mathcal{A}$ span an ellipsoid whose i^{th} principal axis is $w_i \cdot u_i$, where $W = diag(w_1, w_2, \ldots, w_m)$ (singular values) and $\{u_i\}_{i=1}^{m}$ are the columns of U.*

According to this theorem, the principal axes of the ellipsoid created by the time evolution of equation (3) in the tangent space of the reference orbit $\mathbf{x}(t)$ at every time t, are stretched or shrunk, according to the singular values of $w_i > 1$ or $w_i < 1$ respectively for $i = 1, \ldots, m$.

If it so happens that k of the singular values $w_i = 0$ as t grows, then the corresponding principal axes of the ellipsoid vanish and the ellipsoid is less than m – dimensional in the tangent space of the reference orbit because the corresponding deviation vectors of matrix $\mathcal{A}$ have become linearly dependent.

Thus, two distinct cases exist depending on whether the reference orbit $\mathbf{x}(t)$ is chaotic or ordered

1 If the orbit is chaotic, the m deviation vectors become linearly dependent so that $\mathrm{GALI_m}(t) \to 0$ exponentially [27]. Consequently, at least one of the singular values $w_i(t), i = 2, \ldots, m$ becomes zero and $\mathrm{LDI}_m(t) = \prod_{j=1}^{m} w_j(t) \to 0$ (also $\mathrm{LDI}(t) \to 0$) for all $m \geq i$.

2 If the orbit is ordered (i.e. quasiperiodic) lying on a d – dimensional torus, there is no reason [27, 25] for the m deviation vectors to become linearly dependent, as long as $m \leq d$. No principal axis of the ellipsoid is eliminated, since all singular values $w_i,\ i = 1, \ldots, m$ are nonzero and $\mathrm{LDI_m}(t)$ fluctuates around nonzero positive values. On the other hand, for $m \geq d$, the singular values $w_i,\ i = d+1, \ldots, m$ tend to zero following a power law [27], since $m - d$ deviations will eventually become linearly dependent with those spanning the d – dimensional tangent space of the torus [27, 25].

In the remainder of the paper, we apply the LDI indices and numerically demonstrate that

$$\mathrm{LDI}_m = \mathrm{GALI}_m, \quad m = 2, \ldots, 2N \tag{9}$$

for the same choice of m initially linearly independent deviation vectors $\mathbf{v}_i(0),\ i = 1, \ldots, m$. In particular, we present evidence that supports the validity of relation (9) and exploit it to identify rapidly and reliably ordered and chaotic orbits in a 1 – dimensional, N degree of freedom Fermi – Pasta – Ulam lattice under fixed and periodic boundary conditions [4, 3]. We propose that the validity of (9) is due to the fact that both quantities measure the volume of the same ellipsoid, the difference being that, in the case of the LDI, the principal axes of the ellipsoid are orthogonal. As we have not proved it, however, this is a point to which we intend to return in a future publication.

3. APPLICATION TO THE FPU HAMILTONIAN SYSTEM

In this section, we apply the LDI method to the case of a multidimensional Hamiltonian system. Our aim is the comparison of its performance and effectiveness in distinguishing between ordered and chaotic behavior compared with Lyapunov exponents as well as the SALI and GALI methods.

We shall use the N dof Hamiltonian system of the 1D lattice of the Fermi – Pasta – Ulam (FPU) β – model. The system is described by a Hamiltonian function containing quadratic and quartic nearest neighbor interactions

$$H_N = \frac{1}{2}\sum_{j=1}^{N} \dot{x}_j^2 + \sum_{j=0}^{N}\left(\frac{1}{2}(x_{j+1} - x_j)^2 + \frac{1}{4}\beta(x_{j+1} - x_j)^4\right) = E \tag{10}$$

where x_j is the displacement of the j^{th} particle from its equilibrium position, $\dot{x}_j$ is the corresponding conjugate momentum, β is a positive real constant and E is the constant energy of the system.

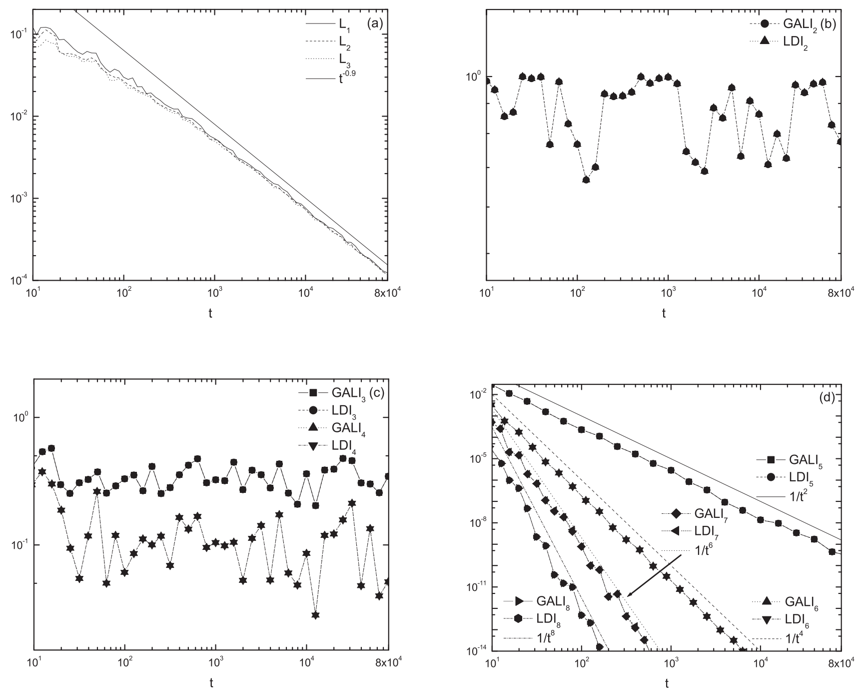


Fig. 1. The case of an ordered orbit:(a) The time evolution of the three maximal positive Lyapunov exponents. (b) The time evolution of GALI_2 and LDI_2. (c) The time evolution of GALI_3, LDI_3 and GALI_4, LDI_4. (d) The time evolution of GALI_m, LDI_m with $m = 5, \ldots, 8$ and the corresponding slopes of the fall to zero. In all panels we have chosen a neighboring orbit at a distance ≈ 2.1 from the OPM of the FPU Hamiltonian (10) with periodic boundary conditions for $N = 4$ and $E = 2$. All axes are logarithmic.

We start by focusing on an ordered case choosing a neighboring orbit of the stable out of phase mode (OPM) [9, 20, 4], which is a simple periodic orbit of the FPU Hamiltonian (10). This solution exists for every N, for fixed as well as periodic boundary condition (PBC)

$$x_{N+1}(t) = x_1(t), \ \forall t \tag{11}$$

and is given by

$$x_j(t) = -x_{j+1}(t), \ \dot{x}_j(t) = 0, \ j = 1, \ldots, N, \forall t. \tag{12}$$

In [9, 4] the stability properties of the OPM mode with periodic boundary conditions were determined using Floquet theory and monodromy matrix analysis and the energy range $0 \leq E(N) \leq E_c^{\text{OPM}}(N)$ over which it is linearly stable was studied in detail.

It is known that for $N = 4$ and $\beta = 1$, the solution (12) with periodic boundary condition (12) is destabilized for the first time at the critical energy $E_c^{\text{OPM}} \approx 4.51$. Below this critical energy, the OPM is linearly stable and is surrounded by a sizable island of stability. By contrast, for $E > E_c^{\text{OPM}}$, the OPM is linearly unstable with no island of stability around it.

In Fig. 1(a), we have calculated the three maximal Lyapunov exponents of a neighboring orbit located at distance ≈ 2.1 away from the OPM at $E = 2 < E_c^{\text{OPM}}$. At this energy, the OPM is linearly stable and thus all Lyapunov Exponents tend to zero following a simple power law. Next, in Fig. 1(b), we compute GALI_2 and LDI_2 for a final integration time $t = 8 \times 10^4$ and observe that GALI_2 and LDI_2 practically coincide fluctuating around non zero values indicating the ordered nature of the orbit. GALI_2 needs 558 seconds of computation time while LDI_2 takes about 912 seconds in a Pentium 4 3.2GHz computer.

In Fig. 1(c), we compute GALI_3, LDI_3 and GALI_4, LDI_4 for the same energy and initial condition. We see once more that GALI_3, LDI_3 and GALI_4, LDI_4 coincide fluctuating around non zero values. The GALI_3 computation now takes about 1044 seconds, LDI_3 about 838 seconds, GALI_4 needs 898 seconds and LDI_4 753 seconds.

Finally, in Fig. 1(d), we present GALI_m, LDI_m with $m = 5, \ldots, 8$ as a function of time for the same energy and initial condition. We observe again that GALI_m and LDI_m with $m = 5, \ldots, 8$, have the same values and tend to zero following a power law of the form $t^{-2(k-N)}$. All these results are in accordance with the formulae reported in [27] and suggest that the torus on which the orbit lies is 4 – dimensional, as expected from the fact that the number of dof of the system is $N = 4$.

In [4] we also studied the stability properties of a different simple periodic orbit of FPU called the SPO1 mode with fixed boundary conditions (FBC). Using monodromy matrix analysis we found that for $N = 5$ and $\beta = 1.04$, the SPO1 mode with FBC is destabilized for the first time at the critical energy $E_c^{\text{SPO1}} \approx 6.4932$.

Thus, in order to study a chaotic case where things are different, we choose initial condition at distance of $\approx 1.27 \times 10^{-4}$ from the SPO1 orbit at the energy $E = 11$, where it is unstable.

In Fig. 2(a), we calculate Lyapunov exponents of the above mentioned orbit and find that the four maximal Lyapunov exponents tend to positive values. This is strong evidence that the nature of the orbit is chaotic. Next, in Fig. 2(b) we calculate GALI_2 and LDI_2 up to $t = 1200$. We see the indices again

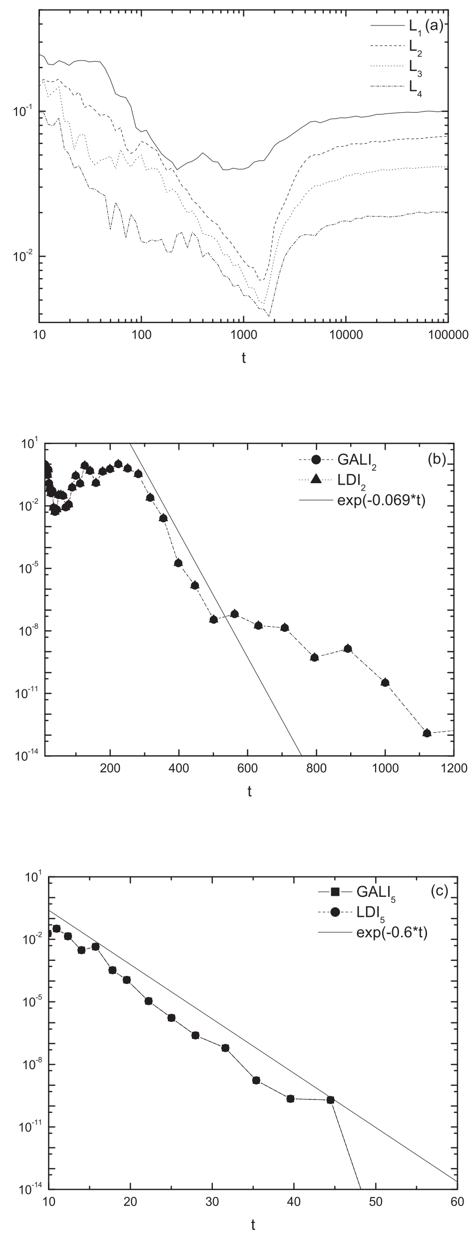


Fig. 2. The case of a chaotic orbit:(a) The time evolution of the four maximal Lyapunov exponents. (b) The time evolution of GALI_2, LDI_2 follows the approximate formula $e^{-(\sigma_1-\sigma_2)t}$ where $\sigma_1 \approx 0.124$ (solid straight line) and $\sigma_2 \approx 0.056$ for $t = 71$. (c) The time evolution of GALI_5, LDI_5 follows the approximate formula $\propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+(\sigma_1-\sigma_4)+(\sigma_1-\sigma_5)]t} \approx e^{-0.069t}$ (solid straight line) where $\sigma_1 \approx 0.197$, $\sigma_2 \approx 0.095$, $\sigma_3 \approx 0.047$, $\sigma_4 \approx 0.026$ and $\sigma_5 \approx 0.022$ for $t \approx 44$. We have used, in all figures, the same orbit of a distance of 1.27×10^{-4} from the SPO1 of the FPU Hamiltonian system (10) with fixed boundary conditions for $N = 5$ and $E = 11$.

coincide and tend to zero as $\propto e^{-(\sigma_1-\sigma_2)t}$ (solid straight line), as predicted by our theory [26, 27]. In this figure, we find $\sigma_1 \approx 0.124$ and $\sigma_2 \approx 0.056$ for time $t = 71$. The corresponding CPU time required for the calculation of all indices does not differ significantly, as they become quite small in magnitude, rather quickly.

Nevertheless, LDI_2 requires less CPU time than GALI_2. In Fig. 2(c), we calculate GALI_5 and LDI_5 for the same energy and initial condition as in the previous panels. We observe now that GALI_5 and LDI_5 coincide falling to zero as $\mathrm{GALI}_5 \propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+(\sigma_1-\sigma_4)+(\sigma_1-\sigma_5)]t}$ (solid straight line) where $\sigma_1 \approx 0.197$, $\sigma_2 \approx 0.095$, $\sigma_3 \approx 0.047$, $\sigma_4 \approx 0.026$ and $\sigma_5 \approx 0.022$ for $t \approx 44$. Clearly, GALI_5 and LDI_5 distinguish the chaotic character of the orbit faster than GALI_2 or LDI_2. This is so, because GALI_2 or LDI_2 reaches the threshold 10^{-8} [25, 26, 27] for $t \approx 750$ while GALI_5 and LDI_5 for $t \approx 35$! The CPU times required for the calculation of GALI_5 and LDI_5 up to $t = 80$ are approximately 1.5 seconds each.

Thus, we conclude from these results that the LDI method performs at least as well as the GALI, predicting correctly the ordered or chaotic nature of orbits in Hamiltonian systems for low dimensions, i.e. at 2, 4 and 5 degrees of freedom. However, in higher dimensional cases, GALI indices become very impractical as they demand the computations of millions of determinants at every time step making the LDI method much more useful.

In order to show the advantages of the LDI method concerning the CPU time, we repeat the above analysis for the same Hamiltonian system (10), but now for $N = 15$ and energy $E = 2$, and for an initial condition very close to the unstable SPO1 [4].

In [4] it has also been shown that for $N = 15$ and $\beta = 1.04$, the SPO1 with fixed boundary conditions destabilizes at the critical energy $E_c \approx 1.55$. Thus, for energies smaller than E_c, SPO1 is linearly stable, while for $E > E_c$ it is unstable and is surrounded by a chaotic region.

In Fig. 3(a) we depict the time evolution of the five maximal Lyapunov exponents which converge to positive values for high enough t suggesting that the neighboring orbit is chaotic. In the second panel of the same figure we present the evolution of GALI_8 and LDI_8 together with the approximate exponential law. We remark once more that the values of the corresponding indices coincide until they become numerically zero. More interestingly, the CPU time required for the calculation of GALI_8 up to $t \approx 100$ is about 186 seconds while for the LDI_8 it takes only one second! This difference is very important, showing why LDI is preferable compared to the corresponding GALI index in Hamiltonian systems of many degrees of freedom.

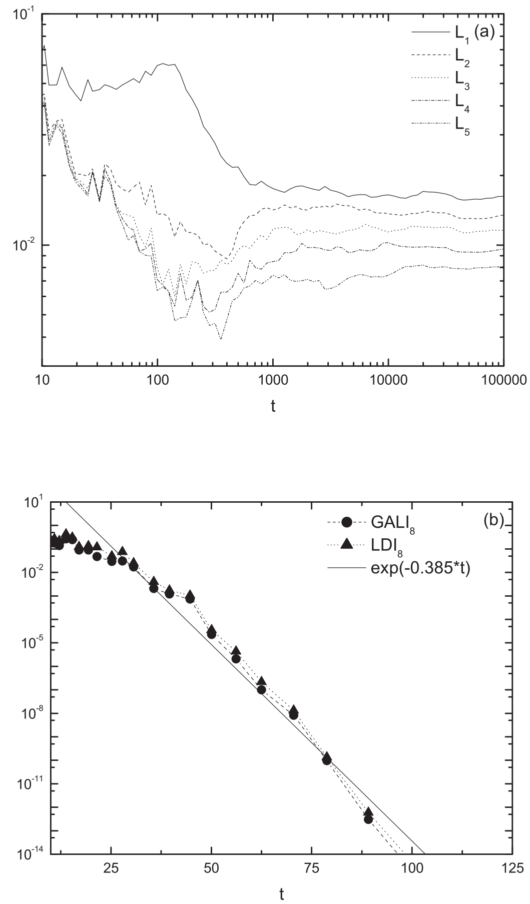


Fig. 3. (a) The time evolution of the five positive Lyapunov exponents. (b) The time evolution of the GALI_8, LDI_8 follows the approximate formula $\propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_8)]t} \approx e^{-0.385t}$ (solid straight line) where $\sigma_1 \approx 0.061$, $\sigma_2 \approx 0.011$, $\sigma_3 \approx 0.006$, $\sigma_4 \approx 0.005$, $\sigma_5 \approx 0.005$, $\sigma_6 \approx 0.004$, $\sigma_7 \approx 0.004$ and $\sigma_8 \approx 0.004$ for time $t \approx 141$. In all panels we have used initial conditions at a distance of 9×10^{-5} from the SPO1 orbit of Hamiltonian system (10) with fixed boundary conditions, $N = 15$ and $E = 2$.

4. CONCLUSIONS

In this paper we have introduced a new method for distinguishing quickly and reliably between ordered and chaotic orbits of multidimensional Hamiltonian systems and argued about its validity justifying it in the ordered and chaotic case. It is based on the recently introduced theory of the Generalized Alignment Indices (GALI). Following this theory, the key point in the distinction between order and chaos is the linear dependence (or independence) of deviation vectors from a reference orbit. Consequently, the method of LDI takes advantage of this property and analyzes m deviation vectors using Singular Value Decomposition to decide whether the reference orbit is chaotic or ordered. If the orbit under consideration is chaotic then the deviation vectors are aligned with the direction of the maximal Lyapunov exponent and thus become linearly dependent. On the other hand, if the reference orbit is ordered then there is no unstable direction and $m = 1, 2, ..., d \leq N$ deviation vectors are linearly independent. As a consequence, the LDI of order m (LDI_m) becomes either zero if the reference orbit is chaotic or it fluctuates around non zero values if the orbit is ordered if $m \leq d$.

After introducing the new method, we presented strong numerical evidence about its validity and efficiency in the interesting case of multidimensional Hamiltonian systems. One first main result is that GALI_m and LDI_m coincide numerically for the same m number of deviation vectors and for the same reference orbit. Moreover, it follows that it is preferable to use the LDI method rather than the equivalent GALI method especially in the multidimensional case of Hamiltonian systems, since the LDI needs considerably less CPU time than the corresponding GALI method for the same number of deviation vectors.

5. ACKNOWLEDGEMENTS

This work was partially supported by the European Social Fund (ESF), Operational Program for Educational and Vocational Training II (EPEAEK II) and particularly the Program PYTHAGORAS II. We thank Dr. Charalambos Skokos and Miss Eleni Christodoulidi for very fruitful discussions on the comparison between the GALI and LDI indices.

References

[1] K. T. Alligood, T. D. Sauer, and J. A. Yorke. *Chaos: An Introduction to Dynamical Systems*. Springer, New York, 1997.

[2] Ch. Antonopoulos. PhD Thesis, University of Patras, Mathematics Department, 2007. In Greek.

[3] Ch. Antonopoulos and T. C. Bountis. Stability of Simple Periodic Orbits and Chaos in a Fermi–Pasta–Ulam Lattice. *Physical Review E*, 73:1, 2006.

[4] Ch. Antonopoulos, T. C. Bountis, and Ch. Skokos. Chaotic Dynamics of N – Degree of Freedom Hamiltonian Systems. *Int. J. Bif. Chaos*, 16(6):1 – 18, 2006.

[5] R. Barrio. *Chaos Sol. Fract.*, 25:711, 2005.

[6] G. Benettin, L. Galgani, A. Giorgilli, and J-M. Strelcyn. *Meccanica*, March 9 1980.

[7] G. Benettin, L. Galgani, A. Giorgilli, and J-M. Strelcyn. *Meccanica*, March 21 1980.

[8] G. Benettin, L. Galgani, and J-M. Strelcyn. *Phys. Rev. A*, 14:2338, 1976.

[9] N. Budinsky and T. Bountis. Stability of Nonlinear Modes and Chaotic Properties of 1D Fermi–Pasta–Ulam Lattices. *Physica D*, 8:445 – 452, 1983.

[10] P. M. Cincotta, C. M. Giordano, and C. Simó. *Physica D*, 182(151), 2003.

[11] P. M. Cincotta and C. Simó. *Astron. Astroph. Suppl. Ser.*, 147(205), 2000.

[12] M. Fouchard, E. Lega, Ch. Froeschlé, and C. Froeschlé. *Celest. Mech. Dyn. Astron.*, 83(205), 2002.

[13] C. Froeschlé, Ch. Froeschlé, and E. Lohinger. *Celest. Mech. Dyn. Astron.*, 56:307, 1993.

[14] C. Froeschlé, R. Gonczi, and E. Lega. *Planet. Space Sci.*, 45:881, 1997.

[15] C. Froeschlé, E. Lega, and R. Gonczi. *Celest. Mech. Dyn. Astron.*, 67(41), 1997.

[16] M. Guzzo, E. Lega, and C. Froeschlé. *Physica D*, 163(1), 2002.

[17] E. Lohinger, C. Froeschlé, and R. Dvorak. *Celest. Mech. Dyn. Astron.*, 56:315, 1993.

[18] N. Ooyama, H. Hirooka, and N. Saitô. Computer Studies on the Approach to Thermal Equilibrium in Coupled Anharmonic Oscillators. II. One Dimensional Case. *J. Phys. Soc. of Japan*, 27(4):815 – 824, 1969.

[19] V. I. Oseledec. *Trans. Moscow Math. Soc.*, 19(197), 1968.

[20] P. Poggi P. and S. Ruffo. Exact Solutions in the FPU Oscillator Chain. *Physica D*, 103:251 – 272, 1997.

[21] Y. B. Pesin. *Russian Math. Surveys*, 32:55, 1977.

[22] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in Fortran 77: The Art of Scientific Computing*. Cambridge University Press, 1986. Second Edition, Volume 1 of Fortran Numerical Recipes.

[23] Zs. Sándor, B. Érdi, A. Széll, and B. Funk. *Celest. Mech. Dyn. Astron.*, 90(127), 2004.

[24] Ch. Skokos. *J. Phys. A*, 34:10029, 2001.

[25] Ch. Skokos, Ch. Antonopoulos, T. C. Bountis, and M. N. Vrahatis. How does the Smaller Alignment Index (SALI) Distinguish Order from Chaos? *Progress of Theoritical Physics Supplement*, 150:439 – 443, 2003.

[26] Ch. Skokos, Ch. Antonopoulos, T. C. Bountis, and M. N. Vrahatis. Detecting Order and Chaos in Hamiltonian Systems by the SALI Method. *Journal of Physics A*, 37:6269 – 6284, 2004.

[27] Ch. Skokos, T. Bountis, and Ch. Antonopoulos. Geometrical Properties of Local Dynamics in Hamiltonian Systems: the Generalized Alignment Index (GALI) Method. *Physica D*, 2007.

[28] N. Voglis and G. Contopoulos. *J. Phys. A*, 27:4899, 1994.

[29] Ch. L. Vozikis, H. Varvoglis, and K. Tsiganis. *Astron. Astroph.*, 359:386, 2000.

ROMAI J., **2**, 2(2006),
15–22

INFORMATION SYSTEM ON THE INTERNET FOR PERFORMANCE SPORTSMEN AND COMPETITION ACTIVITY

Luminita Berechet
"Spiru Haret" Univ., Faculty of Mathematics-Informatics, Bucuresti
luminita berechet@yahoo.com

Abstract We synthesize in the opening the early years of computer science, as a young yet massive propeller of evolution in all fields. Evolution of database systems is reviewed. We nominate some of the application categories of computer technique, in expert systems and other types of data systems dedicated to performance sporting and competition activity, as well as to complementary activities to professional sport. We introduce a software system which integrates information about top competition activity in a sports discipline, to be installed and updated within a dedicated federation Web site. Intermediate results and game events are on-line registered by using measurement equipment connected to the system and log booked for later processing. From the sporting point of view, final main results serve to the record booking, while statistic data will help making the short, medium and/or long term training programs. The application project sustaining the paper is dedicated to tennis and international and/or internal competitions and achieved in Microsoft Access, a flexible and accessible object-oriented, relational database management system. A Web interface is integrated to the database, providing the Internet access to currently updated information.

Keywords:system information, object-oriented relational databases, Internet, performance sports, competition activity.

1. COMPUTER ERA

Since the end of the second world's war, a new tendency conquered irreversibly world's map: solving problems by the use of computer. The arising science was called computer science in the United States and English world, while informatique in France - a hybrid newly derived from information and "automatique". The field was very fast developed, in a rhythm never achieved before in human knowledge zone, by gathering brains, creative force, finance and material resources, inventions, innovations and, not unimportant, mankind's attachment.

Computer era: ˘ revealed the importance of information and of the necessity to fast information processing for evolution; ˘ imposed new human activities, around the creation and management of computer resources (programmers, system engineers, network managers, operators a.s.o.); ˇ redefined traditional human activities by connecting them to the use of computer. In the last over 15 years, Internet became a universal means for human communication, of advertising, banking, shopping and trading, of learning, management a.s.o., thus redefining new fields and ways of developing traditional activities: e-business, e-commerce, e-learning, e-government, e-conomy etc.

Simultaneously, we cross the communication era, with computer systems integrating multimedia, included into vocational domains like art and sports and into creation itself (architecture, painting and music). Lately, also due to the new global trend, computer leads the world towards the knowledge-based society. The latter stated evolution seems most natural, as far as knowledge is a higher, self-assumed and processed instance of information. Romanian academician Mihai Draganescu states that knowledge is information with meaning and information that acts.

In vocational fields like art and sports, where human performances, consequent to capitalization of native abilities by means of outstanding efforts, are individual and personal, computer does not seem a compulsory assistant. Not in the direct line, of developing the performances themselves. However, as an auxilliary tool, implied in the management of resources, planning of activities and results and digital assistance of activities, computer can contribute to human success in the vocational fields too. Eventually, by using multimedia techniques, not only of the top, but very spectacular as well.

By strongly introducing itself as the best information keeper, information processor and information deliverer ever known, computer based systems became a common part of our lives and most certainly a part of late human culture.

2. DATABASES, DATABASES ENVIRONMENTS AND SPECIFIC HARDWARE PLATFORMS

Computer science can be defined as the science responsible for reality modelizing, by the means of computer. As computer technology, digital way of thinking, maturity, as well as software engineering progressed, more applications were developed, more techniques of approaching and achieving information systems in several activities were created, on distinct historical stages.

The database concept was born in the 7th decade of the past century to introduce homogeneous structures of huge amounts of heterogeneous information about distinct categories of persons, objects or activities which share the same properties. Since the '90s, the object oriented paradigm was added to the relational database concept, thus imposing databases as software packages: integrated data together with the management user application interface.

Large and huge databases are best managed on dedicated client-server architectures, on distributed computer networks, including hardware and software database server services. The software developed on the database server is built using the SQL (Structured Query Language) standard language for data retrieving and data management operations. A database on a Microsoft SQL server includes the data, the stored procedures for the data management and the users by groups of access rights.

At the user interface level, the object-oriented concept made possible unique objects dedicated to any problem dealing with databases. Thus, a management application database system object oriented designed integrates data with management procedures, both at the interface control attached to an individual data field level, and at the application level itself. ODBC routines for object attachment on guest machines are applied on the distributed environments. To this purpose, more and more database management environments align to standards of connectivity, building the appropriate interfaces.

Another forward step was achieved by databases attached to complex applications residing on the web. The integrated solution, using or not using an IIS, most commonly includes Active Server Pages which databases can connect to. It lately seems that no regular application can ever stand more without attaching a proper database, as no normal problem arises outside an important amount of data to process. As the knowledge society is being founded, in connection with the newly defined knowledge based economy (if we use Romano Prodi's words while being the President of the European Committee), we state that during the future 5 or 10 years more than 90/100 of the existing computer applications will include a strong bank of data and that at least 50/100 of them will be advertised on the Internet.

3. COMPUTER IN THE SPORTING WORLD

To the planet society level, sports have had a huge span in the glorious period of the ancient helenistic civilization, followed by a centuries long time silence and forgettance. Finally, in the 20th century and especially after the second world's war, sports, like many other fields, acquired new values and channeled multiple forces and resources, as well as an almost general interest, by resurrecting forgotten zones. In sports, such a zone is the performing one, in every branch, which acquired an important mass character on its public adherence, by on-air media transmissions, mostly by television. On the other hand, human evolution brought new interest in improving human physical shape and a lot of knowledge about health and the ways to gain it and to preserve it. Thus far, the continuously increased democracy of life lent to all human creatures the need of day by day sporting, a hobby or an extra-activity that previously was the privilege of the rich persons only.

Within the sports competition and in the continuous sporting training activity, as well as in clubs and economic companies with sports profile/delivery, automation by computer is or may be present in many ways: - Creation of banks of data, for the up-to-date top performances and the historical sports results within the last 20 years or more, grouped hierarchically, by sports, famous sportsmen, years, competition types, traditional competitions etc., like digital libraries or archives, with continuous updating. In distributed environment and published on the Internet, they can be an excellent and open source of information for : sportsmen, trainers, media editorial staff, public ;

- Radio and Television competition transmissions, in which the main parts belong, as human competencies, to commentators and the recording-transmission staff, while technically to the communication field, by gathering strictly sporting comment with electronics and computer technique;

- Planning and organizing stages of the sport training, both individual and collective, between competitions ; planning is a product/sum of the conjugated efforts of trainers and first line/top performers, as consultants, and of computer experts, as designers and performers of the programs ;

- Knowledge based systems to assist the optimum decisions concerning the effort measurement/measuring, the scientific management of the training development, by meeting and by entire program and the performance stage ; the information systems will take into account the scientific theories and the practical experience of the field ever known, stored in knowledge bases ;

- Sportsmen's inter-club movements on Internet book-keeping;

- On-line book-keeping of sporting material, equipment and services on the internal and international market and sporting acquisitions by Internet (e-commerce and e-shopping);

- Common local book keeping of human, material, technical and financial resources in sporting companies and their capitalization.

Nowadays, society evolves by the massive incorporation of knowledge, thus transforming national economies into informational economies. Specifically, one bases professionalism into the productive and managerial act on specializing, globalization and knowledge [4]. Therefore, storing information about all passed events and existent cases and the achievement of information systems using data- and knowledge bases becomes a prime order necessity.

4. INTERNET

Since its boom, Internet has become the most efficient and accessible advertising means. Any private or state institution, of any size and profile, even particular persons, may create its own space of presentation and advertising on the digital international bus of information.

In the past few years, it has also become a custom that big meetings, summits, festivals, contests, competitions, in any field and of any size and level, create and pose their own site on the Internet, thus gathering information about personalities and their activity, mostly in connection to certain moments, for instance around the specified meetings. It seems that the international digital bus might become a real universal archive.

Accordingly, the main part of the sporting Internet sites are the official ones attached to the innumerable punctual world or regional competitions : championships and cups on different sports, either annual, like in : athletics, gymnastics, swimming, ski, artistic skating, football international and european cups a.s.o., or quadriennial, such as the summer or winter editions of the olympic games, the football european and world championships. The actual scored results are presented onto these punctual sites, on one hand, and synthetical information of every sportsman in the competition, on the other hand. Additional data about performance sportsmen and their careers may be found on some personal sites, owned by the personality oneself. However, one can scarcely expect to full information about an entire sport and its performers, even if gathering the data from all sites related.

Therefore, an integral overview on a national sporting branch may be obtained by an official site of a national federation in the mentioned sport. The approach in the present paper refers a project for the data structure attached/connected to a national sport federation for a specific branch, for the performance contests. We refer to lawn tennis for the specific data.

5. THE DATABASE STRUCTURE APPLICATION PROJECT

We use the relational database model. The categories of objects are defined by unique tables, one table for each category of objects. For instance : one sportsmen table, one trainers table, one matches table a.s.o. Each category must relate to at least one other category. The success of the model resides in the normalizing rules, necessary to define and optimally limit the tables, on one hand and in the use of some entity-association diagrams which complete the database structural model.

Into this structure, information refer, on one side, all professional sportsmen from the country and a certain discipline, reknown on the competition arena, and other concern the most important contests they participate to. We designed a flexible structure in order to comprise either national or international level competitions, according to the client wish.

Within the sportsmen category, information include :

- Identity data: Sportsman Identifier, unique in the current application ; Sportsman Identifier in the national sports federation ; Personal Numerical Code ; Surname and Name;

- Official general and competition data: Origin Category, other contest information : applied to tennis and to some type of the competitions, whether the player plays Simple or/and Double ; the Sportsman's Group Position ; the World/National Top Position;

- Official auxiliary personal data : Height, Weight ; for accuracy goal, we log the weights in a history table, weights retained together with the Date Weigh Measurements;

- Additional Personal Data : Studies, Complementary Sports practiced, Extra Hobbies;

- Further information added by grace of the sports consultants.

Sportsmen origin categories are considered to be the clubs, the country regions and, finally, the countries, the latter either compulsory for all sportsmen in the international scheme or necessary for the foreign sportsmen in national competitions.

Complementarily, the database includes information about the trainers related to the recorded sportsmen, in the history of their training relationship (trainer - trains - sportsman). We supposed that a sportsman is trained by one or more trainers during his entire sports career and that trainers train more sportsmen, either concurrently or at least chronologically, on the other side. The application stores professional and personal details about trainers.

For the competition level, we designed a maximal structure configuration, so that the database can include data about international competitions, maybe not exhaustively, but at least of the most important in some performance sportsmen's careers. We define the type and the area of competitions, then the information referring the individual edition of the competition. One can minimize structure span / scope, by recursive design of some of the competition lists. Added to competitions, information about the arbitrators are recorded and, obviously, the relationship between competitions and the arbitrators who led them, among the ones existent into the base.

In tennis or other sports, competitions carry on in meetings where two parts confront themselves, either of which gaining a number of points, so that meetings end at a certain score table, that is after a variable number of minutes, normally or due to a specific recordable reason. The initial reference information about the match include the level and type of the meeting, the names of performing/contesting sportsmen. Additionally, specialized equipment placed on the arena continuously measure several game parameters, such as : service speed and speed of the remainder of strokes for each of the players ; counts the special different strokes and game events, such as : aces, double faults, nets services, forehand and backhand strokes, lobes, smatches, volley-like strokes, long lines and crossings, percentage of time each player spends on the front or back part of his field a.s.o. Measurements are processed by using elementary univariate statistical functions like : average, counter, maximum, minimum, statistic mean/average, updated on-line, during the match time.

Sportsmen entity is referred through the usual mecanism of the couple (primary key in the parent coordinate table - external key into the child subordinate table) (see figure 1), on one hand by the matches table, and finally in the final match results table, the latter actually consisting of the partial instance of the score table at the final moment of the match together with statistical game results. We consider that statistical information collected during the matches may improve sportsmen future performances, if trainer uses them with a dedicated training program software in order to correct the parameters of future training sessions on short, medium and long term.

Sportsmen Matches Primay key External key

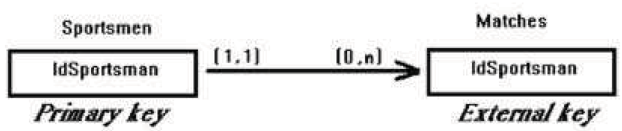


Fig. 1. Entity-association relationship between sportsmen and the competition matches, by highlighting cardinality association.

Relationship between the two categories presented in the figure is typical to the most part of the relationships entity-association used into the database. The relationship cardinality is expressed by the couple formed of the minimum and the maximum number of achievements (instances) of the entity, which can be associated with a single achievement of the association partner. In the figure, in a tennis simple game and on one side/part of the nets can play at least one sportsman but at the same time at most one - semantically just one (the (1, 1) cardinality couple from the sportsmen side), meanwhile the (0, n) cardinality associated to the matches entity reveals the fact that a player recorded into the base may not have played any game or, at the opposite, may have already played into a maximum of n games, where n = 1.

Depending on the association degree, the relationships in the presented model are of the one-to-many type, if only the maximum value of the possible achievements within the binary relationship is specified (view figure 1). Depending on the number of entities participant at the association, the most frequent are the binary associations (between distinct categories), and some of them use recursion (self-adjoint associations).

The input data volume most certainly depends on the human, material and financial force the site owner can offer.

6. HOSTING THE DATABASE ON THE WEB

On one hand, the application user interface to any Web browser includes the site itself, achieved with any usual web programming environment (Dreamweaver, Macromedia Flash). Its aspect and ease of use by quick functionality will fully depend on the artistic skills, realistic and pragmatic vision and experience of the Web designer and, supplementary, on the Web artistic consultant, if any.

On the other hand, the site connects to the above briefly described database. We chose the Active Server Pages solution, as the client application database system is built in a compatible environment, namely Microsoft Access. An initial updated database is provided by the designer. The client of the system is encharged to keep any new data about competitions, sportsmen and trainers up to date, fact which requires an appropriate hardware and software environment at the client's physical site.

If the distributed local network solution is agreed upon, we recommend a further dedicated Internet Information System. An Internet service provider is needed, either in the case of a local stand-alone workstation system or of a local computer network located at the client's updating site.

7. REMARKS

For huge databases (containing more than 500,000 records), we strongly recommend a distributed hardware and software platform, in a dedicated client-server architecture, with an Oracle or Microsoft SQL database server. In the latter mentioned platform, workstations are provided with a Microsoft Access client application. A web server is needed.

References

[1] BDASEIG Group, *Databases. Theoretical and Practical Foundations*, Infomega, Bucharest, 2002.

[2] C*ourse and Laboratory notes in Computer Science and Technology* (lectured and respectively kept by the author at Spiru Haret University, Sports Department, Bucharest, during the university year 2004-2005.

[3] http://www.microsoft.com, on-line documentation regarding Microsoft Access.

[4] DRAGANESCU, M., *Informational and Knowledge Society. Knowledge Society Vectors*, Thematic Study, Romanian Academy, www.academiaromana.ro, 2001.

[5] GHERASIM, Z., ANDRONIE, M., POPESCU BODORIN, N., *Management Computer Technologies in sports activity* , Romania of Tomorrow Foundation, Bucharest, 2004.

ROMAI J., **2**, 2(2006),
23–32

$GL(2,R)-$ORBITS OF THE HOMOGENEOUS DIFFERENTIAL SYSTEMS $\dot{X}_1 = P_4(X_1, X_2)$, $\dot{X}_2 = Q_4(X_1, X_2)$]

Driss Boularas, Angela Păşcanu, Alexandru Şubă

Univ. de Limoges, France, State Univ. of Tiraspol, Chişinău, Republic of Moldova, State Univ. of Moldova, Chişinău, Republic of Moldova

boularas@unilim.fr; suba@usm.md

Abstract For the differential system $\dot{x_1} = P_4(x_1, x_2)$, $\dot{x_2} = Q_4(x_1, x_2)$, where P_4, Q_4 are homogeneous polynomials of degree four, the problem of the classification of $GL(2,R)-$orbits is solved.

Keywords: polynomial differential system, $GL(2,R)-$orbit.
2000 MSC: 34C05, 58F14

1. CENTER-AFFINE TRANSFORMATIONS

Consider the polynomial system

$$\dot{x_1} = P_k(x_1, x_2), \dot{x_2} = Q_k(x_1, x_2), \tag{1}$$

where P_k, Q_k are homogeneous polynomial of degree k $P_k = \sum_{i+j=k} a_{ij} x_1^i x_2^j$, $Q_k = \sum_{i+j=k} b_{ij} x_1^i x_2^j$. Denote by E the space of coefficients $a = (a_{k0}, a_{k-1,1}, \ldots, a_{0k}, b_{k0}, b_{k-1,1}, \ldots, b_{0k})$ of system (1) and by $GL(2,R)$ the group of center-affine transformations of the phase space Ox, $x = (x_1, x_2)^t$. Applying to (1) the transformation $X = qx$, where $X = (X_1, X_2)^t$, $q \in GL(2,R)$, we obtain the system

$$\dot{X}_1 = \sum_{i=0}^{k} a^*_{k-i,i} X_1^{k-i} X_2^i, \ \dot{X}_2 = \sum_{i=0}^{k} b^*_{k-i,i} X_1^{k-i} X_2^i. \tag{2}$$

The coefficients a^* can be expressed as linear combinations of the coefficients a, namely: $a^* = \Lambda_{(q)}(a)$, $det\Lambda_{(q)} \neq 0$. The set $\Lambda = \{\Lambda_{(q)} | q \in GL(2,R)\}$ is a 4-parameter group with respect to the operation of composition. Λ is called the representation of the group $GL(2,R)$ of center-affine transformations of the phase space Ox in the space of coefficients E of system (1). The set $O(a) = \{\Lambda_{(q)}(a) | q \in GL(2,R)\}$ is called the $GL(2,R)-$*orbit* of the point a or of the differential system (1) corresponding to this point. In the space E every $GL(2,R)-$orbit is a 4-parameter surface.

In E consider the vector fields

$$W_l = \sum_{i+j=k} A_{ij}^{(l)}(a)\frac{\partial}{\partial a_{ij}} + B_{ij}^{(l)}(a)\frac{\partial}{\partial b_{ij}}, \quad l = \overline{1,4},$$

where $A_{ij}^{(1)}(a) = (1-i)a_{ij},\ B_{ij}^{(1)}(a) = -ib_{ij}, A_{k0}^{(2)}(a) = b_{k0},\ A_{ij}^{(2)}(a) = b_{ij} - (i+1)a_{i+1,j-1}, B_{k0}^{(2)}(a) = 0,\ \ B_{ij}^{(2)}(a) = -(i+1)b_{i+1,j-1},\ (i,j) \neq (k,0), A_{0k}^{(3)}(a) = 0,\ A_{ij}^{(3)} = -(j+1)a_{i-1,j+1}, B_{0k}^{(3)}(a) = a_{0k},\ B_{ij}^{(3)}(a) = a_{ij}-(j+1)b_{i-1,j+1},\ (i,j) \neq (0,k), A_{ij}^{(4)} = -ja_{ij},\ \ B_{ij}^{(4)} = (1-j)b_{ij}$.
The fields $W_l,\ l = \overline{1,4}$, generate a Lie algebra. The dimension of the orbit $O(a)$ is equal to the dimension of this algebra, i.e. with the rank of the matrix M_k composed of the coordinates of the fields W_l, $l = \overline{1,4}$ [1],

$$v_l(a) = (A_{k0}, A_{k-1,1}, \ldots, A_{0k}, B_{k0}, B_{k-1,1}, \ldots, B_{0k}). \tag{3}$$

Lemma 1.[2] *Let $O(a)$ be a $GL(2,R)-$orbit of the system (1), $k \neq 1$. Then*
1) $dimO(a) = 0$ iff (1) has the form

$$\dot{x_1} = 0,\ \dot{x_2} = 0; \tag{4}$$

2) $dimO(a) \neq 1$, $\forall a \in E$;
3) $dimO(a) > 1$, i.e. $dimO(a)$ is equal with one of the numbers 2, 3 or 4, iff $|P_k(x_1,x_2)| + |Q_k(x_1,x_2)| \not\equiv 0$.

Taking into account Lemma 1 and that $rankM_k \leq 2$, $k = 1, 2$, where

$$M_0 = \begin{pmatrix} a_{00} & 0 \\ b_{00} & 0 \\ 0 & a_{00} \\ 0 & b_{00} \end{pmatrix}, M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01}-a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10}-b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}.$$

It follows that *in the case $k = 0$ ($k = 1$) the dimension of the $GL(2,R)-$orbit of the system (1) is: 0, if $a_{00} = b_{00} = 0$ ($a_{10} = b_{01}, a_{01} = b_{10} = 0$); 2 otherwise.*

A polynomial $K(x,a)$, $x \in R^2$, $a \in E$, is called a *center-affine comitant* [3] of (1) if for $\forall\, q \in GL(2,R)$ the identity $K(qx, \Lambda_{(q)}(a)) \equiv K(x,a)$ holds. When the comitant K does not depend on x, it is called *center-affine invariant* [3] of (1).

In [4]-[11] the invariant conditions of classification of the $GL(2,R)-$orbits are obtained for some differential systems, right-hand sides of which are polynomials of degree at most three. The results of these works lead to

Conjecture. *If the dimension of the $GL(2,R)-$orbit of system (1) is less than 4, then $deg(GCD(P,Q)) \geq k-1$.*

In Sections 2 and 3 the homogeneous quadratic ($k=2$) and cubic ($k=3$) systems with $dimO(a)<4$ are presented in the explicit form. In Section 4 it is shown that this conjecture holds for $k=4$ too.

2. THE HOMOGENEOUS QUADRATIC SYSTEM

Consider the following comitants and invariants from [3]

$$K_1=((2a_{20}+b_{11})x_1+(a_{11}+2b_{02})x_2)/2;$$

$$K_5=-b_{20}x_1^3+(a_{20}-b_{11})x_1^2x_2+(a_{11}-b_{02})x_1x_2^2+a_{02}x_2^3;$$

$$K_7=((4a_{20}^2+4a_{11}b_{20}+b_{11}^2)x_1^2+(4a_{20}a_{11}+2a_{11}b_{11}+8a_{02}b_{20}+4b_{11}b_{02})x_1x_2+ \\ +(a_{11}^2+4a_{02}b_{11}+4b_{02}^2)x_2^2)/4;$$

$$K_9=((-2a_{20}^2b_{02}+a_{20}a_{11}b_{11}+2a_{20}a_{02}b_{20}+a_{20}b_{11}b_{02}-a_{11}^2b_{20}-2a_{11}b_{20}b_{02}+a_{02}b_{02}b_{11})x_1 \\ +(-a_{20}a_{11}b_{02}+2a_{20}a_{02}b_{11}+2a_{20}b_{02}^2-a_{11}a_{02}b_{20}-a_{11}b_{11}b_{02}-2a_{02}b_{20}b_{02}+a_{02}b_{11}^2)x_2)/2;$$

$$I_7=(16a_{20}^3a_{02}-4a_{20}^2a_{11}^2+8a_{20}^2a_{11}b_{02}+8a_{20}^2a_{02}b_{11}-6a_{20}a_{11}^2b_{11}+16a_{20}a_{11}a_{02}b_{20}+ \\ 8a_{20}a_{11}b_{11}b_{02}+32a_{20}a_{02}b_{20}b_{02}-4a_{20}a_{02}b_{11}^2+8a_{20}b_{11}b_{02}^2-2a_{11}^3b_{20}-4a_{11}^2b_{20}b_{02}- \\ 4a_{11}^2b_{11}^2+8a_{11}a_{02}b_{20}b_{11}+8a_{11}b_{20}b_{02}^2-12a_{11}b_{11}^2b_{02}+16a_{02}b_{20}b_{11}b_{02}-2a_{02}b_{11}^3+ \\ 2b_{20}b_{11}^3-4b_{11}^2b_{02}^2)/16;$$

$$I_8=(16a_{20}^3a_{02}-4a_{20}^2a_{11}^2+8a_{20}^2a_{11}b_{02}-8a_{20}^2a_{02}b_{11}-2a_{20}a_{11}^2b_{11}+32a_{20}a_{11}a_{02}b_{20}+ \\ +12a_{20}a_{02}b_{11}^2+8a_{20}b_{11}b_{02}^2-6a_{11}^3b_{20}+12a_{11}^2b_{20}b_{02}-4a_{11}^2b_{11}^2-16a_{11}a_{02}b_{20}b_{11}- \\ -8a_{11}b_{20}b_{02}^2-2a_{11}b_{11}^2b_{02}+32a_{02}^2b_{20}^2+32a_{02}b_{20}b_{11}b_{02}-6a_{02}b_{11}^3+2b_{20}a_{11}^3-4b_{11}^2b_{02}^2)/16;$$

$$I_9=(16a_{20}^3a_{02}-4a_{20}^2a_{11}^2+8a_{20}^2a_{11}b_{02}+24a_{20}^2a_{02}b_{11}+32a_{20}^2b_{02}^2-10a_{20}a_{11}^2b_{11}- \\ -16a_{20}a_{11}b_{11}b_{02}+12a_{20}a_{02}b_{11}^2+8a_{20}b_{11}b_{02}^2+2a_{11}^3b_{20}+12a_{11}^2b_{20}b_{02}-4a_{11}^2b_{11}^2+ \\ +24a_{11}b_{20}b_{02}^2-10a_{11}b_{11}^2b_{02}+2a_{02}b_{11}^3+2b_{20}b_{02}^3-4b_{11}^2b_{02}^2)/16;$$

$$I=27I_8-I_9-18I_7.$$

Theorem 1 ([11], p. 208). *The dimension of the $GL(2,R)-$orbit of the system (1), k=2, is: 0, if $K_1\equiv K_5\equiv 0$; 2, if $K_5(K_1+K_7)\equiv 0$, $K_1+K_5\not\equiv 0$; 3, if $K_5(K_1+K_7)\not\equiv 0$, $K_9+I\equiv 0$; 4, if $K_5(K_9+I)\not\equiv 0$.*

The conditions in Theorem 1 lead us to the following systems of differential equations: *dim* $0:$ equations (5);
dim $2:$

$$\dot{x_1}=x_1\cdot F,\ \dot{x_2}=x_2\cdot F,\ F=\alpha x_1+\beta x_2\not\equiv 0; \tag{5}$$

$$\dot{x_1}=\beta\cdot F,\ \dot{x_2}=\alpha\cdot F,\ F=(\alpha x_1-\beta x_2)^2\not\equiv 0; \tag{6}$$

dim 3 :

$$\dot{x_1} = \alpha x_1^2,\ \dot{x_2} = \beta x_1 x_2,\ \alpha - \beta \neq 0; \tag{7}$$

$$\begin{cases} \dot{x_1} = (b_{11} - \mu b_{20}/\delta)x_1^2,\ \dot{x_2} = x_1(b_{20}x_1 + b_{11}x_2), \\ \quad b_{20}(|b_{11}| + |\mu|) \neq 0; \end{cases} \tag{8}$$

$$\begin{cases} \dot{x_1} = ((\beta a_{11} - \mu a_{02})x_1 + \beta a_{02}x_2)F, \\ \dot{x_2} = (\beta a_{11} - 2\mu a_{02})x_2 F,\ F = (\mu x_1 + \beta x_2)/\beta^2, \\ \quad a_{02}(|a_{11}| + |\mu|) \neq 0; \end{cases} \tag{9}$$

$$\begin{cases} \dot{x_1} = \alpha a_{11} x_1 F,\ \dot{x_2} = (\delta(b_{02} - a_{11})x_1 + \alpha b_{02}x_2)F, \\ F = (\delta x_1 + \alpha x_2)/\alpha^2,\ b_{02} - a_{11} \neq 0; \end{cases} \tag{10}$$

$$\begin{cases} \dot{x_1} = [(\alpha^2 a_{02} + \alpha\beta b_{02} - \delta\beta a_{02})x_1 + \alpha\beta a_{02}x_2] \cdot F, \\ \dot{x_2} = \alpha(\delta a_{02}x_1 + \beta b_{02}x_2) \cdot F,\ F = (\alpha x_2 - \delta x_1)/(\alpha^2\beta), \\ \quad a_{02}(|\alpha a_{02} + \beta b_{02}| + |\alpha^2 + \beta\delta|) \neq 0; \end{cases} \tag{11}$$

$$\begin{cases} \dot{x_1} = \beta[(\alpha a_{02} - \beta b_{02})x_1 + 2\beta a_{02}x_2] \cdot F, \\ \dot{x_2} = [\alpha(\alpha a_{02} + 3\beta b_{02})x_1 + 2\beta^2 b_{02}x_2] \cdot F, \\ F = [2\beta a_{02}x_2 - (\alpha a_{02} + 3\beta b_{02})x_1]/(4\beta^3 a_{02}),\ \alpha a_{02} + \beta b_{02} \neq 0. \end{cases} \tag{12}$$

Thus, Theorem 1 can be reformulated as follows.

Theorem 2. *The system (1), $k = 2$, has the dimension of the $GL(2,R)-$orbit equal to: 0, it has the form (5); 2, it has one of the forms (6), (7); 3, it has one of the forms (8)-(13); 4, otherwise.*

3. THE HOMOGENEOUS CUBIC SYSTEM

Consider the following comitants and invariants from [3]

$$P_1 = ((3a_{30} + b_{21})x_1^2 + 2(a_{21} + b_{12})x_1x_2 + (a_{12} + 3b_{03})x_2^2)/3;$$

$$P_2 = -b_{31}x_1^4 + (a_{30} - b_{21})x_1^3x_2 + (a_{21} - b_{12})x_1^2x_2^2 + (a_{12} - b_{03})x_1x_2^3 + a_{03}x_2^4;$$

$$P_3 = ((-3a_{30}b_{12}+2a_{21}b_{21}+b_{12}b_{21}-3a_{12}b_{30}-9b_{03}b_{30})x_1^2+2(-a_{21}^2+3a_{12}a_{30}+b_{12}^2- \\ -3b_{03}b_{21})x_1x_2 + (-a_{12}a_{21} + 9a_{03}a_{30} + 3a_{21}b_{03} - 2a_{12}b_{12} + 3a_{03}b_{21})x_2^2)/9;$$

$$P_5 = ((9a_{30}^2+b_{21}^2+6a_{21}b_{30})x_1^4+4(3a_{21}a_{30}+a_{21}b_{21}+b_{12}b_{21}+3a_{12}b_{30})x_1^3x_2+2(2a_{21}^2+ \\ 3a_{12}a_{30}+a_{21}b_{12}+2b_{12}^2+4a_{12}b_{21}+3b_{03}b_{21}+9a_{03}b_{30})x_1^2x_2^2+4(a_{12}a_{21}+a_{12}b_{12}+3b_{03}b_{12} \\ +3a_{03}b_{21})x_1x_2^3 + (a_{12}^2 + 9b_{03}^2 + 6a_{03}b_{12})x_2^4)/9;$$

$$J_1 = (-2a_{21}^2 + 6a_{12}a_{30} + 18a_{30}b_{03} - 4a_{21}b_{12} - 2b_{12}^2 + 2a_{12}b_{21} + 2b_{03}b_{21})/9;$$

$$J_2 = (-2a_{21}^2 + 6a_{12}a_{30} + 2a_{21}b_{12} - 2b_{12}^2 - 4a_{12}b_{21} + 6b_{03}b_{21} + 18a_{03}b_{30})/9;$$

$$J_4 = (-2a_{21}^3+9a_{12}a_{21}a_{30}-27a_{03}a_{30}^2-9a_{21}a_{30}b_{03}+a_{21}^2b_{12}+3a_{12}a_{30}b_{12}+9a_{30}b_{03}b_{12}- \\ -a_{21}b_{12}^2+2b_{13}^3-4a_{12}a_{21}b_{21}+9a_{03}a_{30}b_{21}-6a_{21}b_{03}b_{21}+4a_{12}b_{12}b_{21}-9b_{03}b_{12}b_{21}-$$

$-12a_{03}b_{21}^2+12a_{12}^2b_{31}-27a_{03}a_{21}b_{30}-9a_{12}b_{03}b_{30}+27b_{03}^2b_{30}+27a_{03}b_{12}b_{30})/27.$

Theorem 3 ([11], p.222). *The dimension of the $GL(2,R)-$orbit of the system (1), $k=3$, is: 0, for $P_1\equiv P_2\equiv 0$; 2, for $P_2\not\equiv 0$, $P_1\equiv J_2P_5-J_4P_2\equiv P_5\equiv 0$, or $P_2\equiv 0$, $J_1=0$, $P_1\not\equiv 0$; 3, for $P_1P_2\not\equiv 0$, $3P_1P_3-2J_1P_2\equiv 0$, or $P_2P_5\not\equiv 0$, $P_1\equiv J_2P_5-J_4P_2\equiv 0$, or $P_2\equiv 0$, $J_1\neq 0$; 4, for $P_1P_2(3P_1P_3-2J_1P_2)\not\equiv 0$, or $P_1\equiv 0$, $P_2(J_2P_5-J_4P_2)\not\equiv 0$.*

The conditions from Theorem 3 lead us to the following systems of differential equations: *dim* 0 : equations (5);

dim 2 :

$$\dot{x_1}=\pm\beta F,\ \dot{x_2}=\mp\alpha F,\ F=(\alpha x_1+\beta x_2)^3\not\equiv 0; \tag{13}$$

$$\dot{x_1}=x_1F,\ \dot{x_2}=x_2F,\ F=\pm(\alpha x_1+\beta x_2)^2\not\equiv 0; \tag{14}$$

dim 3 :

$$\dot{x_1}=a_{12}x_1x_2^2,\ \dot{x_2}=b_{03}x_2^3,\ a_{12}-b_{03}\neq 0; \tag{15}$$

$$\begin{cases}\dot{x_1}=x_2^2[(\alpha a_{03}+\beta b_{03})x_1+\beta a_{03}x_2]/\beta,\\ \dot{x_2}=b_{03}x_2^3,\ a_{03}(|\alpha|+|b_{03}|)\neq 0;\end{cases} \tag{16}$$

$$\begin{cases}\dot{x_1}=a_{30}x_1^3,\ \dot{x_2}=x_1^2[\delta(b_{21}-a_{30})x_1+\mu b_{21}x_2]/\mu,\\ a_{30}-b_{21}\neq 0;\end{cases} \tag{17}$$

$$\begin{cases}\dot{x_1}=-a_{21}x_1F,\ \dot{x_2}=[\delta(b_{12}-a_{21})x_1-2\alpha b_{12}x_2]\cdot F/(2\alpha),\\ F=x_1(\delta x_1-2\alpha x_2)/(2\alpha),\ |a_{21}|+|b_{12}|\neq 0;\end{cases} \tag{18}$$

$$\begin{cases}\dot{x_1}=(3\alpha b_{30}+\delta b_{21})x_1F,\ \dot{x_2}=[\delta b_{30}x_1+(\delta b_{21}+2\alpha b_{30})x_2]\cdot F,\\ F=(\delta x_1-\alpha x_2)^2/\delta^3,\ b_{30}(|\alpha|+|b_{21}|)\neq 0;\end{cases} \tag{19}$$

$$\begin{cases}\dot{x_1}=[\mu a_{30}x_1+\beta(b_{21}-a_{30})x_2]\cdot F,\ \dot{x_2}=\mu b_{21}x_2F,\\ F=(\mu x_1-\beta x_2)^2/\mu^3,\ a_{30}-b_{21}\neq 0;\end{cases} \tag{20}$$

$$\begin{cases}\dot{x_1}=\beta[(\delta^2a_{03}-\alpha^2a_{21})x_1+2\alpha\delta a_{03}x_2]\cdot F,\\ \dot{x_2}=[2\alpha\delta^2a_{03}x_1+(3\beta\delta^2a_{03}-2\alpha^2\delta a_{03}-\alpha^2\beta a_{21})x_2]\cdot F,\\ F=(\delta x_1-\alpha x_2)^2/(2\alpha^2\beta\delta),\ a_{03}(|\alpha^2a_{21}-3\delta^2a_{03}|+|\alpha^2+\beta\delta|)\neq 0;\end{cases} \tag{21}$$

$$\begin{cases}\dot{x_1}=(\delta b_{21}x_1+\beta b_{30}x_2)F,\ \dot{x_2}=\delta(b_{30}x_1+b_{21}x_2)F,\\ F=(\delta x_1^2-\beta x_2^2)/\delta^2,\ (|b_{30}|+|b_{21}|)(|\beta|+|b_{30}|)(|\beta|+|b_{21}|)\neq 0;\end{cases} \tag{22}$$

$$\begin{cases}\dot{x_1}=-\beta[(\beta a_{21}+\delta a_{03})x_1+2\alpha a_{03}x_2]\cdot F,\\ \dot{x_2}=[-2\alpha\delta a_{03}x_1+(4\alpha^2a_{03}-\beta\delta a_{03}-\beta^2a_{21})x_2]\cdot F,\\ F=(\delta x_1^2-2\alpha x_1x_2-\beta x_2^2)/(2\alpha\beta^2),\\ (|a_{03}|+|a_{21}|)(|\alpha^2+\beta\delta|+|a_{03}\cdot(\beta a_{21}+3\delta a_{03})|)\neq 0.\end{cases} \tag{23}$$

In the explicit form the Theorem 3 can be restated as

Theorem 4. *The system (1), $k=3$, has the dimension of the $GL(2,R)-$orbit equal to: 0, if it has the form (5); 2, if it has one of the forms (14), (15); 3, if it has one of the forms (16)-(24); 4, in other cases.*

4. THE SYSTEM (1), $K = 4$.

Consider the matrix $M_4 = (v_1, v_2, v_3, v_4)^{tr}$, where

$$\begin{gathered} v_1 = (-3a_{40}, -2a_{31}, -a_{22}, 0, a_{04}, -4b_{40}, -3b_{31}, -2b_{22}, -b_{13}, 0), \\ v_2 = (b_{40}, b_{31} - 4a_{40}, b_{22} - 3a_{31}, b_{13} - 2a_{22}, b_{04} - a_{13}, 0, -4b_{40}, \\ -3b_{31}, -2b_{22}, -b_{13}), \\ v_3 = (-a_{31}, -2a_{22}, -3a_{13}, -4a_{04}, 0, a_{40} - b_{31}, a_{31} - 2b_{22}, \\ a_{22} - 3b_{13}, a_{13} - 4b_{04}, a_{04}), \\ v_4 = (0, -a_{31}, -2a_{22}, -3a_{13}, -4a_{04}, b_{40}, 0, -b_{22}, -2b_{13}, -3b_{04}) \end{gathered} \tag{24}$$

(see (4), (3)). Let $M(i_1, i_2, i_3; j_1, j_2, j_3)$ be a minor of the matrix M_4 obtained at the intersection of the rows i_1, i_2, i_3 with the columns j_1, j_2, j_3. Then taking into account Lemma 1, we can assume

$$|P_4(x_1, x_2)| + |Q_4(x_1, x_2)| \not\equiv 0. \tag{25}$$

Case of the null vectors. Taking into account (25) and (26), the systems (1), $k = 4$, for which at least one of the equalities $v_1(a) = 0$, $v_2(a) = 0$, $v_3(a) = 0$, $v_4(a) = 0$ holds become

$$(v_1 = v_3 = 0): \quad \dot{x_1} = a_{13}x_1x_2^3, \ \dot{x_2} = a_{13}x_2^4, \ a_{13} \neq 0; \tag{26}$$

$$(v_2 = 0, v_1 \parallel v_4): \quad \dot{x_1} = a_{04}x_2^4, \ \dot{x_2} = 0, \ a_{04} \neq 0; \tag{27}$$

$$(v_1 = 0): \quad \dot{x_1} = a_{13}x_1x_2^3, \ \dot{x_2} = b_{04}x_2^4, \ a_{13} \neq b_{04}; \tag{28}$$

$$(v_2 = 0): \quad \dot{x_1} = x_2^3(a_{13}x_1 + a_{04}x_2), \ \dot{x_2} = a_{13}x_2^4, \ a_{13} \cdot a_{04} \neq 0; \tag{29}$$

$$(v_2 = v_4 = 0): \quad \dot{x_1} = b_{31}x_1^4, \ \dot{x_2} = b_{31}x_1^3x_2, \ b_{31} \neq 0; \tag{30}$$

$$(v_3 = 0, v_1 \parallel v_4): \quad \dot{x_1} = 0, \ \dot{x_2} = b_{40}x_1^4, \ b_{40} \neq 0; \tag{31}$$

$$(v_4 = 0): \quad \dot{x_1} = a_{40}x_1^4, \ \dot{x_2} = b_{31}x_1^3x_2, \ a_{40} \neq b_{31}; \tag{32}$$

$$(v_3 = 0): \quad \dot{x_1} = b_{31}x_1^4, \ \dot{x_2} = x_1^3(b_{40}x_1 + b_{31}x_2), \ b_{31} \cdot b_{40} \neq 0. \tag{33}$$

Lemma 2. *The systems (27), (28), (31), (32) have the dimension of the $GL(2, R)-$orbit equal to two and the systems (29), (30), (33), (34) equal to three.*

Proof. In the case of a system (27) ((28)) we have that $v_1(a) = v_3(a) = 0$ ($v_2(a) = 0$, and the vectors $v_1(a)$ and $v_4(a)$ are reciprocally parallel). From these and from Lemma 1, p. 3), for systems (27) and (28) it follows that $dimO(a) = 2$.

In the case of a system (30) we have $M(1, 3, 4; 3, 4, 5) = 9a_{04}a_{13}^2 \neq 0$ and in the case (29) the minors $M(2, 3, 4; 3, 4, 5) = 9a_{13}^2(b_{04} - a_{13})$, $M(2, 3, 4; 5, 9, 10) = 3b_{04}(a_{13} - b_{04})(a_{13} - 4b_{04})$ cannot be equal to zero simultaneously.

By substitutions $a_{ij} \longleftrightarrow b_{ji}$, $x_1 \longleftrightarrow x_2$, systems (31)-(34) can be reduced to the following systems (27)-(30) respectively. $\square$

Parallelism. Assume that

$$v_1(a) \neq 0,\ v_2(a) \neq 0,\ v_3(a) \neq 0,\ v_4(a) \neq 0. \tag{34}$$

1) $v_1(a) \parallel v_2(a)$. Let $v_2(a) = \alpha \cdot v_1(a)$, $\alpha \neq 0$. This equality implies the following relations between the coefficients of the systems (1), $k = 4$ (see (25)

$$b_{40} = -3\alpha a_{40},\ b_{31} - 4a_{40} = -2\alpha a_{31},\ b_{22} - 3a_{31} = -\alpha a_{22},\ b_{13} - 2a_{22} = 0,$$

$$b_{04} - a_{13} = \alpha a_{04},\ b_{40} = 0,\ 4b_{40} = 3\alpha b_{31},\ 3b_{31} = 2\alpha b_{22},\ 2b_{22} = \alpha b_{13},\ b_{13} = 0,$$

which, together with (35), lead us to the differential system

$$(v_1 \parallel v_2):\ \ \dot{x_1} = (a_{13}x_1 + a_{04}x_2)x_2^3,\ \ \dot{x_2} = (a_{13} + \alpha a_{04})x_2^4,\ \ \alpha \cdot a_{04} \neq 0. \tag{35}$$

Lemma 3. *The system (36) has the dimension of the $GL(2,R)-$orbit equal to three.*

Proof. $M(1,3,4;3,4,5) = 9a_{13}^2 a_{04}$, $M(1,3,4;4,5,10) = -3a_{04}^2(4\alpha a_{04} + 5a_{13})$. $\square$

2) $v_1(a) \parallel v_3(a)$. Setting $v_3(a) = \alpha v_1(a)$ and taking into account (35), we obtain the systems

$$(v_1 \parallel v_3):\ \ \dot{x_1} = a_{40}x_1(x_1 + \alpha x_2)^3,\ \ \dot{x_2} = a_{40}x_2(x_1 + \alpha x_2)^3,\ \alpha \cdot a_{40} \neq 0; \tag{36}$$

$$(v_1 \parallel v_2):\ \ \begin{cases} \dot{x_1} = a_{40}x_1F,\ \dot{x_2} = [b_{40}x_1 + (a_{40} + \alpha b_{40})x_2]F, \\ F = (x_1 + \alpha x_2)^3,\ \alpha b_{40} \neq 0. \end{cases} \tag{37}$$

Lemma 4. *The $GL(2,R)-$orbit of the system (37) ((38)) has the dimension equal to two (three).*

Proof. In the case of the system (37) we have that $v_2(a) = \alpha \cdot v_1(a)$, $v_4(a) = \alpha \cdot v_2(a)$ and $|P_4(x_1,x_2)| + |Q_4(x_1,x_2)| \not\equiv 0$. For the system (38) the minors $M(1,2,4;1,2,3) = -9\alpha^3 a_{40}^2 b_{40}$ and $M(1,2,4;5,6,7) = -3\alpha^4 b_{40}^2(a_{40} + 4\alpha b_{40})$ can not be simultaneously equal to zero. $\square$

3) $v_4(a) = \alpha v_1(a)$, $\alpha \neq 0$. Taking into account (35), this case leads us to the systems

$$(v_1 \parallel v_4):\ \ \dot{x_1} = a_{22}x_1^2x_2^2,\ \dot{x_2} = b_{13}x_1x_2^3,\ |a_{22}| + |b_{13}| \neq 0; \tag{38}$$

$$(v_1 \parallel v_4):\ \ \dot{x_1} = a_{31}x_1^3x_2,\ \dot{x_2} = b_{22}x_1^2x_2^2,\ |a_{31}| + |b_{22}| \neq 0. \tag{39}$$

Lemma 5. *The $GL(2,R)$ - orbits of the systems (39) and (40) have the dimension equal to three.*

Proof. The system (40) is obtained from the system (39) by performing the substitutions $x_1 \longleftrightarrow x_2,\ a_{ij} \longleftrightarrow b_{ji}$. In the case of the system (39) we have $M(1,2,3;2,3,4) = 2a_{22}^2(b_{13} - 2a_{22}),\ M(1,2,3;8,9,10) = b_{13}^2(a_{22} - 3_{b13})$. □

4) $v_2(a) \parallel v_3(a)$. In the conditions $|v_2(a)| \cdot |v_3(a)| \neq 0$ the relation $v_3(a) = \alpha v_2(a)$ is not realized for any $\alpha \neq 0$. Therefore

Lemma 6. *If the vectors $v_2(a)$ and $v_3(a)$ are not null, then they are linearly independent.*

5) $v_2(a) \parallel v_4(a)$. By the substitutions $a_{ij} \longleftrightarrow b_{ji}$ the equality $v_2(a) = \alpha v_4(a)$ is transformed into the equality $v_3(a) = \alpha v_1(a)$. Thus, this case leads us to a system of the form (37) and to a system obtained from (38) by changing simultaneously x_1 by x_2, x_2 by x_1, a_{ij} by b_{ji} and b_{ij} by a_{ij}, namely to a system

$$(v_2 \parallel v_4): \begin{cases} \dot{x_1} = [(b_{04} + \alpha a_{04})x_1 + a_{04}x_2] \cdot F,\ \dot{x_2} = b_{04}x_2F, \\ F = (\alpha x_1 + x_2)^3,\ \alpha a_{04} \neq 0. \end{cases} \tag{40}$$

Lemma 7. *The $GL(2,R)-$orbit of the system (41) has the dimension equal to three.*

6) $v_3(a) \parallel v_4(a)$. The substitutions $a_{ij} \longleftrightarrow b_{ji}$ reduce this case to the case $v_2(a) \parallel v_1(a)$. We obtain the system

$$(v_3 \parallel v_4): \dot{x_1} = (b_{31} + \alpha b_{40})x_1^4,\ \dot{x_2} = (b_{40}x_1 + b_{31}x_2)x_1^3,\ \alpha \cdot b_{40} \neq 0. \tag{41}$$

Lemma 8. *The dimension of the $GL(2,R)-$orbit of the system (42) is equal to three.*

The $GL(2,R)$-orbits of the dimension two. Let the inequalities (35) hold.

1) $v_1(a) = \alpha v_2(a) + \beta v_3(a),\ \alpha \cdot \beta \neq 0$. Taking into account (25), the given equality can be fulfilled only when $\beta = -4/(25\alpha)$ and $\beta = 2/\alpha$. These cases lead us to the systems

$$\dot{x_1} = 5\alpha F,\ \dot{x_2} = F,\ F = b_{40}(x_1 - 5\alpha x_2)^4 \not\equiv 0, \tag{42}$$

$$\begin{cases} \dot{x_1} = \alpha[(3b_{13} + 2\alpha b_{22})x_1 - 3\alpha b_{13}x_2] \cdot F, \\ \dot{x_2} = 2(-3b_{13}x_1 + \alpha^2 b_{22}x_2)F, \\ F = (x_1 + \alpha x_2)^2(2x_1 - \alpha x_2)/(6\alpha^3),\ |b_{13}| + |b_{22}| \neq 0. \end{cases} \tag{43}$$

Lemma 9. *The dimension of the $GL(2,R)-$orbit of the system (43) ((44)) is equal to two (three).*

Proof. In the case of the system (43) we have that $v_1(a) = \alpha v_2(a) - 4v_3(a)/(25\alpha)$ and $v_4(a) = -4\alpha v_2(a) + v_3(a)/(25\alpha)$ (see also (32)). For (44), $M(2,3,4;1,2,6) = b_{13}(15b_{13}^2 + 4\alpha^2 b_{22}^2 + 8\alpha b_{13}b_{22})/\alpha^6$, $M(2,3,4;1,2,9) = (39b_{13}^3 - 4\alpha^3 b_{22}^3 + 31\alpha b_{13}^2 b_{22})/(2\alpha^3)$ and the resultant of the polynomials $15b_{13}^2 + 4\alpha^2 b_{22}^2 + 8\alpha b_{13}b_{22}$, $39b_{13}^3 - 4\alpha^3 b_{22}^3 + 31\alpha b_{13}^2 b_{22}$ calculated in b_{13} is equal to $186624\alpha^6 b_{22}^6$.

2) $v_4(a) = \alpha v_2(a) + \beta v_3(a)$. By the substitutions $x_1 \longleftrightarrow x_2$, $a_{ij} \longleftrightarrow b_{ji}$, the given equality is transformed into $v_1(a) = \alpha v_2(a) + \beta v_3(a)$, and the systems (43) and (44) become the systems

$$\dot{x_1} = F,\ \dot{x_2} = 5\alpha F,\ F = a_{04}(5\alpha x_1 - x_2)^4 \not\equiv 0; \tag{44}$$

and

$$\begin{cases} \dot{x_1} = 2(\alpha^2 a_{22}x_1 - 3a_{31}x_2) \cdot F, \\ \dot{x_2} = \alpha[-3\alpha a_{31}x_1 + (3a_{31} + 2\alpha a_{22})x_2] \cdot F, \\ F = (\alpha x_1 + x_2)^2(2x_2 - \alpha x_1)/(6\alpha^3),\ |a_{31}| + |a_{22}| \neq 0. \end{cases} \tag{45}$$

respectively.

Lemma 10. *The dimension of the* $GL(2,R)-$*orbit of the system (45) ((46)) is equal to two (three).*

According to Lemmas 1 and 6, the results of the Lemmas 2, 4, 9 and 10, concerning $GL(2,R)-$orbits of the dimension two can be put together in the following theorem.

Theorem 5. *The system (1),* $k=4$, *with the condition (26) has the dimension of the* $GL(2,R)-$*orbit equal to two iff it has one of the forms (27), (28), (31), (32), (37), (43) or (45).*

The $GL(2,R)-$**orbits of the dimension three.** Assume that the inequalities (26), (35) are satisfied and let $v_1(a) = \alpha v_2(a) + \beta v_3(a) + \gamma v_4(a)$, $\alpha\cdot\beta\cdot\gamma \neq 0$. In these conditions the last equality can be fulfilled only in the cases $\gamma = -\alpha\beta$; $\beta = (2\gamma^2 - 5\gamma + 2)/\alpha$ and $\beta = -(4\gamma^2 + 17\gamma + 4)/(25\alpha)$. The first two cases lead us to the systems

$$\begin{cases} \dot{x_1} = \alpha[(a_{22} - 3a_{04}\beta^2)x_1 + 3\beta a_{04}x_2] \cdot F, \\ \dot{x_2} = [3\beta^2 a_{04}x_1 + (3\beta a_{04} + \alpha a_{22} - 6\alpha\beta^2 a_{04})x_2] \cdot F, \\ F = (\beta x_1 + x_2)^3/(3\alpha\beta),\ a_{04}(|a_{22} - 6\beta^2 a_{04}| + |1 + \alpha\beta|) \neq 0; \end{cases} \tag{46}$$

and

$$\begin{cases} \dot{x_1} = \alpha[(6\gamma a_{04} - 3a_{04} - \alpha^2 a_{22})x_1 + 3\alpha\gamma a_{04}x_2] \cdot F, \\ \dot{x_2} = [3\gamma(2 - 5\gamma + 2\gamma^2)a_{04}x_1 + (3\alpha(\gamma^2 + 3\gamma - 1)a_{04} - \alpha^3 a_{22})x_2] \cdot F, \\ F = [(1 - 2\gamma)x_1 + \alpha x_2]^2[(\gamma - 2)x_1 + \alpha x_2]/(3\alpha^5\gamma), \\ |\gamma - 1| + |a_{04}| \cdot |5\alpha^2 a_{22} - 3(\gamma^2 + 14\gamma - 5)a_{04}| \neq 0, \end{cases} \tag{47}$$

respectively and the latest case - to a system of the form (43).

Remark 1. *If* $a_{04} = 0$ $(\gamma - 1 = 5\alpha^2 a_{22} - 3a_{04}(\gamma^2 + 14\gamma - 5) = 0)$, *then the differential system (47) ((48)) has the form of (43), and if* $a_{22} - 6\beta^2 a_{04} = 1 + \alpha\beta = 0$ $(\gamma - 1 = a_{04} = 0)$, *then (47) ((48)) has the form (37).*

Lemma 11. *Each of the systems (47) and (48) has the dimension of the* $GL(2,R)-$*orbit equal to three.*

Proof. For (47) we have $M(2,3,4;1,2,3) = \beta^4 a_{04}(a_{22} - 6\beta^2 a_{04})^2/\alpha$. If $a_{22} = 6\beta^2 a_{04}$, then $M(2,3,4;2,3,8) = -432\beta^6 a_{04}^3(1+\alpha\beta)/\alpha^2 \neq 0$.

For (48), we have $M(2,3,4;4,5,10) = a_{04}^2[5\alpha^2 a_{22} - 3(\gamma^2 + 14\gamma - 5)a_{04}]$. If $a_{04} = 0$, then $M(2,3,4;3,4,8) = -2\gamma^{-3}a_{22}^3(\gamma-1)^2 \neq 0$, and if $5\alpha^2 a_{22} - 3(\gamma^2 + 14\gamma - 5)a_{04} = 0$, then $M(2,3,4;2,4,5) = -3888\gamma \cdot a_{04}^3(\gamma-1)^2/(25\alpha^4) \neq 0$ (see (26)). $\square$

From Lemmas 1-3, 5, 7-11, we obtain

Theorem 6. *The dimension of the $GL(2,R)-$orbit of the system (1), $k=4$, is equal to three iff it has one of the forms (29), (30), (33), (34), (36), (38)-(42), (44), (46)-(48).*

References

[1] Ovsyannikov, L. V., *Group analysis of differential equations*, Nauka, Moscow, 1978. English transl. by Academic, 1982.

[2] Păşcanu, A., Şubă A., *$GL(2,R)-$orbits of the polynomial systems of differential equation*. Bul. Acad. Sci. Rep. Moldova, Math., **3**(46) (2004), 25-40.

[3] Sibirsky, K. S., *Introduction to the algebraic theory of invariants of differential equations*, Shtiintsa, Kishinev, 1982 (in Russian; English transl., 1988).

[4] Braicov, A. V., Popa, M. N., *The $GL(2,R)-$orbits of differential system with homogeneities second order,* The International Conference "Differential and Integral Equations", Odessa, Sept. 12-14, 2000, p. 318.

[5] Boularas, D., Braicov, A. V., Popa, M. N., *Invariant conditions for dimensions of $GL(2,R)-$orbits for quadratic differential system,* Bul. Acad. Sci. Rep. Moldova, Math., **2**(2000), 31-38.

[6] Boularas, D., Braicov, A. V., Popa, M. N., *The $GL(2,R)-$orbits of differential systems with cubic homogeneities,* Bul. Acad. Sci. Rep. Moldova, Math., **1** (2001), 81-82.

[7] Naidenova, E. V., Popa, M. N., *On a classification of orbits for cubic differential systems,* Abstracts of 16th Int. Symp. on Nonlinear Acoustics, Aug. 19-23, 2002, Moskow State University, p. 274.

[8] Naidenova, E. V., Popa, M. N., *$GL(2,R)-$orbits of one cubic system,* Abstract of 11th Conference on Applied and Industrial Mathematics, May 29-31, 2003, Oradea, p. 57.

[9] Staruş, E. V., *Invariant conditions for the dimensions of the $GL(2,R)-$orbits for one differential cubic system,* Bul. Acad. Sci. Rep. Moldova, Math., **3**(2003), 58-70.

[10] Staruş, E. V., *The classification of the $GL(2,R)$-orbits dimensions for the system $s(0,2)$ and a factor system $s(0,1,2)/GL(2,R)$*, Bul. Acad. Sci. Rep. Moldova, Math., **1**(2004), 120-123.

[11] Popa M.N. *Algebraic methods in differential equations,* Series of Applied and Industrial Mathematics **15**, University of Piteşti, 2004 (in Romanian).

ROMAI J., **2**, 2(2006),
33–36

ON A DIFFERENTIAL SUPERORDINATION DEFINED BY SĂLĂGEAN OPERATOR

Adriana Cătaş
Dept. of Mathematics, Univ. of Oradea

Abstract Certain superordinations results related to holomorphic functions of the form $f(z) = z + a_{n+1}z^{n+1} + \dots$, defined on the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ are established.

Keywords: differential subordination, differential superordination, univalence.
2000 MSC: primary 30C80; secondary 30C45, 30A20, 34A40.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}(U)$ be the space of holomorphic functions in the unit disk U of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$. We also let

$$A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

with $A_1 = A$ and for $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \quad f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \quad z \in U\}.$$

If $f, g \in \mathcal{H}(U)$, then f is said to be *subordinate* to g, or g is said to be *superordinate* to f, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function $w \in \mathcal{H}(U)$, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$.

Denote by Ω any set in the complex plane $\mathbb{C}$, let p be an analytic function in the unit disk U and let $\psi(\gamma, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. In [1], properties of functions p that satisfy the differential subordination $\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega$. were determined.

In the present paper we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\}.$$

This problem was introduced in [2], where the conditions on ψ were determined such that

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \quad \text{implies} \quad q(z) \prec p(z),$$

for all p functions that satisfy the left-hand side of the above implication. Moreover, sufficient conditions so that q be the largest function with this property, called the best subordinant of this superordination, were found.

Definition 1. [2] Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfy the first-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z) \tag{1}$$

then p is called a *solution of the differential superordination*. An analytic function q is called a *subordinant of the solution of the differential superordination*, or, simply, a *subordinant* if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be *the best subordinant*. Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with ψ and p given as in Definition 1, suppose (1) is replaced by

$$\Omega \subset \{\psi(p(z), zp'(z); z) \mid z \in U\}. \tag{2}$$

Although this more general situation is a "differential containment", the condition in (2) will also be referred to as a differential superordination, and the definitons of solution, subordinant and best dominant as given above can be extended to this case.

Definition 2. [2] Denote by Q the set of functions f that are analytic and injective on $\bar{U} \backslash E(f)$, where $E(f) = \{\zeta \in \| U : \lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \| U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

We use the following subordination results.

Lemma A. [2] *Let h be a convex function in U with $h(0) = a$, $\gamma \neq 0$ with $\mathbb{R}\gamma \geq 0$, and $p \in \mathcal{H}[a, n] \cap Q$. If $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, $h(z) \prec p(z) + \dfrac{zp'(z)}{\gamma}$, then $q(z) \prec p(z)$, where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt, \quad z \in U$. The function q is convex and is the best subordinant.*

Lemma B. [2] *Let q be a convex function in U and let h be defined by $h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U$, with $\mathbb{R}\gamma \geq 0$. If $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, and $q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$, then $q(z) \prec p(z)$, where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$. The function q is the best subordinant.*

Definition 3 [4]. For $f \in A_n$ and $m \geq 0, m \in \mathbb{N}$, the operator $D^m f$ is defined by $D^0 f(z) = f(z), D^{m+1} f(z) = z[D^m f(z)]', \quad z \in U$.

2. MAIN RESULTS

Theorem 1. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z} \tag{3}$$

be convex in U, with $h(0) = 1$. Let $f \in A_n$ and suppose that $[D^{m+1}f(z)]'$ is univalent and $[D^m f(z)]' \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec [D^{m+1}f(z)]', \quad z \in U, \tag{4}$$

then $q(z) \prec [D^m f(z)]'$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} \cdot t^{\frac{1}{n}-1}dt$,

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}Q(z)\frac{1}{z^{\frac{1}{n}}},\ Q(z) = \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt.$$

The function q is convex and is the best subordinant.

Proof. Let $f \in A_n$. By using properties of the operator $D^m f$ we have $D^{m+1}f(z) = z[D^m f(z)]'$, $z \in U$ and, by differentiation, we obtain $[D^{m+1}f(z)]' = [D^m f(z)]' + z[D^m f(z)]''$, $z \in U$. If we let $p(z) = [D^m f(z)]'$, then it follows $[D^{m+1}f(z)]' = p(z) + zp'(z), z \in U$. Then (4) becomes $h(z) \prec p(z) + zp'(z), z \in U$. By using Lemma A, for $\gamma = 1$, we have $q(z) \prec p(z) = [D^m f(z)]'$, $z \in U$ where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} \cdot t^{\frac{1}{n}-1}dt = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}Q(z)\frac{1}{z^{\frac{1}{n}}}, Q(z) = \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$. The function q is the best subordinant.

Theorem 2. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex in U, with $h(0) = 1$. Let $f \in A_n$ and suppose that $[D^m f(z)]'$ is univalent and $\dfrac{D^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If*

$$h(z) \prec [D^m f(z)]', \quad z \in U, \tag{5}$$

then $q(z) \prec \frac{D^m f(z)}{z}$, $z \in U$, where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} \cdot t^{\frac{1}{n}-1}dt =$$

$$= 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}Q(z)\frac{1}{z^{\frac{1}{n}}}, Q(z) = \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U.$$

The function q is convex and is the best subordinant.

Proof. We let $p(z) = \frac{D^m f(z)}{z}$, $z \in U$ and we obtain $D^m f(z) = zp(z)$, $z \in U$ and by differentiation we obtain $[D^m f(z)]' = p(z) + zp'(z)$, $z \in U$. Then (5) becomes $h(z) \prec p(z) + zp'(z)$, $z \in U$. By using Lemma A we have $q(z) \prec p(z) = \frac{D^m f(z)}{z}, z \in U$, where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}Q(z)\frac{1}{z^{\frac{1}{n}}}$$

$$Q(z) = \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U.$$

The function q is convex and is the best subordinant.

Theorem 3. *Let q be a convex function in U and let h be a function such that $h(z) = q(z) + zq'(z)$, $z \in U$. Let $f \in A_n$ and suppose that $[D^{m+1}f(z)]'$ is univalent in U, $[D^m f(z)]' \in \mathcal{H}[1,n] \cap Q$ and $h(z) = q(z) + zq'(z) \prec [D^{m+1}f(z)]'$, $z \in U$, then $q(z) \prec [D^m f(z)]'$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z \in U$. The function q is convex and is the best subordinant.*

Proof. Let $f \in A_n$. By using properties of the operator $D^m f$ we have $D^{m+1}f(z) = z[D^m f(z)]'$, $z \in U$ and, by differentiation, we obtain $[D^{m+1}f(z)]' = [D^m f(z)]' + z[D^m f(z)]''$, $z \in U$. If we let $p(z) = [D^m f(z)]'$, then the last equality becomes $[D^{m+1}f(z)]' = p(z)+zp'(z)$, $z \in U$. By using Lemma B, we have $q(z) \prec p(z) = [D^m f(z)]'$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z \in U$. The function q is the best subordinant.

Theorem 4. *Let q be a convex function in U and let h be a function such that $h(z) = q(z) + zq'(z)$, $z \in U$. Let $f \in A_n$ and suppose that $[D^m f(z)]'$ is univalent in U, $\dfrac{D^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and*

$$h(z) = q(z) + zq'(z) \prec [D^m f(z)]', \quad z \in U, \tag{6}$$

then $q(z) \prec \dfrac{D^m f(z)}{z}$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z \in U$. The function q is the best subordinant.

Proof. We let $p(z) = \frac{D^m f(z)}{z}$, $z \in U$ and we obtain

$$D^m f(z) = zp(z), \quad z \in U, \tag{7}$$

while by differentiation we have $[D^m f(z)]' = p(z) + zp'(z)$, $z \in U$. Then (6) becomes $q(z) + zq'(z) \prec p(z) + zp'(z)$, $z \in U$. By using Lemma B we have $q(z) \prec p(z) = \frac{D^m f(z)}{z}$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, $z \in U$. The function q is the best subordinant.

Remark. Similar results were obtained in [3] for $n = 1$.

References

[1] S. S. Miller, P. T. Mocanu, *Differential subordinations. Theory and applications*, Marcel Dekker Inc., New York, 2000.

[2] S. S. Miller, P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, **48**, 10(2003), 815-826.

[3] Gh. Oros, G. I. Oros, *Differential superordination defined by Sălăgean operator*, General Mathematics, Sibiu, (to appear).

[4] Gr. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer, Berlin, **1013**(1983), 362-372.

ROMAI J., **2**, 2(2006),
37–44

LOW-DIMENSIONAL QUASIPERIODIC MOTION IN HAMILTONIAN SYSTEMS

Helen Christodoulidi, Tassos Bountis
University of Patras, Greece
mathelegr1@yahoo.com, bountis@math.upatras.gr

Abstract A new method was recently introduced for detecting chaos and order in N degree of freedom Hamiltonian systems: It is called Generalized Alignment Index ($GALI$) and predicts rapidly and accurately if a certain orbit is chaotic or regular, by computing the volume of k unit deviation vectors as they follow the given orbit in the $2N-$dimensional phase space. As is well known, regular orbits of N degrees of freedom Hamiltonian systems lie, in general, on N dimensional tori. It does happen, however, in many cases of physical interest that these tori have dimensions much lower than N. In this paper, we derive the asymptotic behavior of the $GALI_k$ indices for the case of lower dimensional tori and apply our results to the famous Fermi Pasta Ulam lattice.

Keywords: hamiltonian systems, quasiperiodic motion, GALI$_k$ indices, dimension of tori.

1. INTRODUCTION

Hamiltonian systems are very important models of conservative dynamical systems such as planetary motion, charges moving in magnetic fields and particle beams in high energy accelerators [1],[2]. Furthermore, they can be used to describe physically relevant properties of nonlinear lattices, such us periodicity, chaos and statistical behavior [3]-[5],[9], as well as localized oscillations, called discrete breathers [7],[8].

Let us consider an N degree of freedom Hamiltonian system in $\mathbb{R}^{2N}$, described by the Hamiltonian function:

$$H(p_1, ..., p_N, q_1, ..., q_N) = E = const. \tag{1}$$

The equations of motion derived from this Hamiltonian are:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, ..., N. \tag{2}$$

and yield solutions (or orbits) $\{\mathbf{p}(t), \mathbf{q}(t)\}$ of the system evolving in the $2N$ dimensional phase space. Given a value of the total energy E and a specific initial condition $\{\mathbf{p}(0), \mathbf{q}(0)\}$ one is interested to know whether the corresponding orbit will be regular, i.e. quasiperiodic, lying on N dimensional tori characterized by N rationally independent frequencies, or chaotic, evolving in a $(2N-1)$ dimensional region, where solutions are extremely sensitive to the choice of initial conditions [1],[2]. The corresponding variational equations of this system along a given trajectory $\{\mathbf{p}(t), \mathbf{q}(t)\}$ in $\mathbb{R}^{2N}$ are:

$$\frac{d\bar{w}}{dt} = J \cdot M(p(t), q(t))\bar{w}, \tag{3}$$

where $J = \begin{pmatrix} O & I_N \\ -I_N & O \end{pmatrix}$ is the symplectic matrix, I_N is the $N \times N$ identity matrix, and M the Hessian matrix of the Hamiltonian function. We choose k linearly independent vectors $\bar{w}_i, i = 1, ..., k$ of unit magnitude, i.e. $\|\bar{w}_i(0)\| = 1, i = 1, ..., k$ to be the initial conditions of the linear system (3). The solutions of this system are k vectors $\bar{w}_i(t), i = 1, ..., k$ (which need not remain linearly independent) called deviation vectors, whose behavior determines the stability of the orbit. Given k such normalized deviation vectors

$$\hat{w}_i(t) = \frac{\bar{w}_i(t)}{\|\bar{w}_i(t)\|}, \quad i = 1, ..., k, \tag{4}$$

which are linearly independent at $t = 0$, we define the Generalized Alignment Index of order k as the volume of a k-dimensional parallelepiped generated by these vectors and given by the expression

$$GALI_k(t) = \|\hat{w}_1(t) \wedge ... \wedge \hat{w}_k(t)\|, \tag{5}$$

whose evolution in time we wish to follow. For a general description of exterior products and their computation the reader is referred to [11].

In the literature [10] it has been recently proved that the asymptotic behavior of $GALI_k$ for a Hamiltonian system of N degrees of freedom, is given for long times, $t >> 1$, by the formulae:

$$GALI_k(t) \sim e^{-[(\sigma_2-\sigma_1)+...+(\sigma_k-\sigma_1)]t}, \tag{6}$$

if the orbit is chaotic, where $\sigma_1 > ... > \sigma_k$, are the maximal Lyapunov exponents, and

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } 2 \leq k \leq N \\ \frac{1}{t^{2(k-N)-s}}, \text{if } N < k \leq 2N \text{ and } 0 \leq s < k-N \\ \frac{1}{t^{k-N}}, \text{if } N < k \leq 2N \text{ and } s \geq k-N \end{array} \tag{7}$$

if the orbit is regular, where s is the number of deviation vectors that initially lie in the tangent space of the torus. Based on these estimates, a great number of useful results have been reported concerning the detection of chaos as well as the verification that the orbit is regular and lies on an N-dimensional torus [10]. There are also many cases, however, where the regular orbits lie on tori, whose dimension is smaller than N. In such cases the asymptotic formulas (7) above are no longer the same and the theory of $GALI$ indices needs to be extended to explain the new estimates of (algebraic) power law decay, which are observed in numerical computations.

In this paper, we provide such an extension by generalizing the theory of the $GALI$ indices to include the case of tori of dimensionality lower than N. Our theoretical results are in complete agreement with numerical experiments, as we demonstrate by studying the evolution of regular orbits in a Fermi Pasta Ulam Hamiltonian lattice of N particles, moving in one spatial dimension.

2. BEHAVIOR OF GALI$_K$ FOR REGULAR ORBITS

Regular orbits of N degree of freedom Hamiltonian systems typically lie on N-dimensional tori, on which, using action-angle variables we can write Hamilton's equations of motion locally in the form:

$$\dot{J}_i = 0,\ \dot{\theta}_i = \omega_i(J_1, ..., J_N) \tag{8}$$

for $i = 1, ..., N$, where generally it is $\omega_i(J_1, ..., J_N) \neq 0$. In the case where:

$$\omega_i(J_1, ..., J_N) = \left\{ \begin{array}{l} \text{constant} \neq 0, \text{ if } i = 1, ..., m \\ 0, \text{ if } i = m+1, ..., N \end{array} \right\} \tag{9}$$

the orbit lies on an m-dimensional torus, with $m \leq N$. By denoting as ξ_i, η_i, variations of J_i, θ_i, $i = 1, ..., m$, and by c_i, d_i variations of J_i, θ_i, $i = m+1, ..., N$ respectively, equations (3) of this system take the form:

$$\dot{\xi}_i = 0,\ \dot{c}_k = 0,\ \dot{d}_k = 0,\ \dot{\eta}_i = \sum_{j=1}^{m} \omega_{ij}\xi_j \tag{10}$$

for $i = 1, ..., m$, $k = 1, ..., N - m$. By $\Omega = [\omega_{ij}]$ we denote the $m \times m$ Jacobian matrix of $\bar{\omega} = (\omega_1, ..., \omega_m)$, at $\mathbf{J} = (J_1, ..., J_m)$. Every deviation vector has its first $2N - m$ coordinates constant, while the remaining ones are linearly dependent on t. We denote by $\xi = (\xi_1, ..., \xi_m)$, $\mathbf{c} = (c_1, ..., c_{N-m})$, $\mathbf{d} = (d_1, ..., d_{N-m})$ and $\eta = (\eta_1, ..., \eta_m) = (\sum\limits_{j=1}^{m} \omega_{1j}\xi_j t + \eta_{10}, ..., \sum\limits_{j=1}^{m} \omega_{mj}\xi_j t + \eta_{m0}) = \Omega\xi \cdot t + \eta_0$, so:

$$\bar{w}(t) = (\xi, \mathbf{c}, \mathbf{d}, \eta) \tag{11}$$

or:

$$\bar{w}(t) = \bar{w}(0) + (0, ..., 0, \Omega\xi \cdot t) \tag{12}$$

If $\bar{w}(t)$ initially has unit magnitude, its evolution in time becomes:

$$\|\bar{w}(t)\| = \Big(1 + 2 < \eta_0, \Omega\xi > t + \|\Omega\xi\|^2 t^2\Big)^{1/2} \tag{13}$$

which is evidently proportional to t, for long times, i.e.:

$$\|\bar{w}(t)\| \propto t. \tag{14}$$

If the orbit is quasiperiodic, the normalized deviation vectors along the solution tend to fall on the tangent space of the torus, since the only non-vanishing coordinates are those related with η and therefore it holds that:

$$\hat{w}(t) \underset{t\longrightarrow\infty}{\longrightarrow} \frac{1}{\|\Omega\xi\|}(0, ..., 0, \Omega\xi). \tag{15}$$

Let $s = 0$ be the number of deviation vectors, which are initially tangent to the torus. This is the generic case since the complementary space of the m-dimensional tangent space in $\mathbb{R}^{2N}$ is open and dense. Then, the volume of k deviation vectors $\bar{w}_i(t)$, $i = 1, ..., k$, for $t >> 1$, will be approximated by the $k \times k$ determinant, having the maximum degree with respect to t, in order to yield the lowest order term in a long time approximation. Thus estimation for the $GALI_k$ index is given by this determinant divided by the magnitude of each deviation vector, i.e.

$$\prod_{i=1}^{k} \|\bar{w}_i(t)\| \propto t^k. \tag{16}$$

and so,

$$\|\hat{w}_1(t) \wedge ... \wedge \hat{w}_k(t)\| \propto \frac{1}{t^k} \|\bar{w}_1(t) \wedge ... \wedge \bar{w}_k(t)\|. \tag{17}$$

It can be easily seen that if $2 \leq k \leq m$, then $\|\bar{w}_1(t) \wedge ... \wedge \bar{w}_k(t)\|$ is a polynomial of degree k with respect to t and hence $GALI_k$ is estimated to be constant.

If $m+1 \leq k \leq 2N-m$, the $k\times k$ determinant with the maximum degree with respect to t, has m rows filled with terms proportional to t and the remaining $k - m$ rows are filled by either constant terms $\mathbf{c}_i$ and $\mathbf{d}_i$, or by the terms ξ_i of the vector $\bar{w}_i(t) = (\xi_i, \mathbf{c}_i, \mathbf{d}_i, \eta_i)$, $i = 1, ..., k$. In the second case, each row ξ_i will appear twice, because of the term $\eta_i = \Omega\xi_i \cdot t + \eta_{i0}$. Consequently, from the m rows occupied by η_i, only $m - (k - m) = 2m - k$ rows remain proportional to t. So the $k - m$ rows of the determinant will be filled by $\mathbf{c}_i$, $\mathbf{d}_i$ and the last one will be approximated by a polynomial with respect to t of degree k.

If $2N - m + 1 \leq k \leq 2N$, the determinant with the maximum degree with respect to t, has $2N - m$ rows filled by the terms $\mathbf{c}_i$,$\mathbf{d}_i$ and η_i,$i = 1, ..., k$ and the remaining $k - (2N - m)$ terms are left to be filled by ξ_i, implying that $k - (2N - m)$ terms of the form $\Omega\xi_i \cdot t$ will be subtracted from the m rows occupied by η_i, $i = 1, ..., k$. Finally, there remain $m - [k - (2N - m)] = 2N - k$ rows proportional to t.

Summarizing the above and dividing with t^k, we obtain the following asymptotic behavior of $GALI_k$, for $t >> 1$:

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } 2 \leq k \leq m \\ \frac{1}{t^{k-m}}, \text{if } m < k \leq 2N - m \\ \frac{1}{t^{2(N-k)}}, \text{if } 2N - m < k \leq 2N \end{array} \tag{18}$$

which complements what is reported in literature for $m = N$, see (7)[10]. Of course, in the case $m = N$ the results of the two formulae coincide.

3. NUMERICAL VERIFICATION AND APPLICATIONS

In this section we apply the $GALI$ method to a Hamiltonian system whose quasiperiodic orbits lie on $1 \leq m \leq N$ dimensional tori and verify our theoretical results described in the previous section. More specifically we study the Fermi Pasta Ulam chain, described by the Hamiltonian function [6]-[9]:

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N}\left[\frac{1}{2}(q_{i+1} - q_i)^2 + \frac{\beta}{4}(q_{i+1} - q_i)^4\right] \tag{19}$$

with fixed boundary conditions, i.e. $q_0(t) = q_{N+1}(t) = p_0(t) = p_{N+1}(t) = 0$, $\forall t$. In the linear case, $\beta = 0$, the system becomes uncoupled under the transformation to normal mode variables $Q_k = \sqrt{2/(N+1)}\sum_{i=1}^{N} q_i \sin(ki\pi/(N+1)), P_k = \dot{Q}_k$, $k = 1, \ldots, N$. The energy of each oscillator is $E_k = \frac{1}{2}(P_k^2 + {\omega_k}^2 Q_k^2)$ and has frequency $\omega_k = 2\sin(k\pi/2(N+1))$, $k = 1, \ldots, N$. Choosing initial conditions which excite only one normal mode of the linear system, we find, just as observed by Fermi Pasta and Ulam that this mode continues to exist for the nonlinear system (19), for small values of β but different from zero.

We choose randomly k normalized deviation vectors, so that none of them lies in the tangent space of the torus, for an FPU system with $N = 5$ and $\beta = 2$ and plot the results in Figure 1 below. In this case, we observe that only one particle gains the total energy of the system, so the dimension of the torus is one. This is clearly demonstrated by the results of figure 1(a), which are characterized by the slopes:

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } k = 1 \\ \frac{1}{t^{k-1}}, \text{if } 2 \leq k \leq 9 \\ \frac{1}{t^{10-2k}}, \text{if } k = 10 \end{array} \tag{20}$$

Choosing now different initial conditions $p_i = 0, q_i = 0.1$ for the FPU system with 5 degrees of freedom, and $\beta = 1$, we obtain the results shown in Fig. 2

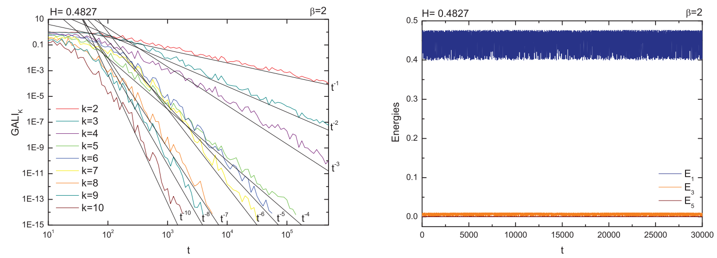


Fig. 1. (a) The evolution of $\mathrm{GALI}_k, k = 2, ..., 10$, for initial conditions near the first normal mode of 5 particle FPU lattice and for $\beta = 2$. (b) The energy of each particle.The motion is quasiperiodic and all deviation vectors align with the orbit, decaying by power laws.

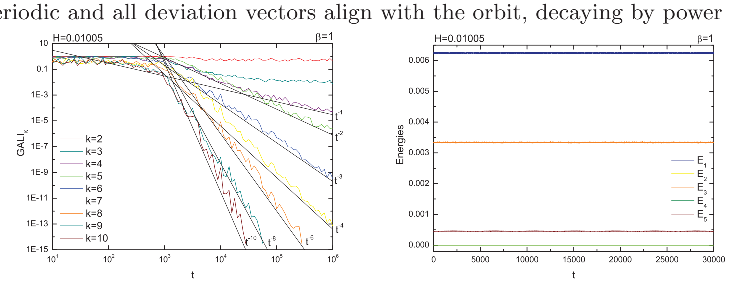


Fig. 2. (a) The evolution of $\mathrm{GALI}_k, k = 2, \ldots, 10$, for k normalized deviation vectors, whose initial conditions are chosen as combinations of 3 normal modes of the linear system at $\beta = 1$. (b) The energy of each particle. This picture implies that the orbit lies on a 3-dimensional torus.

Note that, in this case, the total energy is exchanged quasiperiodically only between the excited E_1, E_3, E_5 modes, so the regular orbit lies on a three dimensional torus, since:

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } 2 \le k \le 3 \\ \frac{1}{t^{k-3}}, \text{if } 4 \le k \le 7 \\ \frac{1}{t^{10-2k}}, \text{if } 8 \le k \le 10 \end{array} \tag{21}$$

Choosing the initial positions randomly with initial momenta equal to zero, for the FPU system, with $\beta = 0.5$, we obtain in Fig. 3 results, which can be interpreted as follows: Since $GALI_2$, $GALI_3$ and $GALI_4$ are constant and all higher order $GALI_k$ decay by power laws, we conclude that the orbit lies on 4-dimensional torus.

In agreement with our analytical formulas, these results show only four normal mode energies are excited, so the regular orbit lies on a four dimensional torus on $\mathbb{R}^{10}$, and

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } 2 \le k \le 4 \\ \frac{1}{t^{k-4}}, \text{if } 5 \le k \le 6 \\ \frac{1}{t^{10-2k}}, \text{if } 7 \le k \le 10 \end{array} \tag{22}$$

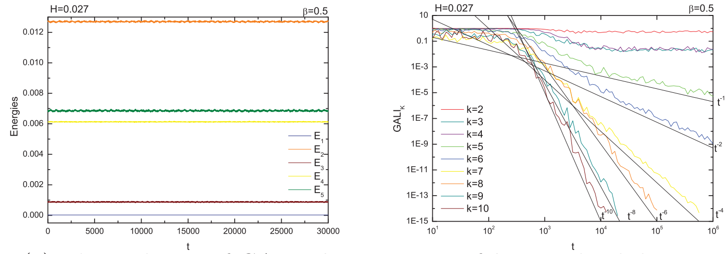


Fig. 3. (a) The evolution of $\mathrm{GALI}_k, k = 2, \ldots, 10$ of k normalized deviation vectors for random initial positions and zero momenta, with $\beta = 0.5$. (b) The evolution of the normal mode energies support the findings of 3(a), i.e. that the orbit quasiperiodic and lies on a 4-dimensional torus.

Finally, choosing random initial conditions, it is easy to find cases, where the generic situation occurs, i.e. the orbit lies on a 5–dimensional torus, as seen in Fig. 4. This is the most general case, because the torus is of maximum

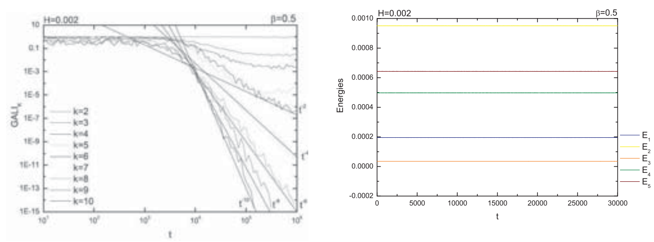


Fig. 4. (a) The evolution of $\mathrm{GALI}_k, k = 2, \ldots, 10$ of k normalized deviation vectors, for random initial conditions, with $\beta = 0.5$. (b) The energy of each particle. This picture implies that the orbit lies on a 5-dimensional torus.

dimension N and hence the analytic estimates for $GALI$ indices, shown below, coincide with those in [10]:

$$GALI_k(t) \sim \begin{array}{l} \text{constant, if } 2 \leq k \leq 5 \\ \frac{1}{t^{10-2k}}, \text{if } 6 \leq k \leq 10 \end{array} \qquad (23)$$

4. CONCLUSIONS

Many important problems of physical interest can be described by N degree of freedom Hamiltonian systems. Notable examples in this category include the solar system, charged particles in magnetic fields and nonlinear lattices of solid state physics [1], [2]. In such problems, it is of the utmost concern to be able to determine domains of phase space, where the motion is regular (quasiperiodic) lying on N-dimensional tori or chaotic, filling regions where the orbits are extremely sensitive to the choice of initial conditions.

Recently, a new method was reported in the literature, called the Generalized Alignment Indices ($GALI$) [10], by which one can determine rapidly and efficiently, whether a given orbit in a Hamiltonian system of N degrees of freedom is regular or chaotic. The results presented in [10], however, were limited to the generic case, where regular motion lies on N-dimensional tori.

In the present paper, we have generalized these results, by deriving asymptotic formulas for the time evolution of the $GALI$ indices, in cases where the tori are m-dimensional with $1 \leq m \leq N$. We have shown that such cases occur typically in Hamiltonian systems and have verified our analytical expressions on a 5-degree of freedom example of a Fermi Pasta Ulam system, in the regime of low total energy.

5. ACKNOWLEDGMENTS

H. Ch. was supported by the State Scholarships Foundation of Greece. We thank Dr. Charalampos Skokos and Dr. Chris Antonopoulos for very fruitful discussions. T. B. also thanks Professor Y. Aizawa and his colleagues for their hospitality at Waseda University, Tokyo, during his visit in April 2007, when some of this work was completed.

References

[1] Lieberman M.A. and Lichtenberg A.J., Regular and Chaotic Dynamics Springer Verlag, 1992.

[2] MacKay R.S. and Meiss J.D. , Hamiltonian Dynamical Systems, Adam Hilger, Bristol, 1987.

[3] Antonopoulos Ch. Bountis T. Skokos Ch., Chaotic Dynamics of N–degree of Freedom Hamiltonian Systems, Int.J.Bif.Chaos, 6 (6) 1777-1793, 2006.

[4] Antonopoulos Ch. Bountis T. Stability of simple periodic orbits and chaos in a Fermi-Pasta-Ulam lattice, Phys. Rev. E 73, 056206,2006.

[5] Antonopoulos Ch., Bountis T., Detecting Order and Chaos by the Linear Dependence Index, ROMAI Journal, this volume.

[6] Berman G.P., Izrailev F.M., Chaos, 15, 015104, 2005.

[7] Flach S. and Gorbach A., Discrete breathers in Fermi-Pasta-Ulam lattices, Chaos 15 (1), 2005.

[8] Flach S., Ivanchenko M.V., Kanakov O.I., q-Breathers in Fermi-Pasta-Ulam chains: existence, localization and stability, Phys. Rev. E, Vol.73, Issue 3, 036618, 2006.

[9] Ford J., Phys. Rep., 213, 271, 1992.

[10] Skokos Ch. et al. Geometrical Properties of Local Dynamics in Hamiltonian Systems: The Generalized Alignment Index (GALI) Method}, Physica D, to appear.

[11] Spivak M., Comprehensive Introduction to Differential Geometry, vol.1, Publ. or Per. Inc., 1999.

ROMAI J., **2**, 2(2006),
45–48

ICA BY MINIMIZATION OF MUTUAL INFORMATION

Doru Constantin
Faculty of Mathematics-Informatics, University of Pitesti
cdomanid@yahoo.com

Abstract An important approach for independent component analysis (ICA) estimation, inspired by information theory, is minimization of mutual information. The motivaton of this approach is that it may be not very realistic in many cases to assume that the data follows the ICA model. Therefore, it was developed an approach that does not assume anything about the data. We intended to have a general-purpose measure of the dependence of the components on the random vector. Using such a measure, we define ICA as a linear decomposition that minimizes that dependence measure. Such an approach can be developed using mutual information, which is a information-theoretic measure of statistical dependence.

1. DEFINING BY MUTUAL INFORMATION

1.1. INFORMATION-THEORETIC CONCEPTS

In the following we present some concepts of information theory. The differential entropy H of a random vector y with density $p(y)$ is defined as

$$H(y) = -\int p(y) \log p(y) \tag{1}$$

A normalized version of entropy is given by negentropy J, which is defined as

$$J(y) = H(y_{gauss}) - H(y), \tag{2}$$

where y_{gauss} is a Gaussian random vector of the same covariance matrix as y. Negentropy is always nonnegative, and zero only for Gaussian random vectors. Mutual information I between m random variables, $y_i, i = 1, ..., m$ is defined as

$$I(y_1, y_2, ..., y_m) = \sum_{i=1}^{m} H(y_i) - H(y). \tag{3}$$

1.2. MUTUAL INFORMATION AS MEASURE OF DEPENDENCE

Mutual information is a natural measure of the dependence between random variables. It is always nonnegative, and zero if and only if the variables are statistically independent. We can use mutual information as the criterion for finding the ICA representation. This approach is an alternative to the model estimation approach. We define the ICA of a random vector x as an invertible transformation

$$s = Bx, \tag{4}$$

where the matrix B is determined so that the mutual information of the transformed components s_i is minimized. Minimization of mutual information can be interpreted as giving the maximally independent components.

2. MUTUAL INFORMATION AND NONGAUSSIANITY

Using the formula for the differential entropy we obtain the expression of mutual information for an invertible linear transformation $y = Bx$

$$I(y_1, y_2, ..., y_m) = \sum_{i=1}^{m} H(y_i) - H(x) - \log|\det B| \tag{5}$$

Next, we constraint that y_i to be uncorrelated and unit variance. This means $E\{yy^T\} = BE\{xx^T\}B^T = I$, which implies

$$\det I = 1 = \det(BE\{xx^T\}B^T = I) = (\det B)(\det E\{xx^T\})(\det B^T) \tag{6}$$

and this implies that $\det B$ must be constant since $\det E\{xx^T\}$ does not depend on B. Moreover, for y_i of unit variance, entropy and negentropy differ only by a constant and the sign. Thus we obtain

$$I(y_1, y_2, ..., y_n) = const - \sum_i J(y_i), \tag{7}$$

where the constant term does not depend on B. This shows the fundamental relation between negentropy and mutual information.

The relation (7) shows that finding an invertible linear transformation B that minimizes the mutual information is roughly equivalent to finding directions in which the negentropy is maximized. Negentropy is a measure of non-Gaussianity. Thus, (7) shows that *ICA estimation by minimization of mutual information is equivalent to maximizing the sum of nonGaussianities of the* *estimates of the independent components*, when the estimation are constrained to be uncorrelated.

It follows that the formulation of ICA as minimization of mutual information gives another rigorous justification of idea of finding maximally nonGaussian directions.

In practice there are some important differences between these two criteria:

1 : Negentropy, and other measure of nonGaussianity, enable the deflationary (one-by-one), estimation of the independent components, since we can look for the maxima of nonGaussianity of a single projection $b^T x$. This is not possible with mutual information or most other criteria, like the likelihood.

2 : A smaller difference is that in using nonGaussianity, we force the estimations of the independent components to be uncorrelated. This is not necessary when using mutual information, because we could use the form in (5) directly.

3. MUTUAL INFORMATION AND LIKELIHOOD

Mutual information and likelihood are intimately connected. To see the connection between likelihood and mutual information, consider the expectation of the log-likelihood in

$$\frac{1}{T} E\{\log L(B)\} = \sum_{i=1}^{n} E\{\log p_i(b_i^T x)\} + \log|\det B|. \tag{8}$$

If the p_i were equal to the actual density functions's of the $b_i^T x$, the first term would be equal to $-\sum_i H(b_i^T x)$. Thus the likelihood would be equal, up to an additive constant given by the total entropy of x, to the negative of mutual information as given in (5).

In practice, the connection may be just as strong, or even stronger. This is because in practice we do not know the distributions of the independent components that are needed in ML estimation. A reasonable approach would be to estimate the density of $b_i^T x$ as part of the ML estimation method, and use this approximation of the density of s_i. Then, the p_i in this approximation of likelihood are indeed equal to the actual density functions $b_i^T x$. Thus, the equivalency would really hold.

In order to approximate mutual information, we would take a fixed approximation of the densities y_i and plug this in the definition of entropy. Denoting the densities functions's by $G_i(y_i) = \log p_i(y_i)$, we could approximate (5) as

$$I(y_1, y_2, ..., y_m) = \sum_i E\{G_i(y_i)\} - \log|\det B| - H(x). \tag{9}$$

Concluding remarks A rigorous approach that is different from maximum likelihood approach is given by minimization of mutual information. Mutual information is a natural information-theoretic measure of dependence, and therefore it is natural to estimate the independent components by minimizing the mutual information of their estimates. Mutual information gives a rigorous justification of the principles of searching for maximally nongaussian directions, and in the end turns out to be very similar to the likelihood as well. Mutual information can be approximated by the same methods that negentropy is approximated.

References

[1] Bishop, C. M. *Neural network for pattern recognition*, Clarendon, 1995.

[2] Cichocki, A., Unbehauen, R., *Neural networks for signal processing and optimization*, New York, Wiley, 1994.

[3] Cover, T. M., Thomas, J. A. *Elements of information theory*, Wiley, New York, 1991.

[4] Diamantaras, K. I., Kung, S. Y., *Principal component neural networks: theory and applications*, Wiley, New York, 1996.

[5] Gardner, W., *Introduction to random processes with applications to signal and systems*, Macmillan, 1986.

[6] Gonzalez, R., Wintz, P., *Digital image processing*, Addison-Wisley, 1987.

[7] Girolami, M., *Self-organising neural networks - independent component analysis and blind source separation*, Springer, Berlin, 1999.

[8] Hyvärinen, A., Karhunen, J., Oja, E., *Independent component analysis*, John Wiley , New York, 2001.

[9] Kay, S., *Modern spectral estimation: theory and application*, Prentince Hall, Englewood Hills, 1988.

[10] Kohonen, T., *Self-organizing maps*, Springer, Berlin, 1995.

[11] Lee, T. W., *Independent component analysis: theory and applications*, Kluwer, Dordrecht, 1998.

[12] Oppenheim, A., Schafer, R., *Discrete-time signal processing*, Prentince Hall, Englewood Hills, 1989.

ROMAI J., **2**, 2(2006),
49–53

THEORETICAL PROPERTIES IN HYPERGRAPHS

Silviu Dumitrescu, Bogdana Pop, Vlad Monescu
"Transilvania" University of Brasov, Computer Science Department

Abstract Well-known relations in graphs are adapted to the hypergraphs. The correctness of these relations using the hyperedge replacement grammars is proved. The number of nodes and hyperedges of an arbitrary hypergraph, paths in string-graphs and their length and also cycles in string-graphs and their length are investigated. In the end, the compatibility between the theoretically properties of hypergraphs and the replacement process of hyperedges are defined, proving these compatibilities for the previous relations.

1. MATHEMATICAL RELATIONS IN HYPERGRAPHS

We show some of the most important properties of hypergraphs. A hypergraph [1, 5] is a special structure, that is why its properties are different from an ordinary graph.

1.1. NUMBER OF NODES AND HYPEREDGES

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be a hypergraph over a fixed and arbitrary set C, of labels. Denote by INT_H, the set of internal nodes of the hypergraph H, meaning the difference between V_H and ext_H.

Let $HRG = (N, T, P, S)$ be a hyperedge replacement grammar with $N \subseteq C$ and $T \subseteq C$, and a derivation in this grammar: $A^\bullet \Rightarrow R \Rightarrow^* H$, $A \in N$. For every $e \in E_R$ we have $lab_R(e)^\bullet \Rightarrow^* H(e)$, meaning that every labeled edge $lab_R(e)$ will have a support $lab_R(e)^\bullet$ and will be replaced by a hypergraph $H(e)$ by applying many productions from P. Therefore, the set of nodes of the hypergraph H consists of the set of nodes of R union with the set of internal nodes of components $H(e)$.

Proposition 1.1 *Let $H \in \mathcal{H}_T$ and $A^\bullet \Rightarrow R \Rightarrow^* H$ be a derivation of H. For every $e \in E_R$ we have that $l_R(e)^\bullet \Rightarrow^* H(e)$. Then*

$$|V_H| = |V_R| + \sum_{e \in E_R} |INT_{H(e)}|,$$
$$|INT_H| = |INT_R| + \sum_{e \in E_R} |INT_{H(e)}|,$$
$$|E_H| = \sum_{e \in E_R} |E_{H(e)}|.$$

Proof: We can prove these by using the induction related to the cardinality of the set E_R, such that $V_H = V_R + \sum_{e \in E_R} INT_{H(e)}$. The verification is immediate if $|E_R| = 0$ then $H = R$ and $V_H = V_R$. The induction step will consider the context independence and from this the derivation definition. Therefore, the external nodes of the derived hypergraph are added to the hypergraph nodes to be derivate. The only eventual new node in the formula are internal nodes of the derived hypergraph.

For the second formula we have $V_H - [ext_H] = V_R - [ext_R] + \sum_{e \in E_R} INT_{H(e)}$ because $[ext_H] = [ext_R]$. Therefore $INT_H = INT_R + \sum_{e \in E_R} INT_{H(e)}$.

The third formula is obvious.

Remark 1.1 *We can define the following functions*

$$size(H) = |V_H| + |E_H|,\ intsize(H) = |INT_H| + |E_H|,$$

with the calculations formulae

$$\begin{aligned} size_H &= |V_R| + \sum_{e \in E_R} intsize(H(e)), \\ intsize(H) &= |INT_R| + \sum_{e \in E_R} intsize(H(e)). \end{aligned}$$

Also, we can define the density function: $dens(H) = |E_H|/|V_H|$ if $|V_H| > 0$ given by the formula

$$dens(H) = \frac{\sum_{e \in E_R} |E_{H(e)}|}{|V_R| + \sum_{e \in E_R} |INT_{H(e)}|}.$$

The proof of these formulae is immediate, if we consider the previous proposition.

1.2. PATHS IN GRAPHS, PATHS OF MINIMUM OR MAXIMUM LENGTH

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be a hypergraph over the set of labels C, arbitrary but fixed. A path that unite v_0 and v_n, two nodes from V_H, represents a sequence $p = v_0 e_1 v_1 e_2 \ldots e_n v_n$ of nodes and edges so that for every $i = 1, n$, v_i and v_{i-1} are incident nodes of the edge e_i. If in a path every node occurs only once we say that the path is simple. The length of a path, denoted by $length(p)$, represents the number of edges that this one contains.

Let 2-graf H. Denote by $PATH_H$ the set of simple paths uniting $begin_H$ and end_H, and by $numpath(H)$ the cardinal of $PATH_H$. Denote by $minpath(H)$ and $maxpath(H)$ the path of minimum or maximum length.

Proposition 1.2 *Let H be a graph-string and the derivation $A^\bullet \Rightarrow R \Rightarrow^* H$. We have the following relations*

$$
\begin{aligned}
numpath(H) &= \sum_{p\in PATH_R} \prod_{e\in p} numpath(H(e)),\\
minpath(H) &= \min_{p\in PATH_R} \sum_{e\in p} minpath(H(e)),\\
maxpath(H) &= \max_{p\in PATH_R} \sum_{e\in p} maxpath(H(e)).
\end{aligned}
$$

Proof: Let p be an arbitrary path in $PATH_R$. Assume that the path passes through the nodes $v_0, \ldots, v_n$. Because $A \in N$ and $A^\bullet$ is a 2-graph, then the hypergraph R will have the edges $e_1, \ldots, e_n$ and e_1 will be the edge that binds nodes $v_0 = begin_H$ and v_1, …, e_i will bind v_{i-1} and v_i, …, e_n will connect v_{n-1} and $v_n = end_H$. Every edge of these will be replaced by hypergraphs. Therefore, for every $e_i \in R$ we have $lab_R(e_i)^\bullet \Rightarrow^* H(e_i)$. Therefore, $numpath(H_p) = \prod_{e\in p} numpath(H(e))$. Because the process is repeating for every path from $PATH_R$ the first relation is demonstrated. The second and the third can be shown considering the global minimum and maximum from the local minimum and maximum on every possible path in R.

1.3. CYCLES IN GRAPHS, NUMBER OF SIMPLE CYCLES, MINIMUM LENGTH AND MAXIMUM LENGTH OF SIMPLE CYCLES

Let $H = (V_H, E_H, att_H, lab_H, ext_H)$ be an hypergraph over a set of labels C, arbitrary but fixed. If in a path from H we have $v_0 = v_n$ then we say that the path is a cycle. A cycle is simple if the nodes are distinct and $n \geq 3$. The length of a cyle is defined as for the paths.

For a 2-graph, H, we say that $CYCLE_H$ is the set of simple cycles, $numcycle(H)$ the cardinal of this set, $mincycle(H)$ and $maxcycle(H)$ the minimum length and the maximum length of these cycles, respectively.

Proposition 1.3 *In a string-graph we have*

$$
\begin{aligned}
&numcycle(H) = \sum_{c\in CYCLE_R} \prod_{e\in c} numpath(H(e)) + \sum_{e\in E_R} numcycle(H(e)),\\
&mincycle(H) = \min\{\min_{c\in CYCLE_R} \sum_{e\in c} minpath(H(e)), \min_{e\in E_R} mincycle(H(e))\},\\
&maxcycle(H) = \max\{\max_{c\in CYCLE_R} \sum_{e\in c} maxpath(H(e)),\\
&\max_{e\in E_R} maxcycle(H(e))\}.
\end{aligned}
$$

Proof: Like in the previous demonstration we consider a cycle c from $CYCLE(R)$. This cycle is $v_0e_1v_1e_2v_2\ldots e_nv_ne_{n+1}v_0$. For all $e_i, i = \overline{1, n+1}$ we have $lab(e_i)^\bullet \Rightarrow^* H(e_i)$. Any path in $H(e_i)$ that begins from v_{i-1} and ends at v_i forms a cycle with edges from c after the elimination of e_i. At these cycles we add the proper cycles (formed only with edges from $H(e_i)$). The algorithm is repeating for every edge from the cycles $c \in CYCLE(R)$ and after that for every cycle from R. The demonstration for the second and third relations is similar.

2. COMPATIBLE FUNCTIONS

We will define the compatible function like a generalization of the functions defined in the previous sections. Therefore, the f_0 function, defined on the hypergraph set and taking values in natural number set is compatible with the derivation process in a hyperedge replacement grammar, if for every hypergraph H and every derivation of it the value $f_0(H)$ can be calculated from the values assumed by the function on subgraphs $H(e)$ that compounds it and are results of the derivation process.

Definition 2.1 *[9] (Compatible function)*

1 *Let $\mathcal{HRG}$ be the class of hyperedges replacement grammars over the set of labels C, arbitrary but fixed, I a finit set of indices, VAL a set of values, $f : \mathcal{H}_C \times I \to VAL$ a function and f' a function defined on triples $(R, assign, i)$, where $R \in \mathcal{H}_C$, $assign : E_R \times I \to VAL$, $i \in I$, with values in VAL. We say then that f is $(\mathcal{HRG}, f')$-* compatible *if, for every HRG=(N,T,P,S)$\in \mathcal{HRG}$, every derivation $A^\bullet \Rightarrow R \Rightarrow^* H$, $A \in N$ and $H \in \mathcal{H}_T$, and for every $i \in I$ we have*

$$f(H, i) = f'(R, f(H(-), -), i)$$

2 *A function $f_0 : \mathcal{H}_C \to VAL$ is called $\mathcal{HRG}$-*compatible *if there exist the functions f, f' and an index i_0 such that $f_0 = f(-, i_0)$ and f is $(\mathcal{HRG}, f')$-compatible.*

We recall that $f(H(-), -)$ represents a notation for the function defined by $f(H(-), -)(e, j) = f(H(e), j)$.

Theorem 2.1 *If in the previous definition $I = \{all, int\}$, $VAL = \mathbf{N}$ and*

$$\begin{aligned}
&f(H, all) = |V_H|; \textit{total number of nodes},\\
&f(H, int) = |INT_H|; \textit{internal number of nodes},\\
&f'(R, assign, all) = |V_R| + \sum_{e \in E_R} assign(e, int),\\
&f'(R, assign, int) = |INT_R| + \sum_{e \in E_R} assign(e, int).
\end{aligned}$$

Then f is $(\mathcal{HRG}, f')$-compatible and if $f_0 = f(-, all)$, then the function number of nodes is $\mathcal{HRG}$-compatible.

Proof: We have $f(H, all) = |V_H| = |V_R| + \sum_{e \in E_R} |INT_{H(e)}| = |V_R| + \sum_{e \in E_R} f(H(e), int) = f'(R, f(H(-), -), all)$. Also, we choose $f_0 = f(-, all)$.

Corollary 2.1 *In a similar way we can assert that the function number of paths, minimum length and maximum length, number of cycles, minimum length and maximum length of simple cycles respectively is $\mathcal{ERG}$-compatible, where $\mathcal{ERG}$ is the 2-graph class.*

References

[1] Dumitrescu S., Pop B., *Hyperedge replacement grammars*, Buletin Ştiinţific, Universitatea din Piteşti, **8** (2002), 69-75.

[2] Dumitrescu S., *Collage grammars*, A Cincea Conferinţă Anuală a SSMR - Braşov, **1** (2001), 104-112.

[3] Dumitrescu S., Pop B., *Gramatici de cod-lanţ*, Buletin Ştiinţific, Universitatea Creştină "Dimitrie Cantemir", Braşov, **3** (2002), 137-139.

[4] Dumitrescu S., *Aspects of context freeness for hyperedge replacement grammars*, Foundations of Computing and Decision Sciences, Poznan, **30**, 2(2005), 91-101.

[5] Habel A., *Hyperedge replacement: grammars and languages*, Springer, Berlin, 1992.

[6] Habel A., Kreowski H. J., Lautemann C., *A Comparison of compatible, finite and inductive graph properties*, Theoretical Computer Science, **110** (1993), 145-168.

[7] Habel A., Kreowski H. J., Vogler W., *Metatheorems for decision problems on hyperedge replacement graph languages*, Acta Informatica, **26** (1989), 657-677.

[8] Salomaa A., *Formal languages*, Academic, New York, 1973.

[9] Thatcher J., *Generalized sequential machine maps*, Journal of Computer and System Sciences, **4** (1970), 339-367.

ROMAI J., **2**, 2(2006),
55–62

A MATLAB PROGRAM BASED FINITE ELEMENT METHOD FOR COMPUTING AND SIMULATION LORENTZ POWER OF MAGNETIC FIELD

Simona Dzitac, Tiberiu Vesselenyi, Ioan Dzitac
Univ. of Oradea
sdzitac@rdslink.ro, tvesselenyi@yahoo.co.uk, idzitac@univagora.ro

Abstract The computation and simulation of the Lorentz force of the magnetic field between two linear conductors of circular section are presented. The model has been constructed in MATLAB, using the finite element method.

Keywords: Lorentz power, finite element method, magnetic field.
2000 MSC: 78M10, 78A55

1. THE REALIZATION OF THE GEOMETRIC MODEL

By realizing the geometric model we define the dimensions and the shape of the two conductors and their position in space (fig. 1).

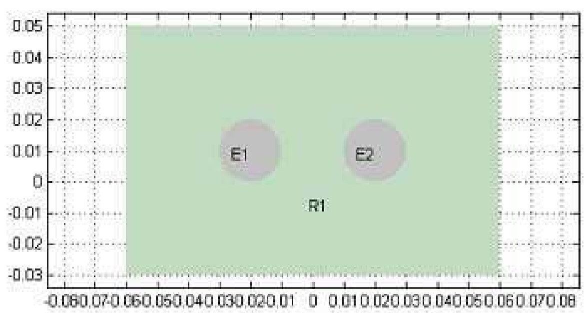


Fig. 1. The geometric model definition.

2. THE REALIZATION OF THE FINITE ELEMENTS NETWORK

After the geometric elements have been defined, the program automatically splits the studied intervals into finite elements (fig. 2). In this case, we use triangular elements. The parameters of the electromagnetic field are computed in the nodes of the network's elements. To increase the precision of the operations, usually one splits the interval in several finite intervals, realizing a smoother splitting as in fig. 2.

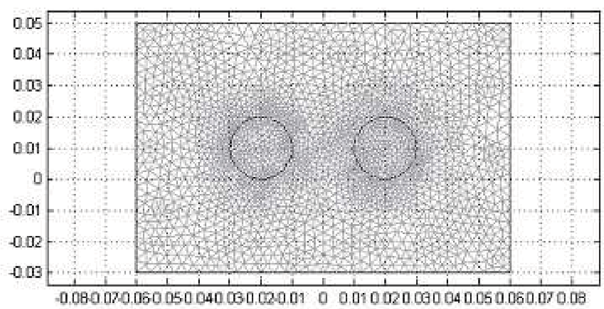


Fig. 2. Finite elements network with 2131 nodes.

3. MODEL PARAMETERS AND THE MATLAB PROGRAM

The model was realized for alternative (AC) and continuous current (CC); in both cases we realized two models with a 40mm distance between the conductors (AC1, CC1) and 80mm (CA2, CC2). For CC we considered o short-circuit current of I = 50000A. For AC the intensity was I = 960A. The conductors are supposed to be made out of copper and the medium between the conductors is air.

Part of the MATLAB program that realizes our simulation reads

```
Program for computing the Lorentz force.
r = 0.01;
Fc=postint(fem,'-2*pi*0.01*0.5*real(Jiz.*conj(Bx))',...  'dl',2);
wt=linspace(0,2*pi,40);
Fv=0.5*real(postint(fem,'-2*pi*0.01*Jiz*Bx','dl',2)*...  exp(-2*j*wt));
figure
plot(wt,Fc+Fv,'k')
xlabel('distanta x '),ylabel('forta Lorentz (Fx)')
```

4. RESULTS OF SIMULATION

4.1. CASE AC1

We can display the results either as electromagnetic field diagram with color codes (fig. 3) or as vectorial diagram (fig. 4).

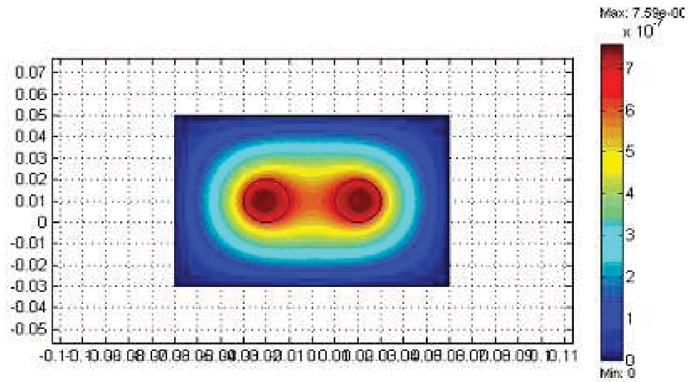


Fig. 3. Electromagnetic field diagram with color codes.

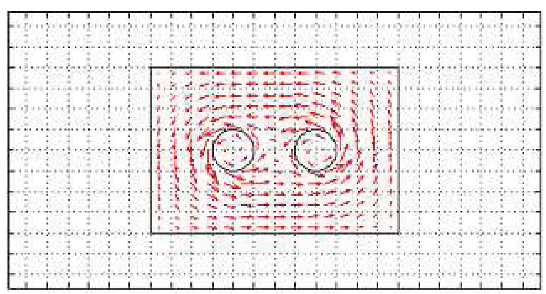


Fig. 4. Diagram for the vectorial representation of the electromagnetic field.

In order to compute the Lorentz force we need an additional program (Annex "Program for computing the Lorentz force"). This program computes the Lorentz force and displays the diagram presented in fig. 5.

After computing the Lorentz force we model the conductor under the influence of Lorentz forces. We consider two distinct cases:
- cantilever conductor at one end for a charge value that corresponds to a 40mm distance between conductors (fig. 6) and 80mm (fig. 7);
- cantilever conductor at two ends for a charge value that corresponds to a 40mm distance between conductors (fig. 8) and 80mm (fig. 9).

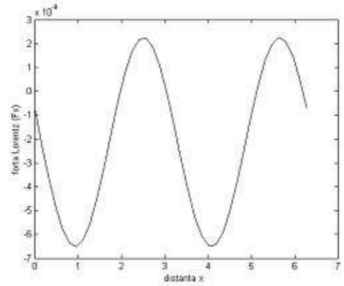


Fig. 5. Lorentz force diagram for AC1.

The modeling was made assuming that the Lorentz force acts as an impulse (short-circuit state) with amplitude computed previously. The model simulates the response in frequency of the conductor at this impulse.

The maximum values for the displacement are:

- cantilever conductor at one end, distance 40mm: maximum displacement = -7,5*10-13;
- cantilever conductor at one end, distance 80mm: maximum displacement = -1,5*10-15;

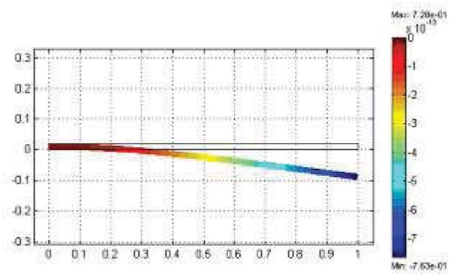


Fig. 6. Cantilever conductor at one end, distance 40 mm.

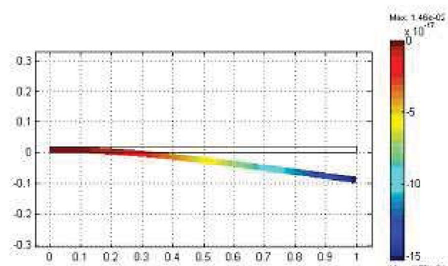


Fig. 7. Cantilever conductor at one end, distance 80 mm.

- cantilever conductor at two ends, distance 40mm: maximum displacement = -1,5*10-14;
- cantilever conductor at two ends, distance 80mm: maximum displacement = -3,2*10-18;

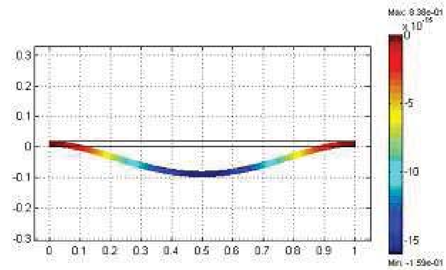


Fig. 8. Cantilever conductor at two ends, distance 40 mm.

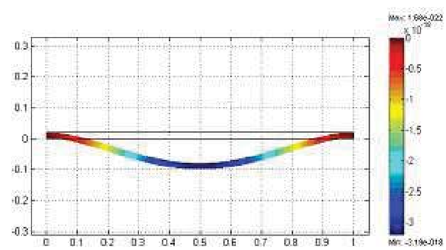


Fig. 9. Cantilever conductor at two ends, distance 80 mm.

4.2. RESULTS OF THE SIMULATION FOR AC2

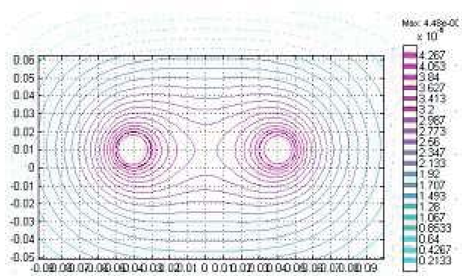


Fig. 10. Field lines diagram.

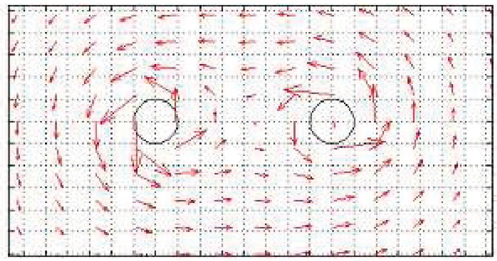


Fig. 11. Vectorial diagram.

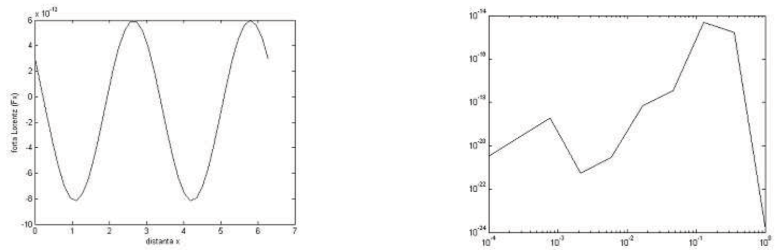


Fig. 12. Lorentz force diagram.

4.3. RESULTS OF THE SIMULATION FOR CC1 AND CC2

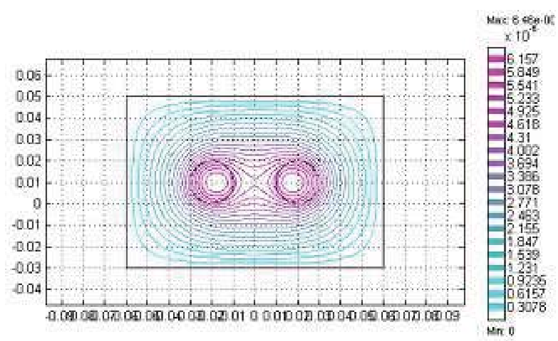


Fig. 13. Field lines diagram.

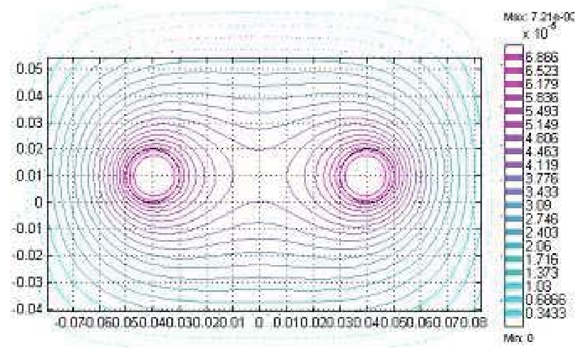


Fig. 14. Field lines diagram.

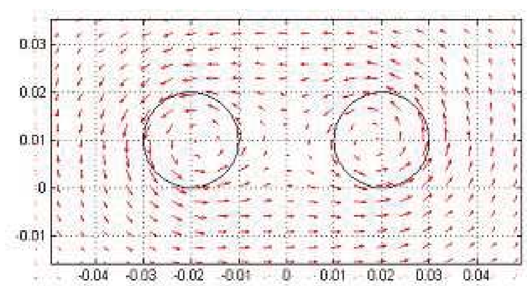


Fig. 15. Vectorial diagram.

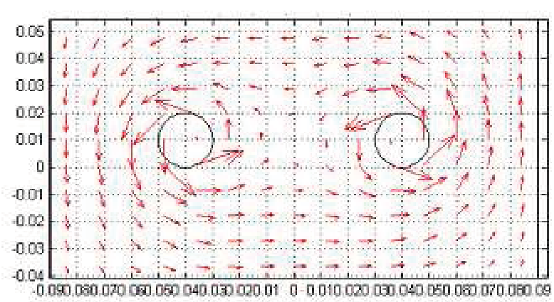


Fig. 16. Vectorial diagram.

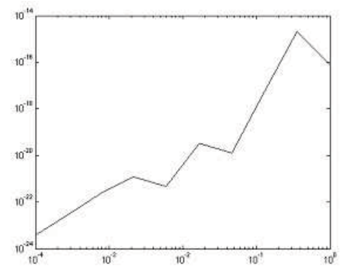


Fig. 17. Lorentz force diagram.

5. CONCLUSIONS

From the results of the simulations we conclude the following:
- the smaller the distance between the conductors, the bigger the forces;
- it is recommended to lean the bar system at two ends, because a simply cantilever bars system is more solicited;
- the distances between the insulators that sustain the bars have to be chosen so that the stress and deformation be miminum;
- under the action of electro-dynamics forces the bars may bend, so that they may diminish the insulation between the phases;
- the leaned bar system rezemat presents much bigger deformations than the cantilever bar system;
- the installations have to resist to the electro-dynamic forces produced by the shock current and it is not necessary to withstand the short-circuit currents permanently.

References

[1] Bathe, K. J., *Finite element procedures in engineering analysis*, Prentice Hall, Englewood Cliffs, NJ, 1982.

[2] Vasilievici, A., Andea, P., *Apparata and electric equipments*, Orizonturi Universitare, Timisoara, 2004.(Romanian)

ROMAI J., **2**, 2(2006),
63–76

DYNAMICAL APPROACH IN BIOMATHEMATICS

Adelina Georgescu, Lidia Palese, Grazia Raguso

Dept. of Mathematics and Computer Sciences, Univ. of Pitesti, Dept. of Mathematics, Univ. of Bari, Italy

palese@dm.uniba.it

Abstract A short survey of topics in biomathematics is followed by a presentation of the mathematical models from biodynamics and medicine studied by the group of nonlinear dynamics and bifurcation of one of the authors (AG). Then a systematic approach of treating these models is formulated. Open problems and topics which can be investigated by this approach are presented. The state of the art in the joint studies on hydrodynamic stability carried out by the first two authors is sketched and possible connections with the field of interest of the third author are revealed.

Keywords: biodynamics, hydrodynamic stability, bifurcation.
2000 MSC: 92C45, 92D25, 92C37, 58K50, 58K60,76E06, 76E25, 76E30, 34CXX, 37Gxx, 37B42, 35B41, 35B42.

1. TOPICS IN BIOMATHEMATICS

Here we discuss the mathematical models associated with biological quantities and mathematical models governing their transformations. Then we briefly present the main topics in biomathematics.

Biomathematics is a branch of applied mathematics dealing with mathematical investigation of geometric, analytical, algebraic, statistical, probabilistic etc aspects involved into the mechanical, physical, chemical phenomena or equilibria occurring in biological systems. Indeed, in order to study biological systems from mathematical point of view, with a **biological characteristic** (object), we associate a **mathematical object**. If this characteristic undergoes changes in time and/or space, this means that in the biological system a motion (change, growth, transformation, phenomenon, process, dynamics) occurs. *It is unanimously assumed that the corresponding **biological phenomenon** is a particular physico-chimical phenomenon.* As a consequence, this motion must obey the general laws of physics and/or chemistry and some specific laws and "material relationships". From this perspective, biology can be viewed as the thermodynamics of biological systems. Here by thermodynamics (also referred to as the third thermodynamics) it is understood the most general science describing the motion and including all basic disciplines and all interdisciplinary area, e.g. mechanics, physics, electromagnetism, M.H.D., classical thermodynamics, chemistry. In the resulting formalism there is no place for soul or other properties specific to living systems only: every biological concept is associated with a concept from particular subdomains of thermodynamics. Naturally, every mathematical model governing a biological modification belongs to such a subdomain and in it only concepts specific to that subdomain occur. The biological aspect appears mainly during the derivation of the model and in the biological interpretation of its solution.

Once formalized mathematically, the biological model becomes a topic of mathematics and it must be studied by specific tools. However, usually, simplifying ideas of biological nature must be incorporated if the resulted model is supposed to be solved at least numerically. For example, a bone may be conceived as a particular solid and, correspondingly, its displacement and/or deformation is governed by a model from mechanics of rigid bodies, or elasticity, or, more general, thermodynamics of elastica, in dependence on the practical needs.

Another example: the blood may be conceived as a particular non-Newtonian fluid the motion of which is governed by some model of fluid mechanics, or, more general, thermodynamics of fluids.

Therefore, every mathematical model of a biological quantity and every mathematical model governing its changes are borrowed from various branches of thermodynamics of inanimate bodies (matter and/or field).

In addition, as already mentioned, with a biological quantity or biological change several mathematical models can be associated, according to the complexity and needs one has in view. For example, for some purposes, it is sufficient to assimilate a bone with a rigid body, in other circumstances a finer model of elastic medium is necessary. Similarly, the blood, urine, saliva, tears, bile, sperm, lymph, cerebrospinal liquid, prespiration, mammary secretion, amniotic liquid can be assimilated with a Newtonian fluid, or a mixture of fluids in dependence on how accurate must be described its motion.

There is a key point in choosing one mathematical model or other for a quantity characterizing a biological, chemical, physical, mechanical, economical a.s.o. system and its change (evolution). This is the energy, and, correspondingly, the energy equation, establishing a balance between the rate of change of the total internal energy of the system and the energy, power, radiation etc. For instance, for pure mechanical systems, total internal energy consists of the kinetic and potential energies and its rate of change is balanced by the power. If the heat is important in the functioning of that system, its total internal energy must contain a new energy related to heat. As a result, the rate of change of this total internal energy is balanced by the mechanical power supplemented by heat and radiation. If the system is an electrically conductor, then the definition of the total internal energy must contain an electrical component too. Correspondingly, its rate of change is balanced by the former terms supplemented by an electrical power. Any time that the energy equation was not true, in the sense that the balance did not hold, it meant that the total internal energy was not defined suitably. It must contain a new part and, correspondingly, a new "power" must occur in the energy equation.

Some biochemists assert and some medical evidence plead for a specific energy for living bodies, but no mathematical model exist for it. This is why we frame our study in the thermodynamics.

In thermodynamics the geometric model (of a material system (substance)) is a continuum. There is an alternative geometric model, namely a totally discrete (disconnected) set. Correspondingly, there is another science dealing with the phenomena occurring in these systems: statistical physics (chemistry etc). There exists a connection between thermodynamics and statistical physics. In this paper we limit ourselves to thermodynamics.

There is a huge number of topics treated in mathematical biology. They are grouped in: mathematical biology in general; physiological, cellular and medical topics; genetics and population dynamics.

The second group treats: biophysics, biomechanics (including biomechanical solid mechanics), developmental biology, pattern formation, cell movement (chemotaxis etc.), neural biology, physiology (general), physiological flows (included in biological fluid dynamics which, in addition, treats biopropulsion in water and in air and other topics), cell biology, biochemistry, molecular biology, kinetics in biochemical problems (pharmacokinetics, enzime kinetics etc., related to chemical kinetics, reaction effects in flows, chemically reacting flows), medical applications, biomedical imaging and signal processing (related to Radon transformation, integral transforms, signal theory), medical epidemiology, plant biology.

The third group treats: genetics (related to genetic algebras), problems related to evolution, protein sequences, DNA sequences, population dynamics, epidemiology, ecology, animal behaviour.

In our opinion, due to the huge diversity of the biosphere, this enumeration reveals only a small part of the possible and necessary topics, namely those which at the time being are of interest for applied mathematicians, physicists and chemists. For others the association of the biological quantities and biological changes with mathematical models is not yet available.

An important conclusion for someone wishing to deal with mathematical biology is, first, to learn about the topics treated by thermodynamics and, second, to fix her/his biological objectives requiring a specific branches of thermodynamics. The first two authors of the paper are fluid dynamicists, therefore they fulfill the first requirement. The third author is a pure mathematician (geometer). Together, we attempt to study dynamics and bifurcation in mathematical models describing various aspects of the cancer.

Finally, we quote the **main subjects** dealt with in a few treatises of mathematics, some of them general, some others concerning only a narrower topic: continuous or discrete population models for single species, continuous models for interacting populations, discrete growth models for interacting populations, reaction kinetics, biological oscillators and switches, Belousov-Zhabotinskii reaction, perturbed and coupled oscillators and black holes, reaction-diffusion, chemotaxis and nonlocal mechanisms, oscillator generated wave phenomena, biological waves (single species models or multi-species diffusion models), travelling waves in reaction-diffusion systems with weak diffusion (analytical technique), spatial pattern formation with reaction/population interaction diffusion mechanisms, animal coat patterns and other practical applications of reaction-diffusion mechanisms, neural models of pattern formation, mechanical models for generating pattern and form in development (including morphogenesis, formation of microvili, cartilage formation), epidemic models and the dynamics of infections diseases, geographic spread of epidemics [1].

To **narrower topics** separate textbooks and proceedings are dedicated. An example is biological and biochemical oscillators [2] including : oscillatory behaviour, excitability, and propagation phenomena on membranes and membrane-like interfaces, stability properties of metabolic pathways with feedback interactions, damping of mitochondrial volume oscillations by propanolol and related compounds, glycolytic oscillations, oscillations in tissues, oscillations in growing cell populations, circadian oscillations.

The large diversity of biological phenomena was described in an enormous number of papers. We estimate this number of order of milions if we take into account that only to cancer more than 1.600.000 works are devoted. Usually, the books are published in series like Lecture Notes in Biomathematics [3] or Biomathematics [4]. A lot of books or chapters treating biological phenomena can be found in series in life sciences [5], or synergetics [6], or chemistry, or physics etc., or even applied mathematics [7].

This is due to the fact that, as we show in Section 2, the equations governing various types of phenomena (mechanical, physical, chemical, economical etc.) are derived from a common trunk and, when approximated, these equations are the same for several distinct phenomena. Indeed, the first approximations contain the same expansion functions. The difference occurs in the coefficients. In particular, frequently, the governing models in biology are presented together with models in economics, e.g. the Goodwin model, the Gompertz model. The Lotka-Volterra models are common to some phenomena in biology and chemistry; the Hodgkin-Huxley model is used in physiology and electric circuits. They are among the simplest and are derived as a result of severe approximations. Of course, the more simplifying hypotheses are assumed, the simpler and applicable to a more general domain are the resulting approximate equations. The largest number of papers on biological phenomena are short papers and they are spread through the existing journals. Only a few of these journals are devoted to biomathematics, e.g. [9], [10]. Most of them are devoted to other domains (especially physics, chemistry and applied mathematics) but contain applications to biology too.

The level and type of mathematics involved in these papers range from the heuristical and elementary mathematics till the most advanced achievements of (the conglomerate called) global analysis. We are interested in those mathematical treatments of models of biodynamics involving (ordinary or partial) differential equations and using techniques of dynamical systems theory and (static, imperfect and dynamic) bifurcation theory.

This paper was conceived as an address to scientists of various orientations willing to form a group intended to carry out analytically, numerically and experimentally some applications of biomathematics in medicine. This group must contain at least applied mathematicians and physicists.

2. MATHEMATICAL MODELS IN BIODYNAMICS

First, one of the most general models of thermodynamics is described. Then the particular models treated by the group of the first author are presented.

One among the most general models governing equilibria and motion of material systems (substances and fields), geometrically modelled as (mathematical) continua $\Omega \subset \mathbf{R}^3$ consists of some differential equations [11], referred to as global *equations of motion*

$$\frac{d}{dt}G(t,\Omega') = \Phi^G + p^G + s^G, \quad t \in \mathbf{R}, \quad \forall \Omega' \subset \Omega \tag{1}$$

constitutive equations (of material), which can be algebraic, differential, integral, integro-differential or, more general, functional

$$\Phi^G = \Phi^G(G), \tag{2}$$

constraints imposed by physical (generically speaking) reasons, e.g.

$$C(G) < 1 \tag{3}$$

initial conditions

$$G(0,\Omega') = G_0(\Omega'), \quad \forall \Omega' \subset \Omega$$

and *boundary conditions*

$$G_{|\partial\Omega} = G_W, \tag{5}$$

where $G(t,\Omega')$ is a global quantity, e.g. the mass, and it is a function of the time t and the subdomain Ω' of the domain Ω (occupied by the physical system (body)). The equation of motion (1) is a balance equation which shows that the rate of change of the global quantity G is balanced by the sum of the flux Φ^G of the quantity G through the frontier $\partial\Omega'$ of Ω', the production p^G of G and the supply s^G of G due to external influences on the part of the system from Ω'. If $p^G = 0$, then the balance equation is said to be a *conservative equation*. The terms in (1) are scalars, e.g. the mass, energy, entropy, or vectors, e.g. momentum, momentum of momentum. Equation (1) models a physical (biological) law, e.g. the second law of Newton, the mass balance or conservation, the total internal energy conservation. The components of (1) are present in all phenomena; of course, some of them can vanish for particular cases. Therefore the equation (1) is the common trunk leading to similar mathematical solutions and, correspondingly, to similar (in some nonevident always) aspects of the modeled phenomenon.

The superscript G shows that for each G , specific Φ^G, p^G and s^G correspond. For instance, if G is the mass, Φ^G, p^G and s^G are the mass flux, mass source and supply of mass respectively. In the case of a mechanical system, if G is the angular momentum, Φ^G, p^G and s^G are the stress terms, zero, and the body force. If G is the total internal energy and the system is mechanical, then $G = E_c + E_p$, where E_c and E_p are the kinetic and potential energy, respectively and $\Phi^G = p^G = 0$, while s^G is the power of the forces acting on the system. If the system is physical and is heat conducting, then $G = E_c + E_p + E_h$, where E_h is part of energy due to heat and $\Phi^G$is the heat flux, $p^G = 0$ and s^G contains supplementary terms due to radiation. If the system is electromagnetic, then, apart from these quantities, influences of the electrical and magnetic fields must be considered. We already remarked that if the system is biological, this system must be considered as a particular mechanical physical, chemical etc. system.

In additional assumptions on the regularity of the functions occurring in equations (1), these equations become partial differential equations. For further simplified assumptions, equations (1) can become ordinary differential equations. For instance, in many cases the mass can be written as $m(t, \Omega') = \int_{\Omega'} \rho(t, \mathbf{x})d\mathbf{x}$, where ρ is the mass density, the momentum can have the expression $\mathbf{M}(t, \Omega') = \int_{\Omega'} \rho(t, \mathbf{x})\mathbf{u}(t, \mathbf{x})d\mathbf{x}$, the total internal energy reads as $E(t, \Omega') = \int_{\Omega'} \rho(t, \mathbf{x})\frac{\mathbf{u}^2}{2}(t, \mathbf{x}) + e(t, \mathbf{x})d\mathbf{x}$, where $\mathbf{u}(t, \mathbf{x})$ is the velocity at the time t and point $\mathbf{x}$ and e is the internal energy. The quantities ρ, $\mathbf{u}$ and e are fields and they are called local quantities. In the adopted formalism [11], the internal energy is defined as the difference from the total internal energy and the kinetic energy (it is in e that possible types of energies, other than those from the inanimate world, would appear. But in this case, other global quantities, specific only to living systems, must be introduced). In regularity conditions for ρ, $\mathbf{u}$ and e the global conservation equations (1) for the mass momentum and energy in a fluid system become local equations, namely partial differential equations, valid for every $t \in \mathbf{R}$, $\mathbf{x} \in \Omega$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{6}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u} = \frac{1}{\rho} \nabla \cdot \mathbf{T} + \frac{1}{\rho} \mathbf{F}, \tag{7}$$

$$\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e = \frac{1}{\rho} \nabla \cdot \mathbf{q} + \mathbf{u} \cdot \frac{1}{\rho} \mathbf{F}, \tag{8}$$

where in (7) and (8) we took into account (6), and (6) and (7) respectively, $\mathbf{T}$ is the stress tensor, $\mathbf{F}$ is the body force and $\mathbf{q}$ is the heat flux. If the fields ρ, $\mathbf{u}$, e, $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{q}$ are homogeneous, i.e. they do not depend on $\mathbf{x}$, then (6), (7) and (8) become the ordinary differential equations $\frac{d}{dt}\rho = 0$, $\frac{d}{dt}\mathbf{u} = \mathbf{F}$, $\frac{d}{dt}e = \mathbf{u} \cdot \mathbf{F}$, i.e. the equations characteristic to the rigid motion (the constant ρ was included in $\mathbf{F}$). The first equation shows that the fluid is incompressible, the second equation is the Newton equation where $\mathbf{F}$ is the resultant of the forces acting upon the system and the third equation is the energy equation from mechanics of rigid bodies. These equations degenerate into algebraic equations characteristic to equilibria when $\mathbf{u}$ and e do not depend on time either.

More general, (1) become ordinary differential equations if the global quantities are homogeneous fields. This is the case of most equations in biomathematics. With them finite-dimensional dynamical systems are associated and their study, analytical as well as numerical, is easier than in the case of partial differential equations.

The equations (6), (7) and (8) (and, in general, (1)) are formally the same for any type of fluids (systems),which can have the property of having the momentum flux of the form $\mathbf{T} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outer normal to Ω'. The difference between various fluids, e.g. the blood, urine or amniotic fluid is mathematically specified by the constitutive equations connecting the fluxes and the global quantities, e.g. excitation and response. For instance, the stress in the blood depend on $\mathbf{D} = [\nabla \mathbf{u} + \nabla \mathbf{u}^T]/2$, the velocity of deformation tensor, as well as some of its derivatives, while the urine can be suitably modeled by $\mathbf{T} = -p\mathbf{I}$, where $\mathbf{I}$ is the unit tensor and p is the static pressure. In general, in (2) $\Phi^G(G)$ are some operators, e.g. differential, integral or functional of the quantities G. These operators cannot be arbitrary; they must obey at most five principles, e.g. objectivity, isotropy. As a result the form of (2) simplifies and (2) become algebraic relationships relating the fluxes (e.g. $\mathbf{T}$, $\mathbf{q}$, e) to $\mathbf{D}$ and other basic quantities and some of their derivatives.

In the constitutive equations and in the constitutive expressions for the fluxes (called the constitutive functions) the coefficients are functions of the temperature T and ρ or T and p taken at equilibrium. Their form is referred to as the *equations of state*, e.g. for energy, entropy. The classical thermodynamic equation of state connects ρ, T, p, e.g. the Gay-Lussac equation.

Consequently, the material is characterized mathematically by constitutive equations and state equations. In the inanimate world the occurring coefficients are measured experimentally or are deduced from the statistical associated models. In living bodies it is only rarely possible to measure these coefficients. For instance, in order to characterize what kind of material is a tumor in its formation and development, when the quoted constitutive equations depend on the time too, we must know how look these equations and so, which are the numerical values of their coefficients.

The constraints (3) are related to the type of the physical, biological etc. quantity. For instance, the concentration cannot be greater than 1, the density is always positive.

The initial conditions (4) connect the phenomenon at the actual time to the past while the boundary conditions liberate the system existing in Ω from the exterior. In a more complex modeling the coefficients in the constitutive equations depend on the time derivative of u and other basic quantities. In this case we say that the *material has a memory.* We think that, in most of the circumstances, the living organs in a body or living beings are materials with memory.

We exemplify these by a few Cauchy problems for some systems of first order ordinary differential equations describing biological phenomena, which were studied by the first author group.

The FitzHugh-Nagumo (F-N) model, the mostly investigated by us, is the Cauchy problem [13], [12]

$$\dot{x}_1 = c(x_1+x_2-x_1^3/3-Acos\omega t), \quad c\dot{x}_2 = -(x_1+bx_2-a), \quad x_1(0) = x_1^o, x_2(0) = x_2^0 \tag{9}$$

where $\cdot \equiv d/dt$, t, the time, is the independent variable, $x, y : \mathbf{R} \to \mathbf{R}$, $x = x(t)$, $y = y(t)$ are the unknown functions, a, b, c, A and ω are real parameters. The problem (9) is also called the Bonhoeffer-Van der Pol (BVP) model and describes the electrical behaviour along a neuronal membrane subject to the action of a periodic external stimulus. It also governs the initiation of the heart beats and follows from the reaction-diffusion equations governing the wave propagation in excitable media

$$\epsilon \frac{d}{dt} x_1 = \epsilon^2 D_1 \triangle x_1 + f(x_1, x_2), \quad \frac{d}{dt} x_2 = \epsilon^2 D_2 \triangle x_2 + g(x_1, x_2) \tag{10}$$

where D_1 and D_2 are the transport coefficients, f and g are reaction terms and ϵ is some parameter. In the case of the cardiac muscle x_1 is the electrical potential. FitzHugh derived (9) from the four-dimensional experimental model of Hodgkin and Huxley and put it as the basis of the axon physiology. The problem (9) generalizes the electrical Van der Pol oscillator. If x_1 and x_2 do not depend on space variables and $f(x_1, x_2) = x_1 + x_2 - x_1^3/3$, and $g(x_1, x_2) = -(x_1 + bx_2 - a)$, the system (10) reduces to (9), for the case of $A = \omega = 0$, i.e. when the forcing is absent.

Mainly we dealt with the case without forcing [12], [13], which, for a, b and c fixed, generates a two-dimensional dynamical system. The static and dynamic bifurcation diagrams were determined by analytic [12], [13] and numerical [14] methods. Two asymptotic dynamics as $\mu = c^{-2} \to 0$ and (a, b) is very close to the curve of the Hopf bifurcation values were studied analytically and numerically [15]. The attention was focused on the relaxation oscillations of the heart and oscillations in two and three times, related to concave limit circles (French canards). When the forcing is present, the dynamical F-N system is three-dimensional. In this case, for specific situations, the chaotic dynamics was studied by reducing the continuous dynamics to a discrete one [13] and by treating numerically the resulting discrete dynamical system [16].

In [17] the static, dynamic and perturbed bifurcation was studied for the Gray-Scott model

$$\dot{u} = a(1 - u) - uv^2 - bu, \quad \dot{v} = a(c - v) + uv^2 + bu - dv, \tag{11}$$

where the unknown functions u and v are the concentrations of the two reactants, while the parameters a, b, c and d are related to the sedimentation time, noncatalyzed conversion, influence of the catalyzer and rate of decomposition of the catalyzer respectively. The Cauchy problem for (11) governs a chemical reaction in the presence of the noncatalyzed enzymes.

Two models in oncology are dealt with in [18]. The first is the lymphocytes-tumor model

$$\dot{x} = \alpha x - xy, \quad \dot{y} = xy - (y/\alpha) - kx + \sigma, \quad x(0) = x_0, y(0) = y_0, \tag{12}$$

where the unknown functions x and y represent the number of malign cells and the number of lymphocytes, respectively and the real parameters α, k and σ are related to the coefficients of the rates of change of the cells (action of the immunitary system on the malign cells), the natural death of the malign cells and the tumor surface interacting with lymphocytes, diffusion of lymphocytes. In the case of a treatment, the number of unknown functions and parameters increases. The second model is a immuno-tumoral model

$$\dot{x} = -x - x^2 + xy, \quad \dot{y} = -(e + b)x + ly - ex^2 + (l + c)xy - b, \quad x(0) = x_0, y(0) = y_0, \tag{13}$$

where the state functions x and y represent the free lymphocytes situated on the tumor surface and the total number of tumoral cells. The parameters have meanings similar to those in (12). For both these models the static and dynamic bifurcation diagrams were determined analytically and the results were represented graphically. It was found that the large number of parameters leads to notable difficulties of the theoretical study and the graphs must be based on the perturbed bifurcation theory. These two features are common to the majority of mathematical models of cancer, which explains the existence of a small number of papers devoted to rigorous mathematical treatment in the field, in spite of the huge quantity of studies devoted to the topic. Keeping only a few parameters means a poor model of the cancer, its evolution and its treatment.

A form specific to biochemistry of a balance equation is the *mass action law*: the velocity of reaction is a sum of two terms. The first is proportional to the product of the concentrations of the reactants while the second is proportional to the product P of the biochemical reaction. More exactly, the rate of change of the concentrations c_1 and c_2 of the reactants has the quoted properties. Therefore the mathematical model of this law is [17]

$$\dot{c}_1 = -k_1 c_1 c_2 + k_{-1} p, \quad \dot{c}_2 = -k_1 c_1 c_2 + k_{-1} p, \quad \dot{p} = k_1 c_1 c_2 - k_{-1} p. \tag{14}$$

An extensive list of references on the mathematical models in biodynamics can be found in [19].

3. DYNAMICS AND BIFURCATION IN SOME BIOLOGICAL MODELS

The synthetic results on the dynamics and bifurcation associated with the Cauchy problem $\mathbf{u}_{t=0} = \mathbf{u}_0$ for the differential vector equation

$$\mathbf{u} = \mathbf{f}(\alpha, \mathbf{u}) \tag{15}$$

are presented in the form of static, imperfect and dynamic bifurcation diagrams. Let us present the main steps to obtain them.

The static bifurcation diagram (sbd) is a graphical representation of the stationary solution set $\mathbf{u}(\alpha)$ in dependence on the (scalar or vector) parameter α.

If $\dim \mathbf{u} + \dim \alpha > 3$, then only sections in this diagram can be represented. But, in this case, the problem of finding all nonequivalent sbd's arises. This problem is solved by considering one component, say $\alpha_1 = \lambda$, of α as a control parameter, all others being assumed small. In addition, λ is supposed to vary near some value λ_0, usually taken as equal to zero. Denote $\tau_1 = \alpha_2, \cdots, \tau_m = \alpha_{m+1}$. Then the sbd existing for $\tau \equiv (\tau_1, \cdots, \tau_m) = \mathbf{0}$ is deformed when τ vary in a neighborhood of $\mathbf{0}$. If the number m of the small scalar parameters is smaller than 5, then only a finite number of nonequivalent sbd's exist. Correspondingly, in the small parameter space τ there are some manifolds $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{D}$ separating some zones. All sbd's corresponding to all τ from one zone are equivalent. Therefore, up to this equivalence, in each zone some bifurcation diagram persists and, consequently, it suffices to draw a single sbd.

Then the *imperfect* or (*perturbed*) bifurcation diagram (ibd) consists of these zones and the corresponding one sbd in each of them.

The *dynamic bifurcation diagram* (dbd) is similar to the ibd. The parameter space is divided into zones (strata) such that the dynamical systems (of a dynamical scheme, i.e. family of dynamical systems) corresponding to the points (parameters) belonging to one zone are topologically equivalent. Then the phase portrait of one dynamical system corresponding to a point in each zone suffices to characterize topologically the dynamic behavior for the entire zone. The configuration represented by strata is called the *parameter portrait*. The dbd consists of the parameter portrait and one phase portrait for each stratum.

A combined dynamics and bifurcation study proceeds in several steps [20]: the stationary solutions (corresponding to equilibria of the dynamical system associated with (15)) are deduced; for each equilibrium the linearized system around that point is written; the eigenvalues are computed. If the real part of all eigenvalues are non null, then the equilibrium is hyperbolic and, by Hartman-Grobman theorem, the nonlinear dynamical system is locally equivalent to the linearized dynamical system. In this case no other study is necessary; if at least one eigenvalue has a null real part, then the equilibrium $\overline{\mathbf{u}}$ is non hyperbolic. Assume that we are in the two-dimensional case for $\mathbf{u}$ and assume that $\overline{\mathbf{u}}$ corresponds to $\overline{\alpha}$; let us transform $\mathbf{u}$ and α such that $\overline{\mathbf{u}}$ and $\overline{\alpha}$ are carried at the origin; let us form the problem (15) for $\overline{\alpha}$, called *the problem at the point*; determine the normal form for the problem (15) at the point; this form indicates which are the corresponding miniversal unfoldings. We determine them by means of the existing theories; let us perform the same study around each non-hyperbolic point; during these investigations some important manifolds occur(the manifold S of the double equilibria, the manifold $\mathcal{H}_{\mathbb{C}}$ of the Hopf bifurcation values, the manifold Q of the double zero eigenvalues (i.e. of the Bogdanov-Takens bifurcations), the manifolds $B-T$ of the homoclinic bifurcations, the manifold B_a of the Bautin bifurcation, manifolds of degenerate bifurcations). All these manifolds are separating in the parameter space the so-called strata. The configuration of strata represents the parameter portrait; for one point of each stratum the phase portrait is represented; if the mentioned manifolds are complicated in geometrical structure, the manifolds $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{D}$ are determined. These manifolds are extra strata in the parameter portrait.

All these steps were used systematically in the studies of the first author's group, in the hope to realize as complete an analysis as possible, using all existing theoretical and numerical approaches. These studies ended by publication of research monographs devoted to a single or at most two models. Later on, it was proved that other new directions of research arose even for the very minutely investigated models. Presently, a lot of open problems revealed are waiting for their solution: the asymptotic behavior for the parameter portrait and the corresponding phase portraits; the perturbed bifurcation diagrams for the above-mentioned manifolds; global bifurcations; the geometric and mechanics classification of the periodic oscillations; the degenerated bifurcations; the French canard phenomenon etc.; for the three-dimensional case the presence of chaos is probable and a systematic and complete study is not conceivable in general.

4. THERMAL CONVECTION

The collaboration of the first two authors in the framework of hydrodynamic stability theory, developed during the last 16 years, mainly concerned the derivation of stability criteria for mechanical equilibria of fluid layers, which can be a premise for interesting applications to biological fluid dynamics. Their studies were devoted to complex fluids subject to several physical influences (thermal, electrical, magnetic fields, concentration, porosity, compressibility, non Newtonicity). Variational methods (extending some of the existing ones), Fourier series combined with backwards integration techniques and a direct method based on the characteristic equation were frequently used in these studies [21].

For most of the fluid flows of interest in real-world applications, the direct use of the methods of functional analysis lead to results very unsatisfactory for users. This is due, in principle, to the fact that some changing-sign terms are estimated by their norm. In addition, some physical effects disappear by projection (and, so, symmetrization of some operators) on the problem spaces. These types of problems remain for dozens of years as a challenge for fluid dynamicists. Such a problem was solved by us by means of several ideas (borrowed from simpler cases of ode's): a change of the governing problem is necessary before applying to it the projection; this change must contain several parameters to be chosen so that the remained symmetric operators preserve all physical parameters; the changing-sign terms must be included in sign-preserving terms by suitable definition of the Liapunov functional; it is necessary an optimal change of the parameters for stability bounds as well as for the simplicity of usually cumbersome computations. Thus, the problem governing the nonlinear stability of the mechanical equilibrium of a horizontal layer Ω of a binary fluid mixture subject to two competing effects temperature and concentration when the Soret and Dufour thermodiffusive currents are present reads [21]

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\nabla p + (\mathcal{R}\theta - s\mathcal{C}\gamma)\mathbf{k} + \Delta\mathbf{v},$$

$$P_r\Big(\frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta\Big) = (1 + N\lambda^2\tau^{-1})\Delta\theta + \mathcal{R}\mathbf{v}\cdot\mathbf{k} + N\lambda\sigma\Delta\gamma, \quad (t,\mathbf{x}) \in (o,\infty)\times\Omega,$$

$$S_c\Big(\frac{\partial\gamma}{\partial t}+\mathbf{v}\cdot\nabla\gamma\Big)=\Delta\gamma+\lambda\sigma^{-1}\tau^{-1}\Delta\theta+\mathcal{C}\mathbf{v}\cdot\mathbf{k}, \tag{16}$$

where t is the time, $\mathbf{v},\theta,\gamma$ are perturbed velocity, temperature and concentration unknown field, while $R,\mathcal{C},s,P_r,N,\lambda,\tau,\sigma,S_c$ are physical parameters. Boundary conditions and the requirement of the solenoidality are imposed energy method yielding results much better than the existing ones.

An inspired handling with algebraic (Young) and integro-differential (Poincaré) inequalities made possible the improvement of the Navier-Stokes spectrum of the bounds for the model [23]

$$\frac{d\mathbf{v}}{dt}+\widetilde{A}\mathbf{v}=R(v), \tag{17}$$

where $\widetilde{A}\mathbf{v}=A\mathbf{v}+M_{\mathbf{u}}\mathbf{v}$, A is related to the projection of the Laplacian on the space N of solenoidal vectors (17), $M_{\mathbf{u}}$ is the projection on N of the linearization of the advective nonlinear term in the Navier-Stokes equations about the basic vector fields $\mathbf{u}$, and R is the projection on N of the nonlinear advective term in perturbation velocity $\mathbf{v}$.

The assumption of normal mode perturbations transformed the pde's into ode's and the boundary conditions became two-point conditions. The trace of the complexity of the fluid and physical effects can be pursuited in the very high order of differentiation in the ode's and in the presence of physical parameters. The complexity of flow can be viewed in the complicated boundary conditions. An example of such a two-point eigenvalue problem is

$$\Big[\big(D^2-a^2\big)^2-M^2D^2\Big]\big(D^2-a^2\big)\,w-b_1a^2w=0,\ z\in[-0.5,0.5],$$
$$W=DW=\Big[\big(D^2-a^2\big)^2-M^2D^2\Big]\,w=0,\ \text{at } z=\pm 0.5. \tag{18}$$

where $W(z)$ is the unknown function and a,M,b_1 are physical parameters. By adapting the direct method in the theory of ode's to, in [24] was determined the secular equation. False secular points, not detected by a straightforward application of a numerical method, were found by a bifurcation analysis of the characteristic manifold.

It is in these bifurcation problems and in the study of dynamics generated by ode's possessing several parameters that the third author can help the first ones. The geometric forms involves are extremely complicated; some of them are fractals. Therefore a specialist in fractal geometry, integral geometry and computation geometry is necessary.

References

[1] Murray, J. D., *Mathematical biology*, Biomathematics **19**, 2nd corrected ed., Springer, Berlin, 1993. (first ed. 1989)

[2] Chance, B., E. K. Rye, A. K. Ghosh, B. Hess, *Biological and biochemical oscillators*, Academic, New York, 1973.

[3] Levin, S. (ed.), *Lecture Notes in Biomathematics*, Springer, Berlin, 1974 - onward.

[4] *Biomathematics*, Springer, Berlin, 1970 - onward.

[5] *Lectures on Mathematics in the Life Sciences*, AMS, Providence, 1968 - onward.

[6] Haken, H. (ed.), *Springer Series in Synergetics*, Springer, Berlin, 1977 - onward.

[7] *Applied Mathematical Sciences, Springer*, New York.

[8] Journal of Mathematical Biology.

[9] Bulletin of Mathematical Biology.

[10] Mathematical Biosciences.

[11] Müller, I., *Thermodynamics*, Plenum, Boston, 1985.

[12] Rocsoreanu, C., A. Georgescu, N. Giurgiteanu, *The FitzHugh-Nagumo model. Bifurcation and dynamics*, Kluwer, Dordrecht, 2000.

[13] Sterpu, M., *Dynamics and bifurcation for two generalized van der Pol models*, Seria Matematica Aplicata si Industriala **8**, Univ. of Pitesti, 2001. (Romanian, extensive English summary)

[14] Giurgiteanu, N., *Computational economical and biological dynamics - DIECBI*, Ed. Europa, Craiova, 1977. (Romanian)

[15] Nistor, Gh., *Contributions to the asymptotic treatment of some problems of nonlinear dynamics (including the FitzHugh-Nagumo model)*, PhD Thesis, Univ. of Pitesti, 2007. (Romanian)

[16] Bazavan, P., *Numerical algorithms in the study of dynamical systems*, Sitech, Craiova, 2005. (Romanian)

[17] Curtu, R., *Dynamics and bifurcations for the Gray-Scott model with uncatalyzed constraints*, Ed. Univ. "Transilvania" of Brasov, 2002. (Romanian; English summary)

[18] Trifan, M., *Dynamics and bifurcation in the mathematical study of the cancer*, Seria Matematica Aplicata si Industriala **22**, Univ. of Pitesti, Pamantul, Pitesti, 2006. (Romanian)

[19] Georgescu, A., *Bifurcation in biodynamics*, Sci. Annals of USAMV Iasi, **46**, 2(2003), 15-34.

[20] Georgescu, A., *Dynamic bifurcation diagrams for some models in economics and biology*, Acta Universitatis Apulensis, "1 Decembrie 1918" University of Alba Iulia, **8** (2004), 156-163.

[21] Georgescu, A., L. Palese, *Stability criteria of fluid flows*, manuscript.

[22] Georgescu, A., L. Palese, A. Radaelli, *On a new method in hydrodynamic stability theory*, Math. Sc. Red. Hot-Line, **4**, 7(2000), 1-16.

[23] Georgescu, A., L. Palese, *Ameliorations des estimations de Prodi pour le spectre*, C. R. Acad. Sc. Paris, **320**, 1(1995), 891-896; *Stability spectrum estimates for confined fluids*, Rev. Roum. Math Pures Appl, **42**, 1-2(1997), 37-51.

[24] Palese, L., A. Georgescu, S. Mitran, *Neutral curves for the MHD Soret-Dufour driven convective instability*, Rev. Roum. Sci. Tech.-Méc. Appl., **45**, 3(2000), 265-275.

ROMAI J., **2**, 2(2006),
77–80

HOPF BIFURCATION IN THE BAZYKIN MODEL I

Raluca-Mihaela Georgescu

Dept. of Mathematics and Computer Sciences, Univ. of Pitesti

Abstract The nature of the Hopf singularity in a predator-prey model proposed by Bazykin in the case when two of the four parameters are kept fixed is presented.

Keywords: Bazykin model, Hopf singularity.
2000 MSC: 37G10, 34C23.

We deal with a particular case of the Bazykin model consisting of a Cauchy problem $x(0) = x_0,\ y(0) = y_0$, for the following system of ordinary differential equations (sode) [5], [2]

$$\left\{ \begin{array}{lcl} \dot{x} & = & x - xy/(1+\alpha x) - \varepsilon x^2, \\ \dot{y} & = & -\gamma y + xy/(1+\alpha x) - \delta y^2, \end{array} \right. \tag{1}$$

where x and y represent the population numbers of the preys and the predators, respectively, therefore $x, y > 0$ and $\alpha,\ \varepsilon,\ \gamma$ and δ are nonnegative parameters describing the behaviour of isolated populations and their interaction. Assume $\gamma = 1,\ \delta = 0$. In this case, the change of the time $dt = (1+\alpha x)d\tau$, turns (1) into [4]

$$\left\{ \begin{array}{lcl} \dot{x} & = & x(1+\alpha x - \varepsilon x - y - \alpha\varepsilon x^2), \\ \dot{y} & = & y(-1-\alpha x + x). \end{array} \right. \tag{2}$$

We are interested only in the Hopf singularities, which, by [4], are $E((1-\alpha)^{-1}, (1-\alpha-\varepsilon)(1-\alpha)^{-2})$ for $-\alpha^2+\alpha-\alpha\varepsilon-\varepsilon = 0$, $\alpha \notin \{0,1\}$ and $\varepsilon \neq 0$. In this case, the equilibrium E becomes $E((1-\alpha)^{-1}, (1-\alpha^2)^{-1})$.

Propoziţia 0.1 *The normal form of (2) at* $E((1-\alpha)^{-1}, (1-\alpha^2)^{-1})$ *for* $-\alpha^2+\alpha-\alpha\varepsilon-\varepsilon = 0$, $\alpha \notin \{0,1\}$ *and* $\varepsilon \neq 0$ *is*

$$\begin{aligned} \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} &= \begin{pmatrix} 0 & -i/\sqrt{1-\alpha^2} \\ i/\sqrt{1-\alpha^2} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \\ &- (w_1^2+w_2^2)\left[a_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + b_1 \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix}\right] + O(w^4), \end{aligned}$$

where $a_1 = -\alpha^2\gamma^2/4$ *and* $b_1 = \beta(-54a^4+41a^2-13)$, $\gamma = \sqrt{(1-\alpha)/(1+\alpha)}$, $\beta = \gamma/\ [12(1+\alpha)]$, *i.e.* E *is a nondegenerated Hopf singularity.*

Proof. By the normal form method in [1], we translate the point E at the origin with the aid of the change $u_1 = x-(1-\alpha)^{-1},\ u_2 = y-(1-\alpha^2)^{-1}$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (2) reads

$$\begin{cases} \dot{u}_1 &= -\dfrac{1}{1-\alpha}u_2 - \dfrac{\alpha^2}{1+\alpha}u_1^2 - u_1u_2 - \dfrac{a^2(1-\alpha)}{1+\alpha}u_1^3, \\ \\ \dot{u}_2 &= \dfrac{1}{1+\alpha}u_1 + (1-\alpha)u_1u_2. \end{cases} \tag{3}$$

The eigenvalues of the matrix defining the linear terms in (3) are $\lambda_1 = \overline{\lambda_2} = i/\sqrt{1-\alpha^2}$ and let $\mathbf{u}_{\lambda_1} = (i,\gamma)^T$ be an eigenvector corresponding to the positive eigenvalue. We have $\mathbf{u}_{\lambda_1} = (0,\gamma)^T + i(1,0)^T$. Then, with the change of the coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{PM}_{\mathbb{C}}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ and $\mathbf{M}_{\mathbb{C}} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, i.e. $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -i\gamma & i\gamma \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (3) achieves the complex form

$$\begin{cases} \dot{v}_1 &= \dfrac{i}{\sqrt{1-\alpha^2}}v_1 + Mv_1^2 - \dfrac{\alpha^2}{2(1+\alpha)}v_1v_2 - Nv_2^2 - \dfrac{\alpha^2(1-\alpha)}{8(1+\alpha)}(v_1+v_2)^3, \\ \\ \dot{v}_2 &= \dfrac{-i}{\sqrt{1-\alpha^2}}v_2 - \overline{N}v_1^2 - \dfrac{\alpha^2}{2(1+\alpha)}v_1v_2 + \overline{M}v_2^2 - \dfrac{\alpha^2(1-\alpha)}{8(1+\alpha)}(v_1+v_2)^3, \end{cases} \tag{4}$$

where $M = \left[(1-2\alpha^2)/(1+\alpha) + i\gamma\right]/4$ and $N = \left[1/(1+\alpha) + i\gamma\right]/4$. The form (4) involves a diagonal matrix of the linear terms. In order to eliminate the second-order nonresonant terms in (4) we determine the transformation $\mathbf{v} = \mathbf{n} + \mathbf{h}_2(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	M	$-\overline{N}$	$i/\sqrt{1-a^2}$	$3i/\sqrt{1-a^2}$	P	Q
1	1	R	R	$-i/\sqrt{1-a^2}$	$i/\sqrt{1-a^2}$	$-S$	S
0	2	$-N$	$\overline{M}$	$-3i/\sqrt{1-a^2}$	$-i/\sqrt{1-a^2}$	$\overline{Q}$	$\overline{P}$

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order vector polynomial in (4), $P = \left[1-\alpha+i(2\alpha^2-1)\gamma\right]/4$, $Q = (1-\alpha+i\gamma)/12$, $R = -\alpha^2/[2(1+\alpha)]$, $S = \alpha^2 i\gamma/2$. We find the transformation

$$\begin{cases} v_1 &= n_1 + Pn_1^2 - Sn_1n_2 + \overline{Q}n_2^2, \\ v_2 &= n_2 + Qn_1^2 + Sn_1n_2 + \overline{P}n_2^2, \end{cases}$$

carrying (4) into

$$\begin{cases} \dot{n}_1 = \dfrac{i}{\sqrt{1-\alpha^2}} n_1 + A n_1^3 + B n_1^2 n_2 + C n_1 n_2^2 + D n_2^3 + O(n^4), \\ \dot{n}_2 = \dfrac{-i}{\sqrt{1-\alpha^2}} n_2 + \overline{D} n_1^3 + \overline{C} n_1^2 n_2 + \overline{B} n_1 n_2^2 + \overline{A} n_2^3 + O(n^4), \end{cases} \tag{5}$$

where $A = \beta \left[(10\alpha^2+3)\sqrt{1-\alpha^2} + i\alpha^2(21\alpha^2-11) \right]$, $B = -\beta \left[3\alpha^2\sqrt{1-\alpha^2} + \right.$ $\left. + i(54\alpha^4 - 41\alpha^2 + 13) \right]$, $C = \beta \left[(7-22\alpha^2)\sqrt{1-\alpha^2} + i\alpha^2(33\alpha^2-19) \right]$, $D = \beta \left[-5\alpha^2\sqrt{1-\alpha^2} + i(5\alpha^2-3) \right]$.

Thus we eliminated the nonresonant second order terms.

Now, we have to reduce the third-order nonresonant terms in (5). This reduces to determine the transformation $\mathbf{n} = \mathbf{s} + \mathbf{h}'_3(\mathbf{s})$, where $\mathbf{n} = (n_1, n_2)^T$ and $\mathbf{s} = (s_1, s_2)^T$, suggested by the Table

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h'_{\mathbf{m},1}$	$h'_{\mathbf{m},2}$
3	0	A	$\overline{D}$	$\frac{2i}{\sqrt{1-\alpha^2}}$	$\frac{4i}{\sqrt{1-\alpha^2}}$	$\frac{i}{24(1+\alpha)}E$	$-\frac{i}{48(1+\alpha)}\overline{G}$
2	1	B	$\overline{C}$	0	$\frac{2i}{\sqrt{1-\alpha^2}}$	-	$-\frac{i}{24(1+\alpha)}\overline{F}$
1	2	C	$\overline{B}$	$-\frac{2i}{\sqrt{1-\alpha^2}}$	0	$\frac{i}{24(1+\alpha)}F$	-
0	3	D	$\overline{A}$	$-\frac{4i}{\sqrt{1-\alpha^2}}$	$-\frac{2i}{\sqrt{1-\alpha^2}}$	$\frac{i}{48(1+\alpha)}G$	$-\frac{i}{24(1+\alpha)}\overline{E}$

Here $X_{\mathbf{m}}$ is the third-order vector polynomial in (5) and $E = (1-\alpha)[(10\alpha^2+3)\sqrt{1-\alpha^2} + i\alpha^2(21\alpha^2-11)]$, $F = (1-\alpha)\left[(7-22\alpha^2)\sqrt{1-\alpha^2} + i\alpha^2(33\alpha^2-19) \right]$, $G = (1-\alpha)\left[-5\alpha^2\sqrt{1-\alpha^2} + i(5\alpha^2-3) \right]$. It follows the transformation

$$\begin{cases} n_1 = s_1 + \dfrac{i}{24(1+\alpha)} E s_1^3 + \dfrac{i}{24(1+\alpha)} F s_1 s_2^2 + \dfrac{i}{48(1+\alpha)} G s_2^3, \\ n_2 = s_2 - \dfrac{i}{48(1+\alpha)} \overline{G} s_1^3 - \dfrac{i}{24(1+\alpha)} \overline{F} s_1^2 s_2 - \dfrac{i}{24(1+\alpha)} \overline{E} s_2^3, \end{cases}$$

carrying (5) into

$$\begin{cases} \dot{s}_1 = \dfrac{i}{\sqrt{1-\alpha^2}} s_1 + B s_1^2 s_2 + O(s^4), \\ \dot{s}_2 = \dfrac{-i}{\sqrt{1-\alpha^2}} s_2 + \overline{B} s_1 s_2^2 + O(s^4). \end{cases} \tag{6}$$

Let us come back to the real state functions by denoting $s_1 = w_1 + iw_2$, $\overline{s_2} = w_1 - iw_2$ [3], to obtain the normal form of (2)

$$\begin{aligned}\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} &= \begin{pmatrix} 0 & -i/\sqrt{1-\alpha^2} \\ i/\sqrt{1-\alpha^2} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \\ &- \quad (w_1^2 + w_2^2) \left[a_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + b_1 \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} \right] + O(w^4),\end{aligned}$$

where $a_1 = \Re e\ B,\ b_1 = \Im m\ B$, i.e. $a_1 = -\alpha^2(1-\alpha)/[4(1+\alpha)]$.

Since $a_1 \neq 0$, it follows [1] that E is a nondegenerated Hopf singularity. Our numerical computations agree with this conclusion. □

References

[1] Arrowsmith, D.K., Place, C.M., *Ordinary differential equations*, Chapman and Hall, London, 1982.

[2] Bazykin, A., Kuznetsov, Y., Khibnik, A, *Bifurcation diagrams of planar dynamical systems*, Research Computing Centre, USSR Academy of Sciences, Pushchino, 1985 (in Russian).

[3] Georgescu, A., Georgescu, R.M., *Normal forms for nondegenerated Hopf singularities and bifurcations*, Proc. Conf. on Applied and Industrial Mathematics - CAIM 2003, Oradea, Romania, 104-112.

[4] Georgescu, R.M., Georgescu, A., *Some results on dynamics generated by Bazykin model*, presented at the International Conference on Mathematical Models and Methods in Biology and Medicine - MMMBM 2005, Bedlewo, Poland; Atti Accad. Peloritana dei Pericolanti; Scienze FMN, **84**, C1A0601003 (2006), 1-10.

[5] Kuznetsov, Y.A., *Elements of applied bifurcation theory*, Springer, New York, 1998.

ROMAI J., **2**, 2(2006),
81–87

A SOLUTION OF THE BOUNDARY INTEGRAL EQUATION OF THE 2D FLUID FLOW AROUND BODIES WITH QUADRATIC ISOPARAMETRIC BOUNDARY ELEMENTS

Luminita Grecu
Faculty of. Engn. and Management of Technological Systems, Univ. of Craiova

Abstract A method for solving the boundary integral equation (b.i.e.) of the 2D fluid flow around bodies using quadratic isoparametric boundary elements of Lagrangian type is presented. The b.i.e. of the 2D fluid flow around bodies has been solved in other papers using the collocation method and linear boundary elements. In order to find a better approximation for the solution, in fact a higher order class function for the unknown, a boundary element method (BEM) that uses quadratic isoparametric boundary elements is presented. So, locally, the approximate function, that is continuous on the boundary, has a quadratic variation and the real boundary is better approximated by curved elements. For evaluating the singular integrals a method based on the definition of the principal value in the Cauchy sense of an integral is used. A computer program is developed for the analysis of general problems and numerical results are presented for some particular cases when exact solutions are known. The numerical results and the analytical solutions are in a very good agreement. The numerical examples presented demonstrate the accuracy and the efficiency of the method proposed.

Keywords: boundary element method, quadratic boundary element, integral equation, Cauchy principal value, subsonic two-dimensional flow.

1. INTRODUCTION

The theory of the b.i.e. for the two-dimensional subsonic ideal compressible fluid flow around the bodies was elaborated by Lazăr Dragoş for example in: [5], [6], [7]. In [5], applying an indirect method with sources distribution of unknown intensities f, it is obtained a singular b.i.e. for the mentioned problem. The author uses a collocation method to solve it. In order to find a better approximation for the solution here we use quadratic isoparametric boundary elements, so the unknown function is approximated by polynomials of Lagrangian type, and the boundary, denoted by C, is assumed to be smooth and closed by curved arcs. Thus the problem is reduced to a linear system of equations. The most important step, of practical importance, is the evaluation of the matrix coefficients, in fact the occurring singular boundary integrals.

For their evaluation in this paper we use a method based on the definition of the principal value in the Cauchy sense of an integral [12].

The b.i.e. of the mentioned theory has the form [5]

$$\left[\left(n_x^0\right)^2+\beta^2\left(n_y^0\right)^2\right] f\left(\overline{x}_0\right)+\frac{1}{\pi}' \int_C f\left(\overline{x}\right) \frac{\left(\overline{x}-\overline{x}_0\right) n_x^0+\beta^2\left(\overline{y}-\overline{y}_0\right) n_y^0}{\left|\overline{x}-\overline{x}_0\right|^2}=2 \beta n_x^0$$

where n_x^0, n_y^0 are the components of the normal unit vector outward the fluid (inward the body) at the point $\overline{x}_0$, situated on C, $\beta=\sqrt{1-M^2}$ (for the subsonic flow, $M=$ Mach number), and f is the unknown function, the intensity of the sources, assumed to satisfy a Hölder condition. The sign " ' " denotes the Cauchy principal value of the integral.

Solving the boundary integral equation (1) we get the sources intensities. Then we can compute the perturbation velocity and the local pressure coefficient.

A collocation method is used, for example in [5], for solving this integral equation. Linear boundary elements are used in [4] for another boundary integral equation equivalent to the same problem

In the boundary element approach used herein, for solving the integral equation (1), we use quadratic isoparametric boundary elements [11]. In order to obtain the discrete equation the boundary is divided into N one-dimensional quadratic boundary elements, denoted by $L_i, i=\overline{1,N}$, each of them with three nodes: two extreme nodes and an interior one, denoted by $\overline{x}_j$, $j=\overline{1,2N}$. Assuming that the discrete equation is satisfied at every node, we have

$$\left[\left(n_x^j\right)^2+\beta^2\left(n_y^j\right)^2\right] f\left(\overline{x} j\right)+\frac{1}{\pi} \sum_{i=1}^N \int_{L_i} f\left(\overline{x}\right) \frac{\left(\overline{x}-\overline{x}_j\right) n_x^j+\beta^2\left(\overline{y}-\overline{y}_j\right) n_y^j}{\left|\overline{x}-\overline{x}_j\right|^2}=2 \beta n_x^j$$

The quadratic isoparametric boundary element uses the same set of basic functions, denoted by N_1, N_2, N_3, for describing the geometry and the unknown function. In the intrinsic system of coordinates, these functions have the expressions

$$N_1(\xi)=\frac{\xi(\xi-1)}{2},\ N_2(\xi)=1-\xi^2,\ N_3(\xi)=\frac{\xi(\xi+1)}{2},\ \xi \in[-1,1]$$

Using a matriceal notation we obtain the following equation

$$\left[\left(n_x^j\right)^2+\beta^2\left(n_y^j\right)^2\right] f\left(\overline{x} j\right)+\frac{1}{\pi} \sum_{i=1}^N\left(\sum_{l=1}^3 a_{ij}^l f_l^i\right)=2 \beta n_x^j,$$

where

$$a_{ij}^{l}=\int\limits_{-1}^{1}N_{l}\frac{\left(\left[N\right]\left\{x^{i}\right\}-x_{j}\right)n_{x}^{j}+\beta^{2}\left(\left[N\right]\left\{y^{i}\right\}-y_{j}\right)n_{y}^{j}}{\left|\left[N\right]\left\{\overline{x}\right\}-\overline{x}_{j}\right|^{2}}J\left(\xi\right)d\xi$$

$[N]=(N_1,N_2,N_3)$ is a line matrix, $\left\{x^i\right\},\left\{y^i\right\}$, are the column matrices made with the global coordinates of the nodes of the boundary element denoted by L_i, and f_l^i , $i=\overline{1,N}, l=1,2,3$, are the nodal values of the unknown function for the three nodes of the element i (the value of the unknown for the node number l of the element number i). Coming back to the global system of notation we obtain the following linear algebraic system

$$[A]\left\{f\right\}=\left\{B\right\}, A\in M_{2N}\left(R\right), \left\{f\right\}, \left\{B\right\}\in R^{2N},\ B_j=2\pi\beta n_x^j.$$

2. EVALUATING THE MATRIX COEFFICIENTS GIVEN BY THE SINGULAR INTEGRALS

In order to find the matrix $[A]$ we need to evaluate the occurring integrals. Some of them are ordinary integrals, but the others are singular integrals. For the singular integrals we used the Cauchy principal value integral method, based on the definition of the Cauchy principal value of a singular integral.

Denoting $m_i=x_1^i+x_3^i-2x_2^i$, $n_i=x_3^i-x_1^i$, $u_{ij}=x_2^i-x_j$, $M_i=y_1^i+y_3^i-2y_2^i$, $N_i=y_3^i-y_1^i$, $U_{ij}=y_2^i-y_j$, $a_i=\frac{m_i^2+M_i^2}{4}$, $aa_i=\frac{n_i^2+N_i^2}{4}, b_i=\frac{m_in_i+M_iN_i}{2}$, $c_{ij}=aa_i+m_iu_{ij}+M_iU_{ij}$, $d_{ij}=n_iu_{ij}++N_iU_{ij}$, $e_{ij}=u_{ij}^2+U_{ij}^2$, $i,j=\overline{1,2N}$. $J\left(\xi\right)=\sqrt{4a_i\xi^2+2b_i\xi+aa_i}$, for $\overline{x}\in L_i$, we get $\left|\overline{x}-\overline{x}_j\right|^2=a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}$,

We start applying the method by using the following expression for the coefficients $a_{ij}^l, l=1,2,3$

$$a_{ij}^{1}=\frac{1}{4}\int\limits_{-1}^{1}\frac{A_{ij}\xi^4+B_i\xi^3+C_{ij}\xi^2+D_{ij}\xi}{a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}}J\left(\xi\right)d\xi,$$

where $A_{ij}=m_in_x^j+\beta^2M_in_y^j$, $B_{ij}=\left(n_i-m_i\right)n_x^j+\beta^2\left(N_i-M_i\right)n_y^j$, $C_{ij}=\left(2u_{ij}-n_i\right)n_x^j+\beta^2\left(2U_{ij}-N_i\right)n_y^j, D_{ij}=-2u_{ij}n_x^j-2\beta^2U_{ij}n_y^j$,

$$a_{ij}^{2}=\int\limits_{-1}^{1}\frac{-A_{ij}\xi^4-B_i\xi^3-C_{ij}\xi^2+B_{ij}\xi+D_{ij}}{a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}}J\left(\xi\right)d\xi,$$

where $A_{ij}=\frac{m_in_x^j+\beta^2M_in_y^j}{2}, B_{ij}=\frac{n_in_x^j+\beta^2N_in_y^j}{2}, C_{ij}=\frac{(2u_{ij}-m_i)n_x^j+\beta^2(2U_{ij}-M_i)n_y^j}{2}$, $D_{ij}=-u_{ij}n_x^j-\beta^2U_{ij}n_y^j$,

$$a_{ij}^3 = \frac{1}{4}\int\limits_{-1}^{1} \frac{A_{ij}\xi^4 + B_i\xi^3 + C_{ij}\xi^2 + D_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

where $A_{ij} = m_i n_x^j + \beta^2 M_i n_y^j$, $B_{ij} = (n_i + m_i)\, n_x^j + \beta^2 (N_i + M_i)\, n_y^j$, $C_{ij} = (2u_{ij} + n_i)\, n_x^j + \beta^2 (2U_{ij} + N_i)\, n_y^j$, $D_{ij} = 2u_{ij} n_x^j + 2\beta^2 U_{ij} n_y^j$.

Let us evaluate these coefficients for $j = 2i - 1$, $j = 2i$, and $j = 2i + 1$, when they are represented by singular integrals.

For $j = 2i - 1$, with singularity at -1, using the definition of the Cauchy principal value, and after some manipulations we deduce

$$\begin{aligned} a_{ij}^1 &= \frac{1}{4}\int\limits_{-1}^{1} \frac{\left(A_{ij}\xi^2 + (B_{ij} - 2A_{ij})\,\xi\right) J(\xi)\, d\xi}{a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i} + \\ &+ \frac{1}{4}\int\limits_{-1}^{1} \frac{(C_{ij} - 2B_{ij} + 3A_{ij})\, J(\xi)\, d\xi}{(a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i)\,(\xi + 1)} \end{aligned}$$

So, only the second integral still possesses a singularity. For its evaluation we use the truncation of the interval's method (that isolates the singularity using a small positive number, denoted by ε) to obtain

$$\begin{aligned} a_{ij}^1 &= \frac{1}{4}\int\limits_{-1}^{1} \frac{\left(A_{ij}\xi^2 + (B_{ij} - 2A_{ij})\,\xi\right) J(\xi)\, d\xi}{a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i} + \\ &+ \frac{1}{4}\int\limits_{-1+\varepsilon}^{1} \frac{(C_{ij} - 2B_{ij} + 3A_{ij})\, J(\xi)\, d\xi}{(a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i)\,(\xi + 1)}. \end{aligned}$$

In this case, for the other coefficients we obtain the expressions

$$a_{ij}^2 = \int\limits_{-1}^{1} \frac{-A_{ij}\xi^2 + (2A_{ij} - B_{ij})\,\xi - 3A_{ij} + 2B_{ij} - C_{ij}}{a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i} J(\xi)\, d\xi,$$

$$a_{ij}^3 = \frac{1}{4}\int\limits_{-1}^{1} \frac{A_{ij}\xi^2 + (B_{ij} - 2A_{ij})\,\xi}{a_i\xi^2 + (b_i - 2a_i)\,\xi + c_{ij} - 2b_i + 3a_i} J(\xi)\, d\xi.$$

For $j = 2i$, as before, we have the following expressions for $a_{ij}^l, l = 1, 2, 3$

$$a_{ij}^1 = \frac{1}{4}\int\limits_{-1}^{1} \frac{A_{ij}\xi^2 + B_{ij}\xi + C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}} J(\xi)\, d\xi,$$

$$a_{ij}^2 = \int\limits_{-1}^{1} \frac{-A_{ij}\xi^2 - B_{ij}\xi - C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}} J(\xi)\, d\xi + \left(\int\limits_{\varepsilon}^{1} + \int\limits_{-1}^{\varepsilon}\right)\left(\frac{B_{ij}}{(a_i\xi^2 + b_i\xi + c_{ij})\,\xi}\right) J(\xi)\, d\xi,$$

$$a_{ij}^3 = \frac{1}{4}\int\limits_{-1}^{1} \frac{A_{ij}\xi^2 + B_{ij}\xi + C_{ij}}{a_i\xi^2 + b_i\xi + c_{ij}} J(\xi)\, d\xi.$$

Similarly, for $j = 2i + 1$, we have

$$a_{ij}^1 = \frac{1}{4}\int\limits_{-1}^{1} \frac{\left(A_{ij}\xi^2 + (B_{ij} + 2A_{ij})\,\xi\right) J(\xi)\, d\xi}{a_i\xi^2 + (b_i + 2a_i)\,\xi + c_{ij} + 2b_i + 3a_i},$$

$$a_{ij}^2 = \int\limits_{-1}^{1} \frac{-A_{ij}\xi^2 - (2A_{ij} + B_{ij})\,\xi - 3A_{ij} - 2B_{ij} - C_{ij}}{a_i\xi^2 + (b_i + 2a_i)\,\xi + c_{ij} + 2b_i + 3a_i} J(\xi)\, d\xi,$$

$$\begin{aligned} a_{ij}^3 &= \frac{1}{4}\int\limits_{-1}^{1} \frac{\left(A_{ij}\xi^2 + (B_{ij} + 2A_{ij})\,\xi\right)\xi}{a_i\xi^2 + (b_i + 2a_i)\,\xi + c_{ij} + 2b_i + 3a_i} J(\xi)\, d\xi + \\ &+ \frac{1}{4}\int\limits_{-1}^{1-\varepsilon} \frac{(C_{ij} + 2B_{ij} + 3A_{ij})\, J(\xi)\, d\xi}{(a_i\xi^2 + (b_i + 2a_i)\,\xi + c_{ij} + 2b_i + 3a_i)(\xi - 1)}. \end{aligned}$$

Denoting by u, v the components of the velocity we deduce (see [6])

$$u(\overline{x}_j) = -\frac{1}{2} f_j n_x^j - \frac{1}{2\pi}\sum_{i=1}^{N}\left(f_1^i b_{ij}^1 + f_2^i b_{ij}^2 + f_3^i b_{ij}^3\right)$$

$$v(\overline{x}_j) = -\frac{1}{2} f_j n_y^j - \frac{1}{2\pi}\sum_{i=1}^{N}\left(f_1^i c_{ij}^1 + f_2^i c_{ij}^2 + f_3^i c_{ij}^3\right)$$

wherefrom

$$b_{ij}^1 = \frac{1}{4}\int\limits_{-1}^{1} \frac{m_i\xi^4 + (n_i - m_i)\,\xi^3 + (2u_{ij} - n_i)\,\xi^2 - 2u_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

$$b_{ij}^2 = -\frac{1}{2}\int\limits_{-1}^{1} \frac{m_i\xi^4 + n_i\xi^3 + (2u_{ij} - m_i)\,\xi^2 - n_i\xi + 2u_{ij}}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

$$b_{ij}^3 = \frac{1}{4}\int\limits_{-1}^{1} \frac{m_i\xi^4 + (n_i + m_i)\,\xi^3 + (2u_{ij} + n_i)\,\xi^2 + 2u_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^1 = \frac{1}{4}\int\limits_{-1}^{1} \frac{M_i\xi^4 + (N_i - M_i)\,\xi^3 + (2U_{ij} - N_i)\,\xi^2 - 2U_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^2 = -\frac{1}{2}\int\limits_{-1}^{1} \frac{m_i\xi^4 + N_i\xi^3 + (2U_{ij} - M_i)\,\xi^2 - N_i\xi + 2U_{ij}}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^3 = \frac{1}{4}\int\limits_{-1}^{1} \frac{M_i\xi^4 + (N_i + M_i)\,\xi^3 + (2U_{ij} + N_i)\,\xi^2 + 2U_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi,$$

where the coefficients of the velocity components are evaluated in the same manner as for the matrix coefficients.

3. NUMERICAL RESULTS

We tested the presented method by applying it to a particular case, when we know an exact solution for the problem. The exact solution for the problem of the uniform ideal incompressible subsonic fluid flow around a circular obstacle, provided in [8], yields the following (dimensionless) expressions for the components of the velocity on the boundary: $u = -\cos 2\theta,\ v = -\sin 2\theta$.

We evaluate the local pressure coefficient, which has the expression: $cp = -\left(u^2 + v^2\right) - 2u$. For the exact solution we get: $cp = -1 + 2\cos 2\theta$.

A computer code in MATHCAD, based on this method, and another one using the exact solution of the problem allow us to compare the numerical solution with the exact one. It can be seen graphically that the diference between them is small enough. In using the truncation method for treating the singularities [1] (because the integrand does not oscillate near the singularity), i.e. the expressions of the coefficients a_{ij}^1, a_{ij}^2 we obtain worse results. Consequently, the method proposed in this paper leads to an error smaller than the truncation method for evaluating the singular integrals. For the discretization of the boundary we choosed 20 nodes while for the truncation parameter we took the value $\varepsilon = 0.09$. The numerical results agree very well with the analytical solutions. This demonstrates the accuracy and the efficiency of the proposed method.

References

[1] Antia, H. M., *Numerical methods for scientists and engineers*, Birkhäuser, Basel, 2002.

[2] Bonne, M., *Bounndary integral equation methods for solids and fluids*, John Wiley and Sons, New York, 1995.

[3] Brebbia, C. A., Telles, J. C. F., Wobel, L. C., *Boundary element theory and application in engineering*, Springer, Berlin, 1984.

[4] Carabineanu, A., *A boundary element approach to the 2D potential flow problem around airfoils with cusped trailing edge*, Comput.Methods Appl. Mech. Engrg., **129**(1996), 213-219.

[5] Dragoş, L., *Mathematical methods in aerodynamics*, Ed. Academiei Române, Bucureşti, 2000.

[6] Dragoş, L., Dinu, A., *A direct boundary integral equations methods to subsonic flow with circulation past thin airfoils in ground effect*, Comput. Methods Appl. Mech. Engrg., **121**(1995) 163.

[7] Dragoş, L., Dinu, A., *A direct boundary integral method for the two-dimensional lifting flow*, Arch. Mech., **47**(1995), 813.

[8] Dragoş, L. *Mecanica fluidelor I*, Ed. Academiei Române, Bucureşti, 1999.

[9] Grecu, L., *A solution of the boundary integral equation of the theory of the infinite span airfoil in subsonic flow with linear boundary elements*, Analele Universităţii Bucureşti, Matematică, Anul LII, 2(2003), 181-188.

[10] Grecu, L., *The truncation method for treating the singularities when solving the singular boundary integral equation of the infinite span airfoil*, International Multidisciplinary Symposium, UNIVERSITARIA SIMPRO 2005, Petroşani, October 2005.

[11] Grecu, L., *The boundary element method applied in fluid mechanics*, Ph.D. Thesis, University of Bucharest, Faculty of Mathematics, 2004.

[12] Lifanov, I.K., *Singular integral equations and discrete vortices*, VSP, Utrecht, TheNetherlands, 1996.

ROMAI J., **2**, 2(2006),
89–92

ASIMPTOTIC BEHAVIOR IN VORTEX PHENOMENA: SIGNIFICANT EVENTS

Adela Ionescu

Faculty of Engng. and Management of Technological Systems, Univ. of Craiova

adela0404@yahoo.com

Abstract The vortex phenomena can be studied both at large and small scale, therefore their applications area is very large, including collection, aggregation and fragmentation of different particles. The turbulent mixing is an important feature of far from equilibrium models. A mixing of a flow implies successive stretching and folding phenomena for system particles, the influence of parameters and initial conditions, and also the issue of significant events - such as rare events - and their physical mean. This paper shows that the turbulent mixing is a basic feature of the vortex phenomena. A comparison between two and three dimensional case is performed.

Keywords: turbulent, mixing, tendril-whirl flow, vortex phenomenon.
2000 MSC: 76F25.

1. THE MIXING CONCEPT

In laminar-turbulent transition theory a *flow* is represented by the map $x = \Phi_t(X)$, with $X = \Phi_t(t=0)(X)$; we say that X *is mapped in* x after a time t. The flow is of class C^k, i.e. $\Phi_t(X) \longrightarrow x$ is a diffeomorphism of class C^k. Moreover, $0 < J < \infty$, when $J = \det\left(\frac{\partial x_i}{\partial X_j}\right)$, or, equivalently, $J = det(D\Phi_t(X))$. Here $D = \frac{d}{dx}$. The basic measure for the deformation with respect to X, is the *deformation gradient*, $\mathbf{F} = (\nabla_X \Phi_t(\mathbf{X}))^T$, $F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right)$, or $\mathbf{F} = D\Phi_t(\mathbf{X})$.

The deformation tensor $\mathbf{F}$ and the associated tensors $\mathbf{C}, \mathbf{C}^{-1}$, (with $\mathbf{C} = \mathbf{F}^T\ \mathbf{F}$) represent the basic quantities in the deformation analysis for the infinitesimal elements. The *length deformation* $\lambda = \lim_{|d\mathbf{X}| \to 0} \frac{|d\mathbf{x}|}{|d\mathbf{X}|}$ and *surface deformation* $\eta = \lim_{|d\mathbf{A}| \to 0} \frac{|d\mathbf{a}|}{|d\mathbf{A}|}$, can be also read as $\lambda = (C : MM)^{\frac{1}{2}}$, $\eta = (\det F) \cdot \left(C^{-1} : NN\right)^{\frac{1}{2}}$, where $\mathbf{M} = d\mathbf{X}/\,|d\mathbf{X}|\,, \mathbf{N} = d\mathbf{A}/\,|d\mathbf{A}|\,.$

In this framework the mixing concept implies the *stretching* and *folding* of the material elements. If in an initial location P there is a material filament dX and an area element dA, the specific length and surface deformations are given by $\frac{D(\ln\lambda)}{Dt} = \mathbf{D} : \mathbf{mm}$, $\frac{D(\ln\eta)}{Dt} = \nabla\mathbf{v} - \mathbf{D} : \mathbf{nn}$, where $\mathbf{D}$ is the deformation tensor. We say that the flow $\mathbf{x} = \Phi_t(\mathbf{X})$ has *a good mixing* if the mean values $D(ln\lambda)/Dt$ and $D(ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations $\mathbf{M}$ and $\mathbf{N}$. The *deformation efficiency in length*, $e_\lambda = e_\lambda(X, M, t)$ of the material element dX, is defined [3,4] by $e_\lambda = \frac{D(ln\lambda)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \leq 1$, while, in the case of an isochoric flow (the jacobian equal 1), the *deformation efficiency in surface*, $e_\eta = e_\eta(X, N, t)$ of the area element dA, by $e_\eta = \frac{D(ln\eta)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \leq 1$.

2. STATISTICAL FEATURES OF THE 3D MIXING

The analysis of e_λ and e_η for a 3D model revealed [1], [3] interesting features associating with a vortex experiment for an aquatic algae (*Spirulina Platensis*). The mechanism [5] has: a small version (with a 15-20mm diameter), and a large version (100-300mm diameter), corresponding to two categories of processing particles – at small and at large scale. The analytical discrete study of the mathematical model associated to the phenomena has confirmed the experimental study. The 3D model reads: $\dot{x_1} = G \cdot x_2, \dot{x_2} = K \cdot G \cdot x_1, \dot{x_3} = c$, where $-1 < K < 1,\ \ c =\ const.$

It is a generalization of the 2D version used in [4], a widespread model for isochoric flows. The third component (corresponding to axis z) represents the rotation velocity, supposed to be constant. The Cauchy problem $x_1(0) = X_1,\ x_2(0) = X_2,\ x_3(0) = X_3$ for the model has a solution $x_i = x_i(X_j)$, i,j=1,2,3 [1,3], where x_i represents the state of the system, at the moment t, with respect to the reference state $X_j, j = 1, 2, 3$ (i.e. it represents the state of the aquatic algae after the vortex experiment).

Let us exhibit the results of the analysis of e_λ and e_η of the material filaments, with the vortex conditions imposed. The studied cases, and the events corresponding to different values of the orientations in length (M_1, M_2, M_3) and in surface (N_1, N_2, N_3) are very few. Their statistical interpretation is realized in [1], including the 2D case. By a *rare event* we mean the event of breaking-up the material filaments, with a corresponding mathematical standpoint in the sudden failing of the running program, or failing the required accuracy. The table represents a synthesis of the work.

3. COMPARISON WITH PERIODIC BEHAVIOR

In [2] there was studied the behavior of a periodic flow: the *tendril-whirl flow (TW)*, introduced by Khakhar, Rising and Ottino (1987). It is a discontinuous succession of *extensional flows and twist maps*.

1.Versors	*2. Versors values*	*3. K*	*4.* $2\cdot\dot{\gamma}$	*5.t*	*6. Remarks*
$2D\,(M_1, M_2)$	$(1,0)$	0,2	0,001		linear
			0,008		linear
			3,0		linear
	$(-1,1)$	0,2	0,008		linear
			0,001	10	sudden growth
$2D\,(N_1, N_2)$	$(1,0)$	0,2	0,008	6	strong discontinuity
		0,5	0,008	2	strong discontinuity
		0,8	0,008		linear
	$(-1,1)$	0,2	0,008	7	rare event
		0,5	0,008		linear
			3,0		linear
$3D\,(M_1, M_2, M_3)$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$	0,2	0,008		linear
			3,0		linear
		0,8	0,008		rare event
			3,0		linear
			$\sqrt{2}$		rare event
	$\left(\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, 0\right)$	0,2	3,0		linear
			$\sqrt{2}$		rare event
		0,8	0,008		rare event
			3,0		linear
	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$	0,2	3,0		linear
			$\sqrt{2}$		rare event
			$\sqrt{3}$		rare event
$3D\,(N_1, N_2, N_3)$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$	0,2	0,01		rare event
	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$	0,2	0,08	8	maxim
			0,8	8	rare event
			$\sqrt{2}$	5	rare event
			$\sqrt{3}$	4	rare event
	$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$	0,8	$\sqrt{2}, \sqrt{3}$		rare events

In the simplest case all the flows are identical and the period of alternation extensional/rotational is also constant. However, this case is complex enough and can be considered as the point of departure for several generalizations (smooth variation, distribution of time periods etc). The physical motivation for this flow is that locally, a velocity field can be decomposed into extension and rotation. Mathematically, by polar theorem, a local deformation can be decomposed into stretching and rotation [4]. In the simplest case of the TW model, the velocity field over a single period is given by its extensional part

$$v_x = -\varepsilon \cdot x, v_y = \varepsilon \cdot y, \quad 0 < t < T_{ext}$$

and rotational part

$$v_r = 0, v_\theta = -\omega\left(r\right), T_{ext} < t < T_{ext} + T_{rot},$$

where T_{ext} denotes the duration of the extensional component and T_{rot} the duration of rotational component. The function $\omega(r)$ is positive and specifies the rate of rotation. Its form is quite arbitrary and its most important aspect is that $\frac{d\omega(r)}{dr} = 0$ for some r. The model consists of vortices producing whorls which are periodically squeezed by the hyperbolic flow leading to the formation of tendrils, and the process repeats. In [2] the efficiency of mixing was evaluated only for the extensional part of TW flow. The computation is less complex, and *the deformations in length and surface are less complex than for three-dimensional (non periodic) flow.* This is due to the fact that in the 3D case there are very few parameters. At the same *random values* for the unit vectors, $\sqrt{2}$, $\sqrt{3}$ etc [2], in the 2D case there seems to be no rare events, the functions e_λ and e_η being linear, while in the 3D case the turbulent mixing occurs. While for the vortex phenomena we have a favorable context of *random distributed events* (events with relative linear behavior, with linear-negative behavior, mixing phenomena and rare events), for the TW model (the extensional component), only the deformation in surface seems to have a non constant behavior (the function e_η is decreasing) [2]. The parameter T_{ext} can be measured in seconds, minutes or even in larger units, depending on the context. The same is available for the 3D flow [1], where the turbulence occurs at *small values of the time units, being in agreement with experiments.* Therefore a further analysis for larger T_{ext} would be useful.

References

[1] Ionescu, A., *The structural stability of biological oscillators. Analytical contributions.* Ph.D. Thesis Bucharest, 2002 (Romanian).

[2] Ionescu, A., *Some qualitative features of turbulent mixing for far from equilibrium phenomena*, Proc. of the ICTAMI2005, Alba Iulia.

[3] Ionescu, A., *The discrete time analysis of the turbulence in excitable media*, Proc. of CAIM 2004, Pitesti.

[4] Ottino, J. M., *The kinematics of mixing: stretching, chaos and transport*, Cambridge University Press, 1989.

[5] Savulescu, St.N., *A special vortex tube for particle processing flows*, Rev. Roum. Sci. Techn.-Mec. Appl., **43**, 5(1998), 611-615.

ROMAI J., **2**, 2(2006),
93–100

A FIXED POINT THEOREM AND SOME APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

Florin Isaia

University "Transilvania" of Brasov

Abstract In this paper, a fixed point theorem for α-condensing maps is proved and two applications of this theorem to nonlinear integral equations without compactness are presented. The main ingredient in the proof of the fixed point theorem is the a priori estimate method which is a consequence of the invariance under homotopy of the degree defined for α-condensing perturbations of the identity. The applications presented here are two existence results. One is for the equation

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))\, ds, \quad t \in [a, b] \tag{1}$$

where $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The other is for the equation

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))\, ds, \quad t \in (a, b) \tag{2}$$

where $\varphi : (a, b) \times \mathbb{R} \to \mathbb{R}$ and $\psi : (a, b) \times (a, b) \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. In both cases, functions φ and ψ satisfy some special growth conditions. The main idea is to transform the integral equations into fixed point problems for some condensing maps $T : C[a, b] \to C[a, b]$, respectively $T : L^p(a, b) \to L^p(a, b)$. Note that the assumptions on functions φ and ψ do not generally ensure the compactness of operator T, therefore the Leray-Schauder degree cannot be used (see K. Deimling [2], Example 9.1, p. 69).

Keywords: nonlinear integral equation, condensing map, topological degree, a priori estimate method

2000 MSC: 45G10, 47H09, 47H10, 47H11, 47H30

1. INTRODUCTION

The topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. Particularly, the a priori estimate method (or the method of a priori bounds) has been often used together with the Brouwer degree, the Leray-Schauder degree or the coincidence degree in order to prove the existence of solutions for some boundary value problems for nonlinear differential equations or nonlinear partial differential equations. See, for example, [7] (Sections V.2 and VI.2), [3] and [4].

In the present paper, the a priori estimate method is used together with the degree for condensing maps in order to prove the existence of solutions $u \in C[a,b]$ for the integral equation (1) under appropriate assumptions on functions φ and ψ. We also prove the existence of solutions $u \in L^p(a,b)$ for the integral equation (2) under appropriate assumptions on functions φ and ψ. The results presented herein are in relation with the results in [5], [6].

2. THE TOPOLOGICAL DEGREE FOR CONDENSING MAPS

For a minute description of the following notions we refer the reader to [2]. In the following, X is a Banach space and $\mathcal{B} \subset \mathcal{P}(X)$ is the family of all its bounded sets.

Definition 2.1 *The function $\alpha : \mathcal{B} \to \mathbb{R}_+$ defined by*

$$\alpha(B) = \inf \left\{ \begin{array}{c} d > 0 : B \text{ admits a finite cover} \\ \text{by sets of diameter} \leq d \end{array} \right\}, \quad B \in \mathcal{B},$$

is called the (Kuratowski-) measure of noncompactness.
Throughout paper, the letter α is used only in this context. We state without proof some properties of this measure.

Proposition 2.1 *The following assertions hold: (a) $\alpha(B) = 0$ iff B is relatively compact; (b) α is a seminorm, i.e. $\alpha(\lambda B) = |\lambda|\,\alpha(B)$ and $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$; (c) $B_1 \subset B_2$ implies $\alpha(B_1) \leq \alpha(B_2)$; $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$; (d) $\alpha(convB) = \alpha(B)$; (e) $\alpha(\overline{B}) = \alpha(B)$.*

Definition 2.2 *Consider $\Omega \subset X$ and $F : \Omega \to X$ a continuous bounded map. We say that F is α-Lipschitz if there exists $k \geq 0$ such that $\alpha(F(B)) \leq k\alpha(B) \quad (\forall)\, B \subset \Omega$ bounded. If, in addition, $k < 1$, then we say that F is a strict α -contraction. We say that F is α-condensing if $\alpha(F(B)) < \alpha(B) \quad (\forall)\, B \subset \Omega$ bounded with $\alpha(B) > 0$. In other words, $\alpha(F(B)) \geq \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict α-contractions $F : \Omega \to X$ is denoted by $SC_\alpha(\Omega)$ and the class of all α-condensing maps $F : \Omega \to X$ is denoted by $C_\alpha(\Omega)$.*

Remark that $SC_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is α-Lipschitz with constant $k = 1$. We also recall that $F : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that $\|Fx - Fy\| \leq k\,\|x - y\| \quad (\forall)\, x, y \in \Omega$ and that F is a strict contraction if $k < 1$. Next, we state without proof some properties of the applications defined above.

Proposition 2.2 *If $F, G : \Omega \to X$ are α-Lipschitz maps with constants k, respectively k', then $F + G : \Omega \to X$ is α -Lipschitz with constant $k + k'$.*

Proposition 2.3 *If $F : \Omega \to X$ is compact, then F is α-Lipschitz with constant $k = 0$.*

Proposition 2.4 *If $F : \Omega \to X$ is Lipschitz with constant k, then F is α-Lipschitz with the same constant k.*

The theorem below asserts the existence and the basic properties of the topological degree for α-condensing perturbations of the identity.

Theorem 2.1 *Let*

$$\mathcal{T} = \left\{ \begin{array}{c} (I - F, \Omega, y) : \Omega \subset X \text{ open and bounded}, \\ F \in C_\alpha\left(\overline{\Omega}\right), \ y \in X \setminus (I - F)(\partial\Omega) \end{array} \right\}$$

be the family of the admissible triplets. There exists one degree function $D : \mathcal{T} \to \mathbb{Z}$ which satisfies the properties:

(D1) (normalization) $D(I, \Omega, y) = 1$ for every $y \in \Omega$;

(D2) (additivity on domain) for every disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and every $y \notin (I - F)\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$ we have

$$D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y);$$

(D3) (invariance under homotopy) $D(I - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ for every continuous, bounded map $H : [0, 1] \times \overline{\Omega} \to X$ which satisfies $\alpha(H([0, 1] \times B)) < \alpha(B) \quad (\forall) B \subset \overline{\Omega}$ with $\alpha(B) > 0$ and every continuous function $y : [0, 1] \to X$ which satisfies

$$y(t) \neq x - H(t, x) \quad (\forall) t \in [0, 1], (\forall) x \in \partial\Omega;$$

(D4) (existence) $D(I - F, \Omega, y) \neq 0$ implies $y \in (I - F)(\Omega)$;

(D5) (excision) $D(I - F, \Omega, y) = D(I - F, \Omega_1, y)$ for every open set $\Omega_1 \subset \Omega$ and every $y \notin (I - F)\left(\overline{\Omega} \setminus \Omega_1\right)$.

Having in hand a degree function defined on $\mathcal{T}$, we study the usability of the *a priori estimate method* by means of this degree. We will obtain a fixed point theorem which will be used in Sections 3 and 4 in the proofs of the main existence results.

Theorem 2.2 *Let $F : X \to X$ be α-condensing and*

$$S = \{x \in X : (\exists) \lambda \in [0, 1] \text{ such that } x = \lambda F x\}.$$

If S is a bounded set in X, so there exists $r > 0$ such that $S \subset B_r(0)$, then

$$D(I - \lambda F, B_r(0), 0) = 1 \quad (\forall) \lambda \in [0, 1].$$

Consequently, F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

Proof. Remark that every affine homotopy of α-condensing maps is an admissible homotopy. Indeed, consider a bounded open set $\Omega \subset X$, the maps $F_1, F_2 \in C_\alpha\left(\overline{\Omega}\right)$ and let $H : [0,1] \times \overline{\Omega} \to X$ be defined by

$$H(t,x) = (1-t)F_1x + tF_2x.$$

For every $B \subset \overline{\Omega}$ with $\alpha(B) > 0$ we have

$$H([0,1] \times B) \subset conv\,(F_1(B) \cup F_2(B))$$

and, using Proposition 2.1,

$$\begin{aligned} \alpha(H([0,1]\times B)) &\leq \alpha(conv\,(F_1(B)\cup F_2(B))) \\ &= \alpha(F_1(B)\cup F_2(B)) \\ &= \max\{\alpha(F_1(B)), \alpha(F_2(B))\} < \alpha(B). \end{aligned}$$

Next, fix $\lambda \in [0,1]$ and consider the affine homotopy between the α-condensing maps $\lambda F, 0 \in C_\alpha(X)$

$$H : [0,1] \times X \to X, \quad H(t,x) = (1-t)\,0x + t\lambda Fx = t\lambda Fx.$$

By the previous argument,

$$\alpha(H([0,1]\times B)) < \alpha(B) \quad (\forall)\, B \subset X \textit{ bounded with } \alpha(B) > 0.$$

If $x \in X$ and $t \in [0,1]$ check $x - H(t,x) = 0$, then $x \in S \subset B_r(0)$. Thus, we can use the properties (D3), (D1) of the degree to obtain
$D(I-\lambda F, B_r(0), 0) = D(I - H(1,\cdot), B_r(0), 0) = D(I - H(0,\cdot), B_r(0), 0) =$
$= D(I, B_r(0), 0) = 1$. Finally, the property (D4) of the degree is used. □

3. THE FIRST EXISTENCE RESULT

Consider equation (1) where $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy the following conditions: (a) there exist $C_1, M_1 \geq 0$, $q_1 \in [0,1)$ such that $|\varphi(t,x)| \leq C_1 |x|^{q_1} + M_1$, for every $(t,x) \in [a,b] \times \mathbb{R}$; (b) there exists $K_1 \in [0,1)$ such that $|\varphi(t,x) - \varphi(t,y)| \leq K_1 |x-y|$ for every $(t,x), (t,y) \in [a,b] \times \mathbb{R}$; (c) there exist $C_2, M_2 \geq 0$, $q_2 \in [0,1)$ such that $|\psi(t,s,x)| \leq C_2 |x|^{q_2} + M_2$, for every $(t,s,x) \in [a,b] \times [a,b] \times \mathbb{R}$.
Under these assumptions, we show that (1) has at least one solution $u \in C[a,b]$.

Define the operators

$$\begin{aligned} F &: C[a,b] \to C[a,b], \quad (Fu)(t) = \varphi(t,u(t)), \ t \in [a,b], \\ G &: C[a,b] \to C[a,b], \quad (Gu)(t) = \int_a^b \psi(t,s,u(s))\,ds, \ t \in [a,b], \\ T &: C[a,b] \to C[a,b], \quad Tu = Fu + Gu. \end{aligned}$$

Then, equation (1) can be written as $u = Tu$, i.e. the existence of a solution for equation (1) is equivalent to the existence of a fixed point for operator T and it is further proved by using

Proposition 3.1 [6]. *The operator $F : C[a,b] \to C[a,b]$ is Lipschitz with constant K_1. Consequently F is α-Lipschitz with the same constant K_1. Moreover, for every $u \in \mathbb{C}[a,b]$, F satisfies the following growth condition*

$$\|Fu\|_{C[a,b]} \le C_1 \|u\|_{C[a,b]}^{q_1} + M_1. \tag{3}$$

Proposition 3.2 [6]. *The operator $G : C[a,b] \to C[a,b]$ is compact. Consequently G is α-Lipschitz with zero constant. Moreover, for every $u \in \mathbb{C}[a,b]$, G satisfies the following growth condition*

$$\|Gu\|_{C[a,b]} \le C_2(b-a)\|u\|_{C[a,b]}^{q_2} + (b-a)M_2, \tag{4}$$

Now, we have the possibility to prove the main result of this section.

Theorem 3.1 *If $\varphi : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $\psi : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy conditions (a), (b),(c), then the integral equation*

$$u(t) = \varphi(t,u(t)) + \int_a^b \psi(t,s,u(s))\,ds, \quad t \in [a,b],$$

has at least one solution $u \in C[a,b]$ and the set of the solutions of equation (1) is bounded in $C[a,b]$.

Proof. Let $F, G, T : C[a,b] \to C[a,b]$ be the above-defined. They are continuous and bounded. Moreover, by Propositions 3.1 and 3.2, F is α-Lipschitz with constant $K_1 \in [0,1)$ and G is α-Lipschitz with zero constant. Then Proposition 2.2 implies that T is a strict α -contraction with constant K_1. Set

$$S = \{u \in C[a,b] : (\exists)\,\lambda \in [0,1] \ \text{such that} \ u = \lambda Tu\}.$$

Next, we prove that S is bounded in $C[a,b]$. Consider $u \in S$ and $\lambda \in [0,1]$ such that $u = \lambda Tu$. From (3) and (4) it follows that

$$\begin{aligned} \|u\|_{C[a,b]} &= \lambda\|Tu\|_{C[a,b]} \le \lambda\left(\|Fu\|_{C[a,b]} + \|Gu\|_{C[a,b]}\right) \\ &\le \lambda\left[C_1\|u\|_{C[a,b]}^{q_1} + C_2(b-a)\|u\|_{C[a,b]}^{q_2} + M_1 + (b-a)M_2\right]. \end{aligned}$$

This inequality, together with $q_1 < 1$, $q_2 < 1$, imply that S is bounded in $C[a,b]$. Consequently, by Theorem 2.2, T has at least one fixed point and the set of the fixed points of T is bounded in $C[a,b]$. □

Remark 3.1 *(i) if the growth condition (a) is formulated for $q_1 = 1$, then the conclusions of Theorem 3.1 remain valid provided that $C_1 < 1$; (ii) if the growth condition (c) is formulated for $q_2 = 1$, then the conclusions of Theorem 3.1 remain valid provided that $(b-a)\,C_2 < 1$; (iii) if the growth conditions (a) and (c) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 3.1 remain valid provided that $C_1 + (b-a)\,C_2 < 1$.*

Remark 3.2 *The conclusions of Theorem 3.1 remain valid provided that equation (1) is replaced by*

$$u(t) = \varphi(t, u(t)) + \int_a^t \psi(t, s, u(s))\,ds, \quad t \in [a,b].$$

Only slight modifications in the proof of Proposition 3.2 are needed.

4. THE SECOND EXISTENCE RESULT

Consider equation (2). Our purpose is to study the existence of a solution in $L^p(a,b)$ of equation (2), with $p \in [1,\infty)$ fixed. Define the operators

$$\begin{aligned} u &\mapsto Fu, \quad (Fu)(t) = \varphi(t, u(t)),\ t \in (a,b), \\ u &\mapsto Gu, \quad (Gu)(t) = \int_a^b \psi(t, s, u(s))\,ds,\ t \in (a,b), \\ u &\mapsto Tu, \quad Tu = Fu + Gu. \end{aligned}$$

Then, equation (2) can be written as $u = Tu$, i.e. the existence of a solution for equation (1) is equivalent to the existence of a fixed point for operator T.

We prescribe the following conditions on functions $\varphi : (a,b) \times \mathbb{R} \to \mathbb{R}$ and $\psi : (a,b) \times (a,b) \times \mathbb{R} \to \mathbb{R}$: (a) for every $x \in \mathbb{R}$, the function $\varphi(\cdot, x) : (a,b) \to \mathbb{R}$ is Lebesgue measurable; (b) for a.e. $t \in (a,b)$, the function $\varphi(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous; (c) there exist $C_1 \geq 0$, $q_1 \in [0,1)$, $f \in L^p(a,b)$, $f \geq 0$, such that $|\varphi(t,x)| \leq C_1 |x|^{q_1} + f(t)$ for a.e. $t \in (a,b)$ and every $x \in \mathbb{R}$; (d) there exists $K_1 \in [0,1)$ such that $|\varphi(t,x) - \varphi(t,y)| \leq K_1 |x-y|$ for a.e. $t \in (a,b)$ and every $x, y \in \mathbb{R}$; (e) for every $x \in \mathbb{R}$, the function $\psi(\cdot,\cdot,x) : (a,b) \times (a,b) \to \mathbb{R}$ is Lebesgue measurable; (f) for a.e. $(t,s) \in (a,b) \times (a,b)$, the function $\psi(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous; (g) there exist $C_2 \geq 0$, $q_2 \in [0,1)$, $g \in L^p((a,b) \times (a,b))$, $g \geq 0$, such that $|\psi(t,s,x)| \leq C_2 |x|^{q_2} + g(t,s)$ for a.e. $(t,s) \in (a,b) \times (a,b)$ and every $x \in \mathbb{R}$; (h) there exist $K_2 > 0$, $r_1 \in [0,p)$, $r_2 > 0$ and $k : (a,b) \times (a,b) \to \mathbb{R}$ measurable such that: (h1) for a.e. $(t_1,s), (t_2,s) \in (a,b) \times (a,b)$ and every $x \in \mathbb{R}$,

$|\psi(t_1,s,x)-\psi(t_2,s,x)| \leq K_2|x|^{r_1}|k(t_1,s)-k(t_2,s)|^{r_2}$ and (h2) for every $\omega \subset\subset (a,b)$, $\lim_{h\to 0}\int_\omega \left(\int_a^b |k(t+h,s)-k(t,s)|^{pr_2/(p-r_1)}ds\right)^{p-r_1} dt = 0$. Under these assumptions, we show that equation (2) has at least one solution $u \in L^p(a,b)$. Our proof is based on the following three propositions [5].

Proposition 4.1 *The operator $F : L^p(a,b) \to L^p(a,b)$ is well-defined, bounded and continuous. Moreover, for every $u \in L^p(a,b)$, operator F satisfies the following growth condition*

$$\|Fu\|_{L^p} \leq 2^{\frac{p-1}{p}}\left[(b-a)^{\frac{1-q_1}{p}} C_1 \|u\|_{L^p}^{q_1} + \|f\|_{L^p}\right]. \tag{5}$$

Proposition 4.2 *The operator $G : L^p(a,b) \to L^p(a,b)$ is well-defined, bounded and continuous. Moreover, for every $u \in L^p(a,b)$, operator G satisfies the following growth condition*

$$\|Gu\|_{L^p} \leq 2^{\frac{p-1}{p}}\left[(b-a)^{\frac{p+1-q_2}{p}} C_2 \|u\|_{L^p}^{q_2} + (b-a)^{\frac{p-1}{p}}\|g\|_{L^p}\right]. \tag{6}$$

Proposition 4.3 *The operator $G : L^p(a,b) \to L^p(a,b)$ is compact.*

Remark 4.1 *The compactness of G is a consequence of condition (h). Note that condition (h) is satisfied if there exist $K_2 > 0$, $r_1 \in [0,p)$, $r_2 > 0$ and $k : (a,b)\times[a,b] \to \mathbb{R}$ continuous such that*

$$|\psi(t_1,s,x)-\psi(t_2,s,x)| \leq K_2|x|^{r_1}|k(t_1,s)-k(t_2,s)|^{r_2},$$

for a.e. $(t_1,s),(t_2,s) \in (a,b)\times(a,b)$ and every $x \in \mathbb{R}$. (Condition (h2) is automatically satisfied.) Moreover, in this case, Proposition 4.3 remains valid for $r_1 = p$ too.

Now, we have the possibility to prove the main result of this section.

Theorem 4.1 *If the functions $\varphi : (a,b)\times\mathbb{R} \to \mathbb{R}$ and $\psi : (a,b)\times(a,b)\times\mathbb{R} \to \mathbb{R}$ satisfy conditions (a)-(h), then the integral equation (2) has at least one solution $u \in L^p(a,b)$ and the set of its solutions is bounded in $L^p(a,b)$.*

Proof. Let $F, G, T : L^p(a,b) \to L^p(a,b)$ be the above-defined operators. They are continuous, bounded and, moreover, by Propositions 4.1, 4.2, 4.3, G is compact. Then G is α-Lipschitz with zero constant (see Proposition 2.3). From (d), it follows that for every $u,v \in L^p(a,b)$, we have $\|Fu-Fv\|_{L^p} \leq K_1\|u-v\|_{L^p}$, which means that F is a Lipschitz map with constant K_1. By Proposition 2.4, F is α-Lipschitz with constant K_1. Proposition 2.2 shows us that T is a strict α-contraction with constant K_1.

Set $S = \{u \in L^p(a,b) : (\exists)\, \lambda \in [0,1] \text{ such that } u = \lambda T u\}$. Next, we prove that S is bounded in $L^p(a,b)$. Consider $u \in S$ and $\lambda \in [0,1]$ such that $u = \lambda T u$. From (5) and (6) it follows that

$$\begin{aligned} \|u\|_{L^p} &= \lambda \|Tu\|_{L^p} \leq \lambda \left(\|Fu\|_{L^p} + \|Gu\|_{L^p}\right) \\ &\leq \lambda 2^{\frac{p-1}{p}} \Bigg[(b-a)^{\frac{1-q_1}{p}} C_1 \|u\|_{L^p}^{q_1} + \|f\|_{L^p} \\ &\quad + (b-a)^{\frac{p+1-q_2}{p}} C_2 \|u\|_{L^p}^{q_2} + (b-a)^{\frac{p-1}{p}} \|g\|_{L^p} \Bigg]. \end{aligned}$$

This inequality, together with $q_1 < 1$, $q_2 < 1$, shows that S is bounded in $L^p(a,b)$. Consequently, by Theorem 2.2, T has at least one fixed point and the set of the fixed points of T is bounded in $L^p(a,b)$. □

Remark 4.2 *(i) if the growth condition (c) is formulated for $q_1 = 1$, then the conclusions of Theorem 4.1 remain valid provided that $2^{\frac{p-1}{p}} C_1 < 1$;*

(ii) if the growth condition (g) is formulated for $q_2 = 1$, then the conclusions of Theorem 4.1 remain valid provided that $2^{\frac{p-1}{p}} (b-a) C_2 < 1$;

(iii) if the growth conditions (c) and (g) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 4.1 remain valid provided that $2^{\frac{p-1}{p}} (C_1 + (b-a) C_2) < 1$.

References

[1] H. Brezis, *Analyse fonctionelle. Théorie et applications*, Masson, Paris, 1987.

[2] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985.

[3] G. Dinca, P. Jebelean, *Une méthode de point fixe pour le p-Laplacien*, C. R. Acad. Sci. Paris, Série I, **324** (1997), 165-168.

[4] G. Dinca, P. Jebelean, J. Mawhin, *Variational and topological methods for Dirichlet problems with p-Laplacian*, Portugaliae Mathematica, **58**, 3(2001), 339-378.

[5] F. Isaia, *An existence result for a nonlinear integral equation without compactness*, PanAmerican Mathematical J., **14**,4(2004), 93-106.

[6] F. Isaia, *On a nonlinear integral equation without compactness*, submitted.

[7] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, CMBS Regional Conference Series in Mathematics, **40**, AMS, Providence, R.I., 1979.

ROMAI J., **2**, 2(2006),
101–108

MOMENTS OF FIRST-PASSAGE PLACES AND RELATED RESULTS FOR THE INTEGRATED BROWNIAN MOTION

Mario Lefebvre

Département de Mathématiques et de Génie Industriel, École Polytechnique, Montréal, Canada

mlefebvre@polymtl.ca

Abstract We consider first-passage problems for the two-dimensional diffusion process $(X(t), Y(t))$, where $Y(t)$ is a Wiener process and $X(t)$ is its integral. Let $T(x, y)$ be the first time $Y^3(t)/X(t)$ leaves a certain region of the second quadrant. With the help of the method of similarity solutions, we obtain an exact solution to the Kolmogorov backward equation, subject to the appropriate boundary conditions, satisfied by the moments of $Y(T(x, y))$. Similarly, the probability that the process $(X(t), Y(t))$ will hit a given part of the boundary is explicitly computed.

Keywords: Kolmogorov backward equation, similarity solutions, hitting time.
2000 MSC: 35Q80, 60J60.

1. INTRODUCTION

First-passage problems are important in many applications. For example, in mathematical finance, one is interested in computing the time it takes some stocks to reach a given price, at which point an *option* is exercised. In biology, people want to determine the time needed for a neuron to *fire a spike*, once it has attained a fixed threshold. Other fields in which first-passage problems appear are those of physics, chemistry and electrical engineering, in particular.

In most cases, first-passage problems are concerned with determining the *time* it takes a stochastic process to reach or cross some boundary. In one dimension, suppose that $B(t)$ is a Brownian motion (starting from 0) and that τ is the time it takes for $B(t)$ to become equal to $a > 0$. Then, one has obviously $B(\tau) = a$. In the case when τ is the time until $B(t)$ leaves the interval $[-b, a]$, where $b > 0$, one might be interested in computing the probability that $B(\tau) = a$. In two or more dimensions, the problem of obtaining the distribution, or at least the moments, of *first-passage places* is generally quite difficult, as it involves solving partial differential equations subject to the appropriate boundary conditions. The author has considered such problems for the most important diffusion processes [3]-[6].

Let $Y(t)$ be a Wiener process with drift coefficient μ and diffusion coefficient σ^2, and $X(t)$ be its integral, so that

$$dX(t) = Y(t)dt, \tag{1}$$

$$dY(t) = \mu\, dt + \sigma dW(t), \tag{2}$$

where $W(t)$ is a standard Brownian motion. In this paper, we will see that sometimes it is possible to obtain relatively simple solutions to certain first-passage problems for the two-dimensional diffusion process $(X(t), Y(t))$ by making use of the method of similarity solutions to solve the Kolmogorov backward equation satisfied by the function of interest.

We define

$$T(x, y) = \inf\{t > 0 : Y^3(t)/X(t) = k_1 \text{ or } k_2 | X(0) = x,\ Y(0) = y\}, \tag{3}$$

where $y^3/x \in (k_2, k_1)$ and $-\infty < k_2 < k_1 < 0$, with $x < 0$ and $y > 0$. The moment generating function of the random variable $T(x, y)$, namely

$$L(x, y; \alpha) := E[e^{-\alpha T(x,y)}], \tag{4}$$

where $\alpha > 0$, satisfies the Kolmogorov backward equation

$$\frac{1}{2}\sigma^2 L_{yy} + \mu L_y + y L_x = \alpha L, \tag{5}$$

where $L_{yy} \equiv \frac{\partial^2 L}{\partial y^2}$, etc. This equation is subject to the boundary conditions

$$L(x, y; \alpha) = 1 \quad \text{if } y^3/x = k_1 \text{ or } k_2. \tag{6}$$

Ideally, we would like to first find the function $L(x, y; \alpha)$ and then invert the Laplace transform to obtain the probability density function of $T(x, y)$. Unfortunately, this is rarely possible for this type of problem and most often we must content ourselves with finding the function $L(x, y; \alpha)$ only, or at least the moments of $T(x, y)$.

The method of similarity solutions consists in assuming that there is a certain relationship between the variables x and y. For instance, because the first-passage time $T(x, y)$ is defined in terms of the ratio y^3/x, we could try a solution of the form

$$L(x, y; \alpha) = N(z; \alpha), \tag{7}$$

where $z := y/x^{1/3}$. Unfortunately, we find that this particular instance of the method of similarity solutions fails in the case of the moment generating function. However, if $\mu = 0$, this method enables us to explicitly compute the moments of $Y(T(x, y))$ and $X(T(x, y))$, as well as the probability that $Y^3(T(x, y))/X(T(x, y)) = k_2$. We will also obtain the expected value of $\ln Y(T(x, y))$. These functions will be computed in Sections 2, 3 and 4, respectively. The paper will then end with some remarks in Section 5.

2. MOMENTS OF $Y(T(X,Y))$ AND $X(T(X,Y))$

As mentioned in Section 1, we would like to explicitly compute the function $L(x, y; \alpha)$. Assuming that (7) holds, then (5) is transformed into

$$\frac{1}{2}\sigma^2 \frac{1}{x^{2/3}} N''(z;\alpha) + \mu \frac{1}{x^{1/3}} N'(z;\alpha) - y \frac{y}{3x^{4/3}} N'(z;\alpha) = \alpha N(z;\alpha) \tag{8}$$

and the boundary conditions would be

$$N(z;\alpha) = 1 \quad \text{if } z^3 = k_1 \text{ or } k_2. \tag{9}$$

For this transformation to be valid, we must be able to express the coefficients of N, N' and N'' in (8) in terms of z. We see that this is not possible, even if we set μ equal to 0.

However, this method works in the case of the moments of $Y(T(x,y))$ (and $X(T(x,y))$). Indeed, the function

$$m_k(x,y) := E[Y^k(T(x,y))] \tag{10}$$

is a solution of the partial differential equation (p.d.e.)

$$\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} m_k(x,y) + \mu \frac{\partial}{\partial y} m_k(x,y) + y \frac{\partial}{\partial x} m_k(x,y) = 0, \tag{11}$$

together with the boundary conditions

$$m_k(x,y) = y^k \quad \text{if } y^3/x = k_1 \text{ or } k_2. \tag{12}$$

We must set μ equal to 0. If we choose σ equal to 1 (for simplicity), then $Y(t)$ is a standard Brownian motion and we find that (11) becomes

$$\frac{1}{2} n_k''(z) - \frac{z^2}{3} n_k'(z) = 0, \tag{13}$$

where

$$n_k(z) := m_k(x,y) \tag{14}$$

(with $z := y/x^{1/3}$). Because the boundary conditions (12) must also be expressed in terms of the variable z, we next define

$$n_k(z) \; (= n_k(y/x^{1/3})) \; = y^k r_k(z). \tag{15}$$

The function $r_k(z)$ is such that

$$r_k(z) = 1 \quad \text{if } z^3 = k_1 \text{ or } k_2 \tag{16}$$

and is a solution of

$$\frac{z^2}{2} r_k''(z) + z\left(k - \frac{z^3}{3}\right) r_k'(z) + \frac{1}{2} k(k-1)\, r_k(z) = 0. \tag{17}$$

The general solution of the second order linear ordinary differential equation (o.d.e.)

$$z^2 r_k''(z) + z(a + bz^3) r_k'(z) + c\, r_k(z) = 0 \tag{18}$$

can be written as

$$r_k(z) = \exp\left\{-bz^3/6\right\} z^{-(a/2)-1} \times \tag{19}$$

$$\left[c_1 M\left(-\frac{1}{3} - \frac{a}{6}, \frac{1}{6}\left(1 + a^2 - 2a - 4c\right)^{1/2}, \frac{b}{3}z^3\right)\right.$$
$$\left. + c_2 W\left(-\frac{1}{3} - \frac{a}{6}, \frac{1}{6}\left(1 + a^2 - 2a - 4c\right)^{1/2}, \frac{b}{3}z^3\right)\right],$$

where c_1 and c_2 are constants, and $M(\cdot,\cdot,\cdot)$ and $W(\cdot,\cdot,\cdot)$ are Whittaker functions (see [1, p. 505]). We may now state the following proposition.

Proposition 2.1 *The k^{th} order moment of the random variable $Y(T(x,y))$ is given by $y^k r_k(z)$, where $r_k(z)$ is given in (19), with $a = 2k$, $b = -2/3$ and $c = k(k-1)$, and in which the constants c_1 and c_2 are* uniquely *determined from the boundary conditions (16).*

Proof. . We obtained an explicit solution to our problem by making use of a particular case of the method of similarity solutions. However, we must make sure that the solution obtained is indeed the one we are looking for.

Now, from the fact that $X(t)$ increases when $Y(t)$ is positive (see (1)) we deduce that $P[T(x,y) < \infty] = 1$. It then follows, using the results in [4, section 9.1], that we can assert that the solution to (11), (12) is *unique*, which completes the proof. □

Corollary 2.1 *The expected value of $Y(T(x,y))$, denoted by $m_1(x,y) = y\, r_1(z)$, where $r_1(z)$ is a solution of*

$$z\, r_1''(z) = [(2/3)z^3 - 1] r_1'(z), \tag{20}$$

can be expressed as

$$E[Y(T(x,y))] = y \exp\left\{y^3/9x\right\} (x^{2/3}/y^2) \times \tag{21}$$

$$\left[c_1 M\left(-\frac{2}{3}, \frac{1}{6}, -\frac{2}{9}\frac{y^3}{x}\right) + c_2 W\left(-\frac{2}{3}, \frac{1}{6}, -\frac{2}{9}\frac{y^3}{x}\right)\right],$$

where c_1 and c_2 are such that $E[Y(T(x,y))] = y$ if $y^3/x = k_1$ or k_2.

Remarks. i) The solution may also be expressed in terms of the incomplete gamma function.

ii) The uniqueness of the solution will be true in the other problems considered in this paper as well.

To complete this section, we will find the moments of the random variable $X(T(x,y))$. Let

$$e_k(x,y) := E[X^k(T(x,y))] \tag{22}$$

We set

$$e_k(x,y) = x^k f_k(x,y) \tag{23}$$

and we assume that $f_k(x,y) = \lambda_k(z)$, with $z = y/x^{1/3}$ as previously. The function $\lambda_k(z)$ is such that $\lambda_k(k_i^{1/3}) = 1$, for $i = 1, 2$, and satisfies the ordinary differential equation

$$\frac{1}{2}\lambda_k''(z) - \frac{1}{3}z^2\lambda_k'(z) + kz\lambda_k(z) = 0. \tag{24}$$

Since the general solution of the o.d.e.

$$\lambda_k''(z) + az^2\lambda_k'(z) + bz\lambda_k(z) = 0 \tag{25}$$

can be written as

$$\lambda_k(z) = \frac{\exp\{-az^3/6\}}{z} \times \tag{26}$$

$$\left[c_1 M\left(-\frac{1}{3} + \frac{b}{3a}, \frac{1}{6}, \frac{a}{3}z^3\right) + c_2 W\left(-\frac{1}{3} + \frac{b}{3a}, \frac{1}{6}, \frac{a}{3}z^3\right)\right],$$

where c_1 and c_2 are constants chosen so that the boundary conditions are satisfied, we can write an explicit expression for $E[X^k(T(x,y))]$.

3. PROBABILITY THAT $Y^3(T(X,Y))/X(T(X,Y)) = K_2$

We know that the two-dimensional diffusion process $(X(t), Y(t))$, starting from a point located in the second quadrant, is certain to eventually leave the region C defined by

$$C = \{(x,y) \in mathbbR^2 : -\infty < k_2 < y^3/x < k_1 < 0\}. \tag{27}$$

We are now interested in computing the probability π that $(X(t), Y(t))$ will leave C through the boundary $y^3/x = k_2$. Note that since $P[T(x,y) < \infty] = 1$, the probability $P[Y^3(T(x,y))/X(T(x,y)) = k_1]$ is simply equal to $1 - \pi$.

The probability π is actually a function $\pi(x,y)$ of the starting point (x,y). It can be shown that, with $\mu = 0$ and $\sigma = 1$, it satisfies the Kolmogorov backward equation

$$\frac{1}{2}\pi_{yy}(x,y) + y\,\pi_x(x,y) = 0, \tag{28}$$

subject to the boundary conditions

$$\pi(x,y) = \begin{cases} 1 & \text{if } y^3/x = k_2, \\ 0 & \text{if } y^3/x = k_1. \end{cases} \tag{29}$$

As in the previous section, we try a solution of the form $\pi(x,y) = \nu(z)$, where $z = y/x^{1/3}$. (28) then simplifies to

$$\frac{1}{2}z^2\nu''(z) - \frac{z^4}{3}\nu'(z) = 0 \quad \overset{z\neq 0}{\Longleftrightarrow} \quad \nu''(z) - \frac{2z^2}{3}\nu'(z) = 0 \tag{30}$$

and the boundary conditions become $\nu(k_1^{1/3}) = 0$, $\nu(k_2^{1/3}) = 1$. We find that the general solution of this o.d.e. may be written as follows

$$\nu(z) = c_1 + c_2\left(2\sqrt{3}\pi - 3\,\Gamma(2/3)\,\Gamma\left(\frac{1}{3}, -\frac{2}{9}z^3\right)\right) \quad \text{for } k_2 < z < k_1, \tag{31}$$

where $\Gamma(\cdot,\cdot)$ is the *incomplete gamma function*, which is defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt \tag{32}$$

if $\mathrm{Re}(a) > 0$. It can be expressed in terms of *confluent hypergeometric functions*. Using the fact that the solution to (28), (29) is *unique*, we can state the proposition that follows.

Proposition 3.1 *The probability $\pi(x,y) := P[Y^3(T(x,y))/X(T(x,y)) = k_2]$ is given by the function in (31), where $z = y/x^{1/3}$ and the constants c_1 and c_2 are uniquely determined from the boundary conditions (29).*

Remark. Notice that we did not have to transform the function $\nu(z)$ to express the boundary conditions in terms of z since $\nu(z)$ is equal to either of two constants on the boundary.

4. EXPECTED VALUE OF $\ln Y(T(X,Y))$

Finally, we obtain an explicit formula for the mathematical expectation $E[\ln Y(T)]$. Let $h(x,y) := E[\ln Y(T(x,y))]$. The function $h(x,y)$ satisfies the same p.d.e. as $m_k(x,y)$ (with $\mu = 0$ and $\sigma = 1$; see (11)

$$\frac{1}{2}h_{yy}(x,y) + yh_x(x,y) = 0. \tag{33}$$

This time, the boundary conditions are

$$h(x,y) = \ln y \quad \text{if } y^3/x = k_1 \text{ or } k_2. \tag{34}$$

Let

$$h(x,y) = g(x,y) + \ln y. \tag{35}$$

We find that

$$\frac{1}{2} g_{yy} + y g_x = \frac{1}{2y^2} \tag{36}$$

(and $g(x,y) = 0$ if $y^3/x = k_1$ or k_2). Assuming that $g(x,y) = \phi(z)$, where $z = y/x^{1/3}$, we reduce the preceding p.d.e. to the o.d.e.

$$\phi''(z) - \frac{2}{3}\, z^2\, \phi'(z) = \frac{1}{2z^2}, \tag{37}$$

which has the general solution

$$\phi(z) = \int_{k_2^{1/3}}^{z} \left[c\, e^{2w^3/9} - \frac{6^{2/3}}{20} w e^{w^3/9} M\left(\frac{1}{3}, \frac{5}{6}, \frac{2w^3}{9}\right) - \frac{w^2}{6} - \frac{1}{2w} \right] dw, \tag{38}$$

where $M(\cdot,\cdot,\cdot)$ is a Whittaker function and the constant c can be found by making use of the boundary condition $\phi(k_1^{1/3}) = 0$. That is,

$$c = \int_{k_2^{1/3}}^{k_1^{1/3}} \left[\frac{6^{2/3}}{20} w e^{w^3/9} M\left(\frac{1}{3}, \frac{5}{6}, \frac{2w^3}{9}\right) + \frac{w^2}{6} + \frac{1}{2w} \right] dw \bigg/ \int_{k_2^{1/3}}^{k_1^{1/3}} e^{2w^3/9}\, dw. \tag{39}$$

Summing up, we have the following proposition.

Proposition 4.1 *The mathematical expectation $E[\ln Y(T(x,y))]$ is given by the formulas (38) and (39), in which $z = y/x^{1/3}$, for $k_2 \le z \le k_1$ (< 0).*

Remark 4.1 *Proceeding as above, we can also compute the mathematical expectation $E[\ln(-X(T(x,y)))]$. This time, the o.d.e. that we have to solve is*

$$\psi''(z) - \frac{2}{3}\, z^2\, \psi'(z) = -2z, \tag{40}$$

where $\psi(z) = \psi(y/x^{1/3}) = E[\ln(-X(T(x,y)))] - \ln(-x)$. We find that its solution that satisfies the boundary conditions $\psi(z) = 0$ if $z^3 = k_1$ or k_2 is quite similar to the one above.

5. CONCLUSION

In this paper, we have explicitly computed the moments of a random variable denoting a first-passage place for an important two-dimensional diffusion process, namely the process $(X(t), Y(t))$, where $Y(t)$ is a Wiener process with zero drift and $X(t)$ is its integral. We also obtained exact solutions to other related problems.

We used the method of similarity solutions to solve the appropriate p.d.e.'s. Because the solutions we were looking for are unique, we could appeal to any method to get the required solutions. Notice, however, that this method (at least the particular case we considered) did not enable us to find the moment generating function of the first-passage time $T(x, y)$. It would surely be interesting to obtain an explicit expression for this function. Similarly, in addition to the moments of $Y(T(x, y))$ and $X(T(x, y))$, we could try to find the distribution of these random variables.

Next, from the two-boundary problems, we could consider the limiting one-boundary problems obtained by letting k_2 decrease to $-\infty$ or k_1 increase to 0.

Finally, because of the importance of the Wiener process, we could try to solve other first-passage time and/or place problems with the help of the method of similarity solutions. In general, such problems in two or more dimensions give rise to really complicated formulas. Here, the solutions obtained are relatively simple, since only in the last section the solution involved an integral difficult to evaluate explicitly.

Acknowledgements

This paper was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover, New York, 1965.

[2] A. Lachal, *Sur l'intégrale du mouvement brownien*, C. R. Acad. Sci. Paris Sér. I Math., **311**, 7(1990), 461-464.

[3] M. Lefebvre, *Moment generating function of a first hitting place for the integrated Ornstein-Uhlenbeck process*, Stochastic Process. Appl., **32**, 2(1989), 281-287.

[4] M. Lefebvre, É. Léonard, *On the first hitting place of the integrated Wiener process*, Adv. in Appl. Probab., **21**, 4(1989), 945-948.

[5] M. Lefebvre, M. Akhbi, *On the mean of first hitting places*, Rev. Roumaine Math. Pures Appl., **47**, 2(2002), 191-204.

[6] M. Lefebvre, J.L. Guilbault, *Hitting place probabilities for two-dimensional diffusion processes*, *Rev. Roumaine Math. Pures Appl.*, **49**, 1(2004), 11-25.

[7] B. Oksendal, *Stochastic differential equations. An introduction with applications*, 6th ed., Springer, Berlin, 2003.

ROMAI J., **2**, 2(2006),
109–113

STRENGTHENING SUBADDITIVITY OF MALLOWS DISTANCE

Bogdan Gheorghe Munteanu
"Transilvania" University, Brasov
b.munteanu@unitbv.ro

Abstract A strengthening subadditivity of the Mallows distance, used in the proof of the classical Central Limit Theorem, is yielded.

Keywords: Mallows distance, probability metric
2000 MSC: 60E15, 60B10.

1. INTRODUCTION

Theorem 1 in [3] asserts that, in specific circumstances, a rate for the convergence is obtained (equation (3) in [3]) by using the fact that $D(T_k)$ is decreasing. Since $D(T_k)$ is bounded below, the difference sequence $D(T_k) - D(T_{k+1})$ must converge to zero, so like in [2] we examine properties of this difference sequence, to show that its convergence implies the convergence of T_k to a normal random variable.

Following [1] we define a new distance

$$D^*(X) = \inf_{m,s^2} d_2^2(X, Z_{m,s^2}).$$

Remark that $D(X) = 2\sigma^2 - 2\sigma\, k \leq 2\sigma^2$, where $k = \int\limits_0^1 F_X^{-1}(x)\phi^{-1}(x)\, \mathrm{d}x$. This follows since F_X^{-1} and ϕ^{-1} are increasing functions, so $k \geq 0$. Hence, using some results from [1], we deduce that

$$D^*(X) = \sigma^2 - k^2 = D(X) - \frac{D(X)^2}{4\sigma^2}$$

such that the convergence of $D(S_n)$ to zero is equivalent to the convergence of $D^*(S_n)$ to zero.

The key result is given by the following proposition proved by author.

Proposition 1.1 *Let X_1 and X_2 be two independent and identically distributed random variables with mean zero and variance $\sigma^2 > 0$. Define $g(u) =$*

$\phi_{\sigma^2}^{-1} \circ F_{(X_1+X_2)/\sqrt{2}}(u)$. *If* $g'(u) \leq c$ *for all* u, *then*

$$D\left(\frac{X_1+X_2}{\sqrt{2}}\right) \leq \left(1-\frac{c}{2}\right) D(X_1) + \frac{cD(X_1)^2}{8\sigma^2} \leq \left(1-\frac{c}{4}\right) D(X_1)\,. \qquad (15.1)$$

Proof. For the random variables X and Y we introduce the functions $g(u) = F_Y^{-1} \circ F_X(u)$ and $h(u) = g^{-1}(u)$. The function $k(x,y) = -\left[x-h(y)\right]^2$ is quasi-monotone and induces the measure $d\mu_k(x,y) = 2h'(y)\mathrm{d}x\mathrm{d}y$. Therefore, by taking $H_1 = \mathbb{P}(X \leq x, Y \leq y)$ and $H_2 = \min(F_X(x), F_Y(y))$ in the Tchen's Lemma in [4] implies that

$$\int \left[x-h(y)\right]^2 \,\mathrm{d}H_1 - \int \left[x-h(y)\right]^2 \,\mathrm{d}H_2 = \int (H_1 - H_2)\,\mathrm{d}\mu_k \;\text{ iff}$$

$$\mathbb{E}\left(X-h(Y)\right)^2 = 2\int h'(y)\left[H_1(x,y) - H_2(x,y)\right]\,\mathrm{d}x\,\mathrm{d}y.$$

Indeed, $\mathbb{E}\left(X^*-h(Y^*)\right)^2 = 0$ because $X^* = h(Y^*)$.
By hypothesis $g'(u) \geq c$ and $h(u) = g^{-1}(u)$; it follows that $h'(y) = \frac{1}{g'(u)} \leq \frac{1}{c}$. Then

$$\begin{aligned}
\mathbb{E}\left(X-h(Y)\right)^2 &\leq \frac{2}{c}\int \left[H_1(x,y) - H_2(x,y)\right]\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{2}{c}\left[\mathrm{Cov}(X^*,Y^*) - \mathrm{Cov}(X,Y)\right]. \qquad (15.2)
\end{aligned}$$

Take Y_1^*, $Y_2^* \sim \mathcal{N}(0,\sigma^2)$ two independent random variables, and put $X_i^* = F_{X_i}^{-1} \circ F_{Y_i}(Y_i^*) = h_i(Y_i^*)$. Then define $Y^* = Y_1^* + Y_2^*$ and take $X^* = F_{X_1+X_2}^{-1} \circ F_{Y_1+Y_2}(Y^*) = h(Y_1^*+Y_2^*)$. Then there exist two real constants a and b such that

$$\begin{aligned}
& d_2^2(X_1,Y_1) + d_2^2(X_2,Y_2) - d_2^2(X_1+X_2, Y_1+Y_2) = \\
= \;& \mathbb{E}\,|X_1^*-Y_1^*|^2 + \mathbb{E}\,|X_2^*-Y_2^*|^2 - \mathbb{E}\,|X^*-Y^*|^2 \\
= \;& \mathbb{E}\left(X_1^*+X_2^*-Y_1^*-Y_2^*\right)^2 - \mathbb{E}\,|X^*-Y^*|^2 \\
\geq \;& \mathbb{E}(X^*)^2 + \mathbb{E}(Y^*)^2 - \mathbb{E}\,|X^*-Y^*|^2 - \mathbb{E}\left(X_1^*+X_2^*\right)^2 - \mathbb{E}\left(Y_1^*+Y_2^*\right)^2 \\
+ \;& \mathbb{E}\left(X_1^*+X_2^*-Y_1^*-Y_2^*\right)^2 \\
= \;& 2\mathrm{Cov}(X^*,Y^*) - 2\mathrm{Cov}(X_1^*+X_2^*, Y_1^*+Y_2^*) \\
\overset{(15.1)}{\geq} \;& c\mathbb{E}\left(X_1^*+X_2^*-h(Y_1^*+Y_2^*)\right)^2 = c\mathbb{E}\left(h_1(Y_1^*)+h_2(Y_2^*)-h(Y_1^*+Y_2^*)\right)^2 \\
\overset{(\text{Proposition 2.1, [2]})}{\geq} \;& c\mathbb{E}\left(h_1(Y_1^*) - aY_1^* - b\right)^2 = cD^*(X_1).
\end{aligned}$$

Recalling that $\mathbb{E}(X+Y)^2 = \mathbb{E}X^2 + \mathbb{E}Y^2$, $aY_1^* + b \sim \mathcal{N}(0,\sigma^2)$ and $D(X) \leq 2\sigma^2$, such that $D^*(X) = D(X) - \frac{D(X)^2}{4\sigma^2} \geq \frac{D(X)}{2}$, we have

$$d_2^2(X_1, Y_1) + d_2^2(X_2, Y_2) - d_2^2(X_1 + X_2, Y_1 + Y_2) \geq c\left(D(X_1) - \frac{D(X_1)^2}{4\sigma^2}\right),$$

i.e.

$$D(X_1) + D(X_2) - D(X_1 + X_2) \geq c\left(D(X_1) - \frac{D(X_1)^2}{4\sigma^2}\right).$$

By multiplying this inequality by $\frac{1}{2}$ and arranging terms, we obtain

$$\begin{aligned} D\left(\frac{X_1 + X_2}{\sqrt{2}}\right) &\leq \left(1 - \frac{c}{2}\right) D(X_1) + \frac{cD(X_1)^2}{8\sigma^2} \\ &= D(X_1) - \frac{c}{2}\left[\underbrace{D(X_1) - \frac{D(X_1)^2}{4\sigma^2}}_{\geq D(X_1)/2}\right] \leq \left(1 - \frac{c}{4}\right) D(X_1) \end{aligned}$$

□

In the following we discuss the strengthening subadditivity. To this aim, we define the scale invariant quantity

$$C(X) = \inf_u \left(\Phi_{\sigma^2}^{-1} \circ F_X\right)'(u) = \inf_{p\in(0,1)} \frac{f_X(F_X^{-1}(p))}{\phi_{\sigma^2}(\Phi_{\sigma^2}^{-1}(p))} = \inf_{p\in(0,1)} \sigma\, \frac{f_X(F_X^{-1}(p))}{\phi(\Phi^{-1}(p))}.$$

Remark 1.1 Since the quantity $C(X)$ is the ratio of two probability densities it follows that $C(X) > 0$.

Example 1.1 *If $X \sim U(0,1)$ then* $C(X) = \frac{1}{\sqrt{12 \sup_x \varphi(x)}} = \sqrt{\frac{\pi}{6}}$.

Indeed, for $X \sim U(0,1)$, $f_X(x) = 1$, $\forall x \in [0,1]$ and $\sigma^2 = \frac{1}{12}$. It follows the Lemma obtained by author.

Lemma 1.1 *If X has a zero mean and unit variance, then* $C(X)^2 \leq \frac{1}{1+\mathrm{median}(X)^2}$.

Proof. By the Mean Value Inequality, for all p, we have

$$\left|\Phi^{-1}(p)\right| = \left|\Phi^{-1}(p) - \underbrace{\Phi^{-1}(1/2)}_{0}\right| \geq C(X)\left|F_X^{-1}(p) - F_X^{-1}(1/2)\right|,$$

so that

$$\begin{aligned} 1 + F_X^{-1}(1/2)^2 &= \int_0^1 F_X^{-1}(p)^2 \,\mathrm{d}p + F_X^{-1}(1/2)^2 = \int_0^1 \left[F_X^{-1}(p) - F_X^{-1}(1/2)\right]^2 \\ &+ \underbrace{2\int_0^1 F_X^{-1}(p) F_X(1/2)\,\mathrm{d}p}_{0} \le \frac{1}{C(X)^2}\int_0^1 \Phi^{-1}(p)^2\,\mathrm{d}p = \frac{1}{C(X)^2}. \end{aligned}$$

□

2. STRENGTHENING SUBADDITIVITY

In order to obtain letter bounds for $D(S_n)$ as $n \to \infty$, we must control the sequence $C(S_n)$. Because F is an increasing function, we have that $C\left((X_1 + X_2)/\sqrt{2}\right) \ge C(X_1)$ for the independent and identically distributed random variables. Then, by induction, we have that $C(S_n) \ge c$, $\forall\, n$, where $C(X) = c$. We consider the powers of two subsequence $T_k = S_{2^k}$. The assumption that $C(S_n) \ge c$, $\forall\, n$, implies that $D(T_k) \le (1 - c/4)^k D(X_1) \le (1 - c/4)^k\, 2\sigma^2$. Now, by generalizing relation (15.1) we have

$$D(T_{k+1}) \le D(T_k)(1 - c/2)\left[1 + \frac{cD(T_k)}{8\sigma^2(1 - c/2)}\right].$$

On the other hand, since $\sum\limits_{k=0}^{\infty} \frac{cD(T_k)}{8\sigma^2} \le c/4 \sum\limits_{k=0}^{\infty} (1 - c/4)^k = 1$, it follows

$$\prod_{k=0}^{\infty}\left[1 + \frac{cD(T_k)}{8\sigma^2(1 - c/2)}\right] \le e^{\sum\limits_{k=0}^{\infty} \frac{cD(T_k)}{8\sigma^2(1-c/2)}} \le e^{\frac{1}{1-c/2}}.$$

Thus, from the successive inequalities

$$D(T_1) \le D(T_0)(1 - c/2)\left[1 + \frac{cD(T_0)}{8\sigma^2(1 - c/2)}\right]$$

$$D(T_2) \le D(T_1)(1 - c/2)\left[1 + \frac{cD(T_1)}{8\sigma^2(1 - c/2)}\right]$$

$$D(T_3) \le D(T_2)(1 - c/2)\left[1 + \frac{cD(T_2)}{8\sigma^2(1 - c/2)}\right]$$

$$D(T_4) \le D(T_3)(1 - c/2)\left[1 + \frac{cD(T_3)}{8\sigma^2(1 - c/2)}\right]$$

....

$$D(T_{k-1}) \leq D(T_{k-2})(1-c/2)\left[1+\frac{cD(T_{k-2})}{8\sigma^2(1-c/2)}\right]$$

$$D(T_k) \leq D(T_{k-1})(1-c/2)\left[1+\frac{cD(T_{k-1})}{8\sigma^2(1-c/2)}\right]$$

we deduce that

$$D(T_k) \leq \underbrace{D(T_0)}_{D(X_1)}(1-c/2)^k e^{\frac{1}{1-c/2}}$$

or $D(S_n) = O(n^t)$ where $t = \log_2(1-c/2)$ and $n = 2^k$.

In conclusion, we obtain the rate of convergence of $D(S_n) = d_2^2(S_n, Z_{\sigma^2})$, where $Z_{\sigma^2} \sim \mathcal{N}(0,\sigma^2)$ and $S_n = (X_1 + ... + X_n)/\sqrt{n}$.

References

[1] Del Barrio, E. et al., *Tests of goodness of fit based on the L_2 - Wasserstein distance*, Ann. Statist. **27(4)**(1999), 1230-1239

[2] Johnson O.T. and Barron A., *Fisher information inequalities and the Central Limit Theorem* , Probability Theory and Related Fields **129(3)**(2004), 391-409.

[3] Munteanu, B., *Rolul distantei Mallows in teorema limita centrala*, Sesiunea de Comunicari Stiintifice cu Patricipare Internationala "Terra Dacica-România Mileniului Trei", Academia Fortelor Aeriene "Henry Coanda" Brasov, 2006. (on CD)

[4] Tchen, A.H., *Inequalities for distributions with given marginals*, Ann. Probab. **8(4)**(1980), 814-827.

ROMAI J., **2**, 2(2006),
115–122

METRIC STRUCTURES AND DEFORMATION ALGEBRAS FOR HARMONIC DYNAMIC EVOLUTION OF CALCIUM OSCILLATIONS

Ileana Rodica Nicola, Vladimir Balan

University Politechnica of Bucharest

nicola_rodica@yahoo.com, vbalan@mathem.pub.ro

Abstract The Calcium oscillations within the living cell can be modelled by a dynamical system governed by a vector field X, depending on parameters which represent biological quantities and rates involved in this process. Emerging from this vector field, the paper describes Riemannian metrics of the ambient space for which X is harmonic, and determines conditions which characterize algebraic properties of the deformation algebra entailed by the correspondent Levi-Civita connection.

Keywords: harmonicity, Riemannian metric, deformation algebra.
2000 MSC: 53C43, 53B20, 17B66.

1. INTRODUCTION.

The vector field which describes the dynamics of the calcium oscillations within the hepatocyte has been subject of intensive recent research [10, 7, 6]. As well, the works on deformation algebras have been substantially stimulated by the notable work [11], leading to further extensive research (e.g, [4]).

The aim of the present paper is to bridge the two streams, by considering the canonic (Levi-Civita) connection attached to a special metric which provides harmonicity for the field. The deformation induced by this connection vs. the flat Euclidean connection is then studied, mainly from algebraic point of view. The geometric and biologic applicative issues of the purely algebraic mathematical results are an open problem which is subject of further concern.

We further describe the vector field which will be investigated throughout the paper. The dynamical system associated with this field describes calcium oscillations in living cells and relies on the mechanism of calcium-induced calcium-release (CICR), that takes into account the calcium-stimulated degradation of inositol 1,4,5-triphosphate ($InsP_3$) by a certain enzyme. We briefly describe the biological process: an external stimulus initiates the synthesis of $InsP_3$, starting an intracellular chain reaction, which culminates with the release of Ca^{2+} from an internal store of the cell, in the cytosol. Two mechanisms are responsable for calcium oscillation: the autocatalitic nature of Ca^{2+}

release in the cytosol and the increased InsP_3 degradation, due to the Ca^{2+}-stimulation of an enzyme.

The dynamical system describes the variation in time of three variables, namely:
- x - the concentration of inositol;
- y - the concentration of calcium in certain internal pool of the cell;
- z - the concentration of free calcium in the cytosol of the cell.

The variation in time of these variables is governed by the following SODE

$$\begin{cases} \dot{x} = \beta V_4 - V_5 - \varepsilon x, \\ \dot{y} = V_2 - V_3 - k_f y, \\ \dot{z} = -V_2 + V_3 + k_f y - kz + V_{in}, \end{cases} \tag{1}$$

where

$$\begin{cases} V_{in} = V_0 + \beta V_1, \\ V_2 = V_{M2} \dfrac{z^2}{K_2^2 + z^2}, \\ V_3 = V_{M3} \dfrac{x^4}{K_x^4 + x^4} \dfrac{y^2}{K_y^2 + y^2} \dfrac{z^m}{K_z^m + z^m}, \\ V_5 = V_{M5} \dfrac{x^p}{K_5^p + x^p} \dfrac{z^n}{K_d^n + z^n}. \end{cases}$$

We further examine the case when the parameters take the following values:

$$\begin{array}{llll} \beta = 0.46, & n = 2, & m = 4, p = 1, & k = 0.1667, \\ k_2 = 0.1, & k_5 = 1, & k_x = 0.1, & k_d = 0.6, \\ \varepsilon = 0.0167, & V_0 = 0.0333, & V_1 = 0.0333, & \\ V_{M2} = 0.1, & V_{M3} = 0.3333, & V_{M4} = 0.0417, & V_{M5} = 0.5. \end{array}$$

2. METRIC STRUCTURES PRODUCING HARMONICITY

Let $D \subset I\!R^n$ be a differentiable manifold. Herein we consider the set $F(D)$ of differentiable functions defined on D and the set $T_s^r(D)$ of all tensor fields of type (r, s) on D, which admit a canonical structure of real vector space and of $F(D)$-module. Particularly, for $T_1^0(D)$ we use the notation the notation $\chi(D)$.

It is well-known that the harmonicity of a vector field $X \subset \chi(D)$, $D \subset I\!R^n$ is equivalent to the nulity of the Laplacian operator of X, $\Delta X = 0$. If we consider the Riemannian metric on $I\!R^n$

$$h = h_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta \in T_2^0(I\!R^n), \tag{2}$$

the harmonicity of X is equivalent to the relation

$$\frac{1}{\sqrt{\det(h_{\alpha\beta})}}\frac{\partial}{\partial x^l}\left(\sqrt{\det(h_{\alpha\beta})}h^{kl}\frac{\partial X}{\partial x^k}\right)=0. \tag{3}$$

For the metric given by the relation (2) we can associate the matrix

$$[h]=(h_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}.$$

The entries of the inverse matrix $[h]^{-1}=(h^{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ define the reciprocal tensor field

$$h^{-1}=h^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\otimes\frac{\partial}{\partial x^\beta}\in T_0^2(D) \tag{4}$$

of the metric h. Then, denoting $\left[h^{-1}\right]=(h^{\alpha\beta})_{\alpha,\beta=\overline{1,n}}=[h]^{-1}$, we have the obvious relations

$$[h]\,[h]^{-1}=I_n\Longleftrightarrow h_{\alpha\beta}h^{\beta\gamma}=\delta^\gamma_\alpha,\alpha,\gamma=\overline{1,n.}$$

In order to find a metric $(h_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ producing the harmonicity for the vector field X, we solve the linear system

$$\sum_{\alpha,\beta=1}^{n}a^i_{\alpha\beta}h^{\alpha\beta}=0,i=\overline{1,3}, \tag{5}$$

where $a^i_{\alpha\beta}=\frac{\partial^2X^i}{\partial x^\alpha\partial x^\beta},\alpha,\beta=\overline{1,3}$, and $x^1=x,x^2=y,x^3=z$.

Obviously the algebraic system (5) has three equations and $n(n+1)/2$ unknown variables. In our case $n=3$ and the system (5) becomes

$$\begin{aligned}
&a^1_{11}h^{11}+a^1_{22}h^{22}+a^1_{33}h^{33}+2a^1_{12}h^{12}+2a^1_{13}h^{13}+2a^1_{23}h^{23}=0\\
&a^2_{11}h^{11}+a^2_{22}h^{22}+a^2_{33}h^{33}+2a^2_{12}h^{12}+2a^2_{13}h^{13}+2a^2_{23}h^{23}=0\\
&a^3_{11}h^{11}+a^3_{22}h^{22}+a^3_{33}h^{33}+2a^3_{12}h^{12}+2a^3_{13}h^{13}+2a^3_{23}h^{23}=0.
\end{aligned} \tag{6}$$

One can easily remark that $a^1_{\alpha\beta}=-a^2_{\alpha\beta},\ \ \forall\alpha,\beta=\overline{1,3},\ a^3_{21}=a^3_{22}=a^3_{23}=0$, hence the system (6) reads

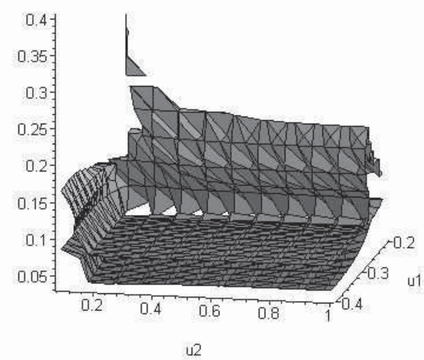
$$\begin{cases}a^1_{11}h^{11}+a^1_{22}h^{22}+a^1_{33}h^{33}+2a^1_{12}h^{12}+2a^1_{13}h^{13}+2a^1_{23}h^{23}=0,\\ a^3_{11}h^{11}+a^3_{33}h^{33}+2a^3_{13}h^{13}=0.\end{cases} \tag{7}$$

In order that h be a Riemannian metric, we have to impose the conditions $h^{11}>0,h^{11}h^{22}-(h^{12})^2>0,\det{[h]}^{-1}>0$. Choosing the components h^{11},h^{12},h^{22} and h^{23} as secondary variables of the algebraic system (7) and fixing $\begin{pmatrix}h^{11}&h^{12}\\h^{12}&h^{22}\end{pmatrix}=I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ we get the main variables h^{13} and h^{33} (which depend on x,y,z).

The conditions $h^{11} > 0$, $\begin{vmatrix} h^{11} & h^{12} \\ h^{12} & h^{22} \end{vmatrix} > 0$, are fulfilled, so h is a Riemannian metric if $h^{33} - (h^{13})^2 > 0$ which describes a zone in $\mathbb{R}^3$ bounded by the surface

$$S_1 : h^{33}(x, y, z) - (h^{13}(x, y, z))^2 = 0.$$

Using the software package Maple 9.5, one can view this surface; its shape is given below.



We conclude that the required metric has the components
$h_{11} = \frac{h^{33}}{h^{33}-(h^{13})^2}, h_{12} = 0, h_{13} = -\frac{h^{13}}{h^{33}-(h^{13})^2}, h_{21} = 0, h_{22} = 1, h_{23} = 0,$
$h_{31} = -\frac{h^{13}}{h^{33}-(h^{13})^2}, h_{32} = 0, h_{33} = \frac{1}{h^{33}-(h^{13})^2}.$

The Levi-Civita connection ∇ associated with h has the Christoffel symbols of second kind

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma s} \left| \alpha\beta, s \right| . \tag{8}$$

with the Christoffel symbols of first kind given by

$$|\alpha\beta, s| = \frac{\partial h_{\beta s}}{\partial x^{\alpha}} + \frac{\partial h_{\alpha s}}{\partial x^{\beta}} - \frac{\partial h_{\alpha\beta}}{\partial x^{s}}.$$

Remark 2.1 *The null components of the Christoffel symbols of second kind have the same indices as the Christoffel symbols of first kind, respectively:* $|12, 2|, |21, 2|, |22, 1|, |22, 2|, |22, 3|, |23, 2|$ *and* $|32, 2|$.

3. THE INDUCED DEFORMATION ALGEBRA

Let $A \in T_2^1(D)$. Defining the product of two vector fields $X, Y \in \chi(D)$ by the relation

$$X * Y = A(X, Y), \tag{9}$$

then the $F(D)$-module $\chi(D)$ becomes an $F(D)$-algebra, called *the algebra associated with* A and denoted by $U(D, A)$. In the following, we shall consider $A = \nabla - \overset{\circ}{\nabla}$, where $\overset{\circ}{\nabla}$ is the trivial connection and ∇ is the Levi-Civita connection associated with h. In this case, the associated $F(D)$-algebra is

$$\nabla(X, Y) = X * Y = X^\alpha Y^\beta A^s_{\alpha\beta} \frac{\partial}{\partial x^s}, \tag{10}$$

where $A^s_{\alpha\beta} = \Gamma^s_{\alpha\beta}$. Further we study the properties of this algebra.

1. Commutativity. The relation $X * Y = Y * X, \quad \forall X, Y \in \chi(D)$ is equivalent to $A^s_{ij} = A^s_{ji}, \quad \forall i, j, s \in \overline{1,3}$, which is obviously true, from the comutativity of Levi-Civita components in lower indices.

2. Associativity. Because of the tensorial character of the associativity condition $X * (Y * Z) = (X * Y) * Z, \ \forall X, Y, Z \in \chi(D)$, this can be specialized for $X = \partial_i, Y = \partial_j$ and $Z = \partial_k$, leading, for all $i, j, k = \overline{1,3}$, to the relations $A^s_{ij} A^t_{sk} = A^s_{jk} A^t_{is} | 2h_{tr} \iff A^s_{ij} \, |sk, r| = A^s_{jk} \, |is, r| \iff |ij, t| \, |sk, r| \, h^{ts} = |jk, t| \, |is, r| \, h^{ts}$. Using Maple computation tehniques, we find that for certain sets of indices, the last relation is not true; hence the algebra is not associative.

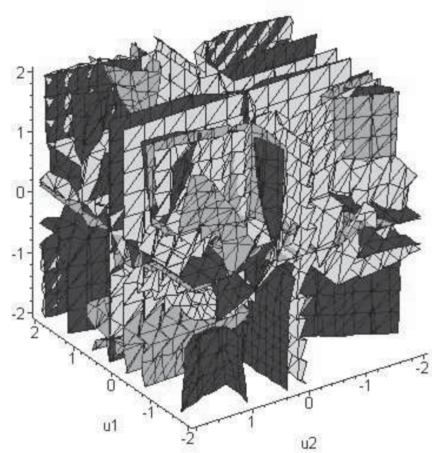
3. Neutral element. In order to prove the existence of a neutral element, taking into account the commutativity, we must have $\exists E \in \chi(D)$ such that $A(E, X) = X, \quad \forall X \in \chi(D) \iff A^i_{jk} E^j \partial_i = \partial_k, \quad \forall k \in \overline{1,3} \iff A^i_{jk} E^j = \delta^i_k, \quad \forall i, k \in \overline{1,3} \iff |jk, s| \, E^j = h_{sk}, \quad \forall k, s \in \overline{1,3}$. One can easily see, that for $(k, s) = (2, 2)$, the last relation is not true. In conclusion, the algebra does not have a neutral element.

4. Idempotency locations for X. Using the software package Maple 9.5 we have plotted the points $(x, y, z) \in I\!R^3$ at which the field X satisfies the relation $X^2 = 0$ or, equivalently, $A^k_{ij} X^i X^j = 0$. The three resulting surfaces describing the vanishing of the field components intersect at the points $p \in I\!R^3$ at which the vector X_p is idempotent w.r.t the deformation algebra structure, as can be seen from the image below.

Obviously a notable point located at the intersection is the equilibrium point

$$p_0 = (x, y, z) = (0.1989819160, 0.2344675015, 0.2916496701)$$

of the field X. We emphasize that due to the mainly rational form of the field component functions, these surfaces present numerous branches and critical



points. This leads to a substantial computational effort, which requires a fast processor and extended RAM for a reasonable simulation output.

5. Zero-divisors. The attempt of finding locations $p \in I\!\!R^3$ where do exist zero-divisors for the algebra, i.e., nontrivial vectors Y_p such that $A_p(X_p, Y_p) = 0$, where X_p is provided by the studied field, leads to considering the system

$$A^k_{ij} X^i Y^j = 0 \tag{11}$$

or, equivalently, $XY = 0$, where X is the vector field which provides the studied SODE (1). The system (11) is linear homogeneous in Y_1, Y_2, Y_3 and admits nontrivial solutions only on the surface

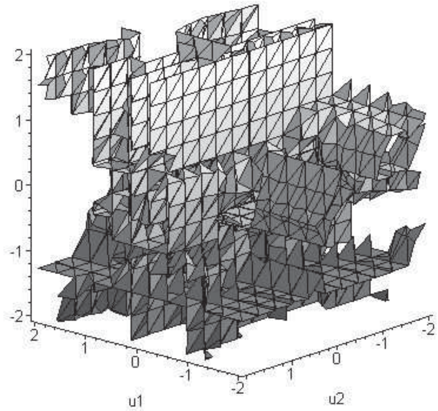
$$\Sigma : \varphi(x, y, z) = 0,$$

implicitly described by the vanishing of the determinant of the system (11),

$$\varphi(x, y, z) = \det[(A^i_{jk} X^k)_{i,j=\overline{1,3}}];$$

its shape is characterized by the image below

Note that at the equilibrium point p_0 of the field X, which belongs to the surface Σ, the system (11) has trivial coefficients and the endomorphism defined by X is the trivial one, having as kernel $T_{p_0} I\!\!R^3$. Hence at any point $p \in \Sigma \backslash \{p_0\}$, the algebra admits 0-divisors, namely X_p and Y^*_p, where Y^* is a nontrivial solution of the homogeneous system of degree of freedom $m \geq 1$.



Conclusions.

Emerging from a field X which describes the dynamics of the calcium oscillations within the hepatocyte, in the three-dimensional Euclidean space were determined those Riemannian metrics for which X becomes a harmonic vector field. For such metrics, the Levi-Civita connection provides a deformation algebra, whose basic algebraic properties were investigated and illustrated making use of Maple 9.5 programming.

Acknowledgements

The present work was partially supported by Grant CNCSIS MEN 1478.

References

[1] M. Berridge, P. Lipp, M. Bootman, *Calcium signalling*, Curr. Biol. **9** (1999), R157-R159.

[2] J. A. M. Borghans, G. Dupont, A. Goldbeter, *Complex intracellular calcium oscillations: A theoretical exploration of possible mechanisms*, Biophys. Chem. **66** (1997), 25-41.

[3] G. Houart, G. Dupont, A. Goldbeter, *Bursting, chaos, birhytmicity originating from self-modulation of the inositol 1,4,5-triphosphate signal in a model for intracellular Ca^{2+} oscillations*, Bull. of Math. Biology, **61** (1999), 507-530.

[4] L. Nicolescu, *The geometry of deformation algebras* , University of Bucharest, 1983 (Romanian).

[5] G. Mircea, M. Neamţu, D. Opriş, *Dynamical systems in economy, mechanics, biology described by differential equations with time-delay*, Mirton Press, Timişoara 2003 (Romanian).

[6] M. Neagu, I. R. Nicola, *Geometric dynamics of calcium oscillations ODE systems*, BJGA **9**, 2(2004),36-67.

[7] I. R. Nicola, C. Udriste, V. Balan, *Applications of dynamical systems with time-delay in the physiology of the hepatocyte*, Proc. of Symposium - Commemoration of Acad. Gh. Vrănceanu (1900-1979), April 27, 2004, Univ. of Bucharest, An. Univ. Bucuresti, **55**, 1(2005), 97-104.

[8] C. Robinson, *Dynamical systems: stability, symbolic dynamics and chaos*, CRC Press, 1995.

[9] J. A. Rottingen, J. G. Iversen, *Rulled by waves ? Intracellular and intercellular calcium signalling*, Acta Physiol. Scand. **169** (2000), 203-219.

[10] C. Udrişte, V. Balan, I. R. Nicola, *Linear stability and Hopf bifurcations for time-delayed intra-cell calcium variation models*, Proc. of the 3-rd International Colloquium "Mathematics in Engineering and Numerical Physics" (MENP-3), October 7-9, 2004, Bucharest, BSG Proceedings **12**, 2005, Geometry Balkan Press, Bucharest, 272-283.

[11] I. Vaisman, *Sur quelques formules du calcul du Ricci global*, Comm. Math. Helv. **41**, 2 (1966), 73-87.

ROMAI J., **2**, 2(2006),
123–128

ASYMPTOTIC SOLUTION OF THE CAUCHY PROBLEM FOR THE PRANDTL EQUATION

Gheorghe Nistor
The Faculty of Mathematics and Computer Sciences, Univ. of Pitesti

Abstract To the Cauchy problem for the famous Prandtl equation the method of the boundary layer type function is applied. The models of an arbitrary order of asymptotic approximation are deduced while the first two such models are solved. The result is compared with the corresponding result obtained by application the inner outer expansions method to the equivalent two- point problem for the Prandtl equation. The existence of the relationship between the two results is revealed.

Keywords: Prandtl equation, method of boundary layer type functions.
2000 MSC: 34E15, 34E05.

1. CLASSICAL PRANDTL MODEL VERSUS CAUCHY PROBLEM FOR THE PRANDTL EQUATION

The classical Prandtl model is a two-point problem for a linear second order ordinary differential equation (ode) of singular perturbations [1]

$$m\ddot{x} + k\dot{x} + cx = 0, \tag{1}$$

$$x(0) = 0, \ \lim_{t\to\infty} x(t) = 0, \tag{2}$$

where $m, \ k, \ c, \in \mathbb{R}$ are parameters, t is the time, $x : \mathbb{R}^+ \to \mathbb{R}$, and $\cdot \equiv \dfrac{d}{dt}$. Among several physical possible interpretations of (1) we mention the mechanical one: (1) represents the Newton second law expressing the balance of in forces and viscous and elastic forces. The parameter m is the mass of a point characterized by the position $x(t)$ at the time t. It is interest to find the asymptotic behaviour of this position as $m \to 0$. The problem (1), (13) is singularly perturbed of boundary layer type because the small parameter multiplies the highest order derivative in the equation (1). In order to deduce the asymptotic behaviour of $x(t)$, in fact it is better to write $x(m,t)$ because x depends on m too, for this model, in 1904, Ludwing Prandtl proposed the inner outer expansion method. Subsequently it was frequently applied especially to fluid dynamics [2]. Indeed, the problem (1), (13) served as a model example for the Navier-Stokes model and then for many other models. It was minutely analyzed in [1] by the inner outer expansion method. In the meantime this method became one of the most important methods in perturbation theory.

The basic facts behind it is the existence of a boundary layer near $t = 0$, inside which the solution varies fast. Hence, for very small t, the order of x differs from that for larger t. So, the asymptotic behaviour of x is derived by means of two asymptotic expansions: one inner and other outer.

The method of matched inner and outer asymptotic expansions is suitable to boundary value problems. It can also viewed as a double scale method [1].

For initial value singular perturbation problems, where several (even infinitely many) layers exist, the most appropriate method is the boundary layer functions method, which was largely studied by the Russian school [3], [4], [5], [6], [7]. In [1], to a Cauchy problem for a first order linear ode with variable coefficients, the inner outer expansion method and the boundary layer type functions method were applied. Complicate relationships between the involved asymptotic expansions were found. In the case of a Cauchy problem for a system of two first order nonlinear ode's it is not clear whether some relationship between the asymptotic expansions involved in the two methods exists. In the following we try to answer this question for the case of the Cauchy problem

$$x(0) = x^0,\ \dot{x}(0) = \dot{x}^{(0)} \tag{3}$$

for the Prandtl equation (1) (equivalent to (1), (13) for an appropriate choice of the initial conditions). To this aim, by the transformations $x \to y,\ m \to \mu,\ \dot{x} \to \dot{y} = z$, the problem (1), (3) becomes

$$\frac{dy}{dt} = z, \tag{4}$$

$$\mu \frac{dz}{dt} = -kz - cy, \tag{5}$$

$$z(0, \mu) = z^0,\ y(0, \mu) = y^0, \tag{6}$$

which is a particular case of the Cauchy problem (24.0) for the equations [5]

$$\begin{cases} \mu \dfrac{dz}{dt} = F(z, y, t), \\ \dfrac{dy}{dt} = f(z, y, t), \end{cases} \tag{7}$$

where $f(z, y, t) = z$ and $F(z, y, t) - kz - cy$. Let us apply to (24.0)-(24.0) the Tihonov method of the boundary layer type functions. To this aim let us decompose y and z in the form

$$\overline{y}(t, \mu) = \overline{y}_0(t) + \mu \overline{y}_1(t) + ...,\ \overline{z}(t, \mu) = \overline{z}_0(t) + \mu \overline{z}_1(t) + ...$$

$$\Pi y(\tau, \mu) = \Pi_0 y(\tau) + \mu \Pi_1 y(\tau) + ...,\ \Pi z(\tau, \mu) = \Pi_0 z(\tau) + \mu \Pi_1 z(\tau) + ...$$

where $\tau = t/\mu$, hence $\dfrac{d}{dt} = \dfrac{d}{d\tau} \cdot \dfrac{d\tau}{dt} = \dfrac{1}{\mu}\dfrac{d}{d\tau}$. Thus the Prandtl equations (3), (24.0) become

$$\mu\frac{d\overline{z}}{dt}(t,\mu) + \frac{d\Pi z}{d\tau}(\tau,\mu) = -k\left[\overline{z}(t,\mu) + \Pi z(\tau,\mu)\right] - c\left[\overline{y}(t,\mu) + \Pi y(\tau,\mu)\right],$$

$$\frac{d\overline{y}}{dt}(t,\mu) + \frac{1}{\mu}\frac{d\Pi z}{d\tau}(\tau,\mu) = \overline{z}(t,\mu) + \Pi z(\tau,\mu).$$

Let us define $\overline{F}(t,\mu) = F\left(\overline{z}(t,\mu), \overline{y}(t,\mu), t\right) = -k\overline{z}(t,\mu) - c\overline{y}(t,\mu)$ and assume that $\overline{F}$ possesses the expansion $\overline{F}(t,\mu) = \overline{F}_0(t) + \mu\overline{F}_1(t) + \mu^2\overline{F}_2(t) + ...$ where

$$\overline{F}_0(t) = \overline{F}(t,0) = -k\overline{z}(t,0) - c\overline{y}(t,0) = -k\overline{z}_0(t) - c\overline{y}_0(t),$$

$$\overline{F}_1(t) = \frac{\partial\overline{F}}{\partial\mu}(t,0) = -k\frac{\partial\overline{z}}{\partial\mu}(t,0) - c\frac{\partial\overline{y}}{\partial\mu}(t,0) = -k\overline{z}_1(t) - c\overline{y}_1(t),$$

$$(2!)\overline{F}_2(t) = \frac{\partial^2\overline{F}}{\partial\mu^2}(t,0) = 2!\left(-k\frac{\partial^2\overline{z}}{\partial\mu^2}(t,0) - c\frac{\partial^2\overline{y}}{\partial\mu^2}(t,0)\right) = 2!\left(-k\overline{z}_2(t) - c\overline{y}_2(t)\right),$$

....

$(n!)\overline{F}_n(t) = \dfrac{\partial^n\overline{F}}{\partial\mu^n}(t,0) = n!\left(-k\overline{z}_n(t) - c\overline{y}_n(t)\right)$, while

$$\Pi F(\tau,\mu) = F\left(\overline{z}(\Pi\mu,\mu) + \Pi z(\tau,\mu),\ \overline{y}(\tau\mu,\mu) + \Pi y(\tau,\mu),\ \tau\mu\right) -$$

$$\overline{F}\left(\overline{z}(\Pi\mu,\mu),\ \overline{y}(\tau\mu,\mu),\ \tau\mu\right) = -k\left[\overline{z}(\tau\mu,\mu) + \Pi z(\tau,\mu)\right] -$$

$$-c\left[\overline{y}(\tau\mu,\mu) + \Pi y(\tau,\mu)\right] + k\overline{z}(\tau\mu,\mu) + c\overline{y}(\tau\mu,\mu).$$

Similarly, assume that ΠF possesses the expansion $\Pi F(\tau,\mu) = \Pi_0 F(\tau) + \mu\Pi_1 F(\tau) + \mu^2\Pi_2 F(\tau)...$, where

$$\begin{aligned}
0!\ \Pi_0 F(\tau) &= \Pi F(\tau,0) = -k\left[\overline{z}(0,0) + \Pi z(\tau,0)\right] - c\left[\overline{y}(0,0) + \Pi y(\tau,0)\right] + \\
&\quad +k\overline{z}(0,0) + c\overline{y}(0,0), \\
1!\ \Pi_1 F(\tau) &= \frac{\partial}{\partial\mu}\Pi F(\tau,0) = -k\left[\frac{\partial\overline{z}}{\partial t}(0,0)\tau + \frac{\partial\overline{z}}{\partial\mu}(0,0)\cdot 1 + \frac{\partial\Pi z}{\partial\mu}(\tau,0)\right] - \\
&\quad -c\left[\frac{\partial\overline{y}}{\partial t}(0,0)\tau + \frac{\partial\overline{y}}{\partial\mu}(0,0)\cdot 1 + \frac{\partial\Pi y}{\partial\mu}(\tau,0)\right] + \\
&\quad +k\left[\frac{\partial\overline{z}}{\partial t}(0,0)\tau + \frac{\partial\overline{z}}{\partial\mu}(0,0)\right] + c\left[\frac{\partial\overline{y}}{\partial t}(0,0)\tau + \frac{\partial\overline{y}}{\partial\mu}(0,0)\right],
\end{aligned}$$

$$\begin{aligned}
2!\ \Pi_2 F(\tau) &= \frac{\partial^2}{\partial^2\mu}\Pi F(\tau,0) = -k\left[\left(\frac{\partial^2\overline{y}}{\partial^2}(0,0)\tau^2\right)\frac{\partial^2\overline{z}}{\partial t\partial\mu}(0,0)\tau + \right.\\
&\quad \left. + \frac{\partial^2\overline{z}}{\partial t\partial\mu}(0,0)\tau + \frac{\partial^2\overline{z}}{\partial\mu^2}(0,0)\tau + \frac{\partial^2\Pi z}{\partial\mu^2}(\tau,0)\right] - \\
&= -k\left[\left(\frac{d^2\overline{z}_0}{dt^2}(0)\tau^2 + \frac{d\overline{z}_1}{dt}(0)\tau + \frac{d\overline{z}_1}{dt}(0)\tau + 2\overline{z}_2(0) + 2\Pi_2 z(\tau)\right)\right] + \\
&\quad + \left[\frac{d^2\overline{y}_0}{dt^2}(0)\tau^2 + \frac{d\overline{y}_1}{dt}(0)\tau + \frac{d\overline{y}_1}{dt}(0)\tau + 2\overline{y}_2(0) + 2\Pi_2 y(\tau)\right].
\end{aligned}$$

Similarly, for f we have $\overline{f}(t,\mu) = f(\overline{z}(t,\mu),\overline{y}(t,\mu),t) = \overline{z}(t,\mu) = \overline{f}_0(t) + \mu\overline{f}_1(t) + \mu^2\overline{f}_2(t) + \ldots$, where

$$\begin{aligned}
&0!\overline{f}_0(t) = \overline{f}(t,0) = \overline{z}(t,0) = \overline{z}_0(t),\\
&1!\overline{f}_1(t) = \frac{\partial\overline{f}}{\partial\mu}(t,0) = \frac{\partial\overline{z}}{\partial\mu}(t,0) = \overline{z}_1(t),\\
&\vdots\\
&n!\overline{f}_n(t) = \frac{\partial^n\overline{f}}{\partial\mu^n}(t,0) = \frac{\partial^n\overline{z}}{\partial\mu^n}(t,0) = \overline{z}_n(t),
\end{aligned}$$

$$\begin{aligned}
\Pi f(\tau,\mu) &= f(\overline{z}(\tau\mu,\mu) + \Pi z(\tau\mu,\mu), \overline{y}(\tau\mu,\mu) + \Pi y(\tau\mu,\mu), \tau\mu) - \\
&\quad - f(\overline{z}(\tau\mu,\mu),\overline{y}(\tau\mu,\mu),\tau\mu) = \overline{z}(\tau\mu,\mu) + \Pi z(\tau,\mu) - \overline{z}(\tau\mu,\mu)\\
&= \Pi z(\tau,\mu) = \Pi_0 f(\tau) + \mu\Pi_1 f(\tau) + \mu^2\Pi_2 f(\tau) + \ldots
\end{aligned}$$

with

$$\Pi_0 f(\tau) = \Pi_0 z(\tau),\ \Pi_1 f(\tau) = \Pi_1 z(\tau), \ldots\ \Pi_n f(\tau) = \Pi_n z(\tau), \ldots$$

Introducing all these expansions in (3), (24.0) we obtain

$$\frac{d\overline{y}_0}{dt}(t)+\mu\frac{d\overline{y}_1}{dt}(t)+...+\frac{1}{\mu}\left(\frac{d\Pi_0 y}{d\tau}(t)+\mu\frac{d\Pi_1 y}{d\tau}(t)+...\right)=$$

$$=\overline{z}_0(t)+\mu\overline{z}_1(t)+...+\Pi_0 z(\tau)+\mu\Pi_1 z(\tau)+...,$$

$$\mu\left[\frac{d\overline{z}_0}{dt}(t)+\mu\frac{d\overline{z}_1}{dt}(t)+...\right]+\left[\frac{d\Pi_0 z}{d\tau}(t)+\mu\frac{d\Pi_1 z}{d\tau}(t)+...\right]$$

$$=-\left\{(kz_0(t+c\overline{y}_0(t)))+\mu\left(k\overline{z}_1(t+c\overline{y}_1(t))\right)+...\right\}-$$

$$-\left\{(k\Pi_0 z(\tau)+c\Pi_0 y(\tau))+\mu\left(k\Pi_1 z(\tau)+c\Pi_1 y(\tau)\right)+...\right\}$$

or, equivalently,

$$\frac{\mu d\overline{y}_0}{dt}(t)+\mu^2\frac{d\overline{y}_1}{dt}(t)+...+\left(\frac{d\Pi_0 y}{d\tau}(t)+\mu\frac{d\Pi_1 y}{d\tau}(t)+...\right)=$$

$$=\mu\overline{z}_0(t)+\mu^2\overline{z}_1(t)+...+\mu\Pi_0 z(\tau)+\mu^2\Pi_1 z(\tau)+...,$$

$$\mu\frac{d\overline{z}_0}{dt}(t)+\mu^2\frac{d\overline{z}_1}{dt}(t)+...+\frac{d\Pi_0 z}{d\tau}(t)+\mu\frac{d\Pi_1 z}{d\tau}(t)+...$$

$$=-\left\{(kz_0(t+c\overline{y}_0(t)))+\mu\left(k\overline{z}_1(t+c\overline{y}_1(t))\right)+...\right\}-$$

$$-\left\{(k\Pi_0 z(\tau)+c\Pi_0 y(\tau))+\mu\left(k\Pi_1 z(\tau)+c\Pi_1 y(\tau)\right)+...\right\},$$

whence the equations of the asymptotic approximation

$$\mu^0\to\begin{cases}\dfrac{d\Pi_0 y}{d\tau}=0, & \\ & \dfrac{d\Pi_0 z}{d\tau}=-k\Pi_0 z(\tau)-c\Pi_0 y(\tau),\\ -k\overline{z}_0(t)-c\overline{y}_0(t)=0, & \end{cases}$$

$$\mu^n\to\begin{cases}\dfrac{dy_{n-1}}{dt}=\overline{z}_{n-1}(t), & \dfrac{d\Pi_n y}{d\tau}=\Pi_{n-1}z(\tau),\\[2ex] \dfrac{dz_{n-1}}{dt}=-k\overline{z}_n(t)-c\overline{y}_n(t)=0, & \dfrac{d\Pi_n z}{d\tau}=-k\Pi_n z(\tau)-c\Pi_n y(\tau).\end{cases}$$

Separating the various asymptotic approximations of the unknown functions z and y and taking into account that $0=F(\overline{z},\overline{y},t)$ implies $-k\overline{z}_0-c\overline{y}_0=0$, whence $\overline{z}_0=-\dfrac{c}{k}\overline{y}_0=0$, we obtain the following models of asymptotic approximation

$$n=0\begin{cases}\overline{z}_0=-\dfrac{c}{k}\overline{y}_0,\\[2ex] \dfrac{d\Pi_0 y}{d\tau}(\tau)=0,\quad \dfrac{dy_0}{dt}(t)=\overline{z}_0(t),\\[2ex] \dfrac{d\Pi_0 z}{d\tau}=-k\Pi_0 z(\tau)-c\Pi_0(\tau).\end{cases}$$

....

$$n=n\begin{cases}\dfrac{d\overline{y}_n y}{dt}(t)=\overline{z}_n(t),\quad \dfrac{dz_{n-1}}{dt}(t)=-k\overline{z}_n(t)-c\overline{y}_n(t),\\[2ex] \dfrac{d\Pi_n y}{d\tau}(\tau)=\Pi_{n-1}z(\tau),\\[2ex] \dfrac{d\Pi_n z}{d\tau}=-k\Pi_n z(\tau)-c\Pi_n y(\tau).\end{cases}$$

At each order functions of t and τ occur, i.e. the systems are coupled. The case $n=0$ leads to $z_0=y_0'=-\dfrac{c}{k}y_0$ whence $y_0=Ae^{-\frac{c}{k}-t}$, while $\dfrac{d\Pi_0 y}{d\tau}(\tau)=0$ implies that $\Pi_0 y$ is constant and the constant is null. Finally it follows $\dfrac{d\Pi_0 z}{d\tau}=-k\Pi_0 z(\tau)$, hence $\Pi_0 z(\tau)=Be^{-k\tau}$. In this way we regained certain terms of the inner and outer expansions, obtained by the inner outer expansions method. Consequently the looked for relationships exist but they can be very complicated if higher order terms are considered.

References

[1] Georgescu, A., *Asymptotic treatment of differential equations*, Chapman & Hall, London, 1995.

[2] van Dyke, M., *Perturbation methods in fluid mechanics*, Academic, New York, 1964; The Parabolic Press, Stanford CA, 1975.

[3] Vasil'eva, A. B., Butuzov, V. F., *Asymptotic expansions of the solutions of singular perturbed equations*, Nauka, Moscova, 1973. (Russian)

[4] Vasil'eva, A. B., Butuzov, V. F., Kalacher, L. V., *The boundary function method for singular perturbation problems*, Studies in Applied Math., **14**, SIAM, Philadelphia, 1995.

[5] Tikhonov, A. N., Vasil'eva, A. B., Sveshnikov, A. G., *Differential equations*, Springer, Berlin, 1985

[6] Mishchenko, E. F., Kolesov, Yu. S., Kolesov, A. Yu., Rozov, N. Kh., *Asymptotic methods in singularly perturbed systems*, Consultants Bureau, New York, 1994.

[7] Mishchenko, E. F., Rozov, N. Kh., *Differential equations with small parameters and relaxation oscillations*, Plenum, New York, 1980.

ROMAI J., **2**, 2(2006),
129–140

STUDY OF TRANSIENT PROCESS IN THE SONIC CIRCUIT OF HIGHT-PRESSURE PIPES USED IN LINE FUEL INJECTION SYSTEMS FOR DIESEL ENGINES

Lucian Pîslaru - Dănescu, Valeriu Nicolae Panaitescu

Dept. Researcher Design, S. C. Mefin S. A. Sinaia, Dept. of Hydraulics and Hydraulic Machinery, Univ. POLITEHNICA of Bucharest

Abstract The modern injection equipment produces a polution level of the emisions which complies with the Europen Norms, i.e. a low fuel consumption level as well as a low level of noise of the Diesel engine. These antagonistic characteristics are achived mainly by optimizing the burning process of the fuel in the burning chanber of the Diesel engine. Obtaining a mixture of optimal aer/fuel ratio depends mainly on a adequate spraying of the fuel such as the drops be as small as possible as well as on the directioning of the pulverised fuel jets by the injector sprayer. For the conventional injection systems, the peak pressure is the the most important measure for the quality of forming the mixture in the burning chamber. Electro-hydraulic analogy, as base of the sonic theory developed by the Romanian scientist George Constantinescu, leads to the possibility of modeling hydraulic systems by electric circuits through sonic resistances, capacities and inductivities. Electro-hydraulic modeling of the high-pressure pipe and injector allows evaluating the adapting condition for optimal adaptation of a chain of sonic qvadripols. By considering the sonic injector circuit at injection phase and writing the transfer functions associated with the sonic quadriples, we are able to obtain the global transfer function, in its operational form. By solving the circuit we can obtain sonic potential differences and also sonic current in operational form. The expressions of pressure and deliveries differences in time range are given as a result of using the Laplace transforms. The presented experimental results show the highest above the pressure peak at injector level, its duration, amplitude of the second peak and attenuation in time domain of the pressure signal.

Keywords: sonic theory, electro-mechanical analogy, compressible fluids
2000 MSC: 74F10, 94C99

1. ELECTRO-HYDRAULIC MODELING; THE ASSOCIATION OF THE HYDRAULIC PHYSICAL MEASURES TO THE ELECTRICAL PHYSICAL MEASURES; THE GOGU CONSTANTINESCU FORMULAS

Consider the equations of the rapidly varying motions of fluids in pipes under pressure, [1],

$$(S_1)\begin{cases} \frac{\rho}{A}\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} + R_u\, q = 0 \\ \frac{A}{\rho\, c^2}\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0 \end{cases} \tag{1}$$

where ρ is the liquid density, A - the current section of a transmission liquid column, q - flow, p - liquid pressure, c - propagating velocity of the perturbation in liquid columns and R_u - unit resistance coefficient in sonic transmissions. Consider the equations of the long electric lines, [1], [4],

$$(S_2)\begin{cases} L_1\,\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} + R_l\, i = 0, \\ C_l\,\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} + G_l\, u = 0, \end{cases} \tag{2}$$

where $u = u(x,t)$ is the line voltage at the x distance from origin, $i = i(x,t)$ is the line current at the x distance from origin, L_l - lineic inductivity, C_l - lineic capacity,R_l - lineic resistance and G_l - lineic conductance. Comparing the systems (S1) and (S2) one can make a formal analogy between electrical and the corresponding hydraulical quantities. Systems (S1) and (S2) coincide if $G_l = 0$. In this case it is possible to achieve quantities in the electro-hydraulic analogy. Table 1 synthesizes the results of the formal comparison of the systems (S1) and (S2).

Notice that in the case of hydraulics circuits, the relation $G_l = 0$ corresponds to the existence of a sonic transmission line without loss of liquid.

Electricity physical measures	i	u	L_l	C_l	R_l
Coresponding physical measures from hydraulics	q	p	ρ/A	$A/\rho c^2$	R_u

Table 1.

2. HIGH PRESSURE PIPE

The transmission of the mechanical power from the pump to the injector proceeds at the sonic speed, (i.e. the sound speed in Diesel oil). The transmission media is Diesel oil modelled as a compressible liquid. The link between the pump and the injector is done by a high pressure pipe of length $L^{'}$. The walls of the high pressure pipe are elastic and the Diesel oil has elastic properties too. It follows the existence of a distributed sonic capacities per unit length of a Diesel gas pipe. Denote by C_1 the sonic capacity of the liquid column whose accumulation volume is V and by C_2 the sonic capacity due to the elasticity of the pipe walls. In fig. 1 we represent the specific capacities $C_1/L^{'}$ and $C_2/L^{'}$, in parallel, and the equivalent sonic capacity C_{SL}, corresponding to the unit length.

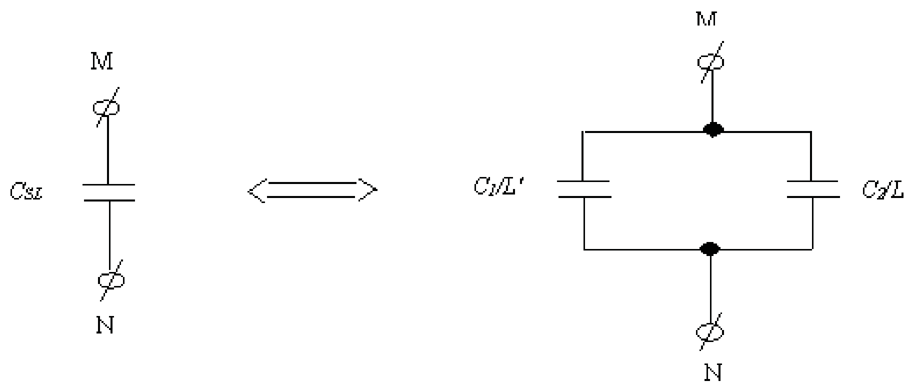


Fig. 1. Distributed sonic capacity.

The sonic capacities have [1],[2] the expressions $C_1 = V/E = L^{'}\Omega/E$, $C_2 = 1.25/E_1 \cdot D_m/e \cdot L^{'}\Omega$, where E is the elasticity module of Diesel oil , $L^{'}$ - the length of the pipe, Ω - the section of the pipe, E_1 - the elasticity module of the material the pipe is made of, D_m - the average diameter of the pipe and e - the thickness of the walls of the pipe. We get $C_{SL} = \frac{C_1+C_2}{L^{'}} = \Omega\ (1/E + 1.25/E_1 \cdot D_m/e)$, where C_{SL} is the distributed sonic capacity per unit length of the high pressure pipe. Sonic capacity C_2 is much (cca 20 - 25 times) smaller than the sonic capacity C_L [2], so that we may consider

$$C_{SL} \approx \Omega/E \tag{3}$$

The inertia of the liquid column determines a distributed sonic inductance per unit length [1], [2]

$$L_{SL} \approx \rho_l/(g\Omega), \tag{4}$$

where is the specific weight of the Diesel oil . Because of the friction between the interior walls of the high pressure pipe and the Diesel oil, we can assume the existence of a sonic resistance [1], [2] distributed per unit length of the liquid column. The expression of the sonic resistance is

$$R_{SL} \approx K^{*}\rho_c/(g\Omega), \tag{5}$$

where K^* is a constant whose value depends on the nature and the speed of the liquid, and ρ_c is the specific mass of the material the pipe is made of.

The high-pressure pipe can be modeled by an infinite cascaded chain of elementary sonic quadruples, with concentrated constants R_{SL}, L_{SL} and C_{SL}. Assuming that the line is homogenous, an elementary quadruple looks like in fig. 2.

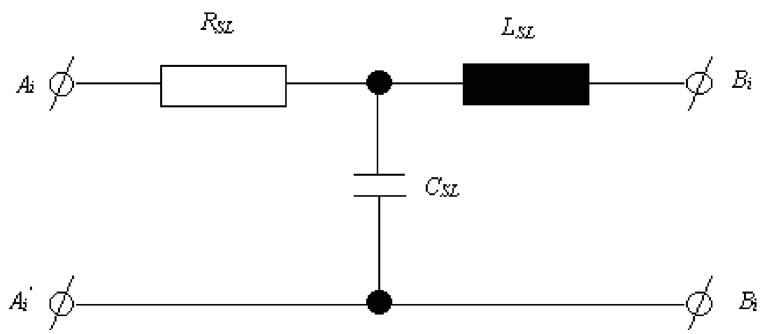


Fig. 2. Electrical equivalent of a sonic circuit associated with an elementary quadruple with concentrated constants R_{SL}, L_{SL} and C_{SL}.

According to the electro-hydraulic modeling, the high pressure pipe can be assimilated with a long electrical line. The electrical equivalent of the sonic circuit associated with the high pressure pipe is represented fig. 3.

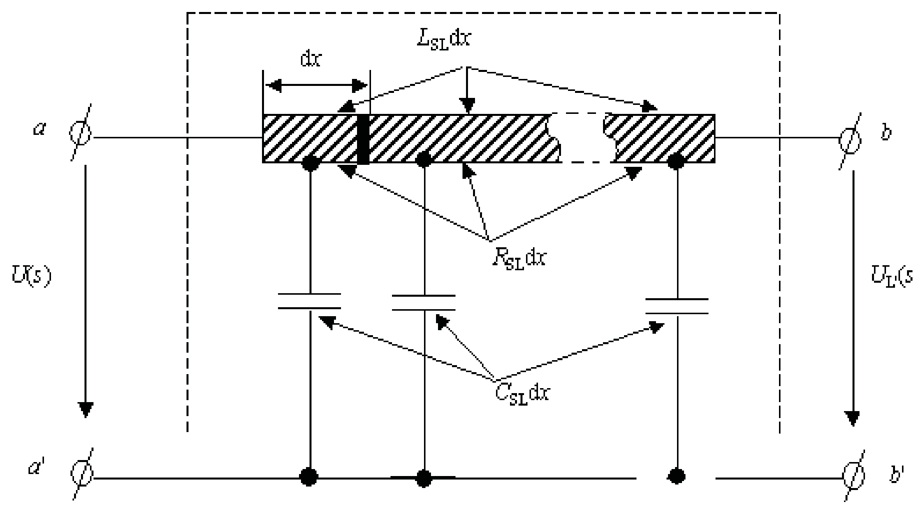


Fig. 3. The electrical equivalent of the sonic circuit associated with the high pressure pipe.

3. THE TRANSFER FUNCTION ASSOCIATED WITH THE SONIC CIRCUIT OF THE HIGH PRESSURE PIPE

Denote by $U(s)$ the sonic input voltage, and by $U_{L'}(s)$ the output sonic voltage of the high pressure pipe, in operational form (fig.3). The transfer function associated with the sonic circuit of the high pressure pipe can be written as $H_{lin}(s) = U_{L'}(s)/U(s)$. The expression of the sonic voltage in operational form in a transversal section of the high pressure pipe at the x distance from the sonic generator has the expression [5], [6]

$$U(x,s) = U(s)\left\{e^{-\gamma x} + \sum_{k=1}^{\infty}(-1)^k \rho_v^k \left[e^{-\gamma(2kL^{'}+x)} - e^{-\gamma(2kL^{'}-x)}\right]\right\}$$

If $x = L^{'}$, where $L^{'}$ is the length of the pipe, we get

$$U_{L'}(s) = U(s)\left\{e^{-\gamma L^{'}} + \sum_{k=1}^{\infty}(-1)^k \rho_v^k(s) \left[e^{-\gamma L^{'}(2k+1)} - e^{-\gamma L^{'}(2k-1)}\right]\right\}$$

and, finally, taking into account the expression of $H_{lin}(s)$,

$$H_{lin}(s) = e^{-\gamma L^{'}} + \sum_{k=1}^{\infty}(-1)^k \rho_v^k(s) \left[e^{-\gamma L^{'}(2k+1)} - e^{-\gamma L^{'}(2k-1)}\right]$$

where γ stands for the propagation constant, and $\rho_v(s)$ is the operational reflexion coefficient. If in the last relation we retain only the first two terms of the series for k = 1, 2, we get

$$H_{lin}(s) \cong e^{-\gamma L^{'}} + \left[\rho_v(s)\left(e^{-\gamma L^{'}} - e^{-3\gamma L^{'}}\right) + \rho_v^2(s)\left(e^{-5\gamma L^{'}} - e^{-3\gamma L^{'}}\right)\right]$$

while if we retain only the first term, for $k = 1$,

$$H_{lin}(s) \cong e^{-\gamma L^{'}} + \rho_v(s)e^{-\gamma L^{'}} - \rho_v(s)e^{-3\gamma L^{'}} \tag{6}$$

4. THE TRANSFER FUNCTION ASSOCIATED WITH THE SONIC CIRCUIT OF THE INJECTOR AT THE INJECTION PHASE

Consider the equivalent of the sonic circuit of the injector initiating the injection (fig. 4). Notice that the equivalent electric circuit of the sonic injector can be achieved exclusively with concentrated sonic elements, R_{ni}, L_i, C_i and R_{di}.

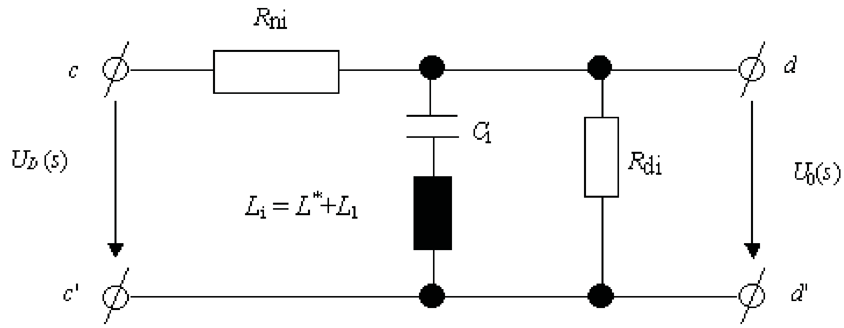


Fig. 4. The electrical equivalent of the sonic circuit of the injector at the initiation of the injection phase.

If the injector is closed, then Diesel gas leaks due to the lack of fitness. These leaks determine the existence of a theoretically infinite sonic resistance, R_{pi}. In this case, the delay line represented by the high pressure pipe ends on an infinite impedance. The needle of the sprayer, the rod and the pressure spring form a sonic circuit $L_i - C_i$ oscillating series. The inertia of the needle and rod determines a sonic inductance L^* and the inertia of the weight of the spring determines a sonic inductance L_1 . The portion between the needle of the injector and the nozzle introduces a sonic resistance, denoted by R_{ni}. The sonic perditance of the nozzle orifices is denoted by S_{di}. These orifices have a constant flow section. The reverse of the sonic perditance Sdi represents the sonic resistance of the nozzle orifices denoted by R_{di}. Denote by $U_{L'}(s)$ the sonic voltage at the output of the high pressure pipe, in operational form, and by $U_0(s)$ the sonic voltage, in operational form, corresponding to the pressure of the Diesel oil at the output of the calibrated orifices of the nozzles of the sonic injector [3]. The operational argument denoted was by s. The transfer function associated with the sonic circuit of the injector has the form

$$H_{inj}(s) = U_0(s)/U_{L'}(s) \tag{7}$$

We determine the expression of the transfer function $H_{inj}(s)$ by transforming the resulting circuit obtained from the electro-hydraulic equivalence of the sonic circuit of the injector. Using the equivalent sonic impedances method, [4], at the first stage, we get the electrical equivalence of the sonic electrical circuit of the injector. This is the first equivalence (fig. 5).

The equivalent sonic impedance Z_1 can be written, in operational form [4] $Z_1(s) = s(L^* + L_1) + (sC_i)^{-1}$. If we consider the sonic impedance Z_{1di}, as equivalent to the sonic impedance Z_1 and R_{di} which are parallel, we obtain the electrical equivalent of the sonic circuit of the injector, i.e. the second equivalence (fig. 6).

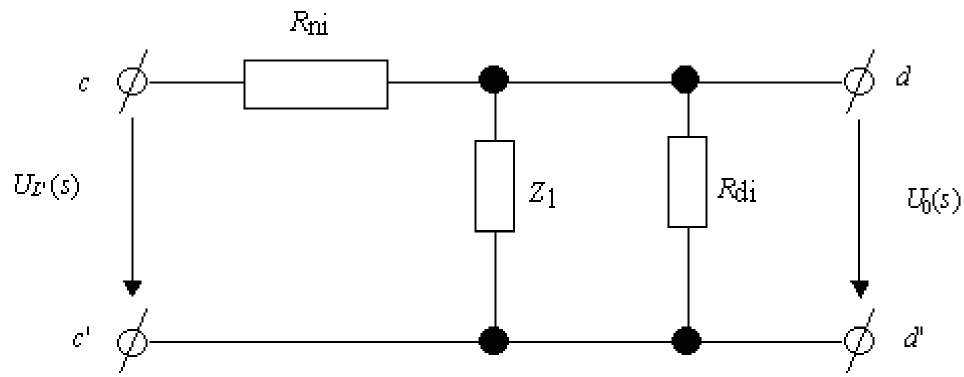


Fig. 5. Electrical equivalent of the sonic circuit of the injector; first equivalence.

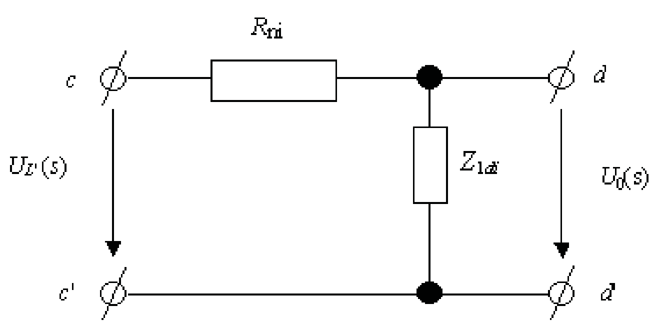


Fig. 6. Electrical equivalent of the sonic circuit of the injector; the second equivalence.

The equivalent sonic impedance Z_{1di} can be written in operational form [4] as $Z_{1di}(s) = Z_1(s)R_{di}/(Z_1(s) + R_{di})$. Using the divisor formula, [4], we get $U_0(s) = Z_{1di}/(R_{ni} + Z_{1di})\ U_{L'}(s)$, whence, by using relation (7), we get the expression of the transfer function associated with the sonic circuit of the injector on the form $H_{inj}(s) = Z_{1di}/(R_{ni} + Z_{1di})$. After successive transformations

and taking into account that $L_i = L^* + L_1$, we get

$$H_{inj}(s) = \frac{T_D s^2 + K_p}{T_1^* s^* + T_2 s + 1} \tag{8}$$

where the following notations have been used: $T_D = R_{di}/(R_{di} + R_{ni})\ C_i L_i = K_p C_i L_i$, $K_p = R_{di}/(R_{di} + R_{ni})$, $T_1^* = C_i L_i$, $T_2 = R_{ni} R_{di}/(R_{di} + R_{ni})\ C_i$. The series circuit $C_i - L_i$ (fig. 4), suggests the apparition of the oscillation phenomenon upon the needle of the sonic injector in the injection phase.

5. MAKING EXPLICIT THE TRANSFER FUNCTION $H_G(S)$ ASSOCIATED WITH THE CHAIN OF SONIC QUADRUPLES PIPE-SONIC INJECTOR

Consider the high pressure pipe and the sonic injector as a quadruples chain connected to the sonic generator in cascade, their interaction being made exclusively on the terminals (fig. 7). The expression of the transfer function $H_g(s)$ can be written [4] as $H_g(s) = H_{lin}(s) \cdot H_{inj}(s)$.

Replacing the expression of $H_{lin}(s)$ and $H_{inj}(s)$ in relations (6) and (8) respectively we obtain, for the injection phase,
$H_g(s) = \left(e^{-\gamma L'} + \rho_v(s)e^{-\gamma L'} - \rho_v(s)e^{-3\gamma L'}\right)\left(\frac{T_D s^2 + K_p}{T_1^* s^2 + T_2 s + 1}\right)$, $H_g(s) = U_0(s)/U(s)$.
Considering the inverse Laplace transforms of the transfer functions $H_{inj}(s)$ and $H_{lin}(s)$ respectively, namely $h_{inj}(t) = L^{-1}\{H_{inj}(s)\}$, $h_{lin}(t) = L^{-1}\{H_{lin}(s)\}$, we may write, [4], $H_g(s) = L\{h_{inj}(t) * h_{lin}(t)\}$ where the convolution product $h_{inj}(t) * h_{lin}(t)$ is given by [4]

$$h_g(t) = h_{inj}(t) * h_{lin}(t) = \int_0^\infty h_{inj}(\tau) h_{lin}(t-\tau) d\tau\ ,$$

where $h_g(t) = L^{-1}\{H_g(s)\}$ and $H_g(s) = L\{h_g(t)\} = \int_0^\infty h_g(t) e^{-st} dt$, s being the operational argument.

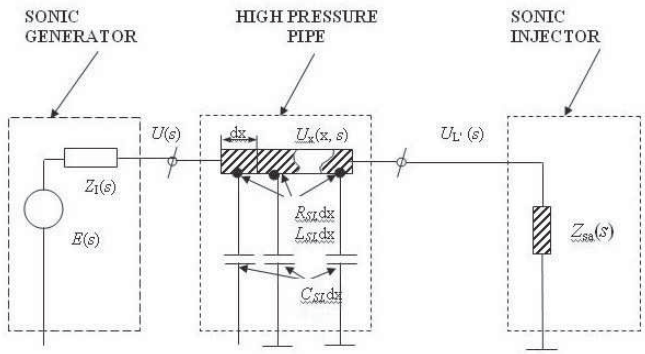


Fig. 7. Electrical equivalent of the sonic circuit associated with a pumping section connected to the injector by a high pressure pipe. Legend: s = operational argument; $E(s)$ = internal sonic voltage of sonic generator, in operational form; $Z_I(s)$ = sonic impedance of sonic generator, in operational form; R_{SL}, L_{SL}, C_{SL} = resistance, inductance, and sonic capacity distributed per unit length of the high pressure pipe; $Z_{sa}(s)$ = equivalent sonic impedance of injector, in operational form; $U(s)$ = the sonic voltage at the input of high pressure pipe, in operational form; $U_{L'}(s)$ = sonic voltage at the output of high pressure pipe, in operational form; $U_x(x,s)$ = sonic voltage in a transversal section of high pressure pipe at the x distance from the sonic generator, in operational form.

6. DETERMINATION OF THE EXPRESSION OF THE SONIC VOLTAGE SIGNAL AT THE INPUT OF THE HIGH PRESSURE PIPE

A pumping section of a Diesel in-line injection pump represents a sonic voltage impulse generator (pressure), (fig. 8). A pumping section (sonic generator) consists of the injection cam, belaying-cleat with reel, the piston of the pumping element, the flow valve and the absorption valve [3], [7]. The high pressure pipe represents a link element of the sonic circuit placed between the absorbing valve and the sonic injector.

Injecting the fuel in the Diesel motor cylinder is a complex phenomenon of transmitting mechanical power from the injection pump to the injector by means of the sonic waves. The transmission media of the sonic waves is the Diesel oil found in the pressure sonic generator, in the high pressure pipe and in the injector. The internal sonic impedance of the sonic generator is written in the operational form as a function of the pressure sonic generator impedances $Z_I^*(s)$, flow valve $Z_{sd}(s)$ and absorption valve $Z_a(s)$, namely

$$Z_I = Z_I^*(s) + Z_{sd}(s) + Z_a(s)$$

According to the divisor formula [4] (fig. 8), the level of the signal at the input of the line is given by the sonic voltage

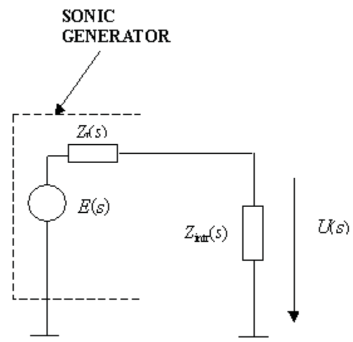


Fig. 8. Electric equivalent of the sonic circuit of a pumping section, connected at a charge of $Z_{intr}(s)$ impedance.

$$U(s) = \frac{Z_{int\,r}(s)}{Z_I^*(s) + Z_{sd}(s) + Z_a + Z_{int\,r}(s)}\,E(s)$$

where $Z_{int\,r}(s)$ is the sonic impedance, in operational form, seen by the sonic generator [4],

$$Z_{int\,r}(s) = \frac{Z_{sa}(s)ch\gamma L' + Z_0(s)sh\gamma L'}{Z_{sa}(s)sh\gamma L' + Z_0(s)ch\gamma L'},$$

$Z_0(s)$ is the operational characteristic sonic impedance of the line (corresponding to the high pressure pipe), and $Z_{sa}(s)$ is the operational sonic impedance of the injector. The operational reflection coefficient, $\rho_v(s)$ has the expression, [4], $\rho_v(s) = [Z_{sa}(s) - Z_0(s)]\,/[Z_{sa}(s) + Z_0(s)]$. If we neglect the leaks of Diesel oil, the propagation constant of the delaying line represented by the high pressure pipe is written as $\gamma = \gamma(s) = \sqrt{(R_{SL} + L_{SL})sC_{SL}}$. Replacing R_{SL}, L_{SL} and C_{SL}, out of the relations (3), (4) and (5) respectively, we get $\gamma = \gamma(s) = \sqrt{(K^*\rho_c + s\rho_l)s/(gE)}$. For a lossless line without leakage ($G_{SL} = 0$ and $R_{SL} = 0$ respectively), the characteristic impedance, in operational form, has the expression

$$Z_0(s) = \sqrt{\frac{R_{SL} + sL_{SL}}{G_{SL} + sC_{SL}}} = \sqrt{\frac{L_{SL}}{C_{SL}}} = \frac{1}{\Omega}\sqrt{\frac{\rho_l E}{g}}\,.$$

Taking into account the electrical equivalent of the sonic circuit of the injector (fig. 5), the impedance of the charge can be written in the form

$$Z_{sa}(s) = R_{ni} + \frac{R_{di}\left(sL_i + \frac{1}{sC_i}\right)}{R_{di} + sL_i + \frac{1}{sC_i}} = R_{ni} + \frac{R_{di}\left(1 + s^2L_iC_i\right)}{1 + sC_iR_{di} + s^2C_iL_i}$$

The case of in-line Diesel injection systems is typical for transmitting relatively great forces in a small amount of time, $\Delta t \cong 2$ ms, by means of the Diesel oil in high pressure pipes to a sonic receptor placed at a distance from the sonic generator. As the liquid is elastic and has a finite mass the transmission is not instantaneous but depends on the speed of the sound in the Diesel oil. The frequency of the liquid column from the high pressure pipe is, in general, several times greater than the frequency at which injections take place. The pressure waves have the time to travel the pipe several times in-between two successive injections, being reflected at the sonic injector's end as well as at the coupling end of the pumping section. The relexion coefficient $\rho_v \approx 0$, can have positive or negative values. In order for the line to be adapted to the charge it is necessary that $\rho_v \approx 0$. In this case the sonic receptor (the injector) absorbs almost the entire energy of the direct wave.

7. EXPERIMENTAL RESULTS

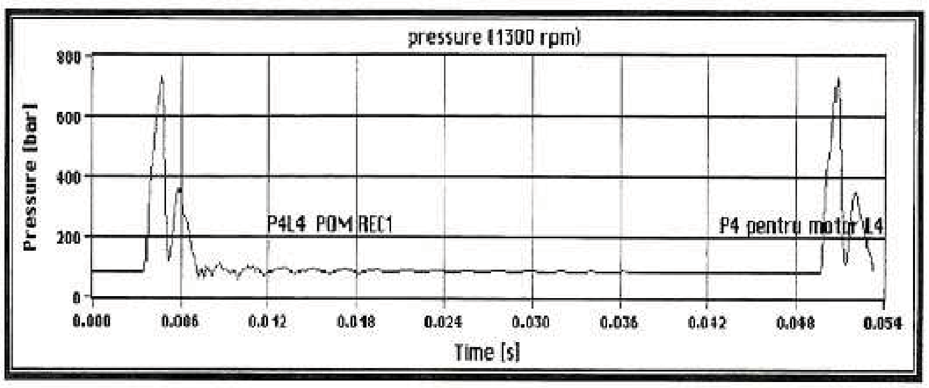


Fig. 9. Pressure diagram at the entry to the high pressure pipe, for the rated power revolution, $n = 1300rpm$.

We report here only two of our experiments. They concern the pressure as a function of time near the coupling where $x = 0$. Namely, the instantaneous pressure is measured at the entry to the high pressure pipe, for the revolution $n = 1300rpm$ corresponding to the rated power (fig. 9), as well as for the revolution $n = 800rpm$, corresponding to the maximum torque (fig. 10). In the other two experiments the pressure was measured at the end close to the injector of the high pressure pipe.

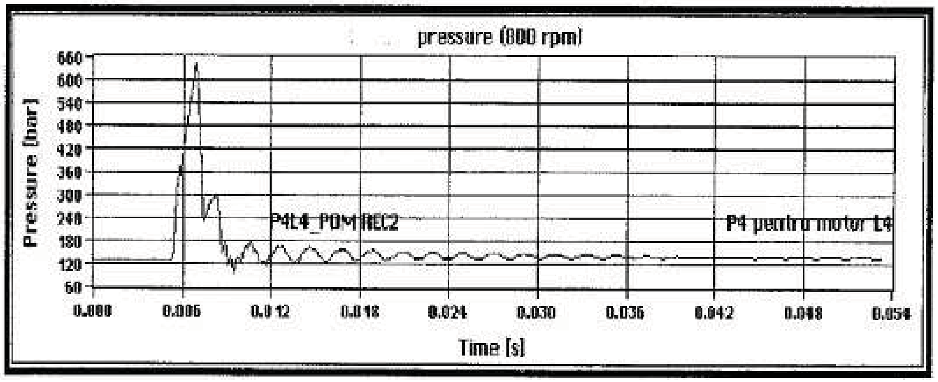


Fig. 10. Pressure diagram at the entry to the high pressure pipe, for the rated power revolution, $n = 800rpm$.

References

[1] Cioc, D., Maruta, AL., *Foundations of sonic theory*, Institute of Constructions of Bucharest, 1972. (Romanian)

[2] Constantinescu, G., *Sonic theory*, Editura Academiei, Bucharest, 1985. (Romanian)

[3] Dumitru, P., *Fuel injection pumps and injectors for cars and tractors*, Editura Ceres, Bucharest, 1988. (Romanian)

[4] Mocanu, C.I., *Theory of electric circuits*, Editura Didactica si Pedagogica, Bucharest, 1979. (Romanian)

[5] Panaitescu, V., Pislaru, D.L., *Electro-hydraulic modeling of the fuel injection systems, based on the George Constantinescu's sonic principle of energy transmision by vibrations*, The 10th National Jubilee Conference, S.P.E.R.I.N., University Politehnica of Bucharest, September 30 , 2000.

[6] Pislaru-Danescu, L., Panaitescu, V., *Development of fuel injection systems by using sonic theory*, Buletinul Institutului Politehnic din Iasi, Engine Constructions, **46 (L)**(2000), 263-266.

[7] Pislaru-Danescu, L., *Contributions to the study of hydraulic system in injection pumps for Diesel engines; applications of the sonic theory*, PhD thesis, Universitatea Politehnica Bucharest, 2005. (Romanian)

[8] Turcoiu, T., Boncoi, I., Time, A., *Injection equipments for engines with internal burning*, Editura Tehnica, Bucharest, 1987. (Romanian)

ROMAI J., **2**, 2(2006),
141–148

ON THE NUMBER OF THE EQUILIBRIA FOR A PENDULUM WITH NEO-HOOKEAN ROD

Marina Pandrea, Nicolae - Doru Stănescu
Department of Applied Mechanics, University of Piteşti

Abstract In this paper we study the pendulum with a non-linear neo-Hookean type sustain rod. In this situation a linear relation is not suitable to connect the force in the rod and the deformation of the rod. In addition, the sustain point of the pendulum has a vertical displacement. This displacement is a function of the instantaneous rod elongation. On the basis of the obtained equation of motion of such a pendulum, we perform a study concerning the number of the equilibrium positions for the pendulum. In our paper, all the possible cases are considered. We find that, in dependence on the values of the parameter, defining the vertical forced oscillations of an end of the rod, there are zero, one or two equilibrium positions.

1. INTRODUCTION

The system proposed for study is drawn in fig. 1. It consists in by the rod AB of length l and negligible mass and the ball of mass m, the ball being situated at the point B .

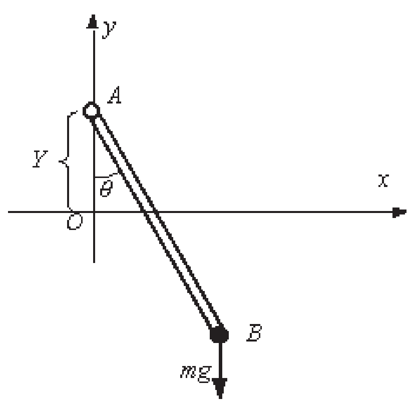


Fig. 1. Mathematical model.

The length of the rod AB in undeformed state is l_0. Its static elongation under the action of the ball weight mg is denoted by z_{st}. The joint from the end A of the rod has a vertical motion under the law $Y = Y\,(t)$, which we assume as known. The elastic force in the rod is assumed to be defined by a potential U, which will be presented further. The system has two degrees of freedom: the length l of the rod AB and the angle θ formed by the rod with the vertical descendent direction.

With the following notation

$$l = l_0 + z_{st} + z; \quad H = \frac{A_0 G}{m l_0}; \quad \omega^2 = \frac{g}{l_0}; \quad \bar{Y} = \frac{Y}{l_0}, \tag{1}$$

where z is the rod elongation relative to the static equilibrium position, G is the shear modulus, and A_0 is the rod cross-sectional area, and assuming that $U = -A_0 G l_0 \left(\frac{\lambda^2}{2} + \frac{1}{\lambda}\right)$ and $\ddot{\bar{Y}} = BH\lambda$, where B is a real constant, one obtains the equations of motion

$$\begin{gathered} \ddot{\lambda} - \lambda\dot{\theta}^2 + H\left(\lambda - B\cos\theta\lambda - \frac{1}{\lambda^2}\right) = \omega_0^2 \cos\theta, \\ \lambda\ddot{\theta} + 2\dot{\lambda}\dot{\theta} + \left(\omega_0^2 + BH\lambda\right)\sin\theta = 0. \end{gathered} \tag{2}$$

2. EQUILIBRIA

Denoting $\xi_1 = \lambda$, $\xi_2 = \theta$, $\xi_3 = \dot{\lambda}$, $\xi_4 = \dot{\theta}$, the system (2) is transformed in a system of four non-linear first order differential equations

$$\begin{gathered} \frac{\mathrm{d}\xi_1}{\mathrm{d}t} = \xi_3; \quad \frac{\mathrm{d}\xi_2}{\mathrm{d}t} = \xi_4; \\ \frac{\mathrm{d}\xi_3}{\mathrm{d}t} = \xi_1\xi_4^2 - H\left[(1 - B\cos\xi_2)\,\xi_1 - \frac{1}{\xi_1^2}\right] + \omega_0^2\cos\xi_2; \\ \frac{\mathrm{d}\xi_4}{\mathrm{d}t} = -\frac{2\xi_3\xi 4}{\xi_1} - \frac{\omega_0^2 + BH\xi_1}{\xi_1}\sin\xi_2. \end{gathered}$$

Its equilibria are obtained at the intersection of the nullclines

$$\begin{gathered} \xi_3 = 0; \quad \xi_4 = 0; \\ \xi_1\xi_4^2 - H\left[(1 - B\cos\xi_2)\,\xi_1 - \frac{1}{\xi_1^2}\right] + \omega_0^2\cos\xi_2 = 0; \\ -\frac{2\xi_3\xi 4}{\xi_1} - \frac{\omega_0^2 + BH\xi_1}{\xi_1}\sin\xi_2 = 0. \end{gathered} \tag{3}$$

There are only two possibilities: $\xi_2 = 0$ or $\xi_2 = \pi$.

3. CASE $\xi_2 = 0$

From the third relation (3) we obtain

$$(1 - B)\,\xi_1^3 - 1 - \frac{\omega_0^2}{H}\xi_1^2 = 0. \tag{4}$$

If $B = 1$, from (4) it follows the equation $-\frac{\omega_0^2}{H}\xi_1^2 = 1$, which has no solution in $\mathbb{R}$; therefore, for $B = 1$ there exists none equilibrium position.

If $B \neq 1$, then dividing the equation (4) by $(1-B)$, performing the transformation

$$\xi_1 = \zeta_1 + \frac{\omega_0^2}{3H(1-B)}, \tag{5}$$

and denoting

$$a = -\frac{\omega_0^4}{3H^2(1-B)^2}; \quad b = -\frac{2\omega_0^6}{27H^3(1-B)^3} - \frac{1}{1-B}, \tag{6}$$

we obtain

$$\zeta_1^3 + a\zeta_1 + b = 0. \tag{7}$$

By the Hudde method, the number of the real roots of the equation (7) depends on the discriminant $\Delta = 4a^3 + 27b^2$. Thus, if $\Delta < 0$, then the equation (7) has three distinct real roots; if $\Delta = 0$, then the equation (7) has three real roots but two of them are equal, and if $\Delta > 0$, then the equation (7) has one and only one real root.

In our case, the discriminant reads $\Delta = \frac{4\omega_0^6}{H^3(1-B)^4} + \frac{27}{(1-B)^2}$. Denote by f the function $f = \zeta_1^3 + a\zeta_1 + b$ and by f' the derivative $f' = 3\zeta_1^2 + a$.

Case $\Delta = 0$. In this case Δ is a sum of two strictly positive terms. We have no equilibrium position.

Case $\Delta < 0$. For the same reasons as in the previous paragraph we obtain no equilibrium points.

Case $\Delta > 0$. In this case $\dfrac{4\omega_0^6}{H^3(1-B)^2} + 27 > 0$. Equation (7) has one and only one real root. If we want to have an equilibrium position, we must impose the condition that the equation (4) has a positive real solution. Recalling the transformation (5) one obtains the relation $\zeta_1 > -\frac{\omega_0^2}{3H(1-B)}$.

Remembering that the derivative f' has two real roots $-\sqrt{-a/3}$, and $\sqrt{-a/3}$, because $a < 0$ (see the first relation (6)), the first root corresponding to a maxim and the second root to a minim of the function f we must have the relation

$$\mathrm{f}\left(-\frac{\omega_0^2}{3H(1-B)}\right) < 0,$$

implying $1 - B > 0$.

Case $\zeta_2 = \pi$ From the third relation (3) it follows

$$(1+B)\xi_1^3 + \frac{\omega_0^2}{H}\xi_1^2 - 1 = 0. \tag{8}$$

If $B = -1$ from (8) we have

$$\frac{\omega_0^2}{H}\xi_1^2 = 1,$$

with the solution

$$\xi_1 = \frac{\sqrt{H}}{\omega_0}.$$

Thus we have an equilibrium position.

If $B \neq -1$, we divide the equation (8) by $(1+B)$ to get

$$\xi_1^3 + \frac{\omega_0^2}{H(1+B)}\xi_1^2 - \frac{1}{1+B} = 0. \tag{9}$$

Performing the transformation

$$\xi_1 = \zeta_1 - \frac{\omega_0^2}{3H(1+B)}, \tag{10}$$

from (9) we obtain the equation

$$\zeta_1^3 - \frac{\omega_0^4}{3H^2(1+B)^2}\zeta_1 + \frac{2\omega_0^6}{27H^3(1+B)^3} - \frac{1}{1+B} = 0. \tag{11}$$

Denote

$$a = -\frac{\omega_0^4}{3H^2(1+B)^2}; \quad b = \frac{2\omega_0^6}{27H^3(1+B)^3} - \frac{1}{1+B}, \tag{12}$$

such that the equation (11) take the form (7).

The comments relative to the number of the real roots of the equation (7) remain valid, with the remark that now the discriminant reads

$$\Delta = -\frac{4\omega_0^6}{H^3(1+B)^4} + \frac{27}{(1+B)^2}.$$

Case $\Delta = 0$. This case implies $\frac{4\omega_0^6}{H^3(1+B)^4} = \frac{27}{(1+B)^2}$, so $B = -1 \pm \sqrt{\frac{4\omega_0^6}{27H^3}}$. If $B = -1 + \sqrt{\frac{4\omega_0^6}{27H^3}}$, then, from (12), we have $a = -\frac{9H}{4\omega_0^2}; b = -\frac{1}{2\sqrt{\frac{4\omega_0^6}{27H^3}}}$. The solutions of the equation $f'(\zeta_1) = 0$ are

$$\zeta_1^{(1)} = -\sqrt{-\frac{a}{3}} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \zeta_1^{(1)} = \sqrt{-\frac{a}{3}} = \sqrt{\frac{3H}{4\omega_0^2}}.$$

On the other hand $f\left(\zeta_1^{(1)}\right) = 0$; therefore the double root of the equation $\mathrm{f}(\zeta_1) = 0$ is $\zeta_1^{(1)}$. Equation $\mathrm{f}(\zeta_1) = 0$ has the roots

$$\zeta_1^{*} = \zeta_1^{(1)} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \zeta_1^{**} = -2\zeta_1^{(1)} = 2\sqrt{\frac{3H}{4\omega_0^2}},$$

the first of them being double.

With the transformation (10), the solutions of the equation (9) read

$$\xi_1^* = -2\sqrt{\frac{3H}{4\omega_0^2}} < 0; \quad \xi_1^{**} = \sqrt{\frac{3H}{4\omega_0^2}} > 0.$$

Again, the root ξ_1^* is double.

We have one equilibrium position given by ξ_1^{**}. If $B = -1 - \sqrt{\frac{4\omega_0^6}{27H^3}}$, then from the relations (12) we have $a = -\frac{9H}{4\omega_0^2}; b = \frac{1}{2\sqrt{\frac{4\omega_0^6}{27H^3}}}$. The solutions of the equation $f'(\zeta_1) = 0$ are

$$\zeta_1^{(1)} = -\sqrt{-\frac{a}{3}} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \zeta_1^{(2)} = \sqrt{-\frac{a}{3}} = \sqrt{\frac{3H}{4\omega_0^2}}$$

We have $f\left(\zeta_1^{(1)}\right) = 0$, therefore the double root of the equation $\mathrm{f}(\zeta_1) = 0$ is $\zeta_1^{(2)}$.

Equation $\mathrm{f}(\zeta_1) = 0$ has the roots

$$\zeta_1^* = -2\zeta_1^{(2)} = -2\sqrt{\frac{3H}{4\omega_0^2}}; \quad \zeta_1^{**} = \zeta_1^{(2)} = \sqrt{\frac{3H}{4\omega_0^2}},$$

the second of them being double.

By the transformation (10), the solutions of the equation (9) read

$$\xi_1^* = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \xi_1^{**} = 2\sqrt{\frac{3H}{4\omega_0^2}}, \tag{13}$$

the second of them being double.

We have one equilibrium position given by $\xi 1^{**}$.

Case $\Delta > 0$. In this case there exists only one root for the equation $\mathrm{f}(\zeta_1) = 0$. We obtain $-\frac{4\omega_0^6}{H^3(1+B)^4} + \frac{27}{(1+B)^2} > 0$; therefore

$$(1+B)^2 > \frac{4\omega_0^6}{27H^3}. \tag{14}$$

If follows that $B \in \left(-\infty, -\frac{2\omega_0^3}{27H^3} - 1\right) \bigcup \left(\frac{2\omega_0^3}{27H^3} - 1, \infty\right), b = \frac{1}{1+B}[\frac{2\omega_0^6}{27H^3(1+B)^2} - 1]$.

If $B < -\frac{2\omega_0^3}{\sqrt{27H^3}} - 1$, we shall prove that $b > 0$. Indeed, in this case $1+B < 0$ and the condition $b > 0$ reads $\frac{2\omega_0^6}{27H^3(1+B)^2} - 1 < 0$, an obvious relation from (14). Therefore $b > 0$.

The roots of the equation $f'(\zeta_1) = 0$ are given by $\zeta_1^{(1)} = -\sqrt{-a/3}$ and $\zeta_1^{(2)} = \sqrt{-a/3}$. In this case f has one negative real root less than $\zeta_1^{(1)}$.

On the other hand $\sqrt{-\frac{a}{3}} = -\frac{\omega_0^2}{3H(1+B)}$. It follows that the transformation (10) reads $\xi_1 = \zeta_1 + \sqrt{-a/3}$ and therefore the equation (8) has one negative real root; we have no equilibrium position.

If $B > \frac{2\omega_0^3}{\sqrt{27H^3}} - 1$, then $b < 0$. Indeed, in this case $1+B > 0$ and the condition $b < 0$ reads $\frac{2\omega_0^6}{27H^3(1+B)^2} - 1 < 0$, an obvious relation from (14). Therefore $b < 0$. It follows that the equation $\mathrm{f}(\zeta_1) = 0$ has exactly one positive real root greater than $\sqrt{-a/3}$.

Since $\sqrt{-\frac{a}{3}} = \frac{\omega_0^2}{3H(1+B)}$ it follows that the transformation (10) reads $\xi_1 = \zeta_1 - \sqrt{-a/3}$. Therefore the equation (8) has exactly one positive real root; we have one equilibrium position.

Case $\Delta < 0$. The equation $\mathrm{f}(\zeta_1) = 0$ has now three distinct real roots. From $\Delta < 0$ it follows $-\frac{4\omega_0^6}{H^3(1+B)^4} + \frac{27}{(1+B)^2} < 0$, so

$$B \in \left(-\frac{2\omega_0^3}{27H^3} - 1, \frac{2\omega_0^3}{27H^3} - 1 \right). \tag{15}$$

The expression of b from (12) reads

$$b = \frac{2\omega_0^6 - 27H^3 (1+B)^2}{27H^3 (1+B)^3}. \tag{16}$$

Denoting $\gamma = 1 + B$ the expression (16) becomes $b = \frac{2\omega_0^6 - 27H^3\gamma^2}{(27H^3\gamma)^3}$ and from (15) one obtains $\gamma \in \left(-\frac{2\omega_0^3}{27H^3}, \frac{2\omega_0^3}{27H^3} \right)$. Remark that $\sqrt{-\frac{a}{3}} = \frac{\omega_0^2}{3H|1+B|} = \frac{\omega_0^2}{3H|\gamma|}$. The expression of b becomes zero for $\gamma_1 = -\sqrt{\frac{2\omega_0^6}{27H^3}}; \gamma_2 = \sqrt{\frac{2\omega_0^6}{27H^3}}$

We have the following six possibilities:

- if $b > 0$ and $\gamma < 0$, then the transformation (20) becomes

$$\xi_1 = \zeta_1 + \sqrt{-a/3}. \tag{17}$$

The function f has one negative real root less than $-\sqrt{-a/3}$, one positive real root situated between 0 and $\sqrt{-a/3}$, and one positive real root greater than $\sqrt{-a/3}$. Recalling now the formula (17) we obtain that the equation (8) has one negative root and two positive real roots; therefore there exist two equilibrium positions;

- if $b < 0$ and $\gamma < 0$, then the transformation (10) has the same form (17). The equation $\mathrm{f}(\zeta_1) = 0$ has one negative real root less than $-\sqrt{-a/3}$, one negative real root situated between $-\sqrt{-a/3}$ and 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has one negative real root and two positive real roots; therefore there exist two equilibrium positions;

- if $b > 0$ and $\gamma > 0$, then the transformation (10) takes the form

$$\xi_1 = \zeta_1 - \sqrt{-a/3}. \tag{18}$$

The equation $\mathrm{f}(\zeta_1) = 0$ has one negative real root less than $-\sqrt{-a/3}$, one positive real root situated between 0 and $\sqrt{-a/3}$, and one positive real root greater than $\sqrt{-a/3}$. From (18) it follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position;

- if $b < 0$ and $\gamma > 0$, then the transformation (10) reads again as (18). The equation $\mathrm{f}(\zeta_1) = 0$ has two negative real roots and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position;

- if $\gamma = -\dfrac{\omega_0^2\sqrt{2}}{\sqrt{27H^3}}$, then $b = 0$, $\gamma < 0$ and the transformation (10) reads again as (17). The equation $\mathrm{f}(\zeta_1) = 0$ has one negative real root less than $-\sqrt{-a/3}$, one root equal to 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two positive real roots; therefore there exist two equilibrium positions;

- if $\gamma = \dfrac{\omega_0^2\sqrt{2}}{\sqrt{27H^3}}$, then $b = 0$, $\gamma > 0$ and the transformation (10) reads as (18). The equation $\mathrm{f}(\zeta_1) = 0$ has one negative real root, one root equal to 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position.

4. CONCLUSIONS

In our paper we studied the pendulum with one neo-Hookean rod. We determined the equations of motion and we presented the number of equilibria as a function of the real parameter B. We obtain that this number can be 0, 1 or 2.

References

[1] Pandrea, N., Stănescu, N.-D., *Mecanica*, Editura Didactică şi Pedagogică, Bucureşti, 2002.

[2] Teodorescu, P., P., *Sisteme mecanice. Modele clasice*, 4 vol., Editura Tehnică, Bucureşti, 1984, 1988, 1997, 2002.

[3] Dragoş, L., *Principiile mecanicii analitice*, Editura Tehnică, Bucureşti, 1976.

[4] Lurie, A. I., *Analytical mechanics*, Springer, Berlin, 2002.

[5] Neimark, Ju. I., Fufaev, N. A., *Dynamics of nonholonomic systems*, Translations of Mathematical Monographs, **33**, AMS, Providence, R. I., 1972.

[6] Bhattacharyya, R., *Behavior of a rubber spring pendulum*, J. of App. Mech., **67**, 2, (2000).

[7] Nayeh, A. H., Mook, D. T., *Nonlinear oscillations*, John Wiley & Sons, New York, 1979.

[8] Olsson, M. G. *Why does a mass on a spring sometimes misbehave?*, Am. J. Phys, **44**, (1976), 1211-1212.

[9] Minorsky, N., *Nonlinear oscillations*, Van Nostrand Reinhold, New York, 1962.

[10] Anicin, B. A., Davidovic, D. M., Babovic, V. M., *On a linear theory of the elastic pendulum*, Eur. J. Phys., **14**, (1993), 132-135.

[11] Lai, H. M., *On the recurrence phenomenon of a resonant spring pendulum*, Am. J. Phys., **52**, (1984), 219-223.

[12] Kane T. R., Kahn M. E., *On a class of two degrees of freedom oscillations*, ASME J. Appl. Mech., **35**, (1968), 547-552.

[13] Ryland II, H. G., Meirovitch, L., *Stability boundaries of a swinging spring with oscillating support*, J. Sound Vib., **51**, (1977), 547-560.

[14] Nunez-Yepez, N., Salas-Brito, A. L., Vargas, C. A., Vicente, L., *Onset of chaos in an extensible pendulum*, Phys. Lett. **A**, **145**, (1990), 101-105.

[15] Stănescu, N.-D., Pârlac, S., Pandrea, M., *About the dynamics and the stability of the pendulum with neo-Hookean rod*, Rev. Roum. Sci. Techn., Méc. Appl., 2005 (in press).

[16] Stănescu, N.-D., Pârlac, S., Pandrea, N., Pandrea, M., *About the dynamics and the sability of the equilibrium points for a nonlinear elastic pendulum*, Proc. of the 12th Int. Congress on Acoustic and Vibrations, Lisbon, 2005.

ROMAI J., **2**, 2(2006),
149–152

NUMERICAL BUILDING OF THE ORTHOGONAL MAPPING OF THE CURVILINEAR QUADRANGLE DOMAIN INTO THE RECTANGLE

Vladimir Patiuc

Moldova State University, Chişinău, Republic of Moldova

Abstract We propose and study the mathematical formulation of the problems of building the orthogonal mapping of the physical curvilinear quadrangle domain into the square or rectangular computation domain. The problem is formulated as a system of two coupled Laplace equations with non-linear boundary conditions, containing both unknown functions. The problem is solved by the finite difference method using the sweep matrix method and Seidel linearization. The program built in FORTRAN was tested on the exact analytical solution.

1. INTRODUCTION

The contemporary level of the computer technique development allows us to formulate and solve the problems of the dynamic prognosis for the conduct of the compound mechanical constructions of complex form. These problems are mathematicaly formulated as a system of partial differential equations of hyperbolic type. One of the effective methods for solving this kind of problems is the method of finite differences (MFD). This very method allows building in the rectangular-type domains the numerical scheme for solving the problem with minimal unwanted effects of physical nature, such as dispersion and dissipation. In a more complex domain of the MFD, encountering the problems of approximation of the boundary conditions, as well as the algorithms and the programs for the problem solving, becomes considerably more complex. That is why it is actual the elaboration of methods to map physical domain having a form of a curvilinear rectangle on the computation domain which has the form of a quadrangle or rectangle.

To the problem of numerical building of the arbitrary domain mapping on the quadrangle has been dedicated a large number of works, the survey of which is presented in [1]. However, algorithms used nowadays are considerably complex and mostly they are used to solve the hydrodynamics problems. In this paper we formulate the problem and propose an effective method for the numerical building of the orthogonal mapping of the curvilinear rectangle on the quadrangle.

After obtaining the orthogonal mapping the problem of calculation of the dynamic concentration of the stresses is solved in the orthogonal system of coordinates. If referred to this system, the elasticity theory equations have a more complex structure compared with the Cartesian system of coordinates. However, due to the rectangular form of the computation domain and the build by the author conservative difference scheme [2, 3], it is possible to effectively solve the problems of the stress concentration in domains of complex form.

2. BUILDING OF THE ORTHOGONAL MAPPING

Let us examine the dynamic problem of the elasticity theory in the domain presented in the left-hand side (fig.1). If we try to solve this problem numerically in the Cartesian system of coordinates Oxy by frontiers of the difference method, we encounter considerable difficulties of approximating the domain and the given boundary conditions.

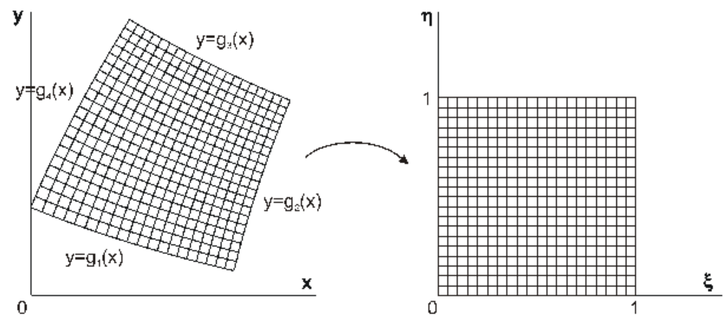


Fig. 1. Physical and computation domains.

Thus we propose the following method: first we perform the conformal or orthogonal mapping of the initial curvilinear domain written in the coordinates Oxy on the quadrangle or rectangular domain written in coordinates $O\xi\eta$. To this aim two problems involving the Laplace equations, which allow to determine the direct and inverse functions of the mapping: $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ and $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ are solved.

In the most general case in order to build the orthogonal mapping in the quadrangle domain written in the coordinates $O\xi\eta$ it is necessary to solve the following system of equations [1]

$$g_{22}\frac{\partial^2 x}{\partial \xi^2} + g_{11}\frac{\partial^2 x}{\partial \eta^2} + g_{11}g_{22}\left(\frac{\partial x}{\partial \xi}\Delta\xi + \frac{\partial x}{\partial \eta}\Delta\eta\right) = 0,$$
$$g_{22}\frac{\partial^2 y}{\partial \xi^2} + g_{11}\frac{\partial^2 y}{\partial \eta^2} + g_{11}g_{22}\left(\frac{\partial y}{\partial \xi}\Delta\xi + \frac{\partial y}{\partial \eta}\Delta\eta\right) = 0,$$

$$g_{11} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2, \quad g_{22} = \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2.$$

This problem is a complex nonlinear problem of mathematical physics. Therefore, in order to elaborate an effective numerical method to solve it, we examine the case $g_{11} = g_{22} = 1$. Then the above system is transformed into the system of two Laplace equations $\Delta x = 0$, $\Delta y = 0$. We mention that the problem solution does not always ensure an injective mapping of the curvilinear domain on the quadrangle.

The definition of the unknown functions $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ in the domain $O\xi\eta$ is presented in the fig. 2. Here $\Delta x = 0$ and $\Delta y = 0$ are the Laplace equations and n_x^k, n_y^k stand for the cosines of the normal to the boundaries of the initial domain $y = g_k(x)$; $n_x^k/n_y^k = -g_k'(x(\xi, \eta))$, $g_k'(x) = dg_k(x)/dx$.

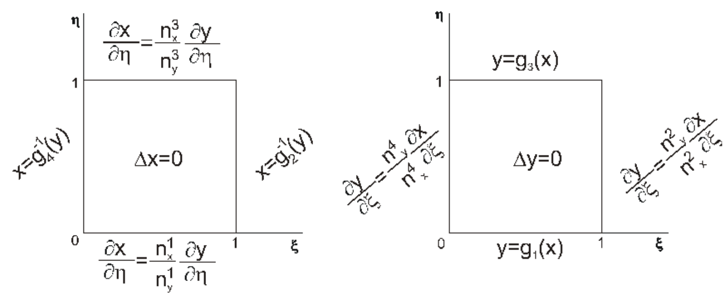


Fig. 2. The definition of the functions $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$.

Up to the first order the boundary conditions are those for the initial physical domain. The boundary conditions of the second order contain derivatives and are obtained by transforming the conditions asserting that on the corresponding boundary the normal derivative to the boundary is equal to zero: $\frac{\partial \xi}{\partial n} = \frac{\partial \xi}{\partial x} n_x^k + \frac{\partial \xi}{\partial y} n_y^k = 0$ on the boundaries with numbers $k = 1$ and 3; $\frac{\partial \eta}{\partial n} = \frac{\partial \eta}{\partial x} n_x^k + \frac{\partial \eta}{\partial y} n_y^k = 0$ on the boundaries with numbers $k = 2$ and 4.

These two problems are coupled, since the unknown functions are part of the nonlinear boundary conditions of both problems. Thus, even in the case when the Laplace equations are used, the problem of determining the unknown functions is a complex nonlinear problem of mathematical physics. But, due to the quadrangle form of the solving domains of these problems, the problems are solved well enough by the method of the finite differences using either the direct sweep method or rapidly converging method of alternating directions. There are some difficulties occurring during the process of the linearization of the boundary conditions, but they are overcome by using the Seidel iterative procedure.

The difference scheme for solving the problem is the following:

$$-\Delta_h x_{ij} = -x_{ij,\bar{\xi}\xi} - x_{ij,\bar{\eta}\eta} = 0, i = \overline{1, N-1}, j = \overline{1, M-1},$$
$$x_{0j} = g_4^{-1}(y_{0j}), x_{Nj} = g_2^{-1}(y_{Nj}), j = \overline{0, M},$$
$$-\frac{2}{h_2} x_{i0,\eta} - x_{i0,\bar{\xi}\xi} = -g_1'(x_{i0}) \left[-\frac{2}{h_2} y_{i0,\eta} - y_{i0,\bar{\xi}\xi} \right], i = \overline{0, N},$$
$$\frac{2}{h_2} x_{iM,\bar{\eta}} - x_{iM,\bar{\xi}\xi} = -g_3'(x_{iM}) \left[\frac{2}{h_2} y_{iM,\bar{\eta}} - y_{iM,\bar{\xi}\xi} \right], i = \overline{0, N};$$

$$-\Delta_h y_{ij} = -y_{ij,\bar{\xi}\xi} - y_{ij,\bar{\eta}\eta} = 0, i = \overline{1, N-1}, j = \overline{1, M-1},$$
$$y_{i0} = g_1(x_{i0}), y_{iM} = g_3(x_{iM}), i = \overline{0, N},$$
$$-\frac{2}{h_1} y_{0j,\xi} - y_{0j,\bar{\eta}\eta} = -\frac{1}{g_4'(x_{0j})} \left[-\frac{2}{h_1} x_{0j,\xi} - x_{0j,\bar{\eta}\eta} \right], j = \overline{0, M},$$
$$\frac{2}{h_1} y_{Nj,\bar{\xi}} - y_{Nj,\bar{\eta}\eta} = -\frac{1}{g_2'(x_{Nj})} \left[\frac{2}{h_1} x_{Nj,\xi} - x_{Nj,\bar{\eta}\eta} \right], j = \overline{0, M}.$$

The generally accepted notation for the left and right difference derivatives is used; $h_1 = 1/N$, $h_2 = 1/M$, $x_{ij} = x(\xi_i, \eta_j)$, $y_{ij} = y(\xi_i, \eta_j)$, $\xi_i = ih_1$, $i = \overline{0, N}$, $\eta_j = jh_2$, $j = \overline{0, M}$. The complex form of the second degree boundary conditions is due to the necessity of building difference approximations of the second oder at all points of the difference net, including the boundary nodes, where the values of the derivatives of the unknown functions are assigned .

The described numerical method for solving of the defined problem was implemented as a Fortran program. The test estimations carried out for solving the known orthogonal mapping [4] $x = \sqrt{\frac{\xi + \sqrt{\xi^2+\eta^2}}{2}}$, $y = \sqrt{\frac{-\xi + \sqrt{\xi^2+\eta^2}}{2}}$ revealed a quite good agreement of the obtained numerical results and the analytical solution.

References

[1] Thompson, J. F., Warsi, Z. U. A., Mastin, C. W., *Numerical grid generation. Foundations and applications*, Elsevier, New York, 1985.

[2] Patsiuk, V. I., *The numerical method for solving the problems of the theory of elasticity in the arbitrary orthogonal co-ordinate system*, Scientific Annals of Faculty of Mathematics and Informatics, State University of Moldova, Chisinau, (1999), 131-137.

[3] Patsiuk, V. I., *The conservative difference scheme for solving the problems of the theory of elasticity in stresses in the arbitrary orthogonal co-ordinate system*, Scientific Annals of Faculty of Mathematics and Informatics, State University of Moldova, Chisinau, **2**, 1(2000), 89-98.

[4] Korn, G. A., Korn, T. M., *Mathematical handbook for scientist and engineers. Definitions, theorems and formulas for reference and review*, Moskow, Nauka, 1973.

ROMAI J., **2**, 2(2006),
153–164

COMPUTATIONAL METHODS FOR FIRST KIND INTEGRAL EQUATIONS

Elena Pelican
Ovidius Univ. of Constanta
epelican@univ-ovidius.ro

Abstract The first kind integral equations model many classes of real-world problems (e.g. backwards heat equation, inverse scattering problems, the hanging cable, geological prospecting, computerized tomography, electric potential problems, etc). That is why it is very important to know how to solve them. In this paper we realize a study on computational methods for solving this type of equations: collocation and projection methods, spline techniques and different types of regularization methods.

Keywords: first kind integral equation, linear least-squares problem, minimal norm solution, spline functions, regularization metods, projection, collocation.
2000 MSC: 45Q05, 45B05, 65F20, 65F22.

1. INTRODUCTION

Let $K : L^2([a,b]) \to L^2([a,b])$ be the (compact) integral operator $Kx(t) = \int_a^b k(t,s)x(s)ds$, and the equation

$$Kx(t) = y(t), \quad \forall\ t \in [0,1]. \tag{1}$$

with square-integrable kernel $k : [a,b] \times [a,b] \to I\!R$, and $y \in L^2([a,b])$. Our task is to find $x(s)$ when the data (function $y(s)$ and the kernel) are known exactly, or only approximately. As in most cases $y \notin R(K)$ (where by $R(K)$ we denoted the range of K), the equation (1) has no longer solution. Thus, if in addition, we suppose that $y \in D(K^+)$, where by $D(K^+) = R(K) \oplus R(K)^\perp$ we denoted the domain for the Moore-Penrose pseudoinverse of the linear compact operator K from (1), we can reformulate (1) as the least-squares problem: find $\bar{x} \in L^2([0,1])$ such that

$$\| K\bar{x} - y \|_{L^2([0,1])} = \min! \tag{2}$$

where $\| f \|_{L^2([0,1])} = (\int_0^1 (f(t))^2 dt)^{\frac{1}{2}}$. It is well-known that, if $y \in D(K^+)$, then the problem (2) has a minimal norm solution, x_{LS}, given by $x_{LS} = K^+y$. This solution also satisfies (in classical sense) the associated normal equation $K^*Kx = K^*y$, where K^* is the adjoint of K.

2. COMPUTATIONAL METHODS

In what follows, we describe the collocation method, the projection one, the spline technique, and Tikhonov regularization method.

2.1. COLLOCATION METHOD

In this case, it is required that the kernel k is continuous. For $n \geq 2$ arbitrary fixed and $T_n = \{t_1, \ldots, t_n\}$ the set of (collocation) points in $[0,1]$ $(0 \leq t_1 < t_2 < \ldots < t_n \leq 1)$, we consider the collocation discretization of (1): find $x \in L^2([0,1])$ such that

$$Kx(t_i) = y(t_i),\ \forall\ i = 1, \ldots, n. \tag{3}$$

If $t_i \in T_n$ we define $k_{t_i} : [0,1] \longrightarrow I\!R$ and $\tilde{y_i}$ by $k_{t_i}(s) = k(t_i, s),\ \forall\ s \in [0,1],\ \tilde{y_i} = y(t_i),\ i = 1, \ldots, n$. Then, the equation (3) can be written (3) can be written as

$$C_n x = \tilde{y}, \tag{4}$$

where $\tilde{y} \in I\!R^n$ and $C_n : L^2 \longrightarrow I\!R^n$ are defined by $C_n z = ((\langle k_{t_1}, z \rightarrow, \ldots, \langle k_{t_n}, z \rightarrow, \ldots, \langle k_{t_n}, z$ If

$$y \in R(K), \tag{5}$$

let x^{LS} be the minimal norm least-squares solution of (1) and let x_n^{LS} be the similar one for (3) (or (4)), given by

$$x^{LS} = K^+ y, \quad x_n^{LS} = C_n^+ \tilde{y}. \tag{6}$$

Assumption CW. There exists a sequence of positive integers $0 < n_1 < n_2 < \ldots < n_p < n_{p+1} < \ldots$ such that $\dim(Y_{n_p}) < \dim(Y_{n_{p+1}}),\ \forall\ p \geq 1$, with $Y_n = \text{span}\{k_t,\ t \in T_n\}$.

Remark 2.1 *The above assumption* **CW** *tells us that the number of linearly independent functions k_t in the subspaces Y_n tends to infinity together with n, but not all the functions in each Y_n are linearly independent, as in the original assumption* [10].

The following result is proved in [11] (Theorem 2.4).

Theorem 2.1 *Under the assumption* **CW**, *if* (5) *holds, and* $\lim_{n \longrightarrow \infty} \Delta_n = 0$, *where by Δ_n we denoted* $\sup_{t \in [0,1]} \left(\inf_{t_i \in T_n} |t - t_i| \right)$, *then*

$$\lim_{n \longrightarrow \infty} \| x_n^{LS} - x^{LS} \| = 0. \tag{7}$$

In [11] it is proven that x_n^{LS} can be computed as

$$x_n^{LS}(t) = \sum_{j=1}^{n} \alpha_j k(s_j, t), \ t \in [0,1], \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the minimal norm solution of the system $A_n \alpha = b_n$, and the entries for matrix A_n and vector b_n are given by

$$(A_n)_{ij} = \int_0^1 k(s_i, t) k(s_j, t) dt, \quad (b_n)_i = y(s_i), \quad i, j = 1, \ldots, n.$$

For the case $y \in R(K) \oplus R(K)^{\perp}$, instead of (1), it is considered normal equation

$$\tilde{Q}x = w, \tag{9}$$

where $\tilde{Q} = K^*K, \quad w = K^*y$. Because of the equality [2] $\tilde{Q}^+ w = K^+ y$, it follows that the equations (1) and (9) have the same minimal norm solution x^{LS} given by (6). Then, we replace (2) by the problem: find $x \in L^2([0,1])$ such that $\sum\limits_{i=1}^{n} \left(\tilde{Q}x(t_i) - w(t_i) \right)^2 = \min!$ In this case, under a assumption similar to as **CW**, Theorem 2.1 still holds [11], where x_n^{LS} is the minimal norm solution for (9).

Also, x_n^{LS} can be computed as $x_n^{LS}(t) = \sum\limits_{j=1}^{n} \alpha_j \tilde{Q}(s_j, t), \ t \in [0,1]$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the minimal norm solution of the (consistent) system $Q_n \alpha = \tilde{w}$ and, in this case, the entries for matrix Q_n and vector $\tilde{w}$ are given by

$$(Q_n)_{ij} = \int_0^1 \tilde{Q}(s_i, t) \tilde{Q}(s_j, t) dt, \quad \tilde{w} = (w(t_1), \ldots, w(t_n)), \quad i, j = 1, \ldots, n.$$

With well-posed problems, better results are obtained as we refine the discretization. However, for the first kind integral equations, refining the discretization causes the discrete problem to more mirror the ill-posed nature of continuous problem. All the above mentioned matrices are rank-defficient, very ill-conditioned, and symmetric. Thus, using a classical direct or iterative method to solve these systems is not a good idea. A class of iterative solvers for relatively dense symmetric linear systems are the Kovarik-like approximative orthogonalization algorithms (see [9]).

Algorithm KOBS Let $A_0 = A$ a symmetric matrix.
for $k = 0, 1, \ldots$ do: $K_k = (I - A_k)(I + A_k)^{-1}, \ A_{k+1} = (I + K_k) A_k$.

Theorem 2.2 *If none of the eigenvalues of A is in the set* $E = \left\{ -\frac{1}{\alpha_j}, j \in N, \ \alpha_0 = 1, \alpha_{j+1} = 2\alpha_j + 1 \right\}$ *then the sequence $(A_k)_{k \geq 0}$ generated as above is well-defined, convergent, and* $\lim\limits_{k \to \infty} A_k = A^+ A$.

In order to avoid the computation of the inverse at each step of the previous algorithm, we shall use a modified version of that one. The inverse $(I+A_k)^{-1}$ will be approximated by ($q \geq 1$ arbitrary fixed) $S(A_k;q) = \sum_{i=0}^{q} a_i(-A_k)^i$, with $a_0 = 0, a_{j+1} = \frac{2j+1}{2j+2} \cdot a_j,\ j > 0$. [9]
Algorithm MKOBS Let $A_0 = A$ be a symmetric matrix with $\sigma(A) \subset [0,\ 1]$. We construct the sequence $(A_k)_{k\geq 0},\ (A_k)_{k\geq 0}$ via

$$K_k = (I - A_k)S(A_k;n_k); \quad A_{k+1} = (I + K_k)A_k. \tag{10}$$

In order to solve the linear least-squares problem of the form $\|Ax - b\| = min!$ the following right hand side (rhs, for short) version of algorithm **MKOBS** was proposed in [9].
Algorithm MKOBS-rhs Let $A_0 = A,\ b^0 = b$; for $k = 0, 1, \ldots$ do

$$K_k = (I - A_k)S(A_k;n_k), \quad A_{k+1} = (I + K_k)A_k, \quad b^{k+1} = (I + K_k)b^k \tag{11}$$

In [9] the following results are proved .

Theorem 2.3 *(i) If the problem* (1) *is consistent, then the sequence* $(b^k)_{k\geq 0}$ *is convergent and*

$$\lim_{k\to\infty} b^k = A^+b = x_{LS} \tag{12}$$

(ii) If the problem (1) *is not consistent, then the sequence* $(A_kb^k)_{k\geq 0}$ *is convergent and*

$$\lim_{k\to\infty} A_kb^k = A^+b = x_{LS} \tag{13}$$

In this case, $\lim_{k\to\infty} \left\|b^k\right\| = \infty$

Remark 2.2 *The last relationship can generate problems. That is why, in practice, it is used a modified version of MKOBS-rhs algorithm.*

Algorithm MKOBS-rhs-1

$$K_k = (I - K_k)(I - \frac{1}{2}A_k), \quad A_{k+1} = (I + K_k)A_k, \quad \alpha^{(k+1)} = (I + K_k)^2\alpha^{(k)} \tag{14}$$

Remark 2.3 *The above algorithm* **MKOBS-rhs-1** *has the same convergence behaviour as described in Theorem* 2.3.

2.2. SPLINE TECHIQUES USING THE PROJECTION METHOD

We shall start by briefly presenting the projection method used to solve the equation (1). Let $n \geq 1$ be arbitrary fixed and $\{v_1, v_2, \ldots, v_n\} \subseteq \overline{R(K)}$ a set of vectors with $\|v_i\| = 1, \ \forall \ i \in N$. We consider the following discretization of the equation (1): find $x \in X_n$ such that

$$\langle Kx, v_i \rangle = \langle y, v_i \rangle, \ \forall \ i = 1, \dots n, \tag{15}$$

where $X_n = span\{K^*v_1, \dots, K^*v_n\}$. If, for any $n \geq 1$, the set $\{v_1, v_2, \dots, v_n\} \subseteq \overline{R(K)}$ is linearly independent, then the discrete problem (15) has a unique solution $x_n \in X_n$ given by [3]

$$x_n = (K^*v_1, K^*v_2, \dots, K^*v_n) Q_n^{-1} (\langle y, v_1 \rangle \langle y, v_2 \rangle, \dots, \langle y, v_n \rangle)^t \tag{16}$$

or, equivalently,

$$x_n = \sum \tag{16}$$

or, equivalently,

$$x_n = \sum_{j=1}^{n} \alpha_j K^* v_j, \tag{17}$$

where system

$$Q_n \alpha = b$$

whith $Q_n = (\langle K^*v_i, K^*v_j \rangle)_{i,j=\overline{1,n}}$, $b = (b_1, \dots, b_n)^t \in I\!R^n$, $b_i = \langle y, v_i \rangle$. Since $K^+y = K^+P_{\overline{RK}}y$, the solution $K^+y = K^+P_{\overline{RK}}y$, the solution of (15) $y \in R(K)$. The following result is proved in [3] (Theorem 2.6).

Theorem 2.4 *Under the above condition of linearly independency, and if $span\{v_1, \dots, v_n, \dots\}$ is dense in $\overline{R(K)}$, then $\lim_{n \to \infty} x_n = x_{LS}$, where x_{LS} is the minimal norm solution of the least-square problem associated with (1).*

Remark 2.4 *In [11] it is proved that the previous theorem still holds even if we do not have the linear independent functions, but under a milder condition, similarly to Assumption* **CW**.

The main idea presented in [4] is to use, in the projection, method using as v_i the spline functions. For this, let $a = x_1 < x_2 < \dots < x_n = b$ be a partition of $[a, b]$. In [4] it is required that y is $b - a$- periodic function. This is not a restrictive condition since we can define the other (eventually needed) values as $y(s_j) = y(s_{j+n}), \ j \leq 0$, and $y(s_j) = y(s_{j-n}), \ j > n$, and the knots x_j with $j < 0$ or $j > n$ are chosen according to the periodicity. We shall denote by $s_{i,2m-1}(x)$ the local polynomial spline of $2m - 1$ degree constructed on knots $x_i, \dots, x_{i+2m}, \ \ i = -2m + 1, \dots, n - 1$. The formulas for the local spline and the algorithms of their stable calculation is given in [1]. In our example we shall use the cubic spline polynomials (so, $m = 2$). Thus, the approximated minimal norm solution will be a linear combination of such splines.

In [4] it is proved that (Theorem 1.2.1) any solution for the initial equation obtained by the computation method is also solution obtained by the projection method. As in most cases, the first method is more tractable to deal with, we shall use this one in numerical experiments.

Remark 2.5 *Another way to approximate the minimal norm solution is using the trigonometric spline.*

2.3. REGULARIZATION METHODS

Even if we formulate (1) in the least-square sense, if K is of infinite rank, we have difficulties with in solving it because the Moore-Penrose inverse $K^+ : D(K^+) = R(K) \oplus R(K)^\perp \to I\!R$ is unbounded, and, as we have a noise in the data, namely

$$\| y - y_\delta \| \leq \delta, \tag{18}$$

one cannot expect the solution of the perturbed least-squares equation to be a good approximation to the exact least-squares $x_{LS} = K^+ y$. This is due to the fact that by its very nature, the initial problem is ill-posed. In order to overcome this shortcoming, it is considered the regularized equation of the normal equation

$$K^* K x_\delta = K^* y_\delta, \tag{19}$$

where adjoint of the operator K. Such (regularized) equations are computationally more tractable, but, in this case, another difficulty arises: to find a good regularization parameter. This task can be an expensive procedure. For example, for the standard Landweber iteration, for an n-point discretization of (1) $2in^2$ operations are required , where i is the number of iterations, which can be quite large; also, for the Tikhonov method the cost is $\frac{n^3}{2} + \frac{jn^3}{6}$.

In what follows, we shall briefly present multilevel schemes which reduce the above mentioned computational cost (for details see [8]).
Auxiliary Results. For the compact operator K, let $\{u_n,\ v_n,\ \mu_n\}$ be the singular system given by the singular value decomposition theorem (for short, the SVD theorem): $\{v_n\}$ is the orthonormal eigenvector system for K^*K with the eigenvectors $\lambda_1^2 \geq \lambda_2^2 \geq \ldots,\ \mu_n = |\lambda_n|^{-1}$, and $u_n = \mu_n K v_n$. It is known that $\{v_n\}$ and $\{u_n\}$ form orthonormal bases in $\overline{R(K^*)}$ and $\overline{R(K)}$ respectively. Also, the Picard Criteria for solvability and stability of (1) states the following [5].

Theorem 2.5 *Equation* (1) *has a solution if and only if*
(i) $y \in N(K^*)^\perp$, and
(ii) $\sum\limits_{n=1}^{\infty} \mu_n^2 \left|(y, u_n)\right|^2 < \infty$.

Under these assumptions, the solution is

$$x = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n. \tag{20}$$

Remark 2.6 *Problems appear when y is perturbed by δy, because, in this case, either for $y + \delta y$ the condition* (ii) *may not hold, or if it does, the series* $\sum\limits_{n=1}^{\infty} \mu_n(\delta y, u_n)$ *may be notable (as $\mu_n \to \infty$). This is due to the fact that $R(K)$ is not closed (or* $\dim R(K) = \infty$*).*

Theorem 2.6 *If $y \in D(K^+)$, then the minimal norm solution (for exact data) is given by*

$$x_{LS} = K^+ = \sum_{n=1}^{\infty} \mu_n (Py, u_n) v_n = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n, \tag{21}$$

where P is the orthogonal projector onto $\overline{R(K)}$.

Remark 2.7 *As in the previous remark, if $R(K)$ is not closed, the perturbed least-squares has the same instability problem.*

Landweber Iteration. Tikhonov Regularization. The aforementioned problems can be solved using the regularization algorithms (the main results can be found in [6]). The Landweber iteration and the Tikhonov regularization methods are defined as

$$x_{n+1}^{\delta} = x_n^{\delta} + \mu(K^* y_{\delta} - K^* K x_n^{\delta}), \; x_0^{\delta} = 0, \; 0 < \mu < \frac{2}{\| K^* K \|} = \frac{2}{\| K \|^2}, \tag{22}$$

and

$$x_{\alpha(\delta)}^{\delta} = [K^* K + \alpha(\delta)]^{-1} K^* y_{\delta}, \tag{23}$$

respectively, where x_α, x_α^δ are the solutions of the regularized equation with exact, and perturbed data respectively. The following estimations hold.

Theorem 2.7

$$\| M, \quad \| x_\alpha - x_\alpha^\delta \| \leq \delta \sqrt{M r(\alpha)}. \tag{24}$$

Remark 2.8 *For the Landweber-Fridman iteration, $M = 1$, $r(n) = \mu n$, and if $n(\delta)$ is chosen such that $\delta^2 \mu n(\delta) \to 0$, $\delta \to 0$, then $x_{n(\delta)}^{\delta} \to x_{LS}$; for the Tikhonov scheme, $M = 1$, $r(\alpha) = \frac{1}{\alpha}$, and if $\frac{\delta^2}{\alpha(\delta)} \to 0$, $\delta \to 0$, then $x_{\alpha(\delta)}^{\delta} \to x_{LS}$.*

The Morozov discrepancy principle chooses the unique $\alpha(\delta)$ with the property $\| Kx_{\alpha(\delta)} - y_\delta = \delta \|$.

For the first kind integral equation, the Landweber iteration is

$$x_n^\delta(s) = x_{n-1}^\delta(s) + \int_a^b k(v,s) \left[y_\delta(t) - \int_a^b k(v,t) x_{n-1}^\delta(t) \text{ dt} \right] \text{ dv},$$

which is solved after being discretized as

$$\tilde{x}_n^\delta = \tilde{x}_{n-1}^\delta + hK_{hh}^t \left[\tilde{y}_{\delta h} - hK_{hh} \tilde{x}_{n-1,h}^\delta \right],$$

where h is the step size of the discretization, and K_{hh} is the discretized kernel with stepsize h. The theory assures us [6], [8] that both

$$\| x_n^\delta(s) - x_{LS} \| \to 0, \ \delta \to 0$$

and

$$\| \tilde{x}_{n,h}^\delta - x_{LS} \| \to 0, \ \delta \to 0,$$

and also, the quadrature error goes to 0. The idea of the multilevel schemes is to monotorize the residual; if the residual does not change much after a coarse-grid correction, then only additional Landweber iteration on the fine grid should be performed. In [8] it is said that α should be not too small to permit magnification of roundoff errors which can be obtained on a grid coarser than H. If this grid is $4h$, letting $H = 2h$, the number of operation is less than in the standard approach.

The standard form of the Tikhonov scheme is

$$(K^*K + \alpha(\delta)I) x_{\alpha(\delta)}^\delta = K^* y_\delta. \tag{25}$$

The zeroth order stabilizer is $f(x) = \| x \|_{L^2}^2$ which applied to a first kind integral equation produces an integro-differential equation with boundary conditions as follows

$$\int_a^b \int_a^b k(v,s) k(v,t) x_{\alpha(\delta)}^\delta(t) \text{ dv dt} + \alpha(\delta) = \int_a^b k(v,s) y_\delta(v) \text{ dv},$$

with $x_{\alpha(\delta)}^\delta(a) = x_1, \ x_{\alpha(\delta)}^\delta(b) = x_2$. For the parameter choice the quasi-optimal method is used i.e. $\alpha_k = \mu\alpha_{k-1}, \ 0 < \mu < 1$. Then the parameter that minimizes $\| x_{\alpha_n(\delta)}^\delta - x_{\alpha_{n-1}(\delta)}^\delta \|$ is chosed. The idea of the Tikhonov multilevel schemes consists in: using n levels, the coarsest level is solved using the discrepancy principle with Choleschy decomposition, and then the higher levels are solved using the discrepancy stopping criterion with an iterative system solver. Thus, the operations number reduces significantly.

3. NUMERICAL EXPERIMENTS

Problem 1.(P1) Consider the the integral equation with the kernel

$$k(s,t) = \frac{1}{\sqrt{(1+(s-t)^2)^3}}, \quad s,t \in [0,1].$$

The problem is a simplified version of a problem arising in the field of electrical potential generated by a known electric field. It was specifically chosen as a model problem to test the algorithms presented, since this kernel is a symmetric function, thus being appropriate for applying the iterative solvers described in Section 2.1.

Collocation points	KOBS iterations	MKOBS iterations
10	18	18
50	16	16
100	16	16
200	15	15

Table 1 P1: KOBS and MKOBS maximum admissible error 10^{-6}.

For $y(s) = \sin(\arctan(1-s)) - \sin(\arctan(-s))$, a solution is $x(t) = 1$ ($y \in R(K)$). The solutions for (P1) with KOBS and MKOBS are very similar fig. 1 and, at the same time, very close to the known solution $x(t) = 1$.

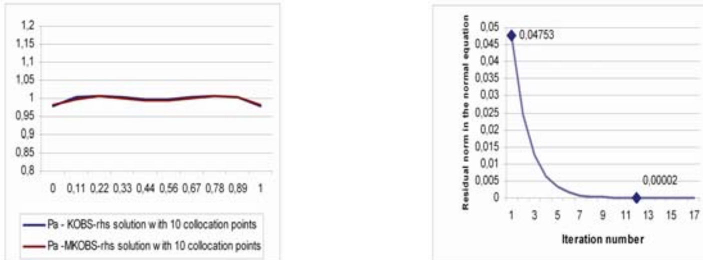
Problem 2. Let the equation (derived from antenna design theory)

$$\int_{-\pi}^{\pi} \cos(st)x(t)\ \mathrm{dt} = 2\pi\left[S((1+s)\pi) + S((1-s)\pi)\right],$$

where $S(s) = \int_0^s \frac{\sin(u)}{u}\ \mathrm{du}$. It has the solution $x(t) = 2\pi\frac{\sin(t)}{t}$. After we transformed this equations from $[-\pi,\pi]$ to $[0,1]$, discretize it, and using the values $h = \sqrt{\frac{12\delta}{\|K_{hh}\|}}$, α is chosen using the Morozov principle on the coarsest grid (stepsize $4h$). In addition, $\alpha \geq 0.00005$ in order to prevent propagation of roundoff errors in the interpolation procedure. The noise is $tr(K_{hh}^t K_{hh})\delta$ The data are presented in Tables 2, 3.

Problem 3. Let the Phillip's equation

$$\int_{-3}^{3} k(t-s)x(t)\ \mathrm{dt} = y(s),\ \ s \in [-6,6]$$



a) KOBS and MKOBS solutions b) Residual norm for MKOBS-rhs

Fig. 1. P1: KOBS and MKOBS with collocation discretization with 10 points.

δ/h	iterations	$\parallel err \parallel_2$
0.0005/0.0078125	21	0.0324145
0.00025/0.0039063	95	0.0241030
0.0001/0.0019531	348	0.0103711

Table 2 Results obtained with a standard Landweber iteration.

δ/h	α	$\parallel err \parallel_2$
0.0005/0.0078125	0.02293	0.0254669
0.00025/0.0039063	0.00717	0.0158831
0.0001/0.0019531	0.00250	0.00739896

Table 3 Results obtained with a multigrid Landweber iteration.

where $k(u) = \begin{cases} 1+\cos(\pi u/3), & |u| \le 3 \\ 0, & |u| \ge 3, \end{cases}$ and

$y(s) = \begin{cases} (6-s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] + \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [0,6] \\ (6+s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] - \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [-6,0] \end{cases}$, with the exact solution $x(t) = \begin{cases} 1+\cos(\pi t/3), & |t| \le 3 \\ 0, & |t| \ge 3. \end{cases}$

The initial value of α is 1, $\mu = 0.5$, a noise $y(s_j)\delta\theta_j$ where θ is a random number chosen from a uniform distribution on $[-1,1]$, and the generalized discrepancy principle. The results are those from Tables 4 and 5.

δ/h	α	$\Vert err \Vert_2$
0.0002/0.015625	0.005722	0.0490729
0.0001/0.0078125	0.002576	0.0322052
0.00002/0.00390625	0.0009570	0.0233487

Table 4 Results obtained by using a standard Tikhonov method with 0th order stabilizer.

δ/tol	α	$\Vert err \Vert_2$
$0.0002/10^{-6}$	0.003725	0.0391557
$0.0001/10^{-6}$	0.001572	0.0260117
$0.00002/10^{-8}$	0.0007053	0.0227495

Table 5 Results obtained by using the multigrid Tikhonov method.

4. ACKNOWLEDGEMENTS

The paper was supported by the PNCDI INFOSOC Grant 131/2004.

References

[1] de Boor, C., *Practical guides on spline*, Radio and Svjaz, Moscow, 1985.

[2] Boullion, T. L., P. L. Odell, *Generalized inverse matrices,* Willey-Interscience, New-York, 1971.

[3] Engl, H. W., *On the convergence of regularization methods for ill-posed linear operator equation,* Improperly Posed Problems and Their Numerical Treatment, Birkhauser, Basel-Boston-Stuttgart, 81-95, Surv. Math. Ind., **3** (1993), 71–143.

[4] Grebennikov, A.I., *Spline approximation method and its applications*, Max Press, Moscow, 2004.

[5] Groetsch, C.W., *Inverse problems in the mathematical sciences*, Vieweg, 1993.

[6] Groetsch, C.W., *The theory of Tikhonov regularization for Fredholm equation of the first kind*, Pitman, Boston, 1984.

[7] Kress. R., *Linear integral equations,* Springer, Berlin, 1989.

[8] Lee, B., *Multilevel approaches to ill-posed equations,* Master Thesis, Simon Frasier University, Burnaby, 1992.

[9] Mohr, M., C. Popa, U. Rude, *A Kovarik type algorithm without matrix inversion for the numerical solution of symmetric least-squares problems*, Lehrstulbericht 05-2, 2005

[10] Nashed, M. Z., G. Wahba, *Convergence rates of approximate least-squares solutions of linear integral and operator equation of the first kind.* Math. of Comp., **28(125)**, (1974).

[11] Pelican, E., *Extensions of projection and collocation methods for first kind integral equations,* Rev. Roumaine Math. Pures Appl., **3(51)**(2006), to appear.

ROMAI J., **2**, 2(2006),
165–174

STRATIFICATION OF $GL(2,\mathbb{R})$-ORBITS OF DIFFERENTIAL FACTOR-SYSTEM $S^2(1,2)/GL(2,\mathbb{R})$ WITH CENTER AT ORIGIN

Elena V. Starus
Institute of Mathematics and Computer Sciences, Academy of Sciences of Moldova
helen@from.md

Abstract According to [3], by means of the centro-affine topological classification of the system $s^2(1,2)$ in the case of the center [4], in the present paper the stratification of $GL(2,\mathbb{R})$-orbits of factor-system $s(1,2)/GL(2,\mathbb{R})$ with center at origin on 19 classes with centro-affine invariant conditions is obtained. For the system $s^2(1,2)$ the first invariant $GL(2,\mathbb{R})$-integral if found in the case when the first set of conditions ensuring the existence of the center at origin holds.

Keywords: Lie algebras, stratification.
2000 MSC: 17B66.

1. GENERAL THEORY

Consider the system of polynomial ordinary differential equations in $\mathbb{R}$

$$\frac{dx^j}{dt} = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\binom{m_i}{k}\overset{i\,j}{a_k}(x^1)^{m_i-k}(x^2)^k \quad (j=1,2), \tag{1}$$

where $\Gamma = \{m_i\}_{i=1}^{l}$ is some finite set of mutually distinct positive integers. Denote the system (1) by $s^2(\Gamma)$ for a specified Γ. The group of centro-affine transformations $GL(2,\mathbb{R})$ is defined by the equalities

$$\bar{x}^1 = \alpha x^1 + \beta x^2,\ \bar{x}^2 = \gamma x^1 + \delta x^2,\quad \left(\Delta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0\right). \tag{2}$$

In [1] it is shown that the four-dimensional Lie algebra $L_4 = \{X_1, X_2, X_3, X_4\}$ corresponds to the linear representation of group $GL(2,\mathbb{R})$ in the space of coefficients and variables of the system (1), given by the operators

$$X_1 = x^1\frac{\partial}{\partial x^1} - D_1,\ X_2 = x^2\frac{\partial}{\partial x^1} - D_2,\ X_3 = x^1\frac{\partial}{\partial x^2} - D_3,\ X_4 = x^2\frac{\partial}{\partial x^2} - D_4,$$

where

$$D_1 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[(m_i-k-1)\overset{i\,1}{a_k}\frac{\partial}{\partial \overset{i\,1}{a_k}} + (m_i-k)\overset{i\,2}{a_k}\frac{\partial}{\partial \overset{i\,2}{a_k}}\right],$$

$$D_2 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[k\left(\overset{i}{a}{}^1_{k-1}\frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k-1}\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right) - \overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i}{a}{}^1_k}\right],$$

$$D_3 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[(m_i-k)\left(\overset{i}{a}{}^1_{k+1}\frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k+1}\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right) - \overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right],$$

$$D_4 = \sum_{i=1}^{s}\sum_{k=0}^{m_i}\left[k\overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i}{a}{}^1_k} + (k-1)\overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right]. \tag{3}$$

Let $a = \left(\overset{1}{a}{}^1_0, \overset{1}{a}{}^1_1, \ldots, \overset{s}{a}{}^2_{m_l}\right) \in E(a)$, where $E(a)$ is the Euclidean space of the coefficients of right-hand sides of (1). Denote by $a(q)$ a point from $E(a)$ corresponding to the system, obtained from the system (1) with coefficients a after transformation $q \in GL(2,\mathbb{R})$.

Definition 1. *The set* $O(a) = \{a(q); q \in GL(2,\mathbb{R})\}$ *is called* $GL(2,\mathbb{R})$*-orbit of the point* a *for system* (1).

Definition 2. *We say that the set* $M \subseteq E(a)$ *is a* $GL(2,\mathbb{R})$-invariant set *if for any point* $a \in M$ *its orbits* $O(a) \subseteq M$.

Since [1]

$$dim_{\mathbb{R}}O(a) = rankM_1, \tag{4}$$

where M_1 is the matrix constructed with the coordinate vectors of the Lie operators (3), it follows that $rankM_1$ can be equal to 4, 3, 2, 1, 0, therefore $dim_{\mathbb{R}}O(a) = 4,\ 3,\ 2,\ 1,\ 0.$

Denote

$$M = \begin{pmatrix} -x^1 & 0 \\ -x^2 & 0 \\ 0 & -x^1 \\ 0 & -x^2 \end{pmatrix}.$$

In some cases, when the matrix (M, M_1) corresponds to some reflection in the space of coefficients and dependent on t variables $x^1,\ x^2$ of the system (1), we denote it by $(\xi(x), \eta(a))$. We recall that ξ is the coefficient of x and η of D.

Consider the manifolds Ψ given in the implicit form in the finite-dimensional space $E(x,a)$. This means that the open set $U \subset E(x,a)$ is given together with the reflection $\psi : U \to \mathbb{R}$ of class $C_\infty(U)$, and $\psi(x_0,a_0) = 0$ for some point $(x_0,a_0) \in U$ such that the set $\psi(U_0)$ is open in $\mathbb{R}$ for any neighborhood $U_0 \subset U$ of the point (x_0,a_0). In these conditions the manifold Ψ can be defined as the locus of $(x,a) \in U$, for which

$$\psi(x,a) = 0 \tag{5}$$

holds. Equality (5) is called an *equation of the manifold* Ψ.

Definition 3. *The manifold Ψ is called* invariant *if for any point $a \in \Psi$ its orbit $O(a) \subseteq \Psi$.*

Definition 4. *The number*

$$r_* = r_*(\xi, \eta) = \max_{(x,a)\in U} rank(\xi(x), \eta(a))$$

is called the general rank of the reflection (ξ, η) on the open set $U \subset E(x, a)$.

Definition 5. *We say that the point $(x, a) \in E(x, a)$ is a* singular point *(of the group $GL(2,\mathbb{R})$ or its Lie algebra L_4), if*

$$rank(\xi(x), \eta(a)) < r_*,$$

and non-singular point *(of the group $GL(2,\mathbb{R})$ or its Lie algebra L_4) if*

$$rank(\xi(x), \eta(a)) = r_*.$$

Definition 6. *The manifold $\Psi \subset U$ is called a* singular manifold *of the group $GL(2,\mathbb{R})$ (or its Lie algebra $L_4(\xi, \eta)$) if all its points are singular and if the reflection (ξ, η) has the rank on Ψ, i.e. for any point $(x, a) \in \Psi$ we obtain $rank(\xi(x), \eta(a)) = r_*(M|\Psi) < r_*$.*

Definition 7. *The manifold $\Psi \subset U$ is called a non-singular manifold of the group $GL(2,\mathbb{R})$ (or its Lie algebra $L_4(\xi, \eta)$) if all its points are non-singular, i.e. if the equality $r_*(M|\Psi) = r_*$ holds.*

By Definitions 6, 7 all invariant manifolds of the group $GL(2,\mathbb{R})$ can be grouped into singular and non-singular invariant manifolds.

From this point of view the classification of dimensions of $GL(2,\mathbb{R})$-orbits of a system of ordinary differential equations can be viewed as a classification of the invariant manifolds of the group $GL(2,\mathbb{R})$, while the non-singular invariant manifolds correspond to the $GL(2,\mathbb{R})$-orbits of maximal dimension.

From the representation theorem [2] it follows

Theorem 1. *If the non-singular manifold of the Lie algebra $L_4(\xi, \eta)$ is given usually by the equation (5), then there exists an invariant $F : E(x, a) \to \mathbb{R}$ of this algebra such that this manifold can be defined by the equality $F(x, a) = 0$.*

Consider first or partial integral of the two-dimensional autonomous polynomial differential system, written in the form (5), satisfying the inequality

$$\left(\frac{\partial\psi}{\partial x^1}\right)^2 + \left(\frac{\partial\psi}{\partial x^2}\right)^2 \not\equiv 0.$$

Definition 8. *An integral $\psi(x, a)$ is called an invariant $GL(2,\mathbb{R})$-integral of a two-dimensional autonomous polynomial differential system, if its corresponding manifold Ψ is an invariant manifold of the group $GL(2,\mathbb{R})$.*

Definition 9. *An invariant $GL(2,\mathbb{R})$-integral $\psi(x,a)$ is called singular (non-singular) if its corresponding invariant manifold Ψ is singular (non-singular) manifold of the group $GL(2,\mathbb{R})$.*

Following [2], we denote any polynomial real differential system $s^l(\Gamma)$ by $E(N,m,k,l)$, where N is the number of coefficients, m the number of the dependent on t variables, k is the order of system and l is the number of equations in system. So, the system (1) can be written as $E(N,2,1,2)$.

By a *factor-system* $s(\Gamma)/GL(2,R)$ of the differential system (1) we understand the system $E(\tilde{\varrho},2,1,2)$, where $\tilde{\varrho}$ is the number of the elements in the algebraic basis of the centro-affine invariants, that is equal to the number

$$\tilde{\varrho} = N - r_* + 1, \tag{6}$$

where r_* is the general rang of the matrix M_1. In other words, the condition of construction of a factor-system $E(\tilde{\varrho},2,1,2)$, where $\tilde{\varrho}$ is given by (6), corresponds to the condition the system $E(N,2,1,2)$ be on the non-singular invariant manifold.

A factor-system $E(\tilde{\varrho},2,1,2)$ contains only $\tilde{\varrho}$ independent parameters as centro-affine invariants, and defines the projection of the system $E(N,2,1,2)$ on the set of centro-affine invariants and comitants.

Following [2], one can check that if we have a non-singular invariant $GL(2,\mathbb{R})$-integral defined by the system $E(N,2,1,2)$ in the space $E(x,a)$, then an integral of the factor-system $E(\tilde{\varrho},2,1,2)$ corresponds to it, and vice-versa.

From [2, 3] are known

Definition 10[2], [3]. *The vector $(\xi(x),\eta(a))$ is called the* tangent vector *to the manifold Ψ at the point (x,a) and represents the class of equivalence of curves which emerge from the point (x,a).*

Definition 11[2], [3]. *The set of tangent vectors $(\xi(x),\eta(a))$ to the manifold Ψ at the point (x,a) forms the* tangent space $T_{(x,a)}\Psi$ *at the point (x,a) to the manifold Ψ, and has the structure of a linear space. Its dimension coincides with the dimension of the manifold Ψ, such that for any two points (x,a), (y,b) of the manifold Ψ the condition $T_{(x,a)}\Psi \cap T_{(y,b)}\Psi = \emptyset$ holds.*

Definition 12. *The union of the tangent spaces $T\Psi$ to the manifold Ψ at all its points is called the* stratification *of the manifold Ψ.*

Taking into consideration the last definitions, the centro-affine invariant topological classification of the phase space of the system (1) can be considered as relevation of the stratification structure on invariant $GL(2,\mathbb{R})$-manifolds of the system (1), i.e. as revelation of the stratification of $GL(2,\mathbb{R})$-orbits of system (1).

2. STRATIFICATION OF $GL(2,\mathbb{R})$-ORBITS OF DIFFERENTIAL FACTOR-SYSTEM $S^2(1,2)/GL(2,\mathbb{R})$ WITH CENTER AT ORIGIN

As an example for the presented theory, consider the system (1) with $\Gamma = \{1,2\}$ in tensor form [4], i.e. the system $s^2(1,2)$

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta \quad (j,\alpha,\beta = 1,2) \tag{7}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices by which the complete convolution holds. Consider the system (7) with the group $GL(2,\mathbb{R})$ given by formulas (2).

The minimal polynomial basis of centro-affine comitants and invariants is constructed for system (7) in [4]. From this basis we need the elements

$$K_1 = a^\alpha_{\alpha\beta}x^\beta,\ K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q \varepsilon_{pq},\ K_9 = a^\alpha_{p\alpha}a^\beta_{q\gamma}a^\gamma_{\beta\delta}x^\delta\varepsilon^{pq},\ I_1 = a^\alpha_\alpha,$$

$$I_2 = a^\alpha_\beta a^\beta_\alpha, I_3 = a^\alpha_p a^\beta_{q\alpha}a^\gamma_{\beta\gamma}\varepsilon^{pq},\ I_4 = a^\alpha_p a^\beta_{q\beta}a^\gamma_{\alpha\gamma}\varepsilon^{pq},\ I_5 = a^\alpha_p a^\beta_{q\gamma}a^\gamma_{\alpha\beta}\varepsilon^{pq},$$

$$I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{q\alpha}a^\delta_{\beta\delta}\varepsilon^{pq},\ I_7 = a^\alpha_{pr}a^\beta_{q\alpha}a^\gamma_{s\beta}a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8 = a^\alpha_{pr}a^\beta_{q\alpha}a^\gamma_{s\delta}a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_9 = a^\alpha_{pr}a^\beta_{q\beta}a^\gamma_{s\gamma}a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_{10} = a^\alpha_p a^\beta_\gamma a^\gamma_\mu a^\delta_{q\alpha}a^\mu_{\beta\gamma}\varepsilon^{pq}, I_{12} = a^\alpha_p a^\beta_{qr}a^\gamma_{s\beta}a^\delta_{\alpha\delta}a^\mu_{\gamma\mu}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_{13} = a^\alpha_p a^\beta_{qr}a^\gamma_{s\gamma}a^\delta_{\alpha\beta}a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},\ I_{15} = a^\alpha_{pr}a^\beta_{qk}a^\gamma_{s\alpha}a^\delta_{l\delta}a^\mu_{\beta\gamma}a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \tag{8}$$

where $\varepsilon^{pq}(\varepsilon^{11} = \varepsilon^{22} = 0,\quad \varepsilon^{12} = -\varepsilon^{21} = 1)$ and $\varepsilon_{pq}(\varepsilon_{11} = \varepsilon_{22} = 0,\ \varepsilon_{12} = -\varepsilon_{21} = 1)$ are the unit bivectors.

Together with system (7) will consider the system (1) with $\Gamma = \{2\}$ in tensor form, i.e. the system $s^2(2)$

$$\frac{dx^j}{dt} = a^j_{\alpha\beta}x^\alpha x^\beta \quad (j,\alpha,\beta = 1,2) \tag{9}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetric in lower indices by which the complete convolution holds.

In [1], by means of (4), corresponding Lie algebra (3) and centro-affine comitants and invariants (8), the classification of dimensions of $GL(2,\mathbb{R})$-orbits is made for the system (9), where the condition for $GL(2,\mathbb{R})$-orbits with maximal dimension is found. For convenience we mention it here.

Lemma 1. *The dimension of the $GL(2,\mathbb{R})$-orbit of system (9) is equal to four if and only if the following condition*

$$K_5(K_9 + \beta) \not\equiv 0 \tag{10}$$

holds, where

$$\beta = 27I_8 - I_9 - 18I_7, \tag{11}$$

and $I_7,\ I_8,\ I_9,\ K_5,\ K_9$ *are given by* (8)

By (4), since $dim_{\mathbb{R}}O(a) = 4$ for system (9) implies $dim_{\mathbb{R}}O(a) = 4$ for system (7), i.e., necessary and sufficient conditions for maximal dimension of the $GL(2,\mathbb{R})$-orbit for system (9) are sufficient for the maximal dimension of the $GL(2,\mathbb{R})$-orbit for system (7). By Lemma 1 these conditions are (10).

Remark 1.

$$\mathrm{Rez}(K_1, K_5) = I_9, \quad \mathrm{Rez}(K_1, K_9) = I_9 - I_7, \tag{12}$$

where $K_1,\ K_5,\ K_9,\ I_7,\ I_9$ are defined in (8).

Taking into account (10) and (12), and Definitions 3 and 7 from the resulting properties it follows that the condition

$$A_1(I_9(I_9 - I_7) \neq 0) \subset A(K_5K_9 \not\equiv 0)$$

defines the non-singular invariant manifold of the group $GL(2,\mathbb{R})$ too.

In [5], for system (7) on the non-singular invariant manifold with condition

$$I_9(I_9 - I_7) \neq 0 \tag{13}$$

is constructed a factor-system $s^2(1,2)/GL(2,\mathbb{R})$

$$\begin{aligned}
\dot{\overline{x}} = {} & \left[\frac{1}{2}I_1 - \frac{I_1I_7 + 2I_{13}}{2I_9} - \frac{I_4I_{15}}{I_9(I_9 - I_7)}\right]\overline{x} - \frac{I_4}{|I_9 - I_7|^{1/2}}\overline{y} + \left[\frac{I_7 + I_9}{2I_9} + \right.\\
& \left. + \frac{I_{15}^2}{I_9(I_9 - I_7)^2}\right]\overline{x}^2 + 2\frac{I_{15}}{|I_9 - I_7|^{3/2}}\bar{x}\bar{y} + \frac{I_9}{I_9 - I_7}\overline{y}^2,\\
\dot{\overline{y}} = {} & \frac{1}{|I_9 - I_7|^{1/2}}\left[\frac{I_4I_{15}^2}{I_9^2|I_9 - I_7|} - \frac{I_4(I_7^2 + I_9^2)}{2I_9^2} + I_5\right]\overline{x} + \\
& + \left[\frac{1}{2}I_1 + \frac{I_1I_7 + 2I_{13}}{2I_9} + \frac{I_4I_{15}}{I_9(I_9 - I_7)}\right]\overline{y} - \frac{1}{|I_9 - I_7|^{1/2}}\left[\frac{I_{15}(I_7 + I_9)}{2I_9^2} + \right.\\
& \left. + \frac{I_{15}^3}{I_9^2(I_9 - I_7)^2}\right]\overline{x}^2 + 2\left[\frac{I_9 - I_7}{2I_9} - \frac{I_{15}^2}{I_9(I_9 - I_7)^2}\right]\bar{x}\bar{y} - \frac{I_{15}}{|I_9 - I_7|^{3/2}}\overline{y}^2,
\end{aligned} \tag{14}$$

for which $K_1 = \overline{x}, K_9 = \overline{y}$, where $I_1,\ I_4,\ I_5,\ I_7,\ I_9,\ I_{13},\ I_{15}, I_{17}, I_{25}, K_1, K_9$ are given by (8).

Theorem 2[6]. *For the center existence at origin for system* (7) *it is necessary and sufficient that conditions* $I_1 = I_6 = 0,\ I_2 < 0$ *and one of the following three sets of conditions*

$$1)\ I_{13} = 0, \quad 2)\ I_3 = 0, \quad 3)\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$$

hold, where $I_1,\ I_2,\ I_3,\ I_4,\ I_5,\ I_{13}$ *are defined by* (8)

In [4], for system (7) possessing a center at the origin, the centro-affine invariant topological classification of the phase space is made, and 32 classes of topologically equivalent systems are found.

By means of this centro-affine invariant topological classification for the system (14) it is proved.

Theorem 3. *If the condition* (13) *holds for the system* (14)*, with the center at the origin, then the centro-affine invariant stratification of the* $GL(2,\mathbb{R})$*-orbit of maximal dimension consists of 19* $GL(2,\mathbb{R})$*-strata with conditions:*

stratum 1 with condition $\gamma\mu \geq 0,\ I_9\gamma > 0,\ I_9(4-\gamma) \leq 0,\ I_{13} = 0, I_4 \neq 0$ *or* $I_{13} = \gamma = 0,\ \mu < 0,\ I_4 \neq 0$*;*
stratum 2 with condition $I_{13} = 0,\ \gamma\mu < 0,\ I_9\gamma > 0,\ I_4 \neq 0$*;*
stratum 3 with condition $I_{13} = 0,\ \gamma(\gamma-6) > 0,\ I_9\gamma < 0,\ I_3I_4\gamma > 0$*;*
stratum 4 with condition $I_{13} = 0,\ \gamma(\gamma-6) > 0,\ I_9\gamma < 0,\ I_3I_4\gamma < 0$*;*
stratum 5 with condition $I_3 = I_{10} = 0,\ I_8 < I_9$*;*
stratum 6 with condition $I_{13} = 0,\ I_9 > 0, 0 \leq \gamma < 4,\ I_4 \neq 0$*;*
stratum 7 with condition $I_{13} = 0,\ I_9 < 0,\ \mu > 0,\ 0 \leq \gamma \leq 4,\ I_4 \neq 0$*;*
stratum 8 with condition $I_{13} = 0,\ I_9 < 0,\ \mu > 0,\ \gamma > 4,\ I_4 \neq 0$
stratum 9 with condition $I_{13} = \mu = 0,\ I_4 \neq 0,\ 0 < \gamma < 3$*;*
stratum 10 with condition $I_{13} = \mu = 0,\ I_4 \neq 0,\ 3 \leq \gamma \leq 4$*;*
stratum 11 with condition $I_{13} = \mu = 0,\ I_4 \neq 0,\ 4 < \gamma < 6$*;*
stratum 12 with condition $I_{13} = 0,\ \mu < 0,\ 0 < \gamma \leq 4, I_4 \neq 0$*;*
stratum 13 with condition $I_{13} = 0,\ \mu < 0,\ 4 < \gamma < 6,\ I_3I_4 > 0$*;*
stratum 14 with condition $I_3 = 0,\ I_8 > I_9,\ \beta < 0$*;*
stratum 15 with condition $I_{13} = 0,\ \mu < 0,\ 4 < \gamma < 6, I_3I_4 < 0$*;*
stratum 16 with condition $I_{13} = \mu = \gamma = 0,\ I_4 \neq 0$*;*
stratum 17 with condition $I_3 = 0,\ I_{13} \neq 0,\ \beta > 0$*;*
stratum 18 with condition $I_3 = \beta = 0,\ I_{13} \neq 0$*;*
stratum 19 with condition $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,\ I_{13} \neq 0$*, where* $\gamma = \frac{3}{I_4^2}[2I_3I_4 + I_2I_9]$*,* $\mu = 4I_4^2 - 3I_2I_9 - 4I_34I_4$*,* β *is given* (11) *and* $I_2,\ I_3,\ I_4,\ I_5,\ I_8,\ I_9,\ I_{10},\ I_{13}$ *by* (8).

According to [4], for system (14), the corresponding topological figure (fig. n) is given on each stratums in Theorem 3, $(n = \overline{1,19})$.

3. FIRST INVARIANT $GL(2,\mathbb{R})$-INTEGRAL OF THE SYSTEM $S^2(1,2)$ IN ONE CASE OF THE CENTER EXISTENCE IN ORIGIN

The following remark holds

Remark 2. *According to [7], from the first set of conditions* $I_1 = I_6 = 0,\ I_2 < 0,\ I_{13} = 0$ *of Theorem 2 it follows, that* $I_4 \neq 0$.

Lemma 2. *If the first set of conditions of Theorem 2 holds, the factor-system* (14) *takes the form*

$$\dot{\overline{x}} = -\frac{I_4}{|I_9 - I_7|^{1/2}}\overline{y} + \frac{I_7 + I_9}{2I_9}\overline{x}^2 + \frac{I_9}{I_9 - I_7}\overline{y}^2,$$

$$\dot{\overline{y}} = \frac{1}{|I_9 - I_7|^{1/2}}\left[I_5 - \frac{I_4\left(I_7^2 + I_9^2\right)}{2I_9^2}\right]\overline{x} + \frac{I_9 - I_7}{I_9}\overline{xy}, \tag{15}$$

where $I_4,\ I_5,\ I_7,\ I_9$ *are given by* (8).

Proof. After the substitution of conditions $I_1 = I_6 = I_{13} = 0$ of Theorem 2 in the syzygy from [7]

$$2I_9I_6 = 2I_4(I_{13} - I_{12}) - I_3(I_1I_7 + 2I_{13} + I_1I_3I_9)$$

we obtain $I_{12} = 0$. After the substitution of the obtained equality and conditions $I_1 = I_6 = I_{13} = 0$ of Theorem 2 in the syzygy from [7]

$$2I_9I_{12} = I_7(I_1I_7 + 2I_{13}) - I_4I_{15} - I_1I_7I_9,$$

we obtain $I_{15} = 0$. After the substitution of conditions $I_1 = I_6 = I_{13} = 0$ of Theorem 2 and the obtained condition $I_{15} = 0$ in the factor-system (14), we obtain the factor-system (15). $\Box$

Theorem 4. *If the conditions*

$$I_1 = I_6 = 0, \quad I_2 < 0, \quad I_{13} = 0, \quad I_4 \neq 0 \tag{16}$$

hold, system (7) *has the following first invariant* $GL(2,\mathbb{R})$*-integral*

$$\mathcal{F} = \frac{I_9^3}{I_7(3I_7 - I_9)(I_9 - I_7)} ln\left|\frac{2I_5I_9^2 - I_4(I_7^2 + I_9^2)}{2I_9^2|I_9 - I_7|^{1/2}}\right| + \frac{|I_9 - I_7|^{1/2}}{I_9}K_9 -$$
$$\frac{I_9^3}{I_7(3I_7 - I_9)(I_9 + I_7)} ln\left|\frac{(-I_5I_9^2 - I_4I_7^2 + 2I_4I_7I_9)[-2I_5I_9^2 + I_4(I_7^2 + I_9^2)]}{2I_9^3(I_9 - I_7)^2} + \right.$$
$$+\frac{I_7(I_9 + I_7)}{I_9(I_9 - I_7)^2}K_9^2 + \frac{I_7(3I_7 - I_9)(I_9 - I_7)}{2I_9^3}K_1^2 -$$
$$\left. -\frac{(I_7 + I_9)(-I_5I_9^2 - I_4I_7^2 + 2I_4I_7I_9)}{I_9^2(I_9 - I_7)^2}K_9\right| = C_1, \tag{17}$$

where $I_4,\ I_5,\ I_7,\ I_9,\ K_1,\ K_9$ *are given by* (8)

Proof. After integrating by Maple 9.5 the system (15), we obtain integral

$$F = \frac{I_9^3}{I_7(3I_7 - I_9)(I_9 - I_7)} ln\left|\frac{2I_5I_9^2 - I_4(I_7^2 + I_9^2)}{2I_9^2|I_9 - I_7|^{1/2}}\right| + \frac{I_9 - I_7}{I_9}K_9 -$$

$$-\frac{I_9^3}{I_7(3I_7-I_9)(I_9+I_7)}ln\left|\frac{(-I_5I_9^2-I_4I_7^2+2I_4I_7I_9)[-2I_5I_9^2+I_4(I_7^2+I_9^2)]}{2I_9^3(I_9-I_7)^2}+\right.$$
$$+\frac{I_7(I_9+I_7)}{I_9(I_9-I_7)}K_9^2+\frac{I_7(3I_7-I_9)(I_9-I_7)}{2I_9^3}K_1^2-$$
$$\left.-\frac{(I_7+I_9)(-I_5I_9^2-I_4I_7^2+2I_4I_7I_9)}{I_9^2(I_9-I_7)^{3/2}}K_9\right|=C_1, \tag{18}$$

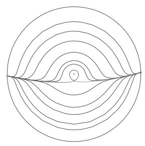


Figure 1

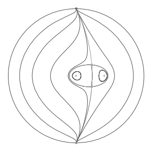


Figure 2

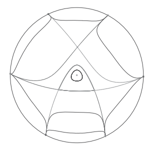


Figure 3

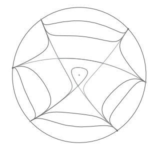


Figure 4

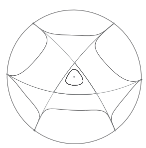


Figure 5

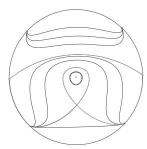


Figure 6

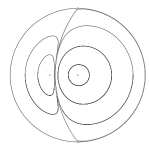


Figure 7

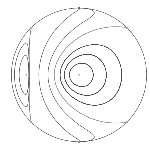


Figure 8

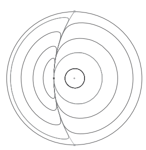


Figure 9

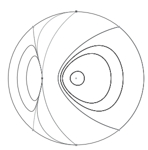


Figure 10

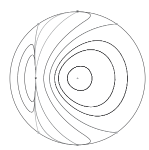


Figure 11

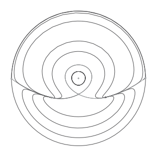


Figure 12

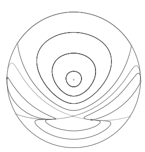


Figure 13

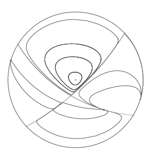


Figure 14

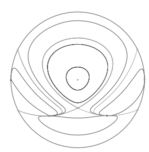


Figure 15

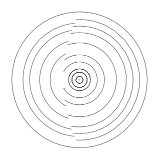


Figure 16

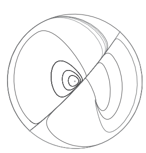


Figure 17

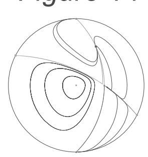


Figure 18

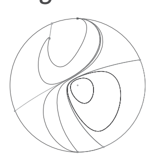


Figure 19

Remark, that integral (18) is not an invariant $GL(2,\mathbb{R})$-integral of system (7), because its terms have different weights. But, as $|I_9 - I_7| = 1$ when conditions (17) holds, the weight of integral (18) can be regulated and, therefore, integral (18) takes the form (17). $\square$

References

[1] Popa M.N., *Application of the algebras to differential systems*, Academy of Sciences of Moldova, Institute of Mathematics and Computer Sciences, Chisinau, 2001. (Russian)

[2] Ovsyannikov L.V., *Group analysis of differential equations*, Nauka, Moscow, 1978. (Russian)

[3] Arnold V.I., *Ordinary differential equatins*, Nauka, Moscow, 1984. (Russian)

[4] Vulpe N.I., *Polynomial basises of comitants of differential systems and their application in the qualitative theory*, Stiinta, Chisinau, 1986. (Russian)

[5] Starus E.V., *The classification of the GL(2,R)-orbit's dimensions for the system* $s(0,2)$ *and a factorsystem* $s(0,1,2)/GL(2,R)$, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 1(44) (2004), 120-123.

[6] Sibirsky K.S., *Algebraic invariants of differential equations and matrices*, Stiinta, Chisinau, 1976. (Russian)

[7] Sibirsky K.S., *Introduction to algebraic theory of invariants of differential equations*, Stiinta, Chisinau, 1982. (Russian; published in English in 1986)

ROMAI J., **2**, 2(2006),
175–176

NUMERICAL DISCRETIZATION OF THE ARBITRARY SHAPED REGION BY MEANS OF LAMÉ EQUATIONS

Galina Ribacova

Moldova State University, Chisinau, Republic of Moldova

Abstract The method for creating the regular two-dimensional grids based on equations of longitudinal plate deformation is presented. This problem is solved numerically by means of finite difference method with the posterior using of the iteration process.

The large number of problems connected with numerical modeling of various physical processes leads to necessity of creation of effective methods of discretization of the computational fields with complicated shape. By now, the numerical grid generation became a common tool for use in the numerical solution of partial differential equations on arbitrary shaped regions. Numerically generated grids obviate the difficulties in description of the arbitrary boundary shape from finite-difference method. With such grids all numerical algorithms (including the finite-difference) are implemented on a square or rectangular computational region regardless of the shape and configuration of the initial physical region. Often, in order to solve this problem, methods based on the application of the elliptical partial differential equations are used to describe the interconnection between the computational (ξ, η) and physical (x, y) regions [1]. In the present article the method of creating regular two dimensional curvilinear grids based on the solution of the problem of longitudinal elastic plate deformation is presented. In this problem a system of partial differential equations of elliptic type, namely – Lamé's equations [2] occur.

In order to formulate the problem let us consider the rectangular elastic plate. Consider the rectangular uniform grid with the grid points (x_i, y_j), $x_i = ih_x, y_j = jh_y, i = \overline{0,n}, j = \overline{0,m},$, ($h_x = l_1/n, h_y = l_2/m$ – the steps of the grid over the corresponding variable, l_1 and l_2 – the dimensions of rectangular plate) marked on this plate. If the plate is subject to longitudinal deformation such that its boundaries take given form (the form of boundaries of the region where the grid must be constructed), then the grid, which was marked on the plate, will be deformed too. As a result of such deformation we obtain the unknown grid. The displacements u and v of the plate points by coordinates x and y respectively satisfy the following system of equations [2]

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial x \partial y} = 0,$$
$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 u}{\partial x \partial y} = 0, \tag{1}$$

where μ is Poisson's ratio, the choice of which has an influence upon the grid lines.

The equations (1) can be solved numerically by means of finite difference method on the rectangular grid that has been introduced above. To this aim the equations (1) must be completed with boundary conditions, i.e. the shape of boundaries of initial region is to be known. Then the displacements of boundary points are given, i.e. the following values are known

$$u\,(0,y)\,,\ v\,(0,y)\,,\ u\,(l_1,y)\,,\ v\,(l_1,y)\,,\ y \ \in\ [0,l_2]\,,$$
$$u\,(x,0)\,,\ v\,(x,0)\,,\ u\,(x,l_2)\,,\ v\,(x,l_2)\,,\ x \ \in\ [0,l_1]\,.$$

Denote by $u_{ij} = u(x_i,y_j)$and $v_{ij} = v(x_i,y_j)$ the values of the unknown functions at the grid points. Then the finite difference approximation of equations (1) is the following

$$u_{ij,x\overline{x}} + \frac{1-\mu}{2} u_{ij,y\overline{y}} + \frac{1+\mu}{4}\left(v_{ij,\overleftarrow{\overline{y}}\,x} + v_{ij,y\overleftarrow{\overline{x}}}\right) = 0,$$
$$v_{ij,y\overleftarrow{\overline{y}}} + \frac{1-\mu}{2} v_{ij,x\overleftarrow{\overline{x}}} + \frac{1+\mu}{4}\left(u_{ij,\overleftarrow{\overline{y}}\,x} + u_{ij,y\overleftarrow{\overline{x}}}\right) = 0, \tag{2}$$

where $i = \overleftarrow{1,n-1}, j = \overleftarrow{1,m-1}$(we here use the generally accepted designation for finite difference derivatives [3]). The created finite difference scheme (2) approximates the initial differential problem (1) with second order relative to h_x and h_y and represents the system of linear algebraic equations of dimensions $2 \times (n-1) \times (m-1)$. The values $v_{0j}, u_{0j}, v_{nj}, u_{nj}, j = \overleftarrow{0,m}$ and $v_{i0}, u_{i0}, v_{im}, u_{im}, i = \overleftarrow{0,n}$ are determined from boundary conditions. Taking into account the large dimensions of the system, its solution must be found by means of the iterative method [3].

The developed algorithm is easy to realize and can be applied to discretize regions with complicated geometrical structures.

References

[1] J. F. Thompson, *Elliptic grid generation*, in Numerical grid generation, J. F. Thompson (ed.), North Holland, New York, 1982, 79-106.

[2] S. P. Timoshenko, J. Gudier, *Elasticity theory*, Nauka, Moscow, 1975 (in Russian).

[3] A. A. Samarskii, *Theory of difference schemes*, Nauka, Moscow, 1977 (in Russian).

ROMAI J., **2**, 2(2006),
177–194

THE STRUCTURE-PHENOMENOLOGICAL STUDY OF TWO-PHASE LIQUID SYSTEMS

E. Yu. Taran, Yu. V. Pridatchenko, V. A. Gryaznova

Faculty of Mechanics and Mathematics, National Taras Shevchenko University of Kiev, Ukraine

taran@univ.kiev.ua

Abstract The structure-phenomenological theory of stress state in arbitrary gradient flows of dilute suspensions of ellipsoidal particles with the Newtonian carrier fluid is constructed. The use of the dynamical method of Landau in the structural part of the theory allows us to obtain general rheological equation of such suspensions before the examination of rotational dynamics of suspended particle in gradient flows of the suspension carrier fluid. As an illustration, the dilute suspension of Brownian ellipsoidal particles is studied. Dynamics of suspended particles in such a suspension is defined not only by hydrodynamical forces but also by rotational Brownian motion. The obtained rheological equation of such a suspension is used to study its rheological behaviour in a simple shear flow. As a result, it is proved that such a suspension behaves as an elasticoviscous fluid presenting the effect of Weissenberg and pseudoplastic dependence of suspension effective viscosity on the rate of shear.

1. INTRODUCTION

Advances in mathematical modelling of the flow of liquid media depend to a large extent on the correct choice of their models. Such a choice essentially depends on the structure of a real liquid medium and on its properties.

This paper describes the procedure of application of structure phenomenological approach proposed for the first time in [1, 2] to study two-phase liquid systems. To this aim, we derive the constitutive equations for stress in dilute suspensions of rigid axially symmetric elongated particles with a Newtonian carrier fluid.

As a model of suspensions, we use a structure continuum with two internal microparameters, namely the orientation vector and vector of the relative angular velocity of suspended particles. According to [1, 2], the constitutive rheological equation for stress which arise in gradient flows of suspensions is postulated phenomenologically. Its phenomenological rheological constants are found theoretically using the results obtained within the framework of structural studies of suspensions by application of the Einstein energy method [3] or Landau dynamic method [4].

The fundamentals of the structure-phenomenological approach are presented in Sec.1 of this paper to the study of the dilute suspension of uniaxial dumbbells. The choice of such a very schematic hydrodynamic model of suspended particles makes possible to describe concisely and adequately the procedure of application of the structure-phenomenological approach with the use of energy method of Einstein [3] to its structural part.

In Section 2 combining the results derived phenomenologically with the results obtained in the structural part of study by the use of dynamical method of Landau [4] gives the possibility to construct for the first time the general rheological equation for dilute suspension of ellipsoidal particles before the examination of the rotational dynamics of suspended particles in gradient flows of suspensions.

The structure-phenomenological method proposed in [1, 2] was approbated by its repeated use to derive rheological equations for dilute suspensions of axisymmetric model particles of any models used in the structural rheology of suspensions, angular position of which can be described uniquely by a single unit vector [5]. In this paper, the approbation of this method is illustrated by identical coincidence of rheological equation for dilute suspension of Brownian ellipsoidal particles obtained in Section 3 with corresponding rheological equation obtained in [6] by another method.

2. FUNDAMENTALS OF STRUCTURE PHENOMENOLOGICAL STUDY OF DILUTE SUSPENSIONS

Dilute suspensions of axisymmetric undeformable particles are considered in the paper to describe the structure-phenomenological method to study two-phase liquid systems. It is assumed that: 1) the suspended particles are rigid, and have the same form and dimensions; 2) the characteristic dimension d of suspended particles is much less than the characteristic length $\bar{l}$ of the suspension macroflow region but it is much longer than the characteristic dimension l of molecules of the Newtonian carrier fluid of the suspension, i.e.

$$l << d << \bar{l}; \tag{1}$$

3) no-slip condition holds on the surface of suspended particles; 4) the motion of the carrier fluid with respect to the suspended particles is slow; 5) the volume concentration of suspended particles is small; suspension is diluted; 6) suspended particles possess zero buoyancy.

In the structural part the interaction of suspended particles with an arbitrary gradient flow of the Newtonian carrier fluid is considered. The concepts of the Einstein energy method employed in the first structural rheological study of dilute suspension of beads [3] and results obtained by Kuhn & Kuhn [7] in their structural rheological studies of dilute suspension of uniaxial dumbbells are used. The hypothesis $l << d$ and property 3 allow us to consider such an interaction as a hydrodynamic one. For the sake of simplicity, in this section the uniaxial dumbbell with undeformable axis of length L is used as a hydrodynamic model of suspended particles. It is assumed that the dumbbell axis exhibit no hydrodynamic resistance, and dumbbell pointlike centers of hydrodynamic interaction, which are placed at the ends of the axis, interact with the carrier fluid as a spherical particles (beads) of radius $\overline{a}$. This means that, if the ends of the dumbbell axis are flown around by the Newtonian carrier fluid with velocities $U_i^{(k)}(k = 1, 2)$, these ends are subject to forces $\xi U_i^{(k)}(k = 1, 2)$ exerted by the carrier fluid, where $\xi = 6\pi\mu\overline{a}$.

It is assumed that the suspension is diluted to such an extent that the direct interaction between suspended particles, and the hydrodynamic interaction between them through the carrier fluid, may be neglected.

The above-mentioned assumptions lead to the resultant vector F_i and moment $M_i^{(hf)}$ of the hydrodynamic forces acting on the dumbbell as

$$F_i = -2\xi v_{0i}, \tag{2}$$

$$M_i^{(hf)} = (1/2)\xi L^2 \varepsilon_{ijk} n_j (d_{ks} n_s - N_k), \tag{3}$$

where (3), $d_{ks} = (1/2)(v_{k,s}+v_{s,k})$, $N_k = \dot{n}_k - \omega_{km} n_m$, $\omega_{km} = (1/2)(v_{k,m}-v_{m,k})$.

The inertia forces and their moments are usually neglected in the rheology of suspensions.Because of this, the equations of motion of the uniaxial dumbbell under the action of hydrodynamic forces are

$$F_i = 0, \tag{4}$$

$$M_i^{(hf)} = 0. \tag{5}$$

Then (2) and (4) imply that the translational velocity v_{0i} of a suspended particle with respect to the carrier fluid is equal to zero, i.e. the suspended particles execute only a rotary motion. The equation of the rotary motion of the dumbbell particles

$$d_{ik} n_k - d_{km} n_k n_m n_i - N_i = 0 \tag{6}$$

is obtained by the vector product of (5) by n_i and taking into account (3).

In the framework of this structural part, the rate of mechanical energy dissipation per unit volume of the suspension

$$\Phi = 2\mu d_{km} d_{km} + n_0(\xi L^2/2)(\langle N_i N_i\rangle - \; - \; 2d_{ij}\langle N_i n_j\rangle + d_{ij} d_{ik}\langle n_j n_k\rangle) \tag{7}$$

is computed too. The first term in (7) is the rate of mechanical energy dissipation per unit volume of the Newtonian carrier fluid of the suspension in the absence of suspended particles, while the second one is the rate of the mechanical energy dissipation on flowing around beads of n_0 model suspended particles contained in the unit volume of the suspension, which is calculated by the formula $n_0\xi\sum_{k=1}^{2}\langle U_i^{(k)} U_i^{(k)}\rangle$.

Transition from microcharacteristics of a separate suspended particle to macrocharacteristic of suspension in (7) takes place during averaging of the function Φ over elementary volume of the suspension containing a sufficiently large number of suspended particles. In (7), the result of the spatial averaging is presented. The angular brackets $\langle\rangle$ in (7) denote the averaging which yet should be made in the phase space of coordinates of the orientation vector n_i of suspended dumbbell particles with the use of the distribution function F of the angular positions of the vector n_i, which satisfies the equation

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial n_i}(F\dot{n}_i) = 0. \tag{8}$$

In order to construct the rheological equation for stress in the suspension of dumbbell particles within the framework of structure-phenomenological approach, such a suspension should be modeled by a structural continuum. The possibility of such modeling is provided by the hypothesis $d << \bar{l}$. Analysis of the results obtained in this structural part of the present study allows us to choose the orientation vector n_i of suspended dumbbell particles and vector N_i defining their relative angular velocity with respect to the suspension carrier fluid as internal microparameters of the structural continuum modeling the considered dilute suspension. The form of the phenomenological rheological equation for stress T_{ij} in the suspension

$$T_{ij} = t_{ij} + n_0\langle\tau_{ij}(d_{km}; n_l; N_p)\rangle \tag{9}$$

is determined by the structure of the expression for the rate of mechnical energy dissipation per unit volume of the considered suspension defined by (7). The averaging in (9) denoted by the brackets $\langle\rangle$ should be carried out as in (7) in the space of coordinates of the orientation vector n_i of suspended dumbbell particles with the use of the distribution function F, which satisfies (8). The term t_{ij} is the stress arising in gradient flows of the Newtonian carrier fluid of suspension in the absence of suspended particles. The other term in (9) is the stress caused by the presence of n_0 suspended particles per unit volume of the suspension. Taking into account that uniaxial dumbbell is symmetric with respect to the center of the axis L, the function τ_{ij} has not to change its sign while replacement of n_i by $-n_i$. This property defines the final form of arguments of the function τ_{ij}: d_{km}; $n_l n_q$; $N_p n_l$.

The explicit form of the phenomenological function τ_{ij} can be determined by comparing the expression for the rate of mechanical energy dissipation per unit volume of the suspension, provided by formula [8]

$$\Phi = t_{ij}d_{ij} + n_0\langle\tau_{ij}\rangle d_{ij} + n_0\langle N_i\varepsilon_{ijk}n_j M_k^{(hf)}\rangle, \tag{10}$$

with the expression (7), obtained within the framework of the structural approach. It follows that τ_{ij} should be a polynomial of its arguments, linear in

d_{km} and N_p, namely

$$\tau_{ij} = (\mu_0 + \mu_1 d_{km} n_k n_m)\delta_{ij} + \mu_2 n_i n_j + \mu_3 d_{km} n_k n_m n_i n_j$$
$$+\mu_4 d_{ij} + \mu_5 d_{ik} n_k n_j + \mu_6 d_{jk} n_k n_i + \mu_7 n_i N_j + \mu_8 n_j N_i,$$

derived by using results obtained in [9].

Taking into account that $t_{ij} = -p\delta_{ij} + 2\mu d_{ij}$, from (9) it follows that the stress tensor T_{ij} in the dilute suspension of dumbbell particles should be defined by the phenomenological rheological equation

$$T_{ij} = -p\delta_{ij} + 2\mu d_{ij} + n_0(\mu_0 + \mu_1 d_{km}\langle n_k n_m\rangle)\delta_{ij}$$
$$+\mu_2\langle n_i n_j\rangle + \mu_3 d_{km}\langle n_k n_m n_i n_j\rangle + \mu_4 d_{ij} \tag{11}$$
$$+\mu_5 d_{ik}\langle n_k n_j\rangle + \mu_6 d_{jk}\langle n_k n_i\rangle + \mu_7\langle n_i N_j\rangle + \mu_8\langle n_j N_i\rangle,$$

where the unknown phenomenological constants $\mu_i (i = \overline{0,8})$, obtained from term-by-term comparison of (7) and (10), are

$$\mu_o = \mu_1 = ... = \mu_4 = \mu_6 = \mu_7 = 0, \quad \mu_5 = -\mu_8 = \xi L^2/2.$$

with due regard for the formula $\langle\tau_{ji} n_j\rangle - \langle\tau_{ij} n_j\rangle = \langle\varepsilon_{ijk} n_j M_k^{(hf)}\rangle$ [8]. Then, (11) becomes

$$T_{ij} = -p\delta_{ij} + 2\mu d_{ij} + n_0(\xi L^2/2)(d_{ik}\langle n_k n_j\rangle - \langle n_j N_i\rangle). \tag{12}$$

Equations (12), (8) and (6) form the closed set of equations defining the stress state in the dilute suspension of dumbbell particles, the rotary motion of which is determined solely by hydrodynamic forces. When the dynamics of suspended particles is defined in addition by other forces, (6) ought to be changed [10].

3. GENERAL RHEOLOGICAL EQUATION FOR DILUTE SUSPENSIONOF ELLIPSOIDAL PARTICLES

The rheological equation (11) is phenomenological. It can be generalized as

$$T_{ij} = (a_0 + a_1 d_{km}\langle n_k n_m\rangle)\delta_{ij} + a_2\langle n_i n_j\rangle + a_3 d_{km}\langle n_k n_m n_i n_j\rangle + a_4 d_{ij} \tag{13}$$
$$+a_5 d_{ik}\langle n_k n_j\rangle + a_6 d_{jk}\langle n_k n_i\rangle + a_7\langle n_i N_j\rangle + a_8\langle n_j N_i\rangle,$$

where a's are new phenomenological rheological coefficients.

The equation (13) can be used instead of (11) to obtain (12). Phenomenological coefficients a's are found similar to the coefficients μ's in (11).

In [5] it was proved that (13) is notable by the fact that it is a phenomenological rheological equation for dilute suspensions of axisymmetric model particles of any models used in the structural rheology of suspensions, angular position of which can be described uniquely with a single unit vector n_i. In this section (13) is used to obtain the general rheological equation for dilute suspension of axisymmetric ellipsoidal particles with axis of symmetry $2a$ and equatorial diameter $2b(a > b)$. Due to the use of the Landau dynamic method [4], instead of the Einstein energy method [3] in the structural part of present theory, the phenomenological coefficients a's in (13) are determined theoretically before the examination of the rotational dynamics of suspended ellipsoidal particles in gradient flows of suspension.

According to [2], first we find the stress tensor σ_{ij} in the carrier fluid of suspension on the surface of the sphere S surrounding an ellipsoidal suspended particle, the center of which coincides with the center of particle and radius R considerably exceeds its dimensions. The use of the Jeffery results [11], who found disturbances of flow of the Newtonian carrier fluid induced by an ellipsoid suspended in it, allows us to determine the stress σ_{ij} on the surface of the sphere S in moving coordinate system $Ox_1x_2x_3$ with axes Ox_1, Ox_2, Ox_3 coinciding with principal axes of ellipsoidal particle

$$
\begin{aligned}
\sigma_{ij} &= -p\delta_{ij} + 2\mu d_{ij} + \\
&\quad +10\mu\left(\frac{5}{R^2}\Phi\delta_{ij} + \frac{4x_ix_j}{R^7} - \frac{x_i}{R^5}\frac{\partial\Phi}{\partial x_j} - \frac{x_j}{R^5}\frac{\partial\Phi}{\partial x_i}\right), \\
\Phi &= A_{km}x_kx_m, \\
A_{11} &= \frac{d_{11}}{6\beta_0''}, \quad A_{12} = \frac{\alpha_0 d_{12} + b^2\beta_0'(\omega_{12}+\omega_3)}{2\beta_0' B}, \\
A_{13} &= \frac{\alpha_0 d_{13} + b^2\beta_0'(\omega_{13}-\omega_2)}{2\beta_0' B}, \\
A_{21} &= \frac{\beta_0 d_{21} + a^2\beta_0'(\omega_{21}-\omega_3)}{2\beta_0' B}, \\
A_{22} &= \frac{d_{22}}{4b^2\alpha_0'} + \frac{d_{11}(\beta_0''-\alpha_0'')}{12b^2\beta_0''\alpha_0'}, \quad A_{23} = \frac{d_{23}}{4b^2\alpha_0'}, \\
A_{31} &= \frac{\beta_0 d_{31} + a^2\beta_0'(\omega_{31}+\omega_2)}{2\beta_0' B}, \quad A_{32} = \frac{d_{32}}{4b^2\alpha_0'}, \\
A_{33} &= \frac{d_{33}}{4b^2\alpha_0'} + \frac{d_{11}(\beta_0''-\alpha_0'')}{12b^2\beta_0''\alpha_0'}, \\
B &= a^2\alpha_0 + b^2\beta_0.
\end{aligned}
$$

In accordance with the structural theory used by Landau [4] while studying dilute suspensions, we take as a tensor defining the stress state in the suspension being considered here the tensor σ_{ij}, which is averaged over the volume of sphere S surrounding the suspended particle. Passing from integration over the volume of sphere to integration over its surface, we find the necessary stress tensor in the dilute suspension of ellipsoidal particles

$$\langle\sigma_{11}\rangle_{vol} = -p + \left(2\mu + \frac{4\mu V}{3ab^2\beta_0''}\right) d_{11}, \langle\sigma_{22}\rangle_{vol} = -p + \left(2\mu + \frac{2\mu V}{ab^4\alpha_0'}\right) d_{22}$$

$$+\frac{2\mu V(\beta_0'' - \alpha_0'')}{3ab^4\beta_0''\alpha_0'}, \langle\sigma_{33}\rangle_{vol} = -p + \left(2\mu + \frac{2\mu V}{ab^4\alpha_0'}\right) d_{33} + \frac{2\mu V(\beta_0'' - \alpha_0'')}{3ab^4\beta_0''\alpha_0'}, \langle\sigma_{12}\rangle_{vol}$$

$$= \left(2\mu + \frac{4\mu\alpha_0 V}{ab^2\beta_0' B}\right) d_{12} + \frac{4\mu V b^2(\omega_{12} + \omega_3)}{ab^2 B}, \langle\sigma_{21}\rangle_{vol} = \left(2\mu + \frac{4\mu\beta_0 V}{ab^2\beta_0' B}\right) d_{21}$$

$$+\frac{4\mu V a^2(\omega_{21} - \omega_3)}{ab^2 B}, \langle\sigma_{13}\rangle_{vol} = \left(2\mu + \frac{4\mu\alpha_0 V}{ab^2\beta_0' B}\right) d_{13} + \frac{4\mu V b^2(\omega_{13} - \omega_2)}{ab^2 B}, \langle\sigma_{31}\rangle_{vol}$$

$$= \left(2\mu + \frac{4\mu\beta_0 V}{ab^2\beta_0' B}\right) d_{31} + \frac{4\mu V a^2(\omega_{31} + \omega_2)}{ab^2 B}, \langle\sigma_{23}\rangle_{vol} = \left(2\mu + \frac{2\mu V}{ab^4\alpha_0'}\right) d_{23}, \langle\sigma_{32}\rangle_{vol}$$

$$= \left(2\mu + \frac{2\mu V}{ab^4\alpha_0'}\right) d_{32}. \tag{14}$$

The coefficients $a_i (i = \overline{0,8})$ given by (13) are found now by comparing the components T_{ij} of the stress tensor in the suspension, which is postulated phenomenologically, with the corresponding components $\langle\sigma_{ij}\rangle_{vol}$ obtained in the structural part of the theory. To this aim, it is necessary to pass, first, in (13) to the moving coordinate system $Ox_1x_2x_3$ connected with ellipsoidal suspended particle. In such a transition $n_1 = 1$, $n_2 = 0$, $n_3 = 0$, $\dot{n}_1 = 0$, $\dot{n}_2 = \omega_3$, $\dot{n}_3 = -\omega_2$, and the components of the stress tensor T_{ij} defined by (13) become

$$\begin{aligned}
T_{11} &= a_0 + a_1 d_{11} + a_2 + (a_3 + a_4 + a_5 + a_6) d_{11}, \\
T_{22} &= a_0 + a_1 d_{11} + a_4 d_{22}, \\
T_{33} &= a_0 + a_1 d_{11} + a_4 d_{33}, \\
T_{12} &= (a_4 + a_6) d_{12} + a_7(\omega_3 + \omega_{12}), \\
T_{21} &= (a_4 + a_5) d_{21} + a_8(\omega_3 - \omega_{21}), \\
T_{13} &= (a_4 + a_6) d_{13} + a_7(-\omega_2 + \omega_{13}), \\
T_{31} &= (a_4 + a_5) d_{31} + a_8(-\omega_2 - \omega_{31}), \\
T_{23} &= a_4 d_{23}, \quad T_{32} = a_4 d_{32}.
\end{aligned} \tag{15}$$

The comparison of (14) and (15) yields

$$
\begin{aligned}
a_0 &= -p, \quad a_1 = \frac{2\mu V(\beta_0'' - \alpha_0'')}{3ab^4\beta_0''\alpha_0'}, \\
a_2 &= 0, \quad a_3 = \frac{2\mu V}{ab^2}\left[\frac{\alpha_0'' + \beta_0''}{b^2\beta_0''\alpha_0'} - \frac{2(\alpha_0 + \beta_0)}{\beta_0'(a^2\alpha_0 + b^2\beta_0)}\right], \\
a_4 &= 2\mu\left(1 + \frac{V}{ab^4\alpha_0'}\right), \\
a_5 &= \frac{4\mu V}{ab^2}\left(\frac{\beta_0}{\beta_0'(a^2\alpha_0 + b^2\beta_0)} - \frac{1}{2b^2\alpha_0'}\right), \\
a_6 &= \frac{4\mu V}{ab^2}\left(\frac{\alpha_0}{\beta_0'(a^2\alpha_0 + b^2\beta_0)} - \frac{1}{2b^2\alpha_0'}\right), \\
a_7 &= \frac{4b^2\mu V}{ab^2(a^2\alpha_0 + b^2\beta_0)}, \quad a_8 = -\frac{4a^2\mu V}{ab^2(a^2\alpha_0 + b^2\beta_0)}.
\end{aligned} \tag{16}
$$

The use of the results in [11] allows us to compute the values of the functions $\alpha_0, \beta_0, \alpha_0', \beta_0', \alpha_0'', \beta_0''$ and to obtain

$$
\begin{aligned}
ab^2\alpha_0 &= 2 - 2A, \quad ab^2\beta_0 = A, \\
ab^4\alpha_0' &= \frac{2p_0 - 3A}{4(p_0^2 - 1)}, \quad ab^4\beta_0' = \frac{3A - 2}{p_0^2 - 1}, \\
ab^2\alpha_0'' &= \frac{(4p_0^2 - 1)A - 2p_0^2}{4(p_0^2 - 1)}, \\
ab^2\beta_0'' &= \frac{2p_0^2 - (2p_0^2 + 1)A}{p_0^2 - 1},
\end{aligned} \tag{17}
$$

where $p_0 = a/b$ and

$$A = \frac{p_0^2}{p_0^2 - 1} - \frac{p_0 \ln(p_0 + \sqrt{p_0^2 - 1})}{(p_0^2 - 1)^{3/2}}$$

for $p_0 > 1$.

Equations (16) and (17) show that rheological constants $a_1, a_3, a_4, \ldots$ in (13) depend only on the dynamic viscosity coefficient μ of the suspension carrier fluid, volume concentration V of suspended particles and on the axial ratio p_0 of ellipsoid of revolution modelling the axisymmetrical suspended particles of the real suspension.

The rheological equation defined by (13) with coefficients given by (16) and (17) is the general rheological equation for dilute suspension of ellipsoidal particles with Newtonian carrier fluid. This equation needs to be complemented by constitutive equation for the internal microparameters n_i and N_i in order to obtain the rheological equation of dilute suspensions of ellipsoidal particles in any special case.

In our framework of the structure-phenomenological approach, the constitutive equation for n_i and N_i of the structural continuum modelling the real dilute suspension is obtained from the equation of rotary dynamics of suspended ellipsoidal particles in gradient flows of the suspension carrier fluid.

4. RHEOLOGICAL EQUATION FOR DILUTE SUSPENSION OF BROWNIAN ELLIPSOIDAL PARTICLES

As an illustration of use of (13) with coefficients a's defined by (16) and (17) as the general rheological equation for dilute suspensions of ellipsoidal particles, we obtain here the rheological equation for dilute suspension of Brownian ellipsoidal particles. Dynamics of such suspended particles is defined not only by hydrodynamic forces, which arise in gradient flows of the suspension carrier fluid, but it is also defined by the rotational Brownian motion. According to [12], the effective radius $r = \sqrt[3]{ab^2}$ of such particles satisfies the condition $10^{-8}m < r < 10^{-6}m$ if the suspension carrier fluid is water.

The constitutive equation for n_i and N_i of the structural continuum which models the dilute suspension of Brownian ellipsoidal particles is obtained as a result of the vector multiplication by the vector n_i of the equation of rotary motion of suspended ellipsoidal particles in gradient flows of the suspension carrier fluid, which take the form $M_i^{(hf)} + M_i^{(Bf)} = 0$, irrespective of the moment of inertia of suspended particles as it usually takes place in the rheology of suspensions. Taking into account that $M_i^{(Bf)} = -kT\varepsilon_{ilm} n_l \frac{\partial \ln F}{\partial n_m}$, and

$$[M_i^{(hf)} \times n_i] = W(\lambda(d_{ik}n_k - d_{km}n_k n_m n_i) - N_i), \tag{18}$$

we obtain [2] the constitutive equation

$$N_i = \lambda(d_{ik}n_k - d_{km}n_k n_m n_i) + D_r \left(n_i n_k \frac{\partial \ln F}{\partial n_k} - \frac{\partial \ln F}{\partial n_i} \right) \tag{19}$$

for the internal microparameters n_i and N_i. In (18) and (19), $\lambda = (p_0^2 - 1)/(p_0^2 + 1)$; $D_r = kT/W$ and

$$W = 4\nu\mu \frac{p_0^4 - 1}{p_0^2 \left[\dfrac{2p_0^2 - 1}{2p_0^2\sqrt{p_0^2 - 1}} \ln \dfrac{p_0 + \sqrt{p_0^2 - 1}}{p_0 + \sqrt{p_0^2 - 1}} - 1 \right]},$$

if $p_0 > 1$ [13]; here, $\nu = 4\pi ab^2/3$.

The substitution of N_i, defined by (19), into (13) and taking into account (16) yields the rheological equation of the dilute suspension of Brownian ellipsoidal suspended particles

$$T_{ij} = -p\delta_{ij} + 2\mu(1 + \frac{V}{ab^4\alpha_0'})d_{ij} + +12\mu D_r \frac{V}{ab^2}\frac{a^2-b^2}{a^2\alpha_0 + b^2\beta_0}(\langle n_i n_j\rangle - \frac{1}{3}\delta_{ij}) +$$

$$+2\mu\frac{V}{ab^2}\left[\frac{\alpha_0''}{b^2\alpha_0'\beta_0''} + \frac{1}{b^2\alpha_0'} - \frac{4}{\beta_0'(a^2+b^2)}\right] d_{km}\langle n_k n_m n_i n_j\rangle +$$

$$+2\mu\frac{V}{ab^2}\left[\frac{2}{\beta_0'(a^2+b^2)} - \frac{1}{b^2\alpha_0'}\right](d_{jk}\langle n_k n_i\rangle + d_{ik}\langle n_k n_j\rangle). \qquad (20)$$

The distribution function F of angular positions of ellipsoidal suspended particles, used in (20) for averaging, is defined by the equation

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial n_i}((\omega_{ik}n_k + \lambda(d_{ik}n_k - d_{km}n_k n_m n_i))F) =$$

$$= D_r(\Delta F - 2n_k\frac{\partial F}{\partial n_k} + n_k n_m\frac{\partial^2 F}{\partial n_k \partial n_m}) \qquad (21)$$

obtained from (8) by taking into account (19).

Equation (20) coincides with the rheological equation for the dilute suspension of the Brownian ellipsoidal particles obtained in [6] by another method. This fact confirms the validity of our structure-phenomenological theory of the stress state in the dilute suspensions of ellipsoidal particles presented here. Such a coincidence confirms in particular the status of the equation (13) with coefficients given by (16) and (17) as a general rheological equation for the dilute suspension of ellipsoidal particles with the Newtonian carrier fluid.

5. RHEOLOGICAL BEHAVIOUR OF DILUTE SUSPENSION OF BROWNIAN ELLIPSOIDAL PARTICLES

The use of the constitutive equations (20) and (21) shows that the dilute suspension of Brownian ellipsoidal particles behave as an elasticoviscous fluid in the simple shearing flow

$$v_x = 0, \qquad v_y = Kx, \qquad v_z = 0 \qquad (K = const) \qquad (22)$$

presenting the effect of Weissenberg and pseudoplastic dependence of the suspension effective viscosity on the rate of shear K.

The computation of the components T_{xy}, T_{xx}, T_{yy}, T_{zz} of the tensor T_{ij} by means of (20) allows us to obtain the expressions of the effective viscosity of the suspensions $\mu_a = T_{xy}/K$ and the differences of normal stresses $\sigma_1 = T_{yy} - T_{zz}, \quad \sigma_2 = T_{xx} - T_{zz}$ in the simple shear flow (22) of the suspension,

$$\mu_a = \mu\left(1+\frac{V}{ab^4\alpha_0'}\right)+6\mu\frac{D_r}{K}\frac{V}{ab^2}\frac{a^2-b^2}{a^2\alpha_0+b^2\beta_0}\langle\sin 2\phi\sin^2\theta\rangle+\mu\frac{V}{ab^2}(\frac{\alpha_0''}{b^2\alpha_0'\beta_0''}+\frac{1}{b^2\alpha_0'}$$

$$-\frac{4}{\beta_0'(a^2+b^2)})\langle\sin^2 2\phi\sin^4\theta\rangle+2\mu\frac{V}{ab^2}\left(\frac{2}{\beta_0'(a^2+b^2)}-\frac{1}{b^2\alpha_0'}-\right)\langle\sin^2\theta\rangle; \quad (23)$$

$$\sigma_1 = 12\mu\frac{D_r}{K}\frac{V}{ab^2}\frac{a^2-b^2}{a^2\alpha_0+b^2\beta_0}\left[\langle\sin^2\phi\sin^2\theta\rangle-\langle\cos^2\theta\rangle\right]$$

$$+\mu\frac{V}{ab^2}[\left(\frac{\alpha_0''}{b^2\alpha_0'\beta_0''}+\frac{1}{b^2\alpha_0'}-\frac{4}{\beta_0'(a^2+b^2)}\right)\times(2\langle\cos\phi\sin^3\phi\sin^4\theta\rangle-\langle\sin 2\phi\sin^2\theta\cos^2\theta\rangle)$$

$$+\left(\frac{2}{\beta_0'(a^2+b^2)}-\frac{1}{b^2\alpha_0'}\right)\langle\sin 2\phi\sin^2\theta\rangle]; \quad (24)$$

$$\sigma_2 = 12\mu\frac{D_r}{K}\frac{V}{ab^2}\frac{a^2-b^2}{a^2\alpha_0+b^2\beta_0}\left[\langle\cos^2\phi\sin^2\theta\rangle-\langle\cos^2\theta\rangle\right]$$

$$+\mu\frac{V}{ab^2}[(\frac{\alpha_0''}{b^2\alpha_0'\beta_0''}+\frac{1}{b^2\alpha_0'}-\frac{4}{\beta_0'(a^2+b^2)})\times(2\langle\sin\phi\cos^3\phi\sin^4\theta\rangle$$

$$\langle\sin 2\phi\sin^2\theta\cos^2\theta\rangle)+(\frac{2}{\beta_0'(a^2+b^2)}-\frac{1}{b^2\alpha_0'})\langle\sin 2\phi\sin^2\theta\rangle]. \quad (25)$$

In Eqs. (23)-(25), ϕ and θ are the angles of the spherical coordinate system, in which $n_x = \cos\phi\sin\theta$, $n_y = \sin\phi\sin\theta$, $n_z = \cos\theta$.

The computations of averaged values in (23)-(25) require the knowledge of the distribution function of axes of suspended ellipsoidal particles over all possible angular positions ϕ and θ. This function is found by solving (21), which in the steady case, that is at $\partial F/\partial t = 0$, in spherical coordinate system (ϕ, θ), takes the form

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2} = \sigma[\frac{1}{2}(1+\lambda\cos 2\phi)\frac{\partial F}{\partial\phi}+\frac{\lambda}{4}\sin 2\phi\sin 2\theta\sin\theta\frac{\partial F}{\partial\theta}-$$

$$\frac{1}{2}\lambda\sin 2\phi(3\sin^3\theta-2\sin\theta+2)F] \quad (26)$$

for the simple shear flow (22) of suspension. In (26), we have $\sigma = K/D_r$.

The solution of (26) is obtained in the form of the double series

$$F(\phi,\theta)=\sum_{j=0}^{\infty}\lambda^j\left[\frac{1}{2}\sum_{n=0}^{j}a_{n0,j}P_{2n}(\cos\theta)+\right.$$
$$\left.+\sum_{n=1}^{j}\sum_{m=1}^{n}\left(a_{nm,j}\cos 2m\phi+b_{nm,j}\sin 2m\phi\right)P_{2n}^{2n}(\cos\theta)\right].$$

For coefficients $a_{n0,j}, a_{nm,j}, b_{nm,j}$ we have obtained recurrence relations, which allow us to find the distribution function F with any degree of accuracy.

The computations of the averaged values in (23)-(25) according to formula

$$\langle a(\phi,\theta)\rangle=\int_0^{2\pi}d\phi\int_0^{\pi}a(\phi,\theta)F(\phi,\theta)d\theta$$

allow us to obtain numerical values of the effective viscosity μ_a and differences of normal stresses σ_1 and σ_2 for the aqueous dilute suspension of the ellipsoidal Brownian particles with effective radius $r=10^{-7}m$ that are shown in figs.1, 2.

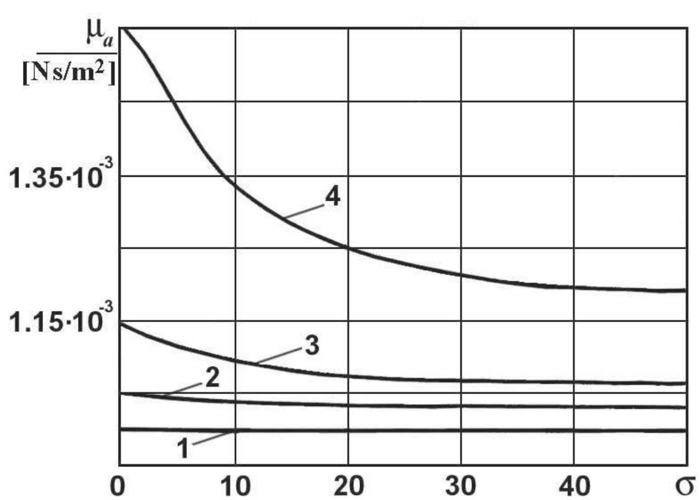


Fig. 1. Dependence of μ_a on $\sigma=K/D_r$ for the dilute suspension ($V=0.01$) in water $\mu=0.001\,\mathrm{N\,s/m^2}$ of ellipsoidal particles with effective radius $r=10^{-7}$ m; curves 2-4 corresponds to values $p_0=4,10,25$; curve 1 corresponds to the viscosity of the suspension carrier fluid in the absence of suspended particles.

The obtained results indicate that the dilute suspension of the Brownian ellipsoidal particles behaves similarly to visco-elastic fluids. So, it reveals the dependence of μ_a on the shear rate K (fig.1), which is characteristic to viscous non-Newtonian pseudoplastic fluids. The expression (23) of μ_a coincides with the expression for the effective viscosity of considered dilute suspension of Brownian ellipsoids obtained in [14] within the frames of structural theory with the use of Einstein energy method. As a result, numerical values of μ_a obtained above are in complete agreement with numerical values of characteristic viscosity $(\mu_a - \mu)/(\mu V)$ of the considered suspension which were calculated in [15] by method of averaging, similar to above mentioned one, and later were confirmed experimentally in [16].

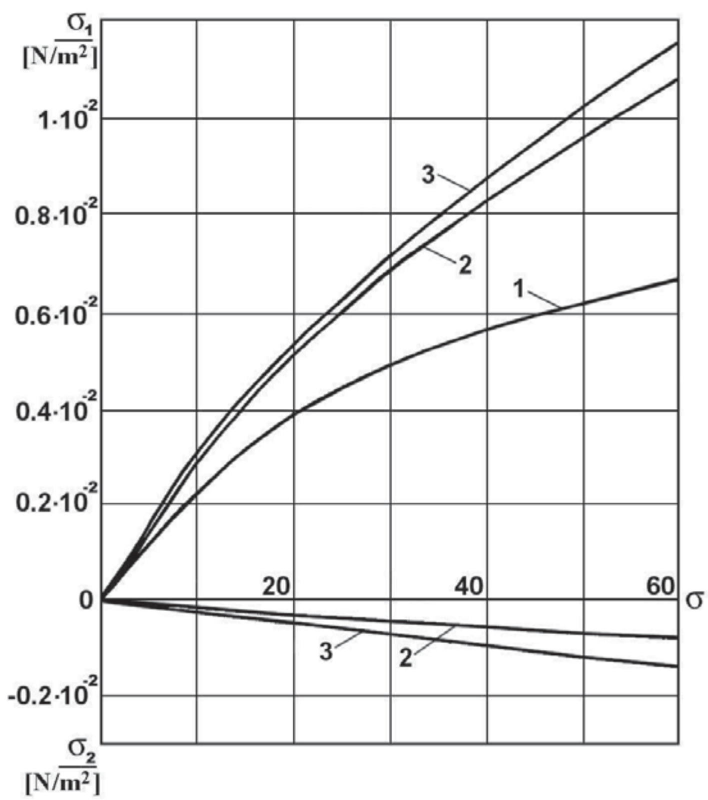


Fig. 2. Dependence of $\sigma_1 = T_{yy} - T_{zz}$ and $\sigma_2 = T_{xx} - T_{zz}$ on $\sigma = K/D_r$ for the dilute suspension ($V = 0.01$) in water $\mu = 0.001\,\mathrm{N\,s/m^2}$ of ellipsoidal particles with effective radius $r = 10^{-7}m$ at temperature $T = 300^o, {}^oK$; curves 1, 2, 3 correspond to values $p_0 = 4, 10, 25$.

The considered suspension reveals the Weissenberg effect in the shearing flow (22), namely the presence of nonzero differences of the normal stresses σ_1 and σ_2, which depend on the shear rate K (fig.2). The presence of the Weissenberg effect in dilute suspensions detected here theoretically still requires an experimental verification.

6. CONCLUSIONS

The advantage of the structure-phenomenological method to study dilute suspensions consists in the capability to combine the continual modeling of the suspension with the possibility to connect the macrorheological characteristics of the suspensions with parameters describing their microstructural and physical properties.

Further investigations show that this method is particularly efficient to obtain rheological equations for dilute suspensions of undeformable particles with non-Newtonian isotropic and anisotropic carrier fluids.

7. NOMENCLATURE

a_0, a_2 are phenomenological constants in (13), [N/m^2];

$a_1, a_i (i = \overline{3, 8})$ are phenomenological constants in (13), [N s/m^2];

$\overline{a}$ is the radius of the dumbbell beads, [m];

$2a$, $2b$ are the axis of revolution and equatorial diameter of ellipsoid of revolution modeling suspended particles, [m];

d is the characteristic dimension of suspended particles, [m];

d_{ij} is the deformation rate tensor, [s^{-1}];

D_r is the coefficient of the rotational Brownian diffusion of suspended particles in the carrier fluid, [s^{-1}];

F is the distribution function of the angular positions of the orientation vector n_i of suspended ellipsoidal particles, dimensionless;

F_i is the resultant vector of the hydrodynamic forces acting on a dumbbell particle [N];

$k = 1.38 \cdot 10^{-23}$J/K is the Boltzmann constant;

K is the rate of shear, [s^{-1}];

l is the characteristic dimension of molecules of the suspension carrier fluid, [m];

$\bar{l}$ is the characteristic dimension of suspension macroflow region, [m];

L is the axis of the suspended dumbbell particles, [m];

$M_i^{(hf)}$, $M_i^{(Bf)}$ are angular momenta of hydrodynamic forces and forces of rotary Brownian motion acting on suspended ellipsoidal particle, [N m];

n_0 is the number of suspended particles per unit volume of the suspension, [m^{-3}];

n_i is the unit vector defining orientation of axisymmetric suspended particles modeled among them by dumbbells and ellipsoids of revolution, dimensionless;

N_i is the vector defining relative angular velocity of suspended particles with respect to the suspension carrier fluid, [s^{-1}];

p is the pressure, [N/m^2];

p_0 is the axial ratio of ellipsoid of revolution modeling the axisymmetrical suspended particles, dimensionless;

P_{2n} are the Legendre polynomials;

P_{2n}^{2m} are the Legendre associated functions.

r is the effective radius of suspended ellipsoidal particles, [m];

R is the radius of the sphere S surrounding an ellipsoidal suspended particle, [m];

S is the sphere surrounding an ellipsoidal suspended particle used to define the stress in the suspension;

t is the time, [s];

T is the absolute temperature, [K];

t_{ij} is the stress tensor in the Newtonian carrier fluid of the suspension in the absence of suspended particles, [N/m^2];

T_{ij} is the stress tensor in the dilute suspension of ellipsoidal particles modeled by a structure continuum, [N/m^2];

$T_{xy}, T_{xx}, T_{yy}, T_{zz}$ are the components of the tensor T_{ij}, [N/m^2];

u_i is the unit vector defining the orientation of suspended particles, directed along the axis L in the case of dumbbell particles and along the axis $2a$ in the case of ellipsoidal particles, [m/s];

$U_i^{(k)} (k = 1, 2)$ are the velocities of the flow around ends of a dumbbell particle by the suspension carrier fluid, [m/s];

v_{0i} is the translational velocity of the dumbbell center with respect to the carrier fluid, [m/s];

v_i is the velocity vector, [m/s];

v_x, v_y, v_z are the components of the velocity vector v_i, [m/s];

$v_{i,k}$ is the velocity gradient tensor, [s^{-1}];

V is the volume concentration of suspended particles, dimensionless;

W is the rotational friction coefficient of ellipsoidal suspended particles in the Newtonian carrier fluid, [N m s];

x_1, x_2, x_3 are coordinates in the moving coordinate system $Ox_1x_2x_3$, axes of which are coinciding with principal axes of ellipsoidal particle, [m];

$\langle\rangle$ denotes the averaging in the phase space of coordinates of orientation vector n_i;

$\langle\rangle_{vol}$ denotes the averaging over the volume placed inside the sphere S;

$\alpha_0, \beta_0, \alpha_0'', \beta_0''$ are the functions determined in (11), [m^{-3}];

α_0', β_0' are the functions determined in (11), [m^{-5}];

δ_{ij} is the Kronecker delta, dimensionless;

Δ is the Laplacian;

ε_{ikm} is the Levi-Civita tensor, dimensionless;

λ is the parameter depending on the geometric characteristics of ellipsoidal particles, dimensionless;

μ is the dynamic viscosity coefficient of the carrier fluid, [$\mathrm{N\,s/m^2}$];

μ_a is the effective viscosity of the suspension, [$\mathrm{N\,s/m^2}$];

μ_0, μ_2 are phenomenological constants in (11), [N m];

$\mu_1, \mu_i (i = \overline{3, 8})$ are phenomenological constants in (11), [N m s];

ν is the volume of ellipsoidal suspended particle, [m^3];

ξ is the translational drag coefficient of the dumbbell beads in the Newtonian carrier fluid, [N s/m];

σ is the dimensionless shear rate;

σ_{ij} is the stress tensor in the suspension carrier fluid on the surface of the sphere S surrounding an ellipsoidal suspended particle, [N/m^2];

σ_1, σ_2 are the differences of normal stresses in the simple shearing flow, [N/m^2];

ϕ, θ are the angles of the spherical coordinate system, [rad];

Φ is the rate of mechanical energy dissipation per unit volume of the suspension, [N/(s m^2)] ;

ω_{ik} is the velocity vortex tensor, [s^{-1}];

ω_2, ω_3 are the components of angular velocity of ellipsoidal particle, [s^{-1}].

Subscripts and superscripts

i, k in $v_{i,k}$ denote the derivation of the velocity vector v_i in the direction of coordinate axis k;

hf in $M_i^{(hf)}$ signify hydrodynamic forces;

Bf in $M_i^{(Bf)}$ signify Brownian forces;

the dot over n_i denotes the local time derivation;

vol in $\langle\rangle_{vol}$ signifies volume;

a in μ_a signifies apparent;

r in D_r signifies rotary.

References

[1] Yu. I. Shmakov, E.Yu. Taran, *Structure-continual approach in rheology of polymer materials*, Inzh.-Fiz.Zhurn., **18**(1970), 1019–1024 (in Russian).

[2] Yu. V. Pridatchenko, Yu.I. Shmakov, *Rheological equations of state of weakly concentrated suspensions of rigid ellipsoidal macromolecules*, Zhurn. Prikl. Mekh. Tekhn. Fiz., **2**(1972), 125–129 (in Russian).

[3] A. Einstein, *Eine neue Bestimmung der Moleküldimensionen*, Ann. Physik, **19**(1906), 289–306.

[4] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Addison Wesley, Reading, MA, 1959.

[5] E.Yu. Taran, *Structure-phenomenological rheology of dilute suspensions*, DSc (Physics and Mathematics) Thesis, Kharkov State University, Kharkov, 1994 (in Russian).

[6] V. N. Pokrovsky, *Some problems of microrheology of dispersed systems and polymers*, Kolloid. Zhurn., **30**(1968), 881–887 (in Russian).

[7] W. Kuhn, H. Kuhn, *Die Abhängigkeit der Viskosität vom Strömungsgefälle bei hochverdünten Suspensionen und Lösungen*, Helv. Chim. Acta, **28**(1945), 97–127.

[8] E. Yu. Taran, Yu. V. Pridatchenko, V.S. Volkov, *Mechanics of suspensions of rigid uniaxial dumbbells in an anisotropic fluids*, Zhurn. Prikl. Mekh. Tekhn. Fiz., **35**, 4(1994), 99–107 (in Russian).

[9] A. J. M. Spenser, R. S. Rivlin, T*he theory of matrix polynomials and its application to the mechanics of isotropic continua*, Arch. Rat. Mech. Anal., **2**(1959), 309–336.

[10] E. Yu. Taran, *Rheological constitutive equation for dilute suspensions of rigid dumbbells*, Prikl. Mekh., **13**, 4(1977), 110–115 (in Russian).

[11] G. B. Jeffery, *The motion of ellipsoidal particles immersed in a viscous fluid*, Proc. Roy. Soc., **A102**(1922), 161–169.

[12] A. Peterlin, *Über die Viskosität von verdünten Lösungen und Suspensionen in Abhängigkeit von der Teilchenforn*, Z.Physik, **111**(1938), 232–263.

[13] V. N. Tsvetkov, V. E. Eskin, S. Ya. Frenkel, *Structure of macromolecules in solutions*, Nauka, Moscow, 1964 (in Russian).

[14] N. Saito, *The effect of the Brownian motion on the viscosity of solutions of macromolecules. I. Ellepsoid of revolution*, J. Phys. Soc. Japan, **6**(1951), 296–302.

[15] H. A. Sheraga, *Non-Newtonian viscosity of solutions of ellipsoidal particles*, J. Chem. Phys., **23**(1955), 1526–1532.

[16] J. T. Yang, *Non-Newtonian viscosity of poly-γ-benzyl-L-glutamate solutions*, J. Amer.Soc., **80**(1958), 1783–1788.

ROMAI J., **2**, 2(2006),
195–203

NUMERICAL SOLUTION OF THE BIFURCATION PROBLEM OF THE DESIGN ELEMENTS SUBJECT TO AERODYNAMIC EFFECTS

Petr A.Vel'misov, S. V. Kireev
Ulyanovsk State Technical University, Russia
ksv15@vens.ru

Abstract The conditions for the statical instability (bifurcation) of a plate in a supersonic gas flow and a pipeline are obtained.

Keywords: bifurcation, bending of a plate, supersonic gas flow.
2000 MSC: 74H15

1. BIFURCATION OF THE EQUILIBRIUM OF A PLATE IN A SUPERSONIC GAS FLOW

The mathematical model governing the forms of a plate bent in a supersonic gas flow, consists of the nonlinear integro-differential equation

$$Dw'''' + Nw'' + \alpha w' + f(w) - \theta w'' \int_0^\ell w'^2\,dx = 0, \tag{1}$$

where $\alpha = \frac{\alpha_0 \rho_0 V^2}{\sqrt{M^2-1}}$, $M = \frac{V}{a}$, $D = EJ$, $\theta = \frac{EF}{2\ell(1-\mu^2)}$, and the boundary conditions

$$c_0 w''(b) = g(w'(b)), \quad d_0 w'''(b) = h(w(b)), \quad b = 0, \;\; b = \ell \tag{2}$$

Further we suppose that functions g, h and f have the form

$$g(w'(b)) = \sum_{k=1}^{n} c_k (w'(b))^{2k-1}, \;\; h(w(b)) = \sum_{k=1}^{m} d_k (w(b))^{2k-1},$$

$$f(w) = \sum_{n=0}^{\infty} a_{2n+1} w^{2n+1},$$

where D - the bending stiffness of the plate; N - the compressing (stretching) force; V, ρ_0, a - the velocity of gas, the density and the sound velocity, corresponding to a homogeneous flow; M - the Mach number; $a_j (j = 1 \div \infty)$ -

the coefficients, characterizing the stiffness of the ground; the integral term takes into account the nonlinear effect of the longitudinal force; $\alpha w'$ - a term related to the aerodynamic effect; $\alpha_0 = 1(\alpha_0 = 2)$ corresponds to one-side (two-side) flow along the plate; $w(x)$ - the bend (arrow) of a plate; E - the module of elasticity; μ - a Poisson's coefficient; F - the cross cut area; J - a moment of the cut inertia. Every coefficient used here is a constant. In (2) $c_i, d_j (i = 0 \div n, j = 0 \div m)$ are arbitrary, part of them must be equal to zero; the boundary conditions can be linear or nonlinear depending on the values of these coefficients. The numbers m and n in (2) can be equal to ∞. The bending moment M and cross-cut force Q in cut x is have the form $M = EJw''(x),\ \ Q = EJw'''(x)$.

Consider the equation (1) for $N = 0$, $a_3 \neq 0$, $a_i = 0$ $(i = 0 \div \infty)$, *i.e.*

$$Dw'''' + \alpha w' + a_3 w^3 - \theta w'' \int_0^{\ell} w'^2\, dx = 0 \tag{3}$$

and let us nondimensionalize (3) by letting $x = \ell\bar{x}$, $w = \ell\bar{w}$, where ℓ is the characteristic length, to obtain the equation

$$\bar{w}'''' + \frac{\alpha\ell^3}{D}\bar{w}' + \frac{a_3\ell^6}{D}\bar{w}^3 - \frac{\theta\ell^3}{D}\bar{w}'' \int_0^1 \bar{w}'^2\, d\bar{x} = 0. \tag{4}$$

Assume the following boundary conditions

$$\bar{w}'(0) = 0,\ \bar{w}'''(0) = 0,\ \bar{w}(1) = 0,\ \bar{w}'(1) = 0, \tag{5}$$

at the free $(x = 0)$ and hard fixed $(x = 1)$ endpoints.

The two-point problem (4), (5) was solved numerically, by reducing it to an initial Cauchy problem. The equation (4) is of the fourth order equation, while in (5) there are only two conditions at $x = 0$. Therefore two more initial conditions are needed. Let us write these conditions as $w(0) = \lambda,\ \ w''(0) = \nu$, where λ, ν are parameters. Then $w(1)$, $w'(1)$ are functions λ and ν, say

$$F_1(\lambda,\nu) \equiv w(1,\lambda,\nu),\ \ F_2(\lambda,\nu) \equiv w'(1,\lambda,\nu). \tag{6}$$

We solve the following Cauchy problem

$$\begin{cases} w'''' + \dfrac{\alpha\ell^3}{D}w' + \dfrac{a_3\ell^6}{D}w^3 - \dfrac{\theta\ell^3}{D}w'' \displaystyle\int_0^1 w'^2\, dx = 0, \\ w(0) = \lambda,\ w'(0) = 0,\ w''(0) = \nu,\ w'''(0) = 0. \end{cases} \tag{7}$$

Cauchy problem (4), (5) corresponds to

$$F_1(\lambda,\nu) = 0,\ \ F_2(\lambda,\nu) = 0. \tag{8}$$

Parameters λ, ν will be defined by the Newton's process, by formulas

$$\begin{pmatrix}\lambda_{n+1}\\ \nu_{n+1}\end{pmatrix} = \begin{pmatrix}\lambda_n\\ \nu_n\end{pmatrix} - \begin{pmatrix}\frac{\partial F_1(\lambda_n,\nu_n)}{\partial\lambda} & \frac{\partial F_1(\lambda_n,\nu_n)}{\partial\nu}\\ \frac{\partial F_2(\lambda_n,\nu_n)}{\partial\lambda} & \frac{\partial F_2(\lambda_n,\nu_n)}{\partial\nu}\end{pmatrix}^{-1} \begin{pmatrix}F_1(\lambda_n,\nu_n)\\ F_2(\lambda_n,\nu_n)\end{pmatrix} \tag{9}$$

We shall continue this iterative process until the fulfillment of the conditions

$$|F_1| < \varepsilon^*, \quad |F_2| < \varepsilon^*, \tag{10}$$

where ε^* is the given accuracy of the computation.

Let us introduce the notation

$$w = y_1, \;\; w' = y_2, \;\; w'' = y_3, \;\; w''' = y_4. \tag{11}$$

Then the integro-differential equation in (7) can be written as the system

$$y_1' = y_2, \;\; y_2' = y_3, \;\; y_3' = y_4, \;\; y_4' = -\frac{\alpha\ell^3}{D}y_2 - \frac{a_3\ell^6}{D}{y_1}^3 + \frac{\theta\ell^3}{D}y_3 I, \tag{12}$$

where $I = \int\limits_0^1 w'^2\,dx = \int\limits_0^1 {y_2}^2\,dx$. Put

$$Y = \begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}, \;\; Y_0 = \begin{pmatrix}\lambda\\ 0\\ \nu\\ 0\end{pmatrix}, \;\; F(x,Y) = \begin{pmatrix} y_2\\ y_3\\ y_4\\ -\frac{\alpha\ell^3}{D}y_2 - \frac{a_3\ell^6}{D}{y_1}^3 + \frac{\theta\ell^3}{D}y_3 I\end{pmatrix}. \tag{13}$$

Then Cauchy problem (7) takes the form

$$\begin{cases} Y' = F(x,Y),\\ Y(0) = Y_0.\end{cases} \tag{14}$$

We solved it by the Runge-Kutta method of the sixth order with the pitch error monitoring. The Cauchy problem complexity consists in the occurence of the integral term in the equation. It is necessary to assign the values of function under the integral sign on the entire segment of integration. So, a direct application of the Runge-Kutta method is impossible. The value of the integral in (14) can be obtained from the following iterative process. Let us present this equation as

$$D(w_k) + I(w_k) = 0, \;\; \text{where}$$
$$I(w_k) = -\frac{\theta\ell^3}{D}w_k''\int\limits_0^1 {w_k'}^2\,dx, \;\; D(w_k) = {w_k}^{(4)} + \frac{\alpha\ell^3}{D}{w_k}' + \frac{a_3\ell^6}{D}{w_k}^3, \;\; k = 1, 2, \ldots$$

1 We solve the equation $D(w_1) = 0$ by the Runge-Kutta method

$$k_1 = hF(x, Y),\ k_2 = hF\left(x + \frac{h}{2}, Y + \frac{k_1}{2}\right),$$

$$k_3 = hF\left(x + \frac{h}{2}, Y + \frac{1}{4}(k_1 + k_2)\right),$$

$$k_4 = hF(x+h, Y-k_2+2k_3),\ k_5 = hF\left(x + \frac{2h}{3}, Y + \frac{1}{27}(7k_1 + 10k_2 + k_4)\right),$$

$$k_6 = hF\left(x + \frac{h}{5}, Y + \frac{1}{625}(28k_1 - 125k_2 + 546k_3 + 54k_4 - 378k_5)\right),$$

$$\triangle Y = \frac{1}{6}(k_1 + 4k_3 + k_4) \tag{15}$$

$$Y(x+h) - Y(h) = r + O(h^6), \quad r = \frac{1}{336}(42k_1 + 224k_3 + 21k_4 - 162k_5 - 125k_6)$$

We obtain the values ${w_1}'(x)$ and ${w_1}''(x)(k = 1)$ on an entire piece of integration segment, which are necessary for the calculation of $I(w_1)$;

2 We find $I(w_1)$. Expression $I(w_1)$ contains the integral, which is by means of the Newton-Cotes quadrature formula. Assume that the integral $\int\limits_a^b q(x)dx$ was computed. Divide a segment of integration $[a; b]$ into n identical parts of length $h = (b-a)/n$. Number n is chosen multiple 5, such that the entire interval of integration was decomposed into parts. On these parts we shall approximate the subintegral function $q(x)$ by the interpolation Lagrange polynomial $L_5(x)$ of the fourth degree

$q(x) \approx L_5(x) = \sum\limits_{i=0}^{5} q(x_i)p_i(x)$, where $p_i(x) = \prod\limits_{\substack{j=0,\\ i\neq j}}^{5} \frac{x-x_j}{x_i-x_j}$ - a weight function. Then the integral $\int\limits_a^b q(x)dx$ will be defined as the integrals

$$\int\limits_{x_0}^{x_5} q(x)\,dx \approx \int\limits_{x_0}^{x_5} L_5(x)\,dx = \sum_{i=0}^{5} p_i q(x_i) = 5h(p_0q(x_0) + p_1q(x_1) + p_2q(x_2) + \tag{16}$$

$p_3q(x_3) + p_4q(x_4) + p_5q(x_5))$, where $p_0 = p_5 = 19/288, p_1 = p_4 = 75/288$,

$p_2 = p_3 = 50/288$.

3 We solve the equation $D(w_2) + I(w_1) = 0$ by the Runge-Kutta method and obtain the values ${w_2}'(x)$ and ${w_2}''(x)(k = 2)$ on the entire segment of integration.

4 We find $I(w_2)$, using formula (16) and so on.

We continue the iterative process until the realization of the condition $\max\limits_{x\in[0,1]} |w_k(x) - w_{k-1}(x)| < \varepsilon^*$ is achieved, where ε^* is the same as in the Newton's process; $w_k(x)$ - a bend on k pitch, $w_{k-1}(x)$ - a bend on $(k-1)$ pitch. The numerical realization used the program written by Delphi 7. Initial parameters of the program are: factor at integral, factor at w^3, accuracy of a calculation, perturbation ε and missing initial values of the Cauchy problem $w(0) = \lambda$, $w''(0) = \nu$. The result of the work of this program is the finding of the value of a size and the definition of the form of a bend of a plate at various set values of perturbation ε. By the program the schedules $w(x)$ describing the form of a bend of a plate, are constructed. The static bifurcation diagrams showing dependence of the maximal bend of a plate on the speed of the flow $(\alpha = \alpha_0\rho_0 V)$ are built too. If perturbation $\varepsilon = \lambda - s_0^3$ is formed due to changing of speed of a flow, the branching at a point $\lambda_0 = {s_0}^3$ is supercritical. It means, that $\varepsilon > 0$ is formed by the increase of the speed of a flow. At the same time $\varepsilon > 0$, at the constant speed of a flow, can be received owing to the reduction of the deflection stiffness for which the branching will be subcritical.

Consider the model ($\ell = 1m$, $a = 330km/s$, $\theta = 35\cdot 10^5 N/m$, $\rho_0 = 1.2kg/m^3$, $a_3 = 1$, $D_2 = 19.06\cdot 10^3 Nm^2$, $D_0 = 19.05\cdot 10^3 Nm^2$, $D_3 = 19.04\cdot 10^3 Nm^2$, $\alpha_0 = 2$). For it the static bifurcation diagrams are constructed (fig. 1) (by the written program) and fig. 2 (by Mathcad 2001i Professional), for fixed factors of the deflection stiffness $D_3 < D_0 < D_2$ depending of the change of speed of a flow over critical values $\lambda_3 < \lambda_0 = s_0^3 < \lambda_2$.

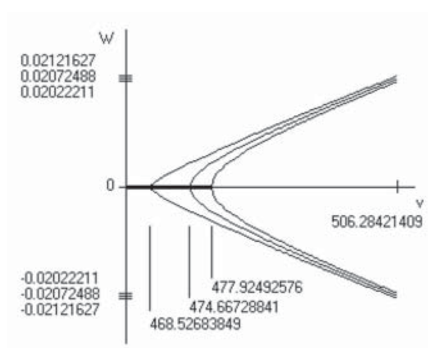


Fig. 1.

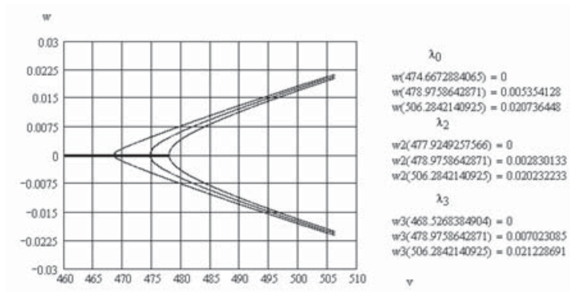


Fig. 2.

In figs. 1 and 2, the left diagrams correspond to D_3, λ_3, $w3$; middle to D_0, λ_0, w; and right to D_2, λ_2, $w2$. In fig. 3 (constructed by Mathcad) and fig. 4 (constructed by the written program) the forms of the deflection of a plate are represented, where $\phi(x)$ corresponds to the positive decision ($\phi(x) = w(x)$), $\psi(x)$ - negative ($\phi(x) = -w(x)$) at D_0,λ_0.

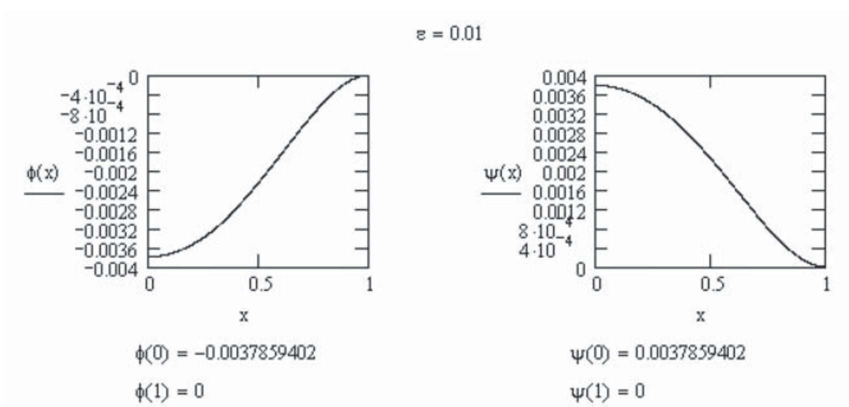


Fig. 3.

2. THE BIFURCATION OF A PIPELINE FORMS

The mathematical model of the deflection forms of the pipeline filled with a moving liquid is the nonlinear integro-differential equation

$$Dw'''' + Nw'' + f(w) - \theta w'' \int_0^{\ell} w'^2 \, dx = 0, \quad N = N_0 + m_* U^2 \tag{17}$$

and boundary conditions (2). In (17) D - the deflection stiffness of the pipeline; $N_0 > 0$ - the compressing ($N_0 < 0$ - stretching) effort; m_* - the specific mass of

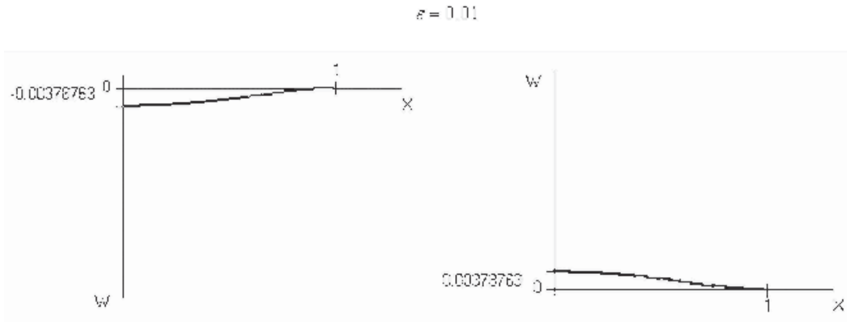


Fig. 4.

liquid; U - the speed of the liquid flow; $a_j (j = 1 \div \infty)$ - the factors describing the stiffness of the basis; the integral term takes into account the nonlinear influence of the longitudinal effort; $w(x)$ - a deflection of the pipeline. All factors included in the equation and boundary conditions are constant.

Consider the nondimensional form of equation (17)

$$\bar{w}'''' + \frac{N\ell^2}{D}\bar{w}'' + \frac{a_3\ell^6}{D}\bar{w}^3 - \frac{\theta\ell^3}{D}\bar{w}'' \int\limits_0^1 \bar{w}'^2 \, d\bar{x} = 0, \tag{18}$$

where $x = \ell\bar{x}$, $w = \ell\bar{w}$, ℓ - characteristic length. We shall study the solutions of the equation (18) for the following boundary conditions

$$\bar{w}(0) = 0, \;\; \bar{w}''(0) = 0, \;\; \bar{w}(1) = 0, \;\; \bar{w}'(1) = 0, \tag{19}$$

corresponding to the cantilever left end and the fixed right end of the pipeline. Further the line above variables will be omitted.

The problem (18), (19) was solved numerically, like the plate problem and by means of the same program. In this case as parameters λ and ν we use $w'(0) = \lambda$, $w'''(0) = \nu$. Then $w(1)$ and $w'(1)$ are functions λ and ν

$$F_1(\lambda, \nu) \equiv w(1, \lambda, \nu), \;\; F_2(\lambda, \nu) \equiv w'(1, \lambda, \nu). \tag{20}$$

We solve the following Cauchy problem

$$\begin{cases} w'''' + \dfrac{N\ell^2}{D}w'' + \dfrac{a_3\ell^6}{D}w^3 - \dfrac{\theta\ell^3}{D}w'' \displaystyle\int\limits_0^1 w'^2 \, dx = 0 \\ w(0) = 0, \; w'(0) = \lambda, \; w''(0) = 0, \; w'''(0) = \nu \end{cases} \tag{21}$$

Cauchy problem (21) will correspond to the boundary problem (18), (19) at the conditions realization (8). Parameters λ, ν we shall define with the help

Newton's process by formulas (9). We shall continue this iterative process until the realization of the condition (10).With the notation

$$y_4' = -\frac{N\ell^2}{D}y_3 - \frac{a_3\ell^6}{D}{y_1}^3 + \frac{\theta\ell^3}{D}y_3 I,\ \ Y_0 = \begin{pmatrix} 0 \\ \lambda \\ 0 \\ \nu \end{pmatrix},$$

$$F(x,Y) = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ -\frac{N\ell^2}{D}y_3 - \frac{a_3\ell^6}{D}{y_1}^3 + \frac{\theta\ell^3}{D}y_3 I \end{pmatrix},$$

the Cauchy problem (21) becomes (14). The problem (14) is solved by the Runge-Kutta method of the sixth order with the pitch error monitoring by formulas (1). Cauchy problem complexity consists in the occurence in the equation integral. As in the above, in order to compute it we use the iterative process described above, in which $D(w_k) = w'''' + \frac{N\ell^2}{D}w_k'' + \frac{a_3\ell^6}{D}w_k^3$.

Consider the model ($\ell = 1m, \theta = 35 \cdot 10^3 N/m$, $N = 1N$, $a_3 = 1$, $D_2 = 449Nm^2$, $D_0 = 450Nm^2$, $D_3 = 451Nm^2$, $m = 10kg$). For it the static bifurcation diagrams are constructed, (fig. 5) (constructed by the written program) and fig. 6 (constructed by Mathcad 2001i Professional), for fixed factors of the deflection stiffness $D_2 < D_0 < D_3$ depending on the change of the fluid flow beyond the critical values $\lambda_2 < \lambda_0 = s_0^2 < \lambda_3$.

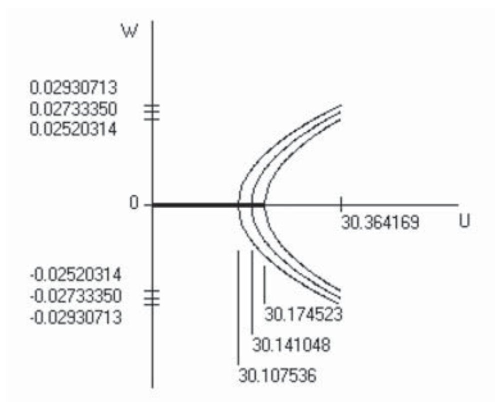


Fig. 5.

In figs. 5 and 6 the left diagrams correspond to D_2, λ_2, $w2$; the middle, to D_0, λ_0, w; and the right to D_3, λ_3, $w3$. In fig. 7 (constructed by Mathcad) and fig. 8 (constructed by the written program) the forms of the deflection of a plate are assigned, where $\phi(x)$ corresponds to the positive decision ($\phi(x) = w(x)$), $\psi(x)$ - negative ($\phi(x) = -w(x)$) at D_0,λ_0.

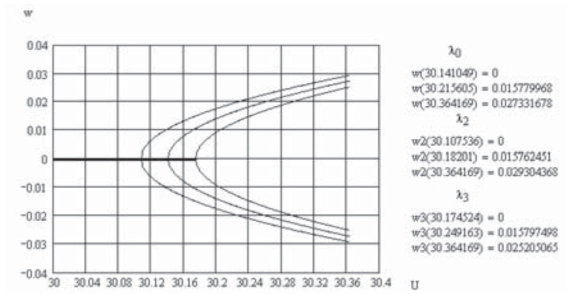


Fig. 6.

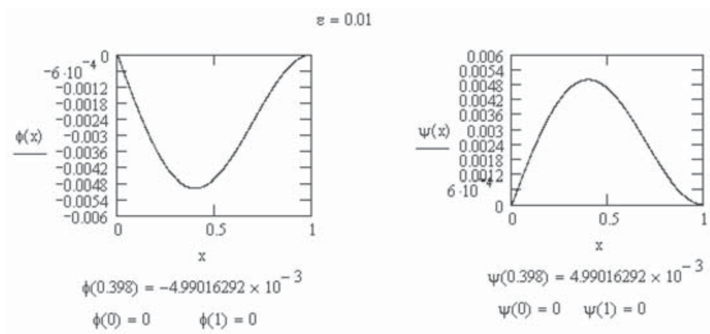


Fig. 7.

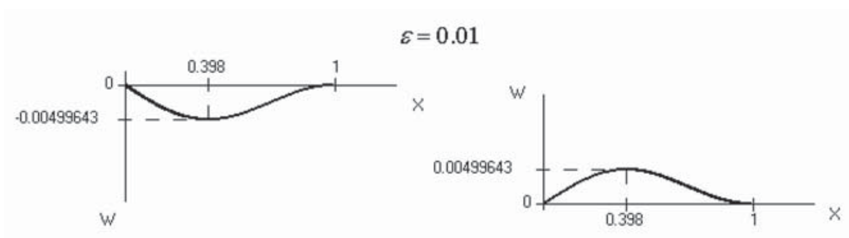


Fig. 8.

ROMAI J., **2**, 2(2006),
205–226

ON THE FUNDAMENTAL MODE

Serban E. Vlad
Oradea City Hall
serban_e_vlad@yahoo.com

Abstract The asynchronous systems are multivalued applications f from the set $\mathbf{R} \to \{0,1\}^m$ functions, called (admissible) inputs, to sets of $\mathbf{R} \to \{0,1\}^n$ functions, called (possible) states. The fundamental (operating) mode of f consists in an input u and a sequence $(\mu^k)_{k\in\mathbf{N}} \in \{0,1\}^n$ of binary vectors so that $\mu^0, \mu^1, \mu^2, ...$ are accessed by all the states $x \in f(u)$ simultaneously in this order, where μ^0 is the initial state and $(\mu^k)_{k\geq 1}$ are 'steady states'.

Keywords: asynchronous system, fundamental mode.
2000 MSC: 93C62, 93D25, 68Q05, 68Q10.

1. INTRODUCTION

The concept of asynchronous system originates in the modeling of the asynchronous circuits from digital electrical engineering. *Asynchronous systems theory* is the theory of modeling such circuits. The uncertainties governing these circuits can be surpassed in at least two ways: by using a three-valued logic and respectively by using a two-valued logic but many-valued functions (i.e. non-deterministic systems), that give for each cause all the possible effects, our choice.

Several papers exist containing equations and inequalities written with $\mathbf{R} \to \{0,1\}$ functions that model the behaviour of the asynchronous circuits. In [1] we present a method of modeling where the fundamental circuit is the 'delay element', i.e. the circuit that computes (inertially, in real time) the identical function $\{0,1\} \to \{0,1\}$. The modeling technique is called *delay theory*. The 'delays', i.e. the models of the delay, elements are one dimensional asynchronous systems that fulfill a certain requirement of stability. They are generalized in [2], [3].

Let f be the asynchronous system that associates the input $u : \mathbf{R} \to \{0,1\}^m$ with the set of states $x \in f(u)$, where $x : \mathbf{R} \to \{0,1\}^n$. The fundamental operating mode of f asks the existence of a sequence $(\mu^k)_{k\in\mathbf{N}} \in \{0,1\}^n$ so that all $x \in f(u)$ run simultaneously through the values $\mu^0, \mu^1, \mu^2, ...$ in this order, where μ^0 is the initial state and $\mu^1, \mu^2, ...$ are final states (steady states). This concept is mentioned in many papers under a non-formalized manner. For instance, in [4] its characterization is: 'inputs are constrained to change only when all the delay elements are stable (i.e. they have the input value equal with the output value)'. 'Note that the fundamental mode excludes' the existence of 'a cycle of oscillations', that is instability. Elsewhere the author refers to the fundamental mode where 'the designer has to make sure that the circuit inputs can change only when the circuit itself is stable and ready to accept them'. The characterization given by L. Lavagno to the fundamental mode, that agrees with other opinions, corresponds to our special case from Section 11.

2. PRELIMINARIES

Definition 2.1 *Denote by* $\mathbf{B}$ *the set* $\{0,1\}$ *together with the order* $0<1$, *the discrete topology and the laws: the complement* $'\overline{}'$, *the intersection* $'\cdot'$, *the union* $'\cup'$ *and the modulo 2 sum* $'\oplus'$. *Then* $\mathbf{B}$ *is called the* binary Boole algebra *or the Boole algebra with two elements.*

Notations 2.1 *For some interval* $I\subset\mathbf{R}$ *and* $x:\mathbf{R}\to\mathbf{B}^n$, *we note with* $x_{|I}$ *the restriction of* x *to* I.

Notations 2.2 *If* x *is constant on the interval* I *and equal to* $\mu\in\mathbf{B}^n$, *we write* $x_{|I}=\mu$, *by identifying the function with the constant.*

Definition 2.2 *Let* $x:\mathbf{R}\to\mathbf{B}^n$ *be some function. We define as* initial value *of* x, *denoted by* $x(-\infty+0)$ *or* $\lim\limits_{t\to-\infty}x(t)$ *to be that vector from* $\mathbf{B}^n$ *satisfying one of the equivalent statements* $\exists t_0\in\mathbf{R},\forall t<t_0,x(t)=x(-\infty+0),\exists t_0\in\mathbf{R},x_{|(-\infty,t_0)}=x(-\infty+0)$. *Dually, the* final value *of* x *is denoted by* $x(\infty-0)$ *or* $\lim\limits_{t\to\infty}x(t)$ *and it is the vector from* $\mathbf{B}^n$ *satisfying one of* $\exists t_f\in\mathbf{R},\forall t\geq t_f,x(t)=x(\infty-0),\exists t_f\in\mathbf{R},x_{|[t_f,\infty)}=x(\infty-0)$.

Notations 2.3 *For some* $d\in\mathbf{R}$, *we denote by* $\tau^d:\mathbf{R}\to\mathbf{R}$ *the translation* $\forall t\in\mathbf{R},\tau^d(t)=t-d$.

Remark 2.1 *For any* x, $x(-\infty+0)$ *and* $x(\infty-0)$ *are uniquely defined since* x *is a function. The initial and final value of* x *and* $x\circ\tau^d$ *coincide* ($\forall t\in\mathbf{R}$, *we have* $(x\circ\tau^d)(t)=x(t-d)$).

Definition 2.3 *For any set* $A\subset\mathbf{R}$, *we define the characteristic function of* A *by* $\chi_A:\mathbf{R}\to\mathbf{B}$,

$$\forall t\in\mathbf{R},\chi_A(t)=\begin{cases}1, if\ t\in A\\ 0, if\ t\notin A\end{cases}$$

Notations 2.4 *Denote by Seq the set of all real unbounded strictly increasing sequences* $t_0<t_1<t_2<...$ *Generally the elements of Seq are denoted by* (t_k).

Definition 2.4 *The function* $x:\mathbf{R}\to\mathbf{B}^n$ *is called an* $n-$dimensional signal *if* $x(-\infty+0)\in\mathbf{B}^n$ *and* $(t_k)\in Seq$ *exist so that*

$$x(t)=x(-\infty+0)\cdot\chi_{(-\infty,t_0)}(t)\oplus x(t_0)\cdot\chi_{[t_0,t_1)}(t)\oplus x(t_1)\cdot\chi_{[t_1,t_2)}(t)\oplus...$$

Notations 2.5 *The set of the $n-$dimensional signals is denoted by $S^{(n)}$. Denote by $S_c^{(n)}$ the set of these $x \in S^{(n)}$, for which $x(\infty - 0)$ exists.*
For some function $F : \mathbf{B}^m \to \mathbf{B}^n$ we put $S_{F,c}^{(m)} = \{u | u \in S^{(m)}, \lim_{t\to\infty} F(u(t))$ exists$\}$.

Definition 2.5 *Let $x \in S^{(n)}$ and the numbers $t_0, t_1 \in \mathbf{R}$ so that $t_0 < t_1$. The restrictions $\gamma = x_{|(-\infty,t_1]}$, $\gamma' = x_{|[t_0,t_1]}$ are called the* transitions *of x from the value $x(-\infty + 0)$ to the value $x(t_1)$, respectively from $x(t_0)$ to $x(t_1)$. The intervals $(-\infty, t_1]$, $[t_0, t_1]$ are called the* support intervals *of the transitions γ, γ'. The number $t_1 - t_0$ is called the* duration of the transition γ'.

Notations 2.6 *Denote by $P^*(S^{(n)}) = \{X | X \subset S^{(n)}, X \neq \emptyset\}$ the set of the non-empty subsets of $S^{(n)}$.*

Definition 2.6 *A function $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ is called an* (asynchronous) system*, given under the explicit form. It associates with the functions $u \in U$ called* (admissible) inputs*, sets of functions $x \in f(u)$, called* (possible) states*.*

Under the implicit form, the asynchronous system consists in one or several equations and/or inequalities where the unknown $x \in S^{(n)}$ depends on $u \in U$.

Definition 2.7 *The system f is* non-anticipatory *if $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$,*

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

3. SYNCHRONOUS ACCESS

Definition 3.1 *By the* synchronous access *of (the states of) $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$, under the input $u \in U$, to the value $\mu \in \mathbf{B}^n$ at $t_0 \in \mathbf{R}$ we mean the property*

$$\forall x \in f(u), x(t_0) = \mu. \tag{1}$$

If it is fulfilled, μ is called synchronously accessible value *and t_0 is called the* access time *(instant) of (the states of) f under the input u to the value μ.*

Theorem 3.1 *Let f be the non-anticipatory system and fix $t_0 \in \mathbf{R}, u \in U$, $\mu \in \mathbf{B}^n$. Then (1) is equivalent to*

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y(t_0) = \mu. \tag{2}$$

Proof. (1)$\Longrightarrow$(2) is obvious.

(2)$\Longrightarrow$(1). From $u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}$ and the non-anticipation of f we infer that $\{x_{|(-\infty,t_0]} | x \in f(u)\} = \{y_{|(-\infty,t_0]} | y \in f(v)\}$. In particular, we have

$$\{x(t_0) | x \in f(u)\} = \{y(t_0) | y \in f(v)\} = \mu,$$

i.e. (1) holds. □

Definition 3.2 *The next special cases of fulfillment of (1)*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu, \tag{3}$$

$$\forall x \in f(u), x_{|[t_0,\infty)} = \mu \tag{4}$$

are called synchronous initial access*, respectively* synchronous final access *of f, under u, to μ. The vector μ is called a* (synchronously) accessible initial *value, respectively* (synchronously) accessible final value *(other terminologies are:* final state *of f, or* steady state *of f. In Abstract and Introduction we indicated the concept of 'steady state' because it is the most popular. We prefer Definition 9 due to its precision and the fact that it highlights the initial-final duality) and* $(-\infty, t_0), [t_0, \infty)$ *are called the* access time intervals *of (the states of) f, under u, to the initial value, respectively to the final value.*

Definition 3.3 *If*

$$\forall x \in f(u), x = \mu \tag{5}$$

we say that f has, under u, a point of equilibrium and μ is called a point of equilibrium *of f. The access time interval of (the states of) f, under u, to μ is, by definition,* $\mathbf{R}$.

Remark 3.1 *The word 'synchronous' in the previous definitions means the fact that the number t_0 does not depend on the choice of $x \in f(u)$.*

The point of equilibrium is a special case of both the synchronously accessible initial value and the synchronously accessible final value of f.

On the other hand, we try to extend, when f is non-anticipatory, the result of Theorem 3.1 to the equivalencies between (3), (4) and respectively

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \tag{6}$$

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \tag{7}$$

(3)$\Longleftrightarrow$(6) is true.

(4)$\Longleftrightarrow$(7) is not true. While (4) shows that all $x \in f(u)$, starting from the time instant t_0, become equal with μ, (7) states that all $x \in f(u)$, starting from t_0, may become equal with μ, if, for example, $u = v$.

It is not the case to try such reasoning for the points of equilibrium.

4. SYNCHRONOUS CONSECUTIVE ACCESSES

Definition 4.1 *Suppose that the system f is non-anticipatory. By the* synchronous consecutive accesses *of (the states of) f, under $u \in U$, to the values $\mu, \mu' \in \mathbf{B}^n$ at the time instants $t_0 < t_1$ it is understood the property*

$$\forall x \in f(u), x(t_0) = \mu \text{ and } x(t_1) = \mu'. \tag{8}$$

Remark 4.1 *In the present paper we are interested in two special cases of synchronous consecutive accesses; the one when in (8) μ is initial value and μ' is final value*

$$\forall x \in f(u), x_{|(-\infty,t_0)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu' \tag{8*}$$

and other when in (8) μ, μ' are both final values*

$$\forall x \in f(u), x_{|[t_0,\infty)} = \mu \text{ and } x_{|[t_1,\infty)} = \mu'. \tag{9}$$

Replace in (8) and (9) the synchronous accesses of x to the final values by (7). After computations and taking into account the non-anticipation of f, we get*

$$\exists v \in U, u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \text{ and } y_{|[t_1,\infty)} = \mu', \tag{10}$$

$$\begin{gathered}\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \text{ and} \\ \text{and } \exists v' \in U, u_{|(-\infty,t_1)} = v'_{|(-\infty,t_1)}, \forall y' \in f(v'), y'_{|[t_1,\infty)} = \mu'\end{gathered} \tag{11}$$

Each of the non-equivalent statements (8) and (10) describe, the accesses of f first to the initial value μ, then to the final value μ', with the difference that in the first case all $x \in f(u)$ stabilize at μ' while in the second case all $x \in f(u)$ may stabilize at μ', for example if $u = v$.*

The non-equivalent statements (9) and (11) give two completely different manners of accessing synchronously first the final value μ, then the final value μ', in the sense that at (9) we have necessarily the triviality $\mu = \mu'$ while at (11) $\mu \neq \mu'$ is possible.

In the properties (8),(8)...,(11) the possibility $\mu = \mu'$ = point of equilibrium exists, with the trivialities that follow from this situation.*

5. TRANSFERS

Definition 5.1 *Suppose that the non-anticipatory system f accesses synchronously, under the input $u \in U$, the values $\mu, \mu' \in \mathbf{B}^n$ at the time instants $t_0 < t_1$, i.e. (8*) is fulfilled. Denote*

$$\Gamma = \{x_{|[t_0,t_1]} | x \in f(u)\}; \tag{12}$$

Γ *is called the* (synchronous) transfer *of (the states of) f, that is made under the input u from μ to μ'. Conversely, if we say that Γ, defined by (12), represents a transfer of f, made under the input u, from $\mu = x(t_0)$ to $\mu' = x(t_1)$ and μ, μ' are independent on the choice of $x \in f(u)$, then we mean that (8*) is true.*

Definition 5.2 *Assume that (10) is true and denote*

$$\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\}. \tag{13}$$

$\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is called the* initial fundamental transfer *of (the states of) f, under u, from the initial value μ to the final value μ'. Conversely, the statement that $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ defined by (13) is an initial fundamental transfer refers to the existence of $t_0 < t_1$ so that (10) is satisfied.*

Definition 5.3 *If (11) is true, denote*

$$\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu' = \{x_{|[t_0,t_1]} | x \in f(u)\}. \tag{14}$$

$\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$ *is called the* non-initial fundamental transfer *of (the states of) f, made under the input u, from the final value μ to the final value μ'. Conversely, the statement that $\mu \overset{u_{|[t_0,t_1)}}{\rightarrow} \mu'$ defined by (14) is a non-initial fundamental transfer means the fulfillment of (11).*

Definition 5.4 *If (5) is fulfilled, denote*

$$(\mu \overset{u}{=} \mu) = \{\mu\}. \tag{15}$$

$\mu \overset{u}{=} \mu$ *is called* trivial fundamental transfer *of (the states of) f, made under the input u, μ being a point of equilibrium. Conversely, when we state that $\mu \overset{u}{=} \mu$ defined by (15) is a trivial fundamental transfer, this means that (5) holds.*

Definition 5.5 *If the synchronous transfer Γ satisfies $\forall \gamma \in \Gamma$, γ is coordinatewise monotonous, then it is called* hazard-free.

Remark 5.1 *At (10), the synchronism of the access of the states to the initial value μ is not necessary in many situations and it was asked for the symmetry of the exposure only.*

At the hazard-free transfers, the condition of monotony seems one of economy and normalization, the coordinates of x do not switch more than necessary, but it has rather a functional meaning.

The trivial fundamental transfers are hazard-free.

6. SIMPLE PROPERTIES OF FUNDAMENTAL TRANSFERS. AN EXAMPLE

Theorem 6.1 *Let f be the non-anticipatory system and fix $t_0, t_1 \in \mathbf{R}, t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$. If (8*) is true then $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is an initial fundamental transfer and if*

$$\exists v \in U, u_{|(-\infty,t_0)} = v_{|(-\infty,t_0)}, \forall y \in f(v), y_{|[t_0,\infty)} = \mu \text{ and } \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$$

then $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ is a non-initial fundamental transfer.

Proof. The first hypothesis makes (10) true for $v = u$ and the second statement makes (11) true for $v' = u$. □

Theorem 6.2 *Let f be non-anticipatory and $\mu \overset{u_{|I}}{\to} \mu'$ a fundamental transfer, where $I \subset \mathbf{R}$ is an interval of the form $(-\infty, t_1)$ or $[t_0, t_1)$.*

a) If $I = (-\infty, t_1)$ and $u' \in U$ is arbitrary with $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$, then $\mu \overset{u'_{|I}}{\to} \mu'$ is an initial fundamental transfer equal with $\mu \overset{u_{|I}}{\to} \mu'$.

b) If $I = [t_0, t_1)$, then $\forall u' \in U$, $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$ implies that $\mu \overset{u'_{|I}}{\to} \mu'$ is a non-initial fundamental transfer equal with $\mu \overset{u_{|I}}{\to} \mu'$.

Proof. a) $\mu \overset{u'_{|(-\infty,t_1)}}{\to} \mu'$ is an initial fundamental transfer, i.e. $\exists t_0 < t_1, \exists v \in U, u'_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu$ *and* $y_{|[t_1,\infty)} = \mu'$ takes place because the hypothesis (10) is true as well as $u_{|(-\infty,t_1)} = u'_{|(-\infty,t_1)}$. We take into account the non-anticipation of f and we get the second statement of the Theorem

$$\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu' = \{x_{|(-\infty,t_1]} | x \in f(u)\} = \{x'_{|(-\infty,t_1]} | x' \in f(u')\} = \mu \overset{u'_{|(-\infty,t_1)}}{\to} \mu'.$$

b) is proved similarly with a). □

Example 6.1 *The system $f : S \to P^*(S)$ defined by the double inequality*

$$\bigcap_{\xi \in [t-1,t)} \overline{u(\xi)} \leq x(t) \leq \bigcup_{\xi \in [t-1,t)} \overline{u(\xi)} \tag{16}$$

models the computation of the logical complement of u, made with a delay of one time unit. Suppose that it is non-anticipatory and denote $u = \chi_{[0,2)}$, $v = \chi_{[0,\infty)}$ for which the inequalities $\bigcap\limits_{\xi \in [t-1,t)} \overline{u(\xi)} \leq x(t) \leq \bigcup\limits_{\xi \in [t-1,t)} \overline{u(\xi)}$, $\bigcap\limits_{\xi \in [t-1,t)} \overline{v(\xi)} \leq y(t) \leq \bigcup\limits_{\xi \in [t-1,t)} \overline{v(\xi)}$ become

$$\chi_{(-\infty,0] \cup [3,\infty)}(t) \leq x(t) \leq \chi_{(-\infty,1) \cup (2,\infty)}(t), \tag{17}$$

$$\chi_{(-\infty,0]}(t) \leq y(t) \leq \chi_{(-\infty,1)}(t). \tag{18}$$

From (18) we infer that $\forall y \in f(v), y_{|(-\infty,0)} = 1$ and $y_{|[1,\infty)} = 0$ and, because $u_{|(-\infty,1)} = v_{|(-\infty,1)}$, we have that $(1 \overset{u_{|(-\infty,1)}}{\to} 0) = (1 \overset{v_{|(-\infty,1)}}{\to} 0)$ is an initial fundamental transfer ((10) is true). From the inequalities (17), (18) we also infer that $\forall y \in f(v), y_{|[1,\infty)} = 0$, $\forall x \in f(u), x_{|[3,\infty)} = 1$, i.e. $0 \overset{u_{|[1,3)}}{\to} 1$ is non-initial fundamental transfer (from Theorem 6.1).

The transitions $\gamma \in 1 \stackrel{u_{|(-\infty,1)}}{\rightarrow} 0$ *and* $\gamma' \in 0 \stackrel{u_{|[1,3)}}{\rightarrow} 1$ *are not monotonous in general. We wonder in what conditions, if we add the (absolute inertia) requests*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t,t+\delta]} x(\xi), \tag{19}$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t,t+\delta]} \overline{x(\xi)}, \tag{20}$$

-where $\delta \geq 0$*- to (16) with* $u = \chi_{[0,2)}$*, i.e. to (17), the monotony is true. Monotony means that* x *switches from 1 to 0 in the interval* $(0,1]$ *and that it cannot switch from 0 to 1 and then from 1 to 0 again in this interval. Let* $0 < t_1 < t_2 < t_3 \leq 1$ *so that* $x(t_1-0)\cdot\overline{x(t_1)} = \overline{x(t_2-0)}\cdot x(t_2) = x(t_3-0)\cdot\overline{x(t_3)} = 1$*. Then we have* $t_2 - t_1 > \delta, t_3 - t_2 > \delta$ *from the fulfillment of (19) and (20, meaning that* $1 > t_3 - t_1 > 2\delta$*. Thus, if* $\delta \geq \frac{1}{2}$*, such* t_1, t_2, t_3 *do not exist and any* $\gamma \in 1 \stackrel{u_{|(-\infty,1)}}{\rightarrow} 0$ *is a monotonous transition. Similarly,* $\delta \geq \frac{1}{2}$ *implies the fact that any* $\gamma' \in 0 \stackrel{u_{|[1,3)}}{\rightarrow} 1$ *is monotonous.*

Another condition is also required here: after having switched from 1 to 0 in the interval $(0,1]$*,* x *is also allowed to switch from 0 to 1 in the interval* $(2,3]$*. This gives* $\delta < 3$*.*

The conclusion is the following: for $\delta \in [\frac{1}{2}, 3)$*, the system* g *that is obtained by intersecting (16), (19), (20), where* $u = \chi_{[0,2)}$*, has the transfers* $1 \stackrel{u_{|(-\infty,1)}}{\rightarrow} 0$*,* $0 \stackrel{u_{|[1,3)}}{\rightarrow} 1$ *hazard-free.*

7. COMPOSITION OF FUNDAMENTAL TRANSFERS

Theorem 7.1 *Let the non-anticipatory system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *satisfy the conditions: i)* U *is closed under translations and under 'concatenation'* $\forall d \in \mathbf{R}, \forall u \in U, u \circ \tau^d \in U$*,* $\forall t \in \mathbf{R}, \forall u \in U, \forall v \in U, u \cdot \chi_{(-\infty,t)} \oplus v \cdot \chi_{[t,\infty)} \in U$*; ii)* $non-anticipation^* \forall t \in \mathbf{R}$*,* $\forall u \in U$*,* $\forall v \in U, (u_{|[t,\infty)} = v_{|[t,\infty)}$ *and* $\{x(t)|x \in f(u)\} = \{y(t)|y \in f(v)\}) \Longrightarrow \{x_{|[t,\infty)}|x \in f(u)\} = \{y_{|[t,\infty)}|y \in f(v)\}$*; iii) time invariance* $\forall d \in \mathbf{R}, \forall u \in U, f(u \circ \tau^d) = \{x \circ \tau^d | x \in f(u)\}$*.*

a) Suppose that $t_0 < t_1$*,* $t_2 < t_3$*,* $u^0, u^1, v^1 \in U$ *and* $\mu, \mu', \mu" \in \mathbf{B}^n$ *are arbitrary with* $\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu$*,* $\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu'$*,* $u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)}$*,* $\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu'$*,* $\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu"$*. We denote* $d = t_1 - t_2$ *and* $\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)}$ *for* $\varepsilon \geq 0$*. We have* $\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|(-\infty,t_0)} = \mu$*,* $\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu"$ *meaning that if* $\mu \stackrel{u^0_{|(-\infty,t_1)}}{\rightarrow} \mu'$ *is initial fundamental and* $\mu' \stackrel{u^1_{|[t_2,t_3)}}{\rightarrow} \mu"$ *is non-initial* *fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\to} \mu"$ *is initial fundamental. In other words, if* $f(u^0)$ *transfers synchronously the initial value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* $\mu"$*, then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the initial value* μ *in the final value* $\mu"$.

b) Suppose that $t_0 < t_1$, $t_2 < t_3$, $u^0, v^0, u^1, v^1 \in U$ *and* $\mu, \mu', \mu" \in \mathbf{B}^n$ *are given so that*

$$u^0_{|(-\infty,t_0)} = v^0_{|(-\infty,t_0)}, \tag{21}$$

$$\forall y \in f(v^0), y_{|[t_0,\infty)} = \mu, \tag{22}$$

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu', \tag{23}$$

$$u^1_{|(-\infty,t_2)} = v^1_{|(-\infty,t_2)}, \tag{24}$$

$$\forall y' \in f(v^1), y'_{|[t_2,\infty)} = \mu', \tag{25}$$

$$\forall x' \in f(u^1), x'_{|[t_3,\infty)} = \mu". \tag{26}$$

With the notation $d = t_1 - t_2$, $\widetilde{v} = v^0$ *and*

$$\widetilde{u}_\varepsilon = u^0 \cdot \chi_{(-\infty,t_1+\varepsilon)} \oplus (u^1 \circ \tau^{d+\varepsilon}) \cdot \chi_{[t_1+\varepsilon,\infty)}, \tag{27}$$

$\varepsilon \geq 0$, *we have*

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)}, \tag{28}$$

$$\forall \widetilde{y} \in f(\widetilde{v}), \widetilde{y}_{|[t_0,\infty)} = \mu, \tag{29}$$

$$\forall \widetilde{x} \in f(\widetilde{u}_\varepsilon), \widetilde{x}_{|[t_3+d+\varepsilon,\infty)} = \mu". \tag{30}$$

This means that if $\mu \overset{u^0_{|[t_0,t_1)}}{\to} \mu'$, $\mu' \overset{u^1_{|[t_2,t_3)}}{\to} \mu"$ *are non-initial fundamental, then* $\mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\to} \mu"$ *is non-initial fundamental (if* $f(u^0)$ *transfers synchronously the final value* μ *in the final value* μ' *and if* $f(u^1)$ *transfers synchronously the final value* μ' *in the final value* $\mu"$ *then* $f(\widetilde{u}_\varepsilon)$ *transfers synchronously the final value* μ *in the final value* $\mu"$*).*

Proof. b) First remark that $\widetilde{u}_\varepsilon$ given by (27) belongs to U, from i).

(28) is satisfied because for any $\varepsilon \geq 0$ we have $t_1 + \varepsilon \geq t_1 > t_0$ and also from the definition of $\widetilde{v}$:

$$\widetilde{u}_{\varepsilon|(-\infty,t_0)} \overset{(27)}{=} u^0_{|(-\infty,t_0)} \overset{(21)}{=} v^0_{|(-\infty,t_0)} = \widetilde{v}_{|(-\infty,t_0)}.$$

(29) is true because it coincides with the hypothesis (22). Let us prove (30). From (24) and from the non-anticipation of f we infer $\{y'_{|(-\infty,t_2]} | y' \in f(v^1)\} = \{x'_{|(-\infty,t_2]} | x' \in f(u^1)\}$ and if we take into account (25), we can see that

$$\{y'(t_2) | y' \in f(v^1)\} = \{x'(t_2) | x' \in f(u^1)\} = \mu' \tag{31}$$

The time invariance of f implies $\{x"|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{x' \circ \tau^{d+\varepsilon}|x' \in f(u^1)\}$, thus

$$\{x'(t_2)|x' \in f(u^1)\} = \{x"(t_1+\varepsilon)|x" \in f(u^1 \circ \tau^{d+\varepsilon})\}. \tag{32}$$

From $\widetilde{u}_{\varepsilon|(-\infty,t_1+\varepsilon)} = u^0_{|(-\infty,t_1+\varepsilon)}$ and the non-anticipation we get $\{\widetilde{x}_{|(-\infty,t_1+\varepsilon]}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x_{|(-\infty,t_1+\varepsilon]}|x \in f(u^0)\}$. In particular we have

$$\{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x(t_1+\varepsilon)|x \in f(u^0)\}. \tag{33}$$

Then

$$\{\widetilde{x}(t_1+\varepsilon)|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} \overset{(33)}{=} \{x(t_1+\varepsilon)|x \in f(u^0)\} \overset{(23)}{=} \mu' =$$
$$\overset{(25)}{=} \{y'(t_2)|y' \in f(v^1)\} \overset{(31)}{=} \{x'(t_2)|x' \in f(u^1)\} =$$
$$\overset{(32)}{=} \{x"(t_1+\varepsilon)|x" \in f(u^1 \circ \tau^{d+\varepsilon})\}. \tag{34}$$

Because

$$\widetilde{u}_{\varepsilon|[t_1+\varepsilon,\infty)} = (u^1 \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)}, \tag{35}$$

(34), (35) and the non-anticipation* of f show that

$$\{\widetilde{x}_{|[t_1+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} = \{x"_{|[t_1+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\}. \tag{36}$$

But the fact that $t_3 + d + \varepsilon > t_1 + \varepsilon$ and

$$\{x"_{|[t_1+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} = \{(x' \circ \tau^{d+\varepsilon})_{|[t_1+\varepsilon,\infty)}|x' \in f(u^1)\} \tag{37}$$

indicate the truth of

$$\{\widetilde{x}_{|[t_3+d+\varepsilon,\infty)}|\widetilde{x} \in f(\widetilde{u}_\varepsilon)\} \overset{(36)}{=} \{x"_{|[t_3+d+\varepsilon,\infty)}|x" \in f(u^1 \circ \tau^{d+\varepsilon})\} = \tag{38}$$
$$\overset{(37)}{=} \{(x' \circ \tau^{d+\varepsilon})_{|[t_3+d+\varepsilon,\infty)}|x' \in f(u^1)\} = \{x'_{|[t_3,\infty)}|x' \in f(u^1)\} \overset{(26)}{=} \mu",$$

hence (30) is proved. □

Definition 7.1 *Use the notation in Theorem 7.1 and suppose that the requests stated in it are fulfilled. We have the following partial law of composition of the fundamental transfers*

$$(\mu \overset{u^0_{|(-\infty,t_1)}}{\rightarrow} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu") = \mu \overset{\widetilde{u}_{\varepsilon|(-\infty,t_3+d+\varepsilon)}}{\rightarrow} \mu",$$

$$(\mu \overset{u^0_{|[t_0,t_1)}}{\rightarrow} \mu') \vee (\mu' \overset{u^1_{|[t_2,t_3)}}{\rightarrow} \mu") = \mu \overset{\widetilde{u}_{\varepsilon|[t_0,t_3+d+\varepsilon)}}{\rightarrow} \mu".$$

8. COMPOSITION OF FUNDAMENTAL TRANSFERS. SPECIAL CASE

Theorem 8.1 *If the system f is non-anticipatory, then the statements are true: a) for any $t_1 < t_2$, $u \in U$ and $\mu, \mu', \mu" \in \mathbf{B}^n$ so that the transfers $\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu'$, $\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu"$ are fundamental, the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu"$ is fundamental; b) if $t_1 < t_2 < t_3$, $u \in U$ and $\mu, \mu', \mu" \in \mathbf{B}^n$ are arbitrary and satisfy the property that the transfers $\mu \overset{u_{|[t_1,t_2)}}{\to} \mu'$, $\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu"$ are fundamental, then the transfer $\mu \overset{u_{|[t_1,t_3)}}{\to} \mu"$ is fundamental.*

Proof. a) The hypothesis states the existence of $t_0 < t_1, v \in U$ and $v' \in U$ so that

$$u_{|(-\infty,t_1)} = v_{|(-\infty,t_1)}, \forall y \in f(v), y_{|(-\infty,t_0)} = \mu \textit{ and } y_{|[t_1,\infty)} = \mu'$$

$$u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|[t_2,\infty)} = \mu"$$

Because $v_{|(-\infty,t_0)} = v'_{|(-\infty,t_0)}$, from the non-anticipation of f we have

$$\{y_{|(-\infty,t_0)} | y \in f(v)\} = \{y'_{|(-\infty,t_0)} | y' \in f(v')\} = \mu$$

thus $u_{|(-\infty,t_2)} = v'_{|(-\infty,t_2)}, \forall y' \in f(v'), y'_{|(-\infty,t_0)} = \mu \textit{ and } y'_{|[t_2,\infty)} = \mu"$, i.e. the transfer $\mu \overset{u_{|(-\infty,t_2)}}{\to} \mu"$ is fundamental.

b) is similar to a). □

Definition 8.1 *In the conditions and with the notation from Theorem 8.1, we have the partial law of composition of the fundamental transfers:*

$$(\mu \overset{u_{|(-\infty,t_1)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_1,t_2)}}{\to} \mu") = \mu \overset{u_{|(-\infty,t_2)}}{\to} \mu",$$

$$(\mu \overset{u_{|[t_1,t_2)}}{\to} \mu') \vee (\mu' \overset{u_{|[t_2,t_3)}}{\to} \mu") = \mu \overset{u_{|[t_1,t_3)}}{\to} \mu".$$

Remark 8.1 *Theorem 8.1 restates the results in Theorem 7.1 under a simplified form. For example at Theorem 7.1 a) we have $u^0 = u^1$ and for this reason the requests of closure of U under the concatenation of the inputs and of non-anticipation* disappear. Similarly $t_1 = t_2$ and for this reason the requests of closure of U under translations and of time invariance disappear too.*

9. THE FUNDAMENTAL MODE

Theorem 9.1 *Consider the non-anticipatory system f and let $u \in U$ be a fixed input. The statements are equivalent:*

a) $(t_k) \in Seq$, $(u^k) \in U$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that*

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

$$\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2,\ \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3,\ \forall x \in f(u^3), x_{|[t_4,\infty)} = \mu^4, ...$$

b) $(t_k) \in Seq$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that the transfers* $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u_{|[t_2,t_3)}} \mu^3$,... *are fundamental;*

c) $(t_k) \in Seq$ *and* $(\mu^k) \in \mathbf{B}^n$ *exist so that the transfers* $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^0 \xrightarrow{u_{|(-\infty,t_2)}} \mu^2$, $\mu^0 \xrightarrow{u_{|(-\infty,t_3)}} \mu^3, ...$ *are initial fundamental.*

Proof. $a) \Longrightarrow b)$ Let (t_k), (u^k) and (μ^k) like at a). Because

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1 \tag{39}$$

is true, $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$ is an initial fundamental transfer. The fact that

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1, \tag{40}$$

$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2, \tag{41}$$

implies that $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$ is non-initial fundamental etc.

$b) \Longrightarrow c)$ (t_k) and (μ^k) exist so that $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u_{|[t_2,t_3)}} \mu^3, ...$ are fundamental. Like in Theorem 8.1 and Definition 8.1

$$\mu^0 \overset{u_{|(-\infty,t_2)}}{\rightarrow} \mu^2 = (\mu^0 \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu^1) \vee (\mu^1 \overset{u_{|[t_1,t_2)}}{\rightarrow} \mu^2),$$

$\mu^0 \overset{u_{|(-\infty,t_3)}}{\rightarrow} \mu^3 = (\mu \overset{u_{|(-\infty_1,t_2)}}{\rightarrow} \mu^2) \vee (\mu^2 \overset{u_{|[t_2,t_3)}}{\rightarrow} \mu^3)$... are initial fundamental.

$c) \Longrightarrow a)$ Consider the sequences (t_k) and (μ^k) like at c). Since $\mu^0 \xrightarrow{u_{|(-\infty,t_1)}} \mu^1$ is initial fundamental there exist $u^0 \in U$ so that (39) holds and because $\mu^0 \xrightarrow{u_{|(-\infty,t_2)}} \mu^2$ is initial fundamental there exist $u^1 \in U$ with (41) true etc. The statement from a) is true. $\square$

Definition 9.1 *We say that* f *is, under the input* u, in the fundamental (operating) mode *if one of the properties a), b), c) from Theorem 9.1 is satisfied.*

Theorem 9.2 *If* f *is non-anticipatory and* $t_0 < t_1, u \in U, \mu, \mu' \in \mathbf{B}^n$ *are fixed, then the fact that* $\forall x \in f(u), x_{|(-\infty,t_0)} = \mu$ *and* $x_{|[t_1,\infty)} = \mu'$ *implies that* f *is, under* u*, in the fundamental mode.*

Proof. The sequences $(t'_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ exist satisfying

$$t'_0 = t_0, t'_1 = t_1, t'_k, k \geq 2 \text{ arbitrary},$$

$$\mu^0 = \mu, \mu^1 = \mu^2 = ... = \mu'.$$

Remark that $\mu \overset{u_{|(-\infty,t'_1)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_2)}}{\longrightarrow} \mu', \mu \overset{u_{|(-\infty,t'_3)}}{\longrightarrow} \mu', ...$ are initial fundamental transfers. □

Remark 9.1 *The fundamental mode may be interpreted as a discrete time symbolic evolution of a deterministic system (i.e. f is uni-valued) of the form*

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1) \overset{u^{k+1}}{\to} ...$$

where the initial fundamental transfer $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$ is identified with the symbolic transfer $x(0) \overset{u^0}{\to} x(1)$ and a non-initial fundamental transfer of rank $k \geq 1$, $\mu^k \overset{u^k_{|[t_k,t_{k+1})}}{\longrightarrow} \mu^{k+1}$ is identified with the symbolic transfer $x(k) \overset{u^k}{\to} x(k+1)$.

In the hypothesis of Theorem 9.2, the symbolic evolution may be considered to be given by a finite sequence

$$\mu^0 = x(0) \overset{u^0}{\to} \mu^1 = x(1) \overset{u^1}{\to} ... \overset{u^k}{\to} \mu^{k+1} = x(k+1),$$

where k can be 0.

Example 9.1 *In Example 6.1 both systems f, g are in the fundamental mode under the inputs u and v.*

Example 9.2 *The deterministic system $f : S \to S$,*

$$\forall u \in S,\ f(u) = \begin{cases} 1, u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...} \\ 0, otherwise \end{cases}$$

satisfies the properties: $u = \chi_{[0,1)\cup[2,3)\cup[4,5)\cup...}$, the unbounded sequence $0 < 2 < 4 < ...$ of real numbers, the family

$$u^0 = \chi_{[0,1)},\ u^1 = \chi_{[0,1)\cup[2,3)},\ u^2 = \chi_{[0,1)\cup[2,3)\cup[4,5)}, ...$$

of inputs and the binary null sequence $0_k \in \mathbf{B}, k \in \mathbf{N}$ exist so that

$$f(u^0)_{|(-\infty,0)} = 0 \text{ and } f(u^0)_{|[2,\infty)} = 0,$$

$$u_{|(-\infty,2)} = u^0_{|(-\infty,2)},\ u_{|(-\infty,4)} = u^1_{|(-\infty,4)}, ...$$

$$f(u^1)_{|[4,\infty)} = 0,\ f(u^2)_{|[6,\infty)} = 0, ...$$

The statements $f(u)_{|(-\infty,2]} = f(u^0)_{|(-\infty,2]},\ f(u)_{|(-\infty,4]} = f(u^1)_{|(-\infty,4]}, \dots$. *are false, since* f *is anticipatory.* f *is not in the fundamental mode under* u.

Theorem 9.3 *Let the non-anticipatory system* f *be in the fundamental mode under* u. *Then the families* $(t_k) \in Seq$ *and* $(u^k) \in U$ *exist so that*

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$$

and for all $k \in \mathbf{N}, f$ *is in the fundamental mode under* u^k.

Proof. From Theorem 9.1 item c), $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ exist so that the transfers $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, \dots$ are initial fundamental, i.e. there exists the sequence $(u^k) \in U$ with

$$u_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0 \textit{ and } x_{|[t_1,\infty)} = \mu^1$$
$$u_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \forall x \in f(u^1), x_{|(-\infty,t_0)} = \mu^0 \textit{ and } x_{|[t_2,\infty)} = \mu^2$$
$$u_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)},\ \forall x \in f(u^2), x_{|(-\infty,t_0)} = \mu^0 \textit{ and } x_{|[t_3,\infty)} = \mu^3 \dots$$

thus $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^0 \overset{u^1_{|(-\infty,t_2)}}{\longrightarrow} \mu^2$, $\mu^0 \overset{u^2_{|(-\infty,t_3)}}{\longrightarrow} \mu^3, \dots$ are initial fundamental and, by Theorem 9.2, f is in the fundamental mode under all u^k, $k \in \mathbf{N}$. □

Theorem 9.4 *Let* f *be non-anticipatory and suppose that* U *has the closure property: for any* $u \in S^{(m)}$ *and any sequences* $(t_k) \in Seq$, $(u^k) \in U$, *from*

$$\forall k \in \mathbf{N}, u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})} \tag{42}$$

we infer $u \in U$. *Then the next statement is true: for any* $(u^k) \in U$ *so that* f *is in the fundamental mode under all* u^k, *a sequence* $(t_k) \in Seq$ *exists so that*

$$\forall k \in \mathbf{N}, u^k_{|(-\infty,t_{k+1})} = u^{k+1}_{|(-\infty,t_{k+1})} \tag{43}$$

implies that f *is in the fundamental mode under the unique* u *satisfying (42).*

Proof. Let $(u^k) \in U$ be a sequence of inputs such that f is in the fundamental mode under all u^k and take an arbitrary $\delta > 0$. There exist $t_0, t_1 \in \mathbf{R}$ and $\mu^0, \mu^1 \in \mathbf{B}^n$ so that $t_0 + \delta < t_1$ and $\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0, \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1$ of $t_2 \in \mathbf{R}$ and $\mu^2 \in \mathbf{B}^n$ so that $t_1 + \delta < t_2$ and $\forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2$ of $t_3 \in \mathbf{R}$ and $\mu^3 \in \mathbf{B}^n$ so that $t_2 + \delta < t_3$ and $\forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3 \dots$. Obviously, by Theorem 6.1, $\mu^0 \overset{u^0_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u^1_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u^2_{|[t_2,t_3)}}{\longrightarrow} \mu^3, \dots$ are fundamental transfers and $(t_k) \in Seq$. Due to (43) the relation (42) may be written for some $u \in S^{(m)}$; it defines a unique function $u \in S^{(m)}$. In addition, we have $u \in U$. By Theorem 6.2, the transfers $\mu^0 \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu^1$, $\mu^1 \overset{u_{|[t_1,t_2)}}{\longrightarrow} \mu^2$, $\mu^2 \overset{u_{|[t_2,t_3)}}{\longrightarrow} \mu^3, \dots$ are fundamental, thus f is in the fundamental mode under u. □

10. A PROPERTY OF EXISTENCE

Theorem 10.1 *Let f be the non-anticipatory system. Suppose that: a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$; b) f satisfies the following property of race-free initialization with bounded initial time, namely $\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$; c) f is absolutely race-free stable with bounded final time, i.e.$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$. Then for any sequence $(u^k) \in U$ of inputs, there exist the time instants $(t_k) \in Seq$ so that f is in the fundamental mode under the input*

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

Proof. Consider some real number $\delta > 0$ and the arbitrary sequence $(u^k) \in U$ of inputs. From b) we infer the existence of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ so that

$$\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0$$

and from c) we have the existence of $\mu^1 \in \mathbf{B}^n$ and $t_1 \in \mathbf{R}$ with $t_1 > t_0 + \delta$ and

$$\forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1.$$

Furthermore, from a) we have that $u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)} \in U$ and from c) the existence of $\mu^2 \in \mathbf{B}^n$ and $t_2 \in \mathbf{R}$ so that $t_2 > t_1 + \delta$ and

$$\forall x \in f(u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2$$

is inferred. The construction of (t_k) and the fact that $(t_k) \in Seq$ are obvious. On the other hand, by a), the obtained $\widetilde{u}$ belongs to U. The statement that f is in the fundamental mode under the input $\widetilde{u}$ is inferred from the equalities

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)}, \widetilde{u}_{|(-\infty,t_2)} = (u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,\infty)})_{|(-\infty,t_2)}, ...$$

□

Theorem 10.2 *If the non-anticipatory system f satisfies the properties:*

a) race-free initialization with bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu;$$

b) absolute race-free stability with bounded final time

$$\forall u \in U, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu',$$

then $\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists \mu' \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$ and $x_{|[t_1,\infty)} = \mu'$, i.e. for any u, some μ, μ' and $t_0 < t_1$ exist so that $\mu \overset{u_{|(-\infty,t_1)}}{\rightarrow} \mu'$ is initial fundamental.

Proof. From the first part of the proof of Theorem 10.1, where $u^0 = u$. □

Theorem 10.3 *Suppose that the non-anticipatory system f is absolutely race-free stable with bounded final time, i.e.*

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu.$$

Then $\forall u \in U$, the vectors $\mu, \mu' \in \mathbf{B}^n$ and the numbers $t_0 < t_1$ exist so that the transfer $\mu \overset{u_{|[t_0,t_1)}}{\longrightarrow} \mu'$ is non-initial fundamental.

Proof. It is sufficient to consider the property: for any $u \in U$, μ and t_0 exist so that $\forall x \in f(u), x_{|[t_0,\infty)} = \mu$; then $\mu' = \mu$ and $t_1 > t_0$ arbitrary imply the conclusion of the theorem. □

11. FUNDAMENTAL MODE, SPECIAL CASE

Definition 11.1 *For any $t_1 \in \mathbf{R}$, the* prefix *of $u \in S^{(m)}$ is the function $u_{t_1} \in S^{(m)}$ given by*

$$u_{t_1}(t) = \begin{cases} u(t), t < t_1, \\ u(t_1 - 0), t \geq t_1. \end{cases}$$

Theorem 11.1 *Let f be the non-anticipatory system and consider the input $u \in U$. For any $(t_k) \in Seq$ and $(\mu^k) \in \mathbf{B}^n$ so that $u_{t_1}, u_{t_2}, u_{t_3}, ... \in U$ and*

$$\forall x \in f(u_{t_1}), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1,$$

$$\forall x \in f(u_{t_2}), x_{|[t_2,\infty)} = \mu^2,\ \forall x \in f(u_{t_3}), x_{|[t_3,\infty)} = \mu^3,\ \forall x \in f(u_{t_4}), x_{|[t_4,\infty)} = \mu^4, ...$$

f is, under the input u, in the fundamental mode.

Proof. Define the sequence $(u^k) \in U$ by $u^k = u_{t_{k+1}}, k \in \mathbf{N}$. Since for any $k \geq 0$ we have $u_{|(-\infty,t_{k+1})} = u^k_{|(-\infty,t_{k+1})}$, the statement of Theorem 9.1 a) holds. □

Corollary 11.1 *Suppose that the non-anticipatory system f and the input $u \in U$ are given. If the sequences $(t_k) \in Seq$, $(\mu^k) \in \mathbf{B}^n$ and $(\lambda^k) \in \mathbf{B}^m$ satisfy*

$$u(t) = \lambda^0 \cdot \chi_{(-\infty,t_1)}(t) \oplus \lambda^1 \cdot \chi_{[t_1,t_2)}(t) \oplus \lambda^2 \cdot \chi_{[t_2,t_3)}(t) \oplus ...$$

$$\lambda^0, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ... \in U \text{ and}$$

$$\forall x \in f(\lambda^0), x_{|(-\infty,t_0)} = \mu^0 \text{ and } x_{|[t_1,\infty)} = \mu^1$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}), x_{|[t_2,\infty)} = \mu^2$$

$$\forall x \in f(\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}), x_{|[t_3,\infty)} = \mu^3$$

$$...$$

then f is, under the input u, in the fundamental mode.

Proof. This is a special case of Theorem 11.1 when $u_{t_1} = \lambda^0$, $u_{t_2} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}$, $u_{t_3} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ...$ □

Remark 11.1 *Theorem 11.1 gives a new perspective on the fundamental mode, when $\forall k \geq 1$ the stabilization of x to the value $x(t_k)$ is a direct consequence of the fact that u has stabilized before t_k to the value $u(t_k - 0)$. Thus, at the time instants $t_1, t_2, t_3, ...$ u and all $x \in f(u)$ are in equilibrium,*

$$\forall k \geq 1, \forall t \geq t_k, u_{t_k}(t) = u(t_k - 0) \text{ and } \forall x \in f(u_{t_k}), x(t) = x(t_k)$$

and we consider the equilibrium be true at the time instant t_0 also under the form

$$\forall t < t_0, u(t) = u(t_0 - 0) \text{ and } \forall x \in f(u_{t_1}), x(t) = x(t_0 - 0)$$

by a suitable choice of t_0.

The situation described in Theorem 11.1 includes the possibilities $\exists k \geq 1, u_{t_k} = u_{t_{k+1}}$ and respectively $\exists k \geq 1, u = u_{t_k}$.

Corollary 11.1 represents that special case of Theorem 11.1, when u is constant in the intervals $(-\infty, t_1)$, $[t_1, t_2)$, $[t_2, t_3), ...$

The next theorem is an adaptation of Theorem 10.1 for the present context.

Theorem 11.2 *The non-anticipatory system f is given and let $H \subset \mathbf{B}^m$ be a non-empty set. If: a) $U = \{\lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...|(\lambda^k) \in H, (t_k) \in Seq\}$; b) f has race-free initial states with bounded initial time $\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$; c) f is relatively race-free stable with bounded final time $\forall u \in U \cap S_c^{(m)}, \exists \mu' \in \mathbf{B}^n, \exists t_1 \in \mathbf{R}, \forall x \in f(u), x_{|[t_1,\infty)} = \mu'$, then for any $(\lambda^k) \in H$, there exist the time instants $(t_k) \in Seq$ so that f is in the fundamental mode under the input $u = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...$*

Proof. We just remark that the closure property from Theorem 10.1 a) is fulfilled and that $\lambda^0, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,\infty)}, \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,\infty)}, ... \in U \cap S_c^{(m)}$ for any $(\lambda^k) \in H$ and any $(t_k) \in Seq$. The proof is similar with that of Theorem 10.1. □

12. ACCESSIBILITY VS. FUNDAMENTAL MODE

Theorem 12.1 *Let* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *be* the non-anticipatory system *and suppose that the next requests are fulfilled: a) for any* $(t_k) \in Seq$ *and any* $(u^k) \in U$ *we have* $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$*; b)* f *has race-free initial states and bounded initial time, i.e.* $\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$*; c) any vector from* $\mathbf{B}^n$ *is final state under an input having arbitrary initial segment* $\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t$, $u_{|(-\infty,t)} = v_{|(-\infty,t)}$ *and* $\forall y \in f(v), y_{|[t',\infty)} = \mu$*. Then some* $\mu^0 \in \mathbf{B}^n$ *exists so that for any sequence* $\mu^k \in \mathbf{B}^n, k \geq 1$ *of binary vectors, a sequence* $(t_k) \in Seq$ *and an input* $\widetilde{u} \in U$ *exist having the property that* $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_2,t_3)}} \mu^3, ...$ *are fundamental transfers.*

Proof. Let $v^0 \in U$ be an arbitrary input. From b) there exist of $\mu^0 \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ depending on v^0 so that $\forall x \in f(v^0), x_{|(-\infty,t_0)} = \mu^0$. Fix the sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ and an arbitrary number $\delta > 0$. The property c) implies the existence of $u^0 \in U$ and $t_1 > t_0 + \delta$ so that

$$v^0_{|(-\infty,t_0)} = u^0_{|(-\infty,t_0)} \text{ and } \forall x \in f(u^0), x_{|[t_1,\infty)} = \mu^1;$$

of $u^1 \in U$ and $t_2 > t_1 + \delta$ so that

$$u^0_{|(-\infty,t_1)} = u^1_{|(-\infty,t_1)} \text{ and } \forall x \in f(u^1), x_{|[t_2,\infty)} = \mu^2;$$

of $u^2 \in U$ and $t_3 > t_2 + \delta$ so that

$$u^1_{|(-\infty,t_2)} = u^2_{|(-\infty,t_2)} \text{ and } \forall x \in f(u^2), x_{|[t_3,\infty)} = \mu^3...$$

Obviously the transfers $\mu^0 \xrightarrow{u^0_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u^1_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u^2_{|[t_2,t_3)}} \mu^3, ...$ are fundamental.

The construction of (t_k) guarantees the fact that this sequence belongs to Seq, thus the input $\widetilde{u}$ defined as $\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus u^2 \cdot \chi_{[t_2,t_3)} \oplus ...$ belongs to U, from a). We have

$$\widetilde{u}_{|(-\infty,t_1)} = u^0_{|(-\infty,t_1)},\ \widetilde{u}_{|(-\infty,t_2)} = u^1_{|(-\infty,t_2)},\ \widetilde{u}_{|(-\infty,t_3)} = u^2_{|(-\infty,t_3)}, ...$$

whence, the transfers $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_2,t_3)}} \mu^3, ...$ equal to $\mu^0 \xrightarrow{u^0_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{u^1_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{u^2_{|[t_2,t_3)}} \mu^3, ...$ by Theorem (6.2) are fundamental. □

Theorem 12.2 *Let the non-anticipatory system $f : U \to P^*(S^{(n)})$ be given and suppose that the conditions:*

a) for any $(t_k) \in Seq$ and any sequence $(u^k) \in U$ of inputs we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$

b) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu$$

c) the vectors from $\mathbf{B}^n$ are accessible final states in the next manner

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$

$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu$$

(we have identified $\lambda \in \mathbf{B}^m$ to the constant input $\lambda \in U$). Then there exists $\mu^0 \in \mathbf{B}^n$ so that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$ of binary vectors, the time instants $(t_k) \in Seq$ and the constants $(\lambda^k) \in \mathbf{B}^m$ exist such that $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_1)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_1,t_2)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_2,t_3)}} \mu^3, ...$ are fundamental transfers, where we denoted

$$\widetilde{u} = \lambda^0 \cdot \chi_{(-\infty,t_1)} \oplus \lambda^1 \cdot \chi_{[t_1,t_2)} \oplus \lambda^2 \cdot \chi_{[t_2,t_3)} \oplus ...$$

Proof. Special case of Theorem 12.1. □

Theorem 12.3 *Suppose that the non-anticipatory system f is given so that:*

a) for any $(t_k) \in Seq$ and any $(u^k) \in U$ we have $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$;

b) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) f has accessible final states in bounded time of the form

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \text{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu.$$

Then there exist $\delta > 0$ and $\mu^0 \in \mathbf{B}^n$ so that for any sequence $\mu^k \in \mathbf{B}^n, k \geq 1$, there exist of $t_0 \in \mathbf{R}$ and $\widetilde{u} \in U$ such that $\mu^0 \xrightarrow{\widetilde{u}_{|(-\infty,t_0+\delta)}} \mu^1$, $\mu^1 \xrightarrow{\widetilde{u}_{|[t_0+\delta,t_0+2\delta)}} \mu^2$, $\mu^2 \xrightarrow{\widetilde{u}_{|[t_0+2\delta,t_0+3\delta)}} \mu^3, ...$ are fundamental transfers.

Proof. Similar with Theorem 12.1. □

Theorem 12.4 *If the non-anticipatory system f satisfies the requests:*

a) f has race-free initial states and bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

b) the vectors from $\mathbf{B}^n$ are accessible final states

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu;$$

then $\forall \mu' \in \mathbf{B}^n, \exists \mu \in \mathbf{B}^n, \exists u \in U, \exists t_0 \in \mathbf{R}, \exists t_1 > t_0, \forall x \in f(u), x_{|(-\infty,t_0)} = \mu$ *and* $x_{|[t_1,\infty)} = \mu'$, *i.e. for any* μ', *there exist* μ, u, t_0 *and* $t_1 > t_0$ *so that* $\mu \overset{u_{|(-\infty,t_1)}}{\longrightarrow} \mu'$ *is initial fundamental.*

Proof. Let $\mu' \in \mathbf{B}^n$ arbitrary, fixed. b) shows the existence of $u \in U$ and $t_1 \in \mathbf{R}$ so that $\forall x \in f(u), x_{|[t_1,\infty)} = \mu'$. Because of a) we infer the existence of $\mu \in \mathbf{B}^n$ and $t_0 \in \mathbf{R}$ that can be chosen $< t_1$ with $\forall x \in f(u), x_{|(-\infty,t_0)} = \mu$. □

Remark 12.1 *In Theorems 12.1-12.4 the next accessibility properties occurred:*

a) any vector from $\mathbf{B}^n$ is final state under an input having arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' > t;$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \textit{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu;$$

b) version of a) where the access in a final state is made under a constant input

$$\forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists \lambda \in U, \exists t' > t,$$

$$\forall y \in f(u \cdot \chi_{(-\infty,t)} \oplus \lambda \cdot \chi_{[t,\infty)}), y_{|[t',\infty)} = \mu;$$

c) version of a) where the access in a final state is made in the next way

$$\exists \delta > 0, \forall \mu \in \mathbf{B}^n, \forall u \in U, \forall t \in \mathbf{R}, \exists v \in U, \exists t' \in (t, t+\delta),$$

$$u_{|(-\infty,t)} = v_{|(-\infty,t)} \textit{ and } \forall y \in f(v), y_{|[t',\infty)} = \mu;$$

d) version of a) where the inputs under which the vectors from $\mathbf{B}^n$ are final states do not have an arbitrary initial segment

$$\forall \mu \in \mathbf{B}^n, \exists u \in U, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \mu.$$

We have the implications

$$\begin{array}{ccccc} b) & \Longrightarrow & a) & \Longrightarrow & d) \\ & & \Uparrow & & \\ & & c) & & \end{array}$$

13. FUNDAMENTAL MODE RELATIVE TO A FUNCTION

Definition 13.1 *Let the system* $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ *and the Boolean function* $F : \mathbf{B}^m \to \mathbf{B}^n$*. If* $\forall t \in \mathbf{R}$, $\forall u \in U$, $\forall v \in U$, *we have*

$$\forall \xi < t, F(u(\xi)) = F(v(\xi)) \Longrightarrow \{x_{|(-\infty,t]} | x \in f(u)\} = \{y_{|(-\infty,t]} | y \in f(v)\}$$

we say that f *is* non-anticipatory relative to the function F.

Definition 13.2 *Suppose that the system* f *is non-anticipatory relative to the function* F *and there exist* $(t_k) \in Seq$, $u, (u^k) \in U$ *and* $\mu^0 \in \mathbf{B}^n$ *such that* $\forall x \in f(u^0), x_{|(-\infty,t_0)} = \mu^0$ *and* $x_{|[t_1,\infty)} = F(u(t_1 - 0))$,

$$\forall k \in \mathbf{N}, \forall \xi \in \mathbf{R}, F(u^k(\xi)) = \begin{cases} F(u(\xi)), \xi < t_{k+1}, \\ F(u(t_{k+1} - 0)), \xi \geq t_{k+1} \end{cases}$$

$$\forall k \geq 1, \forall x \in f(u^k), x_{|[t_{k+1},\infty)} = F(u(t_{k+1} - 0)).$$

Then we say that f *is, under the input* u*,* in the fundamental (operating) mode relative to F.

Remark 13.1 *For* $u \in U$ *and* $t \in \mathbf{R}$, *the functions* $v \in U$ *such that*

$$\forall \xi \in \mathbf{R}, F(v(\xi)) = \begin{cases} F(u(\xi)), \xi < t, \\ F(u(t - 0)), \xi \geq t, \end{cases}$$

act here as prefixes of u*. In other words,* v *is the prefix of* u *relative to* F*. Definition 13.2 pursuits the idea from Theorem 11.1, where* $\mu^k = F(u(t_k - 0)), k \geq 1$*. Note that* $u^k = u_{t_{k+1}}, k \in \mathbf{N}$ *from that theorem are prefixes of* u *relative to* F *too.*

We state now the version of Theorem 10.1 that is valid in this context.

Theorem 13.1 *Let the function* F *and the system* f *that is non-anticipatory relative to* F*. Suppose that the following properties are fulfilled:*

a) for any $(t_k) \in Seq$ *and any sequence* $(u^k) \in U$ *of inputs we have* $u^0 \cdot \chi_{(-\infty,t_0)} \oplus u^1 \cdot \chi_{[t_0,t_1)} \oplus u^2 \cdot \chi_{[t_1,t_2)} \oplus ... \in U$*;*

b) race-free initialization with bounded initial time

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|(-\infty,t)} = \mu;$$

c) $F-$*relative race-free stability with bounded final time*

$$\forall u \in U \cap S^{(m)}_{F,c}, \exists t \in \mathbf{R}, \forall x \in f(u), x_{|[t,\infty)} = \lim_{\xi \to \infty} F(u(\xi)).$$

Then for any sequence $(u^k) \in U \cap S^{(m)}_{F,c}$ *of inputs, the time instants* $(t_k) \in Seq$ *exist so that* f *is in the fundamental mode relative to* F *under the input*

$$\widetilde{u} = u^0 \cdot \chi_{(-\infty,t_1)} \oplus u^1 \cdot \chi_{[t_1,t_2)} \oplus ... \oplus u^k \cdot \chi_{[t_k,t_{k+1})} \oplus ...$$

References

[1] Serban, V. E., *Real time models of the asynchronous circuits: The delay theory in new developments in computer science research*, Susan Shannon (ed.), Nova Science Publishers, Inc., 2005.

[2] Serban, V. E., *Topics in asynchronous systems*, Analele Universitatii din Oradea, Fascicola Matematica, **10** (2003).

[3] Serban, V. E., *Asynchronous pseudo-systems*, Analele Universitatii din Oradea, Fascicola Matematica, **11** (2004).

[4] Lavagno, L., *Synthesis and testing of bounded wire delay asynchronous circuits from signal transition graphs*, Ph.D. Thesis, Electrical Engineering and Computer Sciences, Univ. of California at Berkeley, 1992.

ROMAI J., **2**, 2(2006),
227–255

IMAGINARY UNITS: USE IN ANALYSIS AND APPLICATIONS

Muhamadi Zakhirov
National University of Uzbekistan, Uzbekistan
hurschidbek@yahoo.de

Abstract In his excellent report in 1854, B. Riemann spoke about "three independent units". In this paper it is shown that the system of Beltrami equations possesses solutions with three imaginary units. This situation differs from the system of Cauchy-Riemann equations where one imaginary unit i occurs. Lorenz and Poincaré transformations are built on the basis of these units. Applications to baryon and quark's color charge conservation laws when a particle possesses three, four and even five color states are considered. Computations carried out on the basis of the model reveal that the multiplet of four identical quarks is impossible while of five quarks is completely possible.

1. INTRODUCTION

From the point of view of differential geometry the concept of metric structure is considered as of "higher order" than, for example, the concept of manifold (variety). Selecting a definite tensor field on a special position, we go out the framework of *pure differential manifold*. It is easy to remark a very rich geometrical structure in the similar approach possessed by an ordinary manifold. Basic concepts such as Lie derivatives, differential forms do never concern the metric. On the other hand, as a rule, the group is defined, firstly, by means of exact realizations or representations, as it enables one more concretely to study all properties of the group. However, it is expedient to consider it as an abstract group, or it may exist other useful representations or realizations which are unknown so far. We act just in the same manner. Consider, first, a system of elliptic Beltrami equations, leading to the canonical form of positively defined quadratic form (in the metric of Euclid). First-order differential forms of Beltrami system of equations, integral curves which are mapped onto the parallel lines family in the field of Δ affine plane are selected. If we introduce in the suitable manner the elliptic (ordinary) imaginary unit, in the domain Δ we receive the geometry of complex plane. By introducing hyperbolic imaginary unit, the domain Δ is transformed to the hyperbolic plane of Lobachevsky, leading to the proof of the Riemann's theorem on mappings for hyperbolic systems. Introducing the hyperbolic unit enables one to use the procedure of new doubling of real numbers, quaternions and Cayley octaves.

It is essential to occur the Lorenz and Poincaré transformations. From the point of the view of the new representation, the Dirac's equations and application to the conservation laws of baryon charges and quarks' color charges are considered.

2. NECESSARY DATA

Here necessary data from geometry, algebra and other fields of mathematics are given.

1. Let M be a two-dimensional space (manifold) and let D be a domain of M defined by two coordinates. The vector field v^i defines on $D \subset M$ a one-parameter family of lines which are the family of integral curves of differential equations $v^i du^i = 0,\ i = 1, 2$. The two vectors $\mathbf{v},\ \mathbf{w}$ with components $v^i, w^i$are said to be independent if their skew product $v^1 w^2 - v^2 w^1$ is different from zero $[1, p.15]$. The three independent vector fields are said hexangular (or diagonal) if it is possible to map their lines onto three parallel lines on the affine plane $[1, p.119]$. In order for the three vector fields to be hexangular, it is necessary and sufficient that they have a common integration multiplier $[1, p.120]$.

2. Consider the curve which passes through the point P of the manifold M and is described by the equation $x^i = x^i(\varphi)$. Let $f\left(x^1, x^2\right)$ be a differentiable function on D. At each point of the curve the value of the function is defined. Thus, on the curve, a differentiable function g of the parameter φ, $[[2, p.47]\,, g(\varphi) = f\left(x^1(\varphi), x^2(\varphi)\right),$ occurs.

By the chain rule of differentiation, we obtain $\frac{dg}{d\varphi} = \sum \frac{dx^i}{d\varphi}\frac{\partial g}{\partial x^i}$. This equality is valid for any function g, therefore we can write $\frac{d}{d\varphi} = \sum \frac{dx^i}{d\varphi}\frac{\partial}{\partial x^i}$,where $\frac{dx^i}{d\varphi}$ are the components of the vector $\frac{d}{d\varphi}$. Let a and b be two arbitrary numbers and $x^i = x^i(\psi)$ another curve passing through the point P. Then at this point we have $\frac{d}{d\psi} = \sum \frac{dx^i}{d\psi}\frac{\partial}{\partial x^i}$ and $a\frac{d}{d\varphi} + b\frac{d}{d\psi} = \sum \left(a\frac{dx^i}{d\varphi} + b\frac{dx^i}{d\psi}\right)\frac{\partial}{\partial x^i}$. So, the numbers $a\frac{dx^i}{d\varphi} + b\frac{dx^i}{d\psi}$ can be considered as the components of a new vector which is certainly tangent to a certain curve passing through the point P. Consequently, a curve depending on a parameter, say λ, must exist, such that at the point P we have $\frac{d}{d\lambda} = \sum \left(a\frac{dx^i}{d\varphi} + b\frac{dx^i}{d\psi}\right)\frac{\partial}{\partial x^i}$.

Operators of differentiation along the curves (like $\frac{d}{d\lambda}$) create a vector space. The space of all tangent vectors at the point P and the space of all differentiations along curves passing through the point P are related by a one-to-one correspondence $[2, p.49]$.

3. Let us give two vector fields $\frac{d}{d\varphi},\ \frac{d}{d\psi}$ on the two-dimensional manifold M. Suppose that $\frac{d}{d\varphi}, \frac{d}{d\psi}$ are linearly independent at each points of the domain D, therefore they form a basis of vector fields. It is known that in order for this basis to consist of coordinates it is necessary and sufficient that the fields commuted, i.e. $\left[\frac{d}{d\varphi},\frac{d}{d\psi}\right]=\frac{d}{d\varphi}\frac{d}{d\psi}-\frac{d}{d\psi}\frac{d}{d\varphi}=0\,[2,p.65]$.

4. Basic lemma. *If the linear operator $A{:}W\rightarrow W$ acting on the real or complex space W is unitary, i.e. $A^2=E$, then its eigenvalue are equal to ±1 and it is diagonalizable, i.e. the space W is the direct sum $W=W_+\oplus W_-$. of the eigenspace W_+. corresponding to the eigenvalue +1 and eigenspace W_- corresponding to the eigenvalue -1* $[3(a),p.262]$.

5. E. Vigner's theorem. *The symmetry operations of quantum-physical systems are realized by unitary and anti-unitary operators.*

3. PROBLEM STATEMENT

The problems discussed in the article have been famous in mathematics and physics literatures for a long time. They are the following.

1) Important and difficult problems of gasdynamics include the problem of finding where zones of subsonic and supersonic motion exist. Suppose that there exists a tube symmetric with respect to the axis $0X$, (we consider the planar case). Until a certain moment it narrows and then extends. If it has a flow with high enough subsonic speed V_0 to $-\infty$ (amonte), then according to the basic property of the subsonic flow, the tube narrowing leads to the speed increase: it reaches the sound speed, and after that, the tube extends (according to the property of supersonic flow) and determines a speed increase too. Nozzles are arranged in the same manner to receive supersonic flows. The problem is to compute this flow and partially, to find the line junction through sound speed. So far, the complete solution of the nozzle problem including the proof of existence and yielding the conditions providing uniqueness has not been successful neither by classical theories, nor by simplified models $[4]\,,[5,p.146]$.

2) In Ch. 1 the study concerning the dynamic behaviors gives us the possibility to write the canonical form to which the following equation is reduced $a\,(x,t)\,dt^2-2b\,(x,t)\,dxdt+k\,(x,t)\,dx^2=0$, $\Delta\,(x,t)=b^2-ak\geq0$ with concrete coefficients $a,\ b,\ k$ if this reduction is principally possible. Actually, the proof of the possibility of this reduction, that is the existence proof of the variables $\xi=\xi\,(x,y)\,,\ \tau=\tau\,(x,y)$ with respect to which the abovestated equation assumes canonical form does not exist $[6,p.195,196]$.

3) Now, in function theory, we compare conformal (or quasi-conformal) mappings. In this frame it is possible to state a certain dominance of the ellipticity factor. The author thinks that his substantial theory of functions will be developed sometime in the "parabolic" and "hyperbolic" cases. Maybe a certain *absolute* theory of functions will be developed. Today all these are more related to the theory of partial differential equations and are a bit far from function theory for eigenvalues. These problems were laid down and from time to time they were repeated by the Russian academician M. A. Lavrentev and R. Nevanlinna who, at the end of his life, used to say that he is busy with seeking analogues of analytic function theory for hyperbolic and parabolic cases [7, p. 268 - 270].

4) The electric charge conservation law is used to consider as a result of calibrating invariance, i.e. invariance relative to the group of electromagnetic interactions. On the other hand, relative to the conservation laws, which were to be written down in the dynamic group of specific relativity theory, we can build only pure speculative conclusions. However, there is a base to hope that the conservation laws of baryon and lepton charges can be obtained by means of dynamic group of strong or weak interactions. If the given hypothesis is true, then it only means that we do not know the real group of strong or weak interactions. Two reasons seem to lead to the last statement. First, so far the conservation laws of baryon and lepton charges were not possible to separate from symmetry properties of strong and weak interactions and it is improbable that this will be successful in the future. Second, the symmetry of strong and weak interactions is not exact and breaks down if other interactions are involved. It is not clear how exact conservation laws can be deduced from approximate symmetries. In the meanwhile, all available data seem to suggest that the conservation laws of baryon and lepton charges are conserved exactly [21, p. 31]. In conclusion, I want to say that the separation of the conservation laws of baryon charges from symmetry properties is impossible, as follows from the interesting article by Sarukai [26], [21, p. 31].

4. CURVATURE OF SURFACES AND BELTRAMI MIXED EQUATIONS SYSTEM

1.The functions further considered are definite and continuous in D. Sometimes the existence of continuous higher order partial derivatives of the given function is needed. The system of first order partial differential equations

$$\begin{cases} \partial_x u = \frac{-g_{12}\partial_x v + g_{11}\partial_y v}{\sqrt{g_{11}g_{22}-g_{12}^2}} \equiv \beta\partial_x v + \gamma\partial_y v, \\ -\partial_y u = \frac{g_{22}\partial_x v - g_{12}\partial_y v}{\sqrt{g_{11}g_{22}-g_{12}^2}} \equiv \alpha\partial_x v + \beta\partial_y v, \end{cases} \tag{1}$$

with the ellipticity condition $\alpha\gamma - \beta^2 = 1$, where always $\gamma > 0$, is said the Beltrami equations system. It can be considered as the condition of the fact that the mapping $(x, y) \to (u, v)$ is conformal with respect to the Riemann metric

$$ds^2 = g_{11}(x, y)\, dx^2 + 2g_{12}(x, y)\, dxdy + g_{22}dy^2, \tag{2}$$

i.e. by this mapping any angle α on the plane (x, y) measured by the metrics (2) is transformed to the angle α on the plane (u, v) measured in the standard manner. Two Riemann metrics are said to be conformally equivalent if their coefficients are proportional, up to the factor (multiplier) of proportionality which can be a function of (x, y). Two similar metrics generate one and the same Beltrami system [8, p. 61, 9, p. 150 - 151]. The characteristics of an arbitrary ellipsis are the ratio p≥1 of its semiaxes and the angle θ $(0 \leq \theta \leq \pi)$ formed by its big axis and the axis *OX*. The equations of the ellipsis centered at the origin and with the small semiaxis h and characteristics p and θ read $\gamma dx^2 - 2\beta dxdy + \alpha dy^2 = ph^2$, where

$$\alpha = p\cos^2\theta + \frac{1}{p}\sin^2\theta,\ \beta = \left(p - \frac{1}{p}\right)\cos\theta\sin\theta,\ \gamma = p\sin^2\theta + \frac{1}{p}\cos^2\theta \quad (3)$$

Formulas (3), in due course, were studied and used in developing his theory of quasi-conformal mappings by M. A. Lavrentev [9, p. 3 - 4]. From the geometric point of view and from the point of view of relativity theory it is necessary to decide whether the studied objects (quadratic forms in our case) contain the curvature tensor component or not [10, p. 16]. This problem is solved in the following way.

It is known [11, p. 447] that at each point of a smooth surface there exist two perpendicular tangent lines $l_1,\ l_2$ in the direction of which a normal curvature of the surface reaches its maximal and minimal values $k_1,\ k_2$. Let $l_3,\ l_4$ be two tangent lines. If the co-perpendicular direction tangents at the same point of the surface form the angle θ with the directions $l_1,\ l_2$, then the normal curvatures in the direction $l_3,\ l_4$ are computed by means of Euler's formula

$$g_{11} = k_1\cos^2\theta + k_2\sin^2\theta,\ g_{22} = k_1\sin^2\theta + k_2\cos^2\theta. \quad (4)$$

Moreover, let $g_{12} = (k_1 - k_2)\cos\theta\sin\theta$. Then $g_{11}g_{22} - g_{12}^2 = k_1k_2 = K$, where K is the theorema ergerium (Gaussian curvature) of the surface at the same point. From the last equalities $k_1,\ k_2,\ \theta$ are deduced

$$\begin{aligned} &tg\theta = \tfrac{2g_{12}}{g_{11}-g_{22}} \\ &k_1 = \tfrac{g_{11}+g_{22}}{2} + \tfrac{\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}}{2},\ k_2 = \tfrac{g_{11}+g_{22}}{2} - \tfrac{\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}}{2} \end{aligned} \quad (5)$$

Basic result. If the elements g_{ij} of the Riemann metrics are given, then the main directions and main curvatures of the surface are defined by them in a unique way (5) and, conversely, if k_1, k_2 and θ are given, then (4) provides uniquely g_{ij}. Further we show that the curvilinear system of coordinates is connected with the quadratic form (3). The Christoffel symbols are defined. It is proved that the corresponding components of the Riemann curvature tensor are equal to zero. It is a famous result that the first and second quadratic forms of the surface differ by the multiplier k, where k is a normal curvature of the surface. *When the surface is of hyperbolical type, by the coefficients* α, β, γ *occur imaginary multipliers. It means that the system (1) changes its type from elliptic to hyperbolic.*

5. STEREOGRAPHIC PROJECTION AND BELTRAMI EQUATIONS SYSTEMS

2. Consider the stereographic mapping of the surface S, where by ς is denoted a point of the plane C. First consider the case when S is a sphere [12, p. 30 - 32]. In order not to use endless coordinates $\varsigma = \infty$ for the point (0,0,0,1) on the north pole of the sphere at the stereographic mapping it is comfortable to mark the points on S not with one complex number ς but with the pairs (ξ, η) of complex numbers (not equal to zero simultaneously) which meet the condition $\varsigma = \xi/\eta$. They represent projective (homogeneous) complex coordinates, therefore at an arbitrary, different from zero complex number λ the pairs (ξ, η) and $(\lambda\xi, \lambda\eta)$ describe one and the same point on S. Additional points $\varsigma = \infty$ are given with a final mark in these coordinates, for example (1,0). Thus, S is considered as the realization of a certain complex projective line. The expressions of the correspondence between the points on S and points of the plane written in these complex coordinates read

$$x = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\xi\bar{\xi} + \eta\bar{\eta}},\ y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{i\left(\xi\bar{\xi} + \eta\bar{\eta}\right)},\ z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}}.$$

Note that $x,\ y,\ z$ are zero order homogeneous functions relative to $\xi,\ \eta$ and therefore invariant to the scale changes of $\xi,\ \eta$. The point $P\left(1, x, y, z\right)$ on S represents a certain isotropic direction outgoing from the origin 0 in the space-time. We could choose any other point on the line 0. Let us choose the point of coordinates *(T, X, Y, Z)* obtained by multiplying the coordinates of the point P by $\frac{\xi\bar{\xi}+\eta\bar{\eta}}{\sqrt{2}}$, namely

$$T = \frac{\xi\bar{\xi} + \eta\bar{\eta}}{\sqrt{2}},\ X = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\sqrt{2}},\ Y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{i\sqrt{2}},\ Z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\sqrt{2}}. \tag{6}$$

As the partial derivatives of the functions belong to the tangent plane to the surfaces of these functions, it is possible to represent $\xi,\ \eta$ in the form of partial derivatives of some complex-valued function w, i.e. $w = u + iv, \quad \xi = \partial_x w,\ \eta = \partial_y w$ [13]. Putting these in (6) we obtain

$$T = \frac{\left(\partial_x u\right)^2 + \left(\partial_x v\right)^2 + \left(\partial_y u\right)^2 + \left(\partial_y v\right)^2}{\sqrt{2}},\ X = \sqrt{2}\left(\partial_x u \partial_y u + \partial_x v \partial_y v\right),$$

$$Y = -\sqrt{2}\left(\partial_x u \partial_y v - \partial_y u \partial_x v\right),\ Z = \frac{\left(\partial_x u\right)^2 + \left(\partial_x v\right)^2 - \left(\partial_y u\right)^2 - \left(\partial_y v\right)^2}{\sqrt{2}}. \tag{7}$$

Solving the system (1) relative to $\alpha,\ \beta,\ \gamma$, under the ellipticity condition $\alpha\gamma-\beta^2=1$, we obtain the expressions

$$\alpha=\frac{(\partial_y u)^2+(\partial_y v)^2}{\partial_x u\partial_y v-\partial_y u\partial_x v},\ \beta=-\frac{\partial_x u\partial_y u+\partial_x v\partial_y v}{\partial_x u\partial_y v-\partial_y u\partial_x v},\ \gamma=\frac{(\partial_x u)^2+(\partial_x v)^2}{\partial_x u\partial_y v-\partial_y u\partial_x v},\quad (8)$$

which, introduced in (7), yield

$$\alpha=-\frac{T-Z}{Y},\ \beta=\frac{X}{Y},\ \gamma=-\frac{T+Z}{Y},\qquad (9)$$

such that the ellipticity condition reads $T^2-X^2-Y^2-Z^2=0$.

The abovestated results and invariance of the Beltrami equations system solution relative to the conformal mappings imlpy the following statement.

Every simply-connected domain D on the surface of the sphere S can be homeomorphically mapped onto every simply-connected domain Δ of the Euclidean $u,v-$plane. Moreover, this homeomorphism satisfies the system (1) *with coefficients* (9), *where $T,\ X,\ Y,\ Z$ are functions of $x,\ y$ relative to the local coordinates on D.*

6. AUTOMORPHISM OF VECTOR FIELDS AND BELTRAMI EQUATION SYSTEM

3. The vector field $\mathbf{v}$ on the manifold F is a smooth mapping $\mathbf{v}:F\to TF(TF$- tangent fibration of $F)$ such that $\mathbf{p}\circ\mathbf{v}:F\to F-$ is the identical mapping. Consider the case when F is two-dimensional. Let D be a chart on F, let x,y be local coordinates in D, and let Δ be the domain in (u,v) affine plane. The abovestated mapping $F\to F$ reads

$$u\equiv u\left[x\left(u,v\right),y\left(u,v\right)\right],\ v\equiv v\left[x\left(u,v\right),y\left(u,v\right)\right]\qquad (10)$$

where $u=u\left(x,y\right),\ v=v\left(x,y\right)$ and $x=x\left(u,v\right),\ y=y\left(u,v\right)$ are the inverse functions. If, in addition, we suppose that Δ is a domain of the Euclidean plane, then the following identity must be valid: $du^2+dv^2\equiv du^2+dv^2$ (and $du^2-dv^2\equiv du^2-dv^2$). This corresponds to the identical conformal mapping $\Delta\leftrightarrow\Delta$: $\partial_u u=\partial_v v=1,\ \partial_v u=-\partial_u v=0$ (identical hyperbolical conformal mapping $\Delta\to\Delta:\partial_u u=\partial_v v=1,\ \partial_v u=\partial_u v=0$). Applying this identity to both sides of (10) we obtain the identities

$$\begin{cases}\partial_x u=-\frac{\partial_x u\partial_y u+\partial_y v\partial_x v}{\partial_x u\partial_y v-\partial_y u\partial_x v}\partial_x v+\frac{(\partial_x u)^2+(\partial_x v)^2}{\partial_x u\partial_y v-\partial_y u\partial_x v}\partial_y v\equiv\beta\partial_x v+\gamma\partial_y v\\ -\partial_y u=\frac{(\partial_y u)^2+(\partial_y v)^2}{\partial_x u\partial_y v-\partial_y u\partial_x v}\partial_x v-\frac{\partial_x u\partial_y u+\partial_x v\partial_y v}{\partial_x u\partial_y v-\partial_y u\partial_x v}\partial_y v\equiv\alpha\partial_x v+\beta\partial_y v\end{cases}\qquad (11)$$

The existence of the functions $\alpha=\alpha\left(x,y\right),\ \beta=\beta\left(x,y\right),\ \gamma=\gamma\left(x,y\right)$ follows from the fact that the derivatives of the functions u,v belong to TF, therefore at each point $(x, y) \in D$ any two of them are linearly dependent on the other two. This dependence is expressed by the Beltrami equations system. From (9) it follows

$$\gamma dx - \beta dy = \partial_v y du - \partial_u y dv, \;\; \beta dx - \alpha dy = \partial_v x du - \partial_u x dv$$

where $\partial_u x, \;\partial_v x, \;\partial_u y, \;\partial_v y$ are partial derivatives of the functions $x = x\left(u, v\right),$ $y = y\left(u, v\right)$. The following equalities can be easily investigated

$$\begin{aligned} &\tfrac{1}{\gamma}\left(\gamma dx - \beta dy\right)^2 + \tfrac{1}{\gamma}dy^2 = \tfrac{y_u^2 + y_v^2}{\gamma}\left(du^2 + dv^2\right) = \tfrac{\partial(x,y)}{\partial(u,v)}\left(du^2 + dv^2\right), \\ &\tfrac{1}{\alpha}\left(\beta dx - \alpha dy\right)^2 + \tfrac{1}{\alpha}dx^2 = \tfrac{x_u^2 + x_v^2}{\alpha}\left(du^2 + dv^2\right) = \tfrac{\partial(x,y)}{\partial(u,v)}\left(du^2 + dv^2\right), \end{aligned}$$

where $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ is the Jacobian of the mapping $\Delta \to D$. On the other hand, the form $\gamma dx - \beta dy$ possesses an integration multiplier M such that $M\left(\gamma dx - \beta dy\right) = d\varphi$ and the multiplier N such that Ndy=$d\phi$. Here M and $N = N\left(y\right)$ are chosen such that the mapping $D \supset (x, y) \to (\varphi, \phi) \subset \Delta_1$ be one-to-one, Δ_1 - be a certain domain in the affine plane and φ, ϕ be isothermal coordinates. According to the Riemann theorem, there is a conformal mapping $\Delta \leftrightarrow \Delta_1$. Since the system (1) is invariant to the corresponding conformal mappings and the coefficients are proportional, we have: $M = N = \sqrt{\frac{1}{\gamma}\frac{\partial(\varphi,\phi)}{\partial(x,y)}}$,where $\frac{\partial(\varphi,\phi)}{\partial(x,y)}$ is the Jacobian of the mapping $D \to \Delta_1$ Taking into account all these we can have a representation of the partial derivatives of the mapping $D \to \Delta_1$

$$\partial_x\varphi = M\gamma, \;\; \partial_y\varphi = -M\beta, \;\; \partial_x\phi = 0, \;\; \partial_y\phi = M, \;\text{ or}$$

$$\partial_x\varphi = i_m^2 M, \partial_y\varphi = 0, \partial_x\phi = M\beta, \partial_y\phi = -M\alpha. \tag{12}$$

The partial derivatives (12) of the mapping $\varphi, \;\phi : D \to \Delta_1$ satisfy (11) independently of i_m^2. Changing, in a suitable manner, the multiplier M (multiplied by the Jacobian of the mapping $\Delta_1 \to \Delta$) we obtain various mappings on other topologically equivalent D domains. From (12) it is easy to find the derivatives of inverse functions

$$x = x\left(\varphi, \phi\right), \; y = y\left(\varphi, \phi\right) : \Delta_1 \to D : \quad x_\varphi = \frac{1}{M\gamma}, \; y_\varphi = 0, \; x_\phi = \frac{\beta}{M\gamma}, \; y_\phi = \frac{1}{M}$$

We prove that the basis fields $\varphi = const, \phi = const$ are coordinates too. Indeed, the Lie brackets are $\left[\frac{d}{d\varphi}, \frac{d}{d\phi}\right] = \left(\frac{dx}{d\varphi}\frac{\partial}{\partial x} + \frac{dy}{d\varphi}\frac{\partial}{\partial y}\right)\left(\frac{dx}{d\phi}\frac{\partial}{\partial x} + \frac{dy}{d\phi}\frac{\partial}{\partial y}\right) -$ $- \left(\frac{dx}{d\phi}\frac{\partial}{\partial x} + \frac{dy}{d\phi}\frac{\partial}{\partial y}\right)\left(\frac{dx}{d\varphi}\frac{\partial}{\partial x} + \frac{dy}{d\varphi}\frac{\partial}{\partial y}\right) = \left(\frac{1}{M}\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}\right)\left(\frac{\beta}{M\gamma}\frac{\partial}{\partial x} + \frac{1}{M}\frac{\partial}{\partial y}\right) =$ $= -\left(\frac{\beta}{M\gamma}\frac{\partial}{\partial x} + \frac{1}{M}\frac{\partial}{\partial y}\right)\left(\frac{1}{M}\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}\right) = 0$. The equality to zero follows from the existence of the common integration multiplier M in (12) and the equalities in [14, 15], namely

$$\partial_x M = 0,\ \partial_y M = 2M^2,\ \partial_x \frac{\partial_x \beta + \partial_y \gamma}{\gamma} = 0,\ \frac{\partial_x \beta + \partial_y \gamma}{\gamma} = -2M$$

relative to the introduced basis fields φ, ϕ [15] . The independent fields defined by the integral curves of the equations $\gamma dx - \beta dy = 0,\ \gamma dx - (\beta + 1)\, dy = 0,\ \gamma dx - (\beta - 1)\, dy = 0$, which have a common integration multiplier $M = M(y)$, form a diagonal three-fabric [1, p. 119 - 122].

Using $M(y) = M^*(\varphi)\, N^*(\phi)$ and the results from [14], [15] it follows the theorem on the new representation of solution of the Beltrami equations.

Theorem 1. *Let $u(x, y)$, $v(x, y)$ be the quasi-conformal mapping $D \to \Delta$ corresponding to the system* (1). *Then u, v are represented in the following form: $u = u(\varphi(x, y))$, $v = v(\phi(x, y))$ where the functions u, v are differentiable with respect to their arguments function, and φ, ϕ are the above defined isothermal coordinates.*

7. AFFINE CONNECTEDNESS AND MIXED SYSTEM OPERATORS

3. It is well-known that general theory of relativity uses curved space-time and it seems essential to present plane space as the simplest type of spaces. But, in reality, from the point of view of manifolds' theory, even a plane space is not so simple: as compared with a simple differentiable manifold it has a much richer structure or the affine connectedness is set on it. As usual, if the rectangular system of coordinates are used, this connectedness is not felt because in these coordinates the Christoffel symbols (values) are equal to zero. However, if the laws of physics are formulated in the plane space and described by curvilinear coordinates in which Christoffel's symbols are needed to use, the connectedness becomes visible. At the first glance, this is not necessary but it contains the possibility of generalization. The majority of physical laws involve the Christoffel symbols and not the Riemann's tensor. Consequently, their equations are interpreted and appear identical irrespective of the fact if the manifold is plane or curved. Thus, it is essential to postulate that the mathematical form of the physical laws in the curved space-time of general relativity theory is precisely the same as in curvilinear coordinates of the plane Minkowskian space. This widely accepted postulate is confirmed by an experiment [2, p.261]. Because of the condition that the covariant derivative in the first coordinate basis $\varphi = const, \phi = const$ is equal to zero, it follows

Theorem 2. *Along the fields $\frac{d}{d\varphi}, \frac{d}{d\phi}$ the Christoffel symbols are* [15]

$$\Gamma^1_{11} = \frac{\gamma_x}{\gamma},\ \Gamma^2_{11} = 0,\ \Gamma^1_{12} = -\frac{\beta_x}{\gamma}, \tag{13}$$

$$\Gamma_{12}^{2} = 0,\ \Gamma_{22}^{1} = -\frac{\beta_y}{\gamma},\ \Gamma_{\ 22}^{2} = -\frac{\beta_x + \gamma_y}{\gamma}.$$

The second coordinate basis $\mu = const,\ \nu = const$ is defined, where the partial derivatives of the functions $\mu = \mu(x,y),\ \nu = \nu(x,y)$ are: $\partial_x \mu = N\beta,\ \partial_y \mu = -N\alpha,\ \partial_x \nu = N,\ \partial_y \nu = 0$ and the Christoffel symbols are

$$\Gamma_{11}^{1} = -\frac{\alpha_x + \beta_y}{\alpha},\ \Gamma_{11}^{2} = -\frac{\beta_x}{\alpha},\ \Gamma_{12}^{1} = 0,\ \Gamma_{12}^{2} = -\frac{\beta_y}{\alpha},\ \Gamma_{22}^{1} = 0,\ \Gamma_{22}^{2} = \frac{\alpha_y}{\alpha}.$$

According to the famous result from Riemannian geometry, the geometry of the domain Δ_1 is restored completely after defining the Christoffel symbols [16, p. 350]. It may exist only the following components of the Riemann's tensor different from zero in the two-dimensional surface inserted into R^3 [17, p. 276] $R_{211}^{1} = R_{112}^{1} = -R_{212}^{2} = -R_{221}^{2} = -R_{122}^{2} = -R_{121}^{1}$. However, we obtain the following equality from the formula of the Riemann's tensor definition: $R_{211}^{1} = \frac{\partial}{\partial x}\left[\frac{1}{\gamma}\left(\partial_x \beta + \partial_y \gamma\right)\right] = 0$ for the first and $R_{211}^{1} = \frac{\partial}{\partial y}\left[\frac{1}{\alpha}\left(\partial_x \alpha + \partial_y \beta\right)\right] = 0$ for the second coordinate basis. So, all components of Riemann's tensor are equal to zero relative to the inserted by us system of coordinates. Corespondingly, the coordinate lines of this system depend on x and y and on the parameters.

Let us consider the problem on reduction of the indefinite quadratic form to the canonical form [6]. Let in (2) be $g_{11}g_{22} - g_{12}^2 < 0$. Then $ds^2 = \frac{\partial(u,v)}{\partial(x,y)}\left(\gamma dx^2 - 2\beta dxdy + \frac{\beta^2-1}{\gamma}dy^2\right)$ and $g_{11}g_{22} - g_{12}^2 = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^2\left(\alpha\gamma - \beta^2\right) = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^2(\ -1) < 0$. The surface with the metric tensor (2) is of hyperbolical type, therefore $\frac{\partial(u,v)}{\partial(x,y)} < 0$. Consequently, (2) reduces to the canonical form

$$du^2 - dv^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2 = \frac{\partial(u,v)}{\partial(x,y)}\left(\gamma dx^2 - 2\beta dxdy + \frac{\beta^2-1}{\gamma}dy^2\right) \tag{14}$$

The corresponding system reads

$$\partial_x u = \beta\partial_x v + \gamma\partial_y v - \partial_y u = \frac{\beta^2-1}{\gamma}\partial_x v + \beta\partial_y v \tag{15}$$

According to (12) the differentials of $\varphi,\ \phi -$ with respect to the inserted isothermal coordinates of the last system solutions (15) are represented in the form

$d\varphi = M(\gamma dx - \beta dy), d\phi = M dy, d\varphi^2 + d\phi^2 = M^2\gamma\left(\gamma dx^2 - 2\beta dxdy + \frac{\beta^2+1}{\gamma}dy^2\right).$

On the other hand, $-\left(du^2 + dv^2\right) = \frac{\partial(u,v)}{\partial(x,y)}\left(\gamma dx^2 - 2\beta dxdy + \frac{\beta^2+1}{\gamma}dy^2\right)$,where $\frac{\partial(u,v)}{\partial(x,y)} \prec 0-$ Jacobian mapping $D \to \Delta$.

From the Riemann's theorem on existence and uniqueness of quasi-conformal mappings corresponding to the system (1) it follows that $M = \sqrt{-\frac{1}{\gamma}\frac{\partial(\varphi,\phi)}{\partial(x,y)}}$. Consequently, $du = N\left(\gamma dx - \beta dy\right),\ dv = Ndy$, where

$$N = \sqrt{-\frac{1}{\gamma}\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}}. \tag{16}$$

Putting the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ into the quadratic form (14) we obtain the canonical form of the indefinite form. In addition the u, v - solutions of the system (19) realize the mapping $D \to \bar{\Delta}$, where $\bar{\Delta}$ is the domain Δ with an inverse congruent.

4. Let us write the system (1) in the following form [13], [18]

$$\begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} \partial_x w \\ \partial_y w \end{pmatrix} = i \begin{pmatrix} \partial_x w \\ \partial_y w \end{pmatrix} \Leftrightarrow A\zeta = i\zeta, \tag{17}$$

where $w = u + iv$. The equalities (10) read $-iA\zeta = \zeta$. Obviously, the operator $-iA$ satisfies all conditions of the basic lemma such that from its statement it follows that $-iA\zeta = \pm\zeta$, whence $i^4 = 1$ iff $i^2 = +1,\ i^2 = -1$. The imaginary unit of complex numbers satisfies the condition $i^2 = -1$. However, the imaginary unit which satisfies the equation $i^2 = 1$, for unclear reasons is being omitted. In connection with this, in the following considerations two imaginary units $i_e,\ i_e^2 = -1$ as well as $i_{h,}\ i_h^2 = 1,\ i_h \neq \pm 1$ are inserted. Introducing $i = i_e$ in (10) we obtain $A^2 = -E$, where E is the unit matrix. Indeed,

$$A^2 = \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} \beta^2 - \alpha\gamma & 0 \\ 0 & \beta^2 - \alpha\gamma \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consequently, $\alpha\gamma - \beta^2 = -i_e^2{=}1$ or $\alpha = \frac{\beta^2+1}{\gamma}$. Introducing $i = i_h$ in (11) we obtain $A^2 = E$, i.e.

$$\begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} \beta^2 - \alpha\gamma & 0 \\ 0 & \beta^2 - \alpha\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, $\alpha\gamma - \beta^2 = -1 = -i_h^2$ or $\alpha = \frac{\beta^2-1}{\gamma}$.
Introduce the following notations

$$A = \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2+1}{\gamma} & -\beta \end{pmatrix}, B = \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2-1}{\gamma} & -\beta \end{pmatrix},$$

$$C = \frac{B+A}{2} = \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2}{\gamma} & -\beta \end{pmatrix}, D = \frac{B-A}{2} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma} & 0 \end{pmatrix}.$$

Direct computations yield

$$\begin{gathered} A^2 = -E, B^2 = E, C^2 = 0, D^2 = 0, AB + BA = -C^2 = 0, \\ AC + CA = -B^2 = -E, BC + CB = -A^2 = E \\ AD + DA = E, BD + DB = E, CD + DC = E \end{gathered}$$

In the case $\alpha = \gamma = 1$ and $\beta = 0$ the system (1) turns into the Cauchy-Riemann system and we have

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x, \ -i_e A = \begin{pmatrix} 0 & -i_e \\ i & 0 \end{pmatrix} = \sigma_y, \ AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z,$$

where $\sigma_x, \ \sigma_y, \ \sigma_z$ are the Pauli matrices [19, p. 48]. The non-zero eigenvalues of the matrices C and D will be denoted by $i_p, \ i_d$. Obviously $i_p \neq 0, \ i_d \neq 0$. By (10) the eigenvalues introduced by us satisfy the relations

$$\begin{gathered} i_e^2 = -1, \ i_h^2 = 1, \ i_p^2 = 0, \ i_d^2 = 0, \ i_e i_h + i_h i_e = 0, \ i_e i_p + i_p i_e = -i_h^2 = -1, \\ i_h i_p + i_p i_h = -i_e^2 = 1, \ i_e i_d + i_d i_e = 1 \\ i_h i_d + i_d i_h = 1, \ i_d i_p + i_p i_d = 1. \end{gathered}$$

Remark 1. If in the Dirac's matrices $\sigma_x, \ \sigma_y, \ \sigma_z$ are replaced by the matrices $A, \ B, \ AB$ respectively we obtain matrices which depend on two arbitrary functions β, γ and satisfy all anticommutative relations as Dirac's ordinary matrices [19, p. 59].

Remark 2. If in the stereographic projection we replace $\xi, \ \eta$ by $\xi = \partial_x U + i_m \partial_x V, \ \eta = \partial_y U + i_m \partial_y V$, then the formulas (6), (7) become

$$\begin{aligned} T &= \tfrac{1}{\sqrt{2}} \left((\partial_x U)^2 - i_m^2 (\partial_x V)^2 + (\partial_y U)^2 - i_m^2 (\partial_y V)^2 \right), \\ X &= \sqrt{2} \left(\partial_x U \partial_y U - i_m^2 \partial_x V \partial_y V \right), \\ Y &= -\sqrt{2} \left(\partial_x U \partial_y V - \partial_y U \partial_x V \right), \\ Z &= \tfrac{1}{\sqrt{2}} \left((\partial_x U)^2 - i_m^2 (\partial_x V)^2 - (\partial_y U)^2 + i_m^2 (\partial_y V)^2 \right). \end{aligned}$$

Using (9) we obtain the following system of identities

$$\partial_x U \equiv -\frac{\partial_x U \partial_y U - i_m^2 \partial_x V \partial_y V}{\partial_x U \partial_y V - \partial_y U \partial_x V} \partial_x V + \frac{(\partial_x U)^2 - i_m^2 (\partial_x V)^2}{U_x V_y - U_y V_x} \partial_y V - \partial_y U \equiv$$

$$\frac{(\partial_y U)^2 - i_m^2 (\partial_y V)^2}{\partial_x U \partial_y V - \partial_y U \partial_x V} \partial_x V - \frac{\partial_x U \partial_y U - i_m^2 \partial_x V \partial_y V}{U_x V_y - U_y V_x} \partial_y V$$

$$\alpha = \frac{(\partial_y U)^2 - i_m^2 (\partial_y V)^2}{\partial_x U \partial_y V - \partial_y U \partial_x V}, \ \beta = -\frac{\partial_x U \partial_y U - i_m^2 \partial_x V \partial_y V}{\partial_x U \partial_y V - \partial_y U \partial_x V},$$

$$\gamma = \frac{(\partial_x U)^2 - i_m^2 (\partial_x V)^2}{\partial_x U \partial_y V - \partial_y U \partial_x V},$$

where the imaginary units i_m^2 are defined in (10). If $\alpha,\ \beta,\ \gamma$ are defined as functions of $x,\ y$ and $\alpha\gamma - \beta^2 = i_m^2$, then the system of equations (17) can be considered as a system of first order mixed partial differential equations. The condition of hyperboliticity of (17) is $\alpha\gamma - \beta^2 = -1$ if $T^2 - X^2 - Z^2 + Y^2 = 0$ is the equation of single-sheet hyperboloid. The condition of parabolicity of the system (17) is $\alpha\gamma - \beta^2 = 0$ iff $T^2 - X^2 - Z^2 = 0$ is the equation of a cone. The system of equations (17) is: elliptical for $i_m^2 = i_e^2 = -1$, hyperbolical for $i_m^2 = i_h^2 = 1$, and parabolic for $i_m^2 = i_p^2 = 0$ (parabolic of a degenerated type). The set of these equations is called the *MES (Mixed Equations System)* [20]. It is known that the system (1) realizes the reduction of the quadratic form to the canonical form

$$ds^2 = g_{11} dx^2 + 2 g_{12} dx dy + g_{22} dy^2 = \frac{\partial (u,v)}{\partial (x,y)} ds_1^2 = du^2 + dv^2.$$

Thus, in the condition that $\alpha\gamma - \beta^2 = 1$ and $\frac{\partial(u,v)}{\partial(x,y)} > 0$ the surface is elliptic, while in the condition that $\alpha\gamma - \beta^2 = -1$ and $\frac{\partial(u,v)}{\partial(x,y)} < 0$ the surface is hyperbolic. The Poincaré's indices at the ellipticity points are equal to +1 and the points can be a centre, focus or node while at the hyperboliticity points, the Poincaré's indices are equal to -1 and the points can be only saddles. When the surface is elliptic, the corresponding quasi-conformal mappings realized by the solution of the system (1) are considered as mappings of the first genus. When the surface is hyperbolic (two-sheeted hyperboloid), the corresponding quasi-conformal mappings are realized by the system which is conjugate to the system (1) and the considered mappings are of the second genus. Thus, the system of equations (1) and the conjugate to it systems realize the corresponding quasi-conformal mappings of elliptic and hyperbolic type surfaces.

Remark 3. We proved that the Beltrami equations system realizes a holomorphic mapping of the domain D in S^2 on the domain Δ of the Euclidean plane, where the ellipticity condition $\alpha\gamma - \beta^2 = 1$ is equivalent to the condition $T^2 - X^2 - Y^2 - Z^2 = \Omega = 0$. According to the Lagrange theorem [*3*(b), p. *105*] the following basis exists for the quadratic form (10): $e_i e_j + e_j e_i = 0$, $i = 1, 2, 3, 4$. Then, from the statement [*3(a)*, p. *266,*] Ω can be represented in the form $\Omega = T e_1 + X e_2 + Y e_3 + Z e_4$, where $e_1^2 = 1,\ e_2^2 = -1,\ e_3^2 = -1,\ e_4^2 = -1$. Performing the substitutions $e_1 = i_h$ and $e_2 = i_e = \mathbf{i}$, $e_3 = \mathbf{j}$, $\quad e_4 = \mathbf{k}$, where **i, j, k** - imaginary units of quaternions, we obtain

$$\Omega\overline{\Omega} = -(T^2 - X^2 - Y^2 - Z^2) = 0.$$

The conditions $\mathbf{i_h j} + \mathbf{j i_h} = 0, \mathbf{i_h k} + \mathbf{k i_h} = 0$ follow from the statements that for $\mathbf{i} = i_e$ the operators A and B generate two projection operators: $P_i = \frac{-i_e A + E}{2}$, $P_{i_h} = \frac{-i_h B + E}{2}$, which act in the space W, i.e.$-iAW_- = W_-$, $-i_h W_+ = W_+$. Similarly, with raport to the vectors $\mathbf{j}$ and $\mathbf{k}$ we have $P_j = \frac{-\mathbf{j}A+E}{2}$, $\quad P_k = \frac{-\mathbf{k}A+E}{2}$.

The operator C (as well as the operator D) is nilpotent and acts in the space W. Then W decomposes into the direct sum of invariant subspaces W_+ and W_-, on each of which the operator C induces the cyclic operator $i_h B$ and $i_e A$, i.e.

$$-i_e A W_- = W_-, \quad -i_h B W_+ = W_+, \quad W = W_+ \oplus W_- \quad [3(b), p. 152],$$

$(-i_e A)^2 = E$, $(-i_h B)^2 = E$. In this way we clarified the three imaginary units on which B. Riemann spoke. The last part of the Riemann's work is named "Application to space". Referred to as a many-dimensional space in this work it is called "manifold" by Riemann. However, when he speaks about "space", he understands the space of the real world. Listing the sufficient and necessary coditions to define metric relations in "space", namely the equality to zero of the equation of the curvature measure at each point for each two-dimensional direction, constancy of curvature measures in space and "independence of line length" of their place, he concluds that the transition or changes of the position are considered complex quantities expressed by three imaginary units. In the case of Cauchy-Riemann equations system, i.e. when $\alpha = \gamma = 1$, $\beta = 0$ we obtain $X = 0$, $-T + Z = Y \quad T + Z = -Y \Rightarrow \quad Z = 0$ and $T = Y$, i.e. in this case, only one imaginary unit acts in the space of dimensionality. Therefore, in this case it is possible to use one and only one imaginary unit $i = i_e$, contrary to the Beltrami equations system where three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are needed.

In the case of one-sheeted hyperboloid the following theorem holds.

Theorem 3. *Let the function $w = u(\varphi) + i_e v(\phi)$ realizes a quasi-conformal mapping of the domain $D \subset M$ onto the domain Δ of the affine plane as the solution of the Beltrami equations system. Then the function $\chi = u(\varphi) + i_h v(\phi)$ realizes the h – quasi-conformal mapping of the domain $D \subset M$ onto the same domain Δ of the affine plane as the solution of the hyperbolic system of MES (hyperbolic analogue of Beltrami equations system). The hyperbolical analogue of the Cauchy-Riemann system reads: $u_x = v_y$, $u_y = v_x$.*

Let $\varsigma = x + i_h y$ be the hyperbolic complex variable. The function $\chi = f(\varsigma) = u(x, y) + i_h v(x, y)$ is called the hyperbolical holomorphic function if $f_{\bar{\varsigma}}(\varsigma) = 0$. The last equation is just another form of the hyperbolic analogue of the Cauchy-Riemann equations system and it means that f does not depend on $\bar{\varsigma}$. Using the Green formula, the integral theorem of Cauchy is easily proved: $\oint_{\partial D} f(\varsigma)d\varsigma = 0$ where ∂D is the oriented wall of the domain D, $d\varsigma = dx + i_h dy$ [20]. The Euler's identity on the Lobachevsky plane holds $e^{i_h\delta} = ch\delta + i_h sh\delta$.

8. ON REPRESENTATION OF MES SOLUTIONS

5. Consider now the second statement of the basic lemma. From the statement of the basic lemma and from the definition of the direct sum [3] it follows

Theorem 4. *The solutions of the system* (1) *are represented in the unique form as follows* $w = u+i_e v+i_h t$, *where* $u+i_e v$ *corresponds to the eigenvalue* i_e, *but* $i_h t$ *- to the eigenvalue* i_h *of the operator* A. *The obtained representations will be put to the system (17) for the definition of* t

$$\begin{pmatrix} \beta & \gamma \\ \frac{-\beta^2-1}{\gamma} & -\beta \end{pmatrix} \begin{pmatrix} \partial_x u + i_e \partial_x v + i_h \partial_x t \\ \partial_y u + i_e \partial_y v + i_h \partial_y t \end{pmatrix} = i_e \begin{pmatrix} \partial_x u + i_e \partial_x v + i_h \partial_x t \\ \partial_y u + i_e \partial_y v + i_e \partial_y t \end{pmatrix} \tag{18}$$

After simple computations and comparisons of the coefficients of the imaginary units $i_e,\ i_h,\ i_e i_h$ and the absolute term, we obtain the system of equations for $u + i_e v$ which coincides with the Beltrami equations system

$$\partial_x u = \beta\partial_x v + \gamma\partial_y v,\ -\partial_x v = \beta\partial_x u + \gamma\partial_y u. \tag{19}$$

and for t, $-i_e\partial_x t = \beta\partial_x t + \gamma\partial_y t$. For our aims, this last equation makes a sense if we substitute $t = p + i_e q$, where $p,\ q$ are the functions defined and possessing continuous partial derivatives in the domain D. Then

$$-\partial_x p = \beta\partial_x q + \gamma\partial_y q,\ \partial_x q = \beta\partial_x p + \gamma\partial_y p. \tag{20}$$

The system (20) can be considered as conjugate to the Beltrami equations system. We remind that the solution of the conjugate system realizes the mappings with a negative Jacobian, i.e. they change the domain orientation. So, the general solution of (18) is represented in the form

$$\zeta = u + i_e v + i_h p + i_e i_h q, \tag{21}$$

which can be considered as another doubling of complex numbers. As it is known, an ordinary doubling of complex numbers brings to the concept of quaternions. The set of the introduced by us the imaginary units $i_e,\ i_h,\ i_p,\ i_d$ forms the Lie algebra, while the set $\pm 1,\ \pm i_e,\ \pm i_h,\ \pm i_e i_h$ forms the group [12]. The Lie brackets, Lie algebras and the set of imaginary units $i_e,\ i_h,\ i_e i_h$ satisfy the following relation: $[i_e, i_h] = 2i_e i_h,\ [i_h, i_e i_h] = -2i_e,\ [i_e i_h, i_e] =$

$2i_h$. Throughout this paper $i_e i_h$ is considered as the tensor product of i_e by i_h [13]. All these statements make a sense even for the Cauchy-Riemann equations system which corresponds to the case $\beta = 0,\ \gamma = 1$ in (18), (19). If ς has the form $\varsigma = (u + i_h p) + i_e(v + i_h q)$, then its components satisfy the following MES system

$$\begin{cases} \partial_x p = \beta\partial_x u + \gamma\partial_y u, \\ \partial_x u = \beta\partial_x p + \gamma\partial_y p, \end{cases} \text{and} \begin{cases} \partial_x q = \beta\partial_x v + \gamma\partial_y v, \\ \partial_x v = \beta\partial_x q + \gamma\partial_y q. \end{cases}$$

The last system of equations belongs to the hyperbolical MES. Putting these received forms into the MES we obtain

$$\begin{pmatrix} \beta - i_e & \gamma \\ -\alpha & -\beta - i_e \end{pmatrix} \begin{pmatrix} \partial_x u + i_e\partial_x v + i_h\partial_x p + i_e i_h\partial_x q \\ \partial_y u + i_e\partial_y v + i_h\partial_y p + i_e i_h\partial_y q \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} \beta\partial_x u + \gamma\partial_y u = -\partial_x v & \beta\partial_x p + \gamma\partial_y p = \partial_x q \\ \beta\partial_x v + \gamma\partial_y v = \partial_x u & \beta\partial_x q + \gamma\partial_y q = -\partial_x p \end{pmatrix}$$

(or equivalent to it the system

$$\begin{pmatrix} \alpha\partial_x u + \beta\partial_y u = \partial_y v & \alpha\partial_x p + \beta\partial_y p = \partial_y q \\ \alpha\partial_x v + \beta\partial_y v = -\partial_y u & \alpha\partial_x q + \beta\partial_y q = -\partial_y p \end{pmatrix})$$

and

$$\begin{pmatrix} \beta - i_h & \gamma \\ -\alpha & -\beta - i_h \end{pmatrix} \begin{pmatrix} \partial_x u + i_h\partial_x p + i_e\left(\partial_x v + i_h\partial_x q\right) \\ \partial_y u + i_h\partial_y p + i_e\left(\partial_y v + i_h\partial_y q\right) \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} \beta\partial_x u + \gamma\partial_y u = \partial_x p & \beta\partial_x v + \gamma\partial_y v = \partial_x q \\ \beta\partial_x p + \gamma\partial_y p = \partial_x u & \beta\partial_x q + \gamma\partial_y q = \partial_x v \end{pmatrix}$$

(or equivalent to it the system

$$\begin{pmatrix} \alpha\partial_x u + \beta\partial_y u = -\partial_y p & \alpha\partial_x v + \beta\partial_y v = -\partial_y q \\ \alpha\partial_x p + \beta\partial_y p = -\partial_y u & \alpha\partial_x q + \beta\partial_y q = -\partial_y v \end{pmatrix})$$

Comparing them we obtain $\begin{pmatrix} \beta\partial_x(p - v) + \gamma\partial_y(p - v) = 0 & \partial_x(p + v) = 0 \\ \beta\partial_x(q - u) + \gamma\partial_y(q - u) = 0 & \partial_x(q + u) = 0 \end{pmatrix}$

(or equivalent to it system $\begin{pmatrix} \alpha\partial_x(p - v) + \beta\partial_y(p - v) = 0 & \partial_y(p + v) = 0 \\ \alpha\partial_x(q - u) + \beta\partial_y(q - u) = 0 & \partial_y(q + u) = 0 \end{pmatrix}$)

We obtain the following from the last equations

$$\gamma dx - \beta dy = 0 \text{ or } \beta dx - \alpha dy = 0,$$

$$u = \frac{\sigma(s) + \lambda(x)}{2},\ p = \frac{\sigma(s) - \lambda(x)}{2},\ v = \frac{\tau(s) + \lambda(x)}{2},\ q = \frac{\tau(s) - \lambda(x)}{2}$$

(or, equivalently, $u = \frac{\mu(t)+v(y)}{2},\ p = \frac{\mu(t)-v(y)}{2},\ v = \frac{\xi(t)+\varsigma(y)}{2},\ q = \frac{\xi(t)-\varsigma(y)}{2}$).

9. PSEUDOOCTAVES AND THEIR APPLICATIONS

Consider the hypercomplex numbers with seven imaginary units $i,\ j,\ k,\ i_h,\ ii_h,\ ji_h,\ ki_h$, where $i,\ j,\ k$ are the imaginary units of quaternions, i_h is the introduced above hyperbolical imaginary unit, $ii_h,\ ji_h,\ ki_h$ are tensor products of the corresponding imaginary units. The imaginary unit $ii_h \equiv i_e i_h$ and $ji_h = -i_h j,\ ki_h = -i_h k,\ (ji_h)^2 = ji_h ji_h = 1,\ (ki_h)^2 = ki_h ki_h = 1$. It is known that the algebra of the imaginary units of quaternions and the algebra of each triple of imaginary units $i, i_h,\ ii_h;\ j, i_h, ji_h;\ k, i_h, ki_h$ are isomorphic [3, p. 283]. In the same way, by using the correspondence $i \to i,\ j \to j,\ k \to k,\ i_h \to E,\ ii_h \to I,\ ji_h \to J,\ ki_h \to K$, where $i,\ j,\ k,\ E,\ I,\ J,\ K$ [23, p. 38-47] are the imaginary units of Caylay octaves, it is possible to establish the isomorphism between the algebra of Caylay octaves' imaginary units and the abovestated seven imaginary units. The obtained hypercomplex numbers form the algebra with division and it is isomorphic to the Caylay octaves algebra [23, p. 38-47]. In the following we call them pseudooctaves. Thus, pseudooctaves are inserted, as Caylay octaves, by the procedure of quaternions' doubling, only in our case with the imaginary unit i_h.

Let $\Theta = x_1 i + x_2 j + x_3 k + x_4 i_h + x_5 ii_h + x_6 ji_h + x_7 ki_h \in R_4^7$ [17, p. 62] be a pseudooctave. Consider the function $f(\Theta) = 2\frac{R}{\Theta}$. From the pseudooctave imaginary units' property we have: $\Theta^2 = \sum\limits_1^3 x_l^2 - \sum\limits_4^7 x_l^2$, $df(\Theta) = -2\frac{R}{\Theta^2}d\Theta$, whence $df d\bar{f} = 4R^2\frac{\sum\limits_1^3 dx_l^2 - \sum\limits_4^7 dx_l^2}{(\sum\limits_1^3 x_l^2 - \sum\limits_4^7 x_l^2)^2}$. Choosing in the corresponding manner the coefficients x_l, it is possible to receive different types of different surfaces' metrics, partially the metrics of the sphere in the Poincare model of the Lobachevsky geometry (two-sheeted hyperboloid) [17, p. 97]. In the case of the one-sheeted hyperboloid the metrics of Lobachevsky geometry is $dl^2 = 4R^2\frac{dx^2 - dy^2}{(R^2-(x^2-y^2))^2}$, where in the formula Θ it is put $x_1 = x,\ x_4 = y,\ x_2 = x_3 = x_5 = x_6 = x_7 = 0$, such that the equation of one-sheeted hyperboloid becomes: $t^2 - \left(x^2 - y^2\right) = R^2$. The pseudo Euclidean space R_2^4 is a four-dimensional space over the field R of real numbers, besides it is set orientations in R_2^4, bilinear scalar product with a signature (++–). The obtained space is a vector space. In the four-dimensional space R_2^2 exists a basis consisting of a unit and three linearly independent vectors $1,\ i_e,\ i_h,\ i_e i_h$. In other words, any vector $U \in R_2^2$ can be written in the only form

$$U = T1 + Xi_e + Yi_h + Zi_e i_h,$$

where $T,\ X,\ Y,\ Z$ are the components of the vector U. From all elements of the vector space only the element 0 possesses components equal to zero.

The determinant of the nonsingular matrix of transformation of one vector to another by means of which the orientation of the vectors to U is defined, can be positive or negative. The scalar product of the vectors $U_1,\ U_2$ from R_2^2 is a real number. Taking into consideration (16), the scalar product of the vectors $U_1 = T_1 + X_1 i_e + Y_1 i_h + Z_1 i_e i_h,\ U_2 = T_2 + X_2 i_e + Y_2 i_h + Z_2 i_e i_h$is defined as

$$\frac{1}{2}\left(U_1 U_2^T + U_2 U_1^T\right) = T_1T_2 + X_1X_2 - Y_1Y_2 - Z_1Z_2 \Rightarrow \left\|\overline{U}\right\| = T^2 + X^2 - Y^2 - Z^2.$$

The transformation of the group of special relativity theory can be written in the matrix form [14], [21, p.252]

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{10} & a_x \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{20} & a_y \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{30} & a_z \\ \lambda_{01} & \lambda_{02} & \lambda_{03} & \lambda_{00} & a_t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} x^1 \\ y^1 \\ z^1 \\ t^1 \\ 1 \end{pmatrix}. \tag{22}$$

The entries of λ are the components of homogeneous Lorenz transformations. The group (22) is the Poincaré group, where $a_t,\ a_x,\ a_y,\ a_z$ are the movements in the corresponding direction. Consider the transformations which do not change the time similar coordinate but change space similar coordinates, therefore the quadratic form (in used by us notation) $T^2 - X^2 - Y^2 - Z^2$ is not changed. Such a transformation reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i_e & i_h & i_e i_h \\ 0 & i_h & i_e i_h & i_e \\ 0 & i_e i_h & i_e & i_h \end{pmatrix} \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} T_1 \\ X_1 \\ Y_1 \\ Z_1 \end{pmatrix},$$

implying $T_1 = T,\ X_1 = i_e X + i_h Y + i_e i_h Z,\ Y_1 = i_h X + i_e i_h Y + i_e Z,\ Z_1 = i_e i_h X + i_e Y + i_h Z$. Using the relation (16) we obtain $T_1^2 - X_1^2 - Y_1^2 - Z_1^2 = T^2 - X^2 - Y^2 - Z^2$. It is easy to show that the scalar product $T_1\bar{T}_1 + X_1\bar{X}_1 + Y_1\bar{Y}_1 + Z_1\bar{Z}_1 = T^2 - X^2 - Y^2 - Z^2$ is Hermitian, where $\bar{T}_1,\ \bar{X}_1,\ \bar{Y}_1,\ \bar{Z}_1$ are Hermitian conjugates, i.e. $\bar{X}_1 = -i_e X - i_h Y - i_e i_h Z$ etc. In our case, in our notation (22), we have

$$U_r \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & i_h a_t \\ 0 & i_e & i_h & i_e i_h & a_x \\ 0 & i_h & i_e i_h & i_e & a_y \\ 0 & i_e i_h & i_e & i_h & a_z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_k \\ X_k \\ Y_k \\ Z_k \\ 1 \end{pmatrix} = \begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_{k+1} \\ Z_{k+1} \\ 1 \end{pmatrix},\ k \in N. \tag{23}$$

This time the Hermitian scalar product reads

$$T_{k+1}\bar{T}_{k+1}+X_{k+1}\bar{X}_{k+1}+Y_{k+1}\bar{Y}_{k+1}+Z_{k+1}\bar{Z}_{k+1} = T_k^2-X_k^2-Y_k^2-Z_k^2-a_t^2+a_x^2+a_y^2+a_z^2, \tag{24}$$

where the vector $(T_{k+1},\ X_{k+1},\ Y_{k+1},\ Z_{k+1}) \subset R_4^8$. It is easily seen that $U_{rs} = U_rU_s$. If (24) is equal to zero and $T_k^2 - X_k^2 - Y_k^2 - Z_k^2 = c = const$, then $a_t^2 - a_x^2 - a_y^2 - a_z^2 = c = const$. The converse statement holds too. As it is known, in relativistic theories, the Poincaré groups serve as symmetry groups. The equalities (23), (24) show that the state of symmetry is relativistically invariant and possesses an infinite collection of orthogonal states [21, p. 225]. It is essential that in quantum mechanics the physical (real) state of the system is defined not by the position and speed, but by a vector in a Hilbert space (in our case the space of pseudooctaves). Equivalence of all directions in quantum theory as well as in classical theory means the existence of all states arising from any states at rounding (action of the operator U). However, in quantum theory all states arising from a certain state at rounding can be represented as linear compositions of certain basis states. (In our case $1,\ i_e,\ i_h,\ i_ei_h,\ a_t,\ a_x,\ a_y, a_z$ serve as basis states, while the mentioned linear combination is an equality (23)).

10. APPLICATIONS

We mention some famous definitions, statements and facts from quantum mechanics, theory of elementary particles and the relativity theory necessary in the following. Comments to them yielded by our researchers are simultaneously given.

I). The introduced by us operators of the Beltrami equations system and its hyperbolical analogue are one-dimensional (complex-dimensional) representations of the abstract Lie group. The operators A and B form the cyclic groups:$\left\{\pm A, \pm A^2, \pm A^3, \pm A^4 = \pm E\right\}$, $\left\{\pm B, \pm B^2 = E\right\}$, where E is the unit operator. Obviously, $A^{2s} = (\,-1)^s\, E, B^{2s} = E$, where $s \in Z$. It is known [24, p. 337 - 341] that the group of permutations has two one-dimensional representations:

1) symmetric representation, in which all permutations are represented by the unit operator E, i.e. $\mathrm{P}|\psi \succ = +\psi$ for all permutations;

2) antisymmetric representation for which all even permutations are realized with the operator E, but all odd permutations with the operator $-E$, i.e.:

$\mathrm{P}|\psi = (\,-1)^s\,\psi$,where s is even, if P is even; s is odd, if P is odd.

The space of physical states N of identical quantum-mechanical systems are either symmetric space H_+^N or antisymmetric space H_-^N. The particles, the space of physical states of which is H_+^N are called bosons, but the particles, the space of physic states of which is H_-^N are called fermions. Whereof follows that the operators A, B from MES can serve as creating the spaces $\mathrm{H}_-^N,\ \mathrm{H}_+^N$ correspondingly. Earlier we have noted that the operators $B,\ -i_eA, AB$ corresponding to the Cauchy-Riemann equations system completely coincide with the famous Pauli matrices $\sigma_x \sigma_y, \sigma_z$. Thus, fundamental concepts of quantum mechanics are connected with the Cauchy-Riemann equations system and with MES.

II). The Dirac's equations system for electrons reads [25, p. 484-499, 267-270]

$$(p_t + p_z)\,\psi_1 + (p_x - ip_y)\,\psi_2 - mc\psi_3 = 0, (p_t - p_z)\,\psi_2 + (p_x + ip_y)\,\psi_1 + mc\psi_4 = 0,$$
$$(p_t - p_z)\,\psi_3 + (p_x - ip_y)\,\psi_4 - mc\psi_1 = 0, (p_t + p_z)\,\psi_4 + (p_x + p_y)\,\psi_3 + mc\psi_2 = 0,$$

where $p_t,\ p_x,\ p_y, p_z$ are the components of the energy-momentum vector, m is the mass of the particle, c is the light speed, $\psi_1,\ \psi_2,\ \psi_3, \psi_4$ is the wave function of the particle with half-integer spin. The MES reads

$$(\beta - i_e)\,\partial_x W + \gamma \partial_y W = 0, \tag{25}$$

$$\alpha \partial_x W + (\beta + i_e)\,\partial_y W = 0, \tag{26}$$

for any continuous differentiable function and also for the wave function. We put

$$\alpha = \frac{p_t + i_h p_z + i_e i_h mc}{p_y},\ \beta = \frac{p_x}{p_y},\ \gamma = \frac{p_t - i_h p_z - i_e i_h mc}{p_y},\ (p_y \neq 0)\,,$$

$\alpha\gamma - \beta^2 = 1 \Leftrightarrow p_t^2 - p_x^2 - p_y^2 - p_z^2 = (mc)^2$.

Denote

$$\partial_x W = \psi_4,\ \partial_y W = \psi_3,\ i_e i_h \partial_x W = \psi_2,\ i_e i_h W = \psi_1. \tag{27}$$

This representation makes a sense if (21) holds. Taking into account (27) in (25), (26) we obtain

$$\begin{cases} (p_x + i_e p_y)\,\partial_x W + (p_t - i_h p_z)\,\partial_y W = -mc\psi_4, \\ (p_t + i_h p_z)\,\partial_x W + (p_x - i_e p_y)\,\partial_y W = mc\psi_3. \end{cases} \tag{28}$$

Supposing $\partial_x W = \psi_1,\ \partial_y W = \psi_2$ and solving (26) relative to $mc\psi_1,\ mc\psi_2$ we obtain

$$\begin{cases} (p_t + i_h p_z)\,\psi_1 + (p_x - i_e p_y)\,\psi_2 - i_h mc\psi_3 = 0, \\ (p_t - i_h p_z)\,\psi_2 + (p_x + i_e p_y)\,\psi_1 + i_h mc\psi_4 = 0, \\ (p_t - i_h p_z)\,\psi_3 + (p_x - i_e p_y)\,\psi_4 - i_h mc\psi_1 = 0, \\ (p_t + i_h p_z)\,\psi_4 + (p_x + i_e p_y)\,\psi_3 + i_h mc\psi_2 = 0, \end{cases}$$

Obviously,

$$\det\begin{pmatrix} p_x+i_ep_y & p_t+i_ep_z+i_ei_hmc \\ -(p_t-i_ep_z-i_ei_hmc) & -(p_x-i_ep_y) \end{pmatrix} = p_t^2-p_x^2-p_y^2-p_z^2-(mc)^2=0$$

$$iff\ \det\Lambda=0,\ \det\begin{pmatrix} p_x-i_ep_y & p_t+i_ep_y+i_ei_hmc \\ -(p_t-i_hp_z-i_ei_hmc) & -(p_x+i_ep_y) \end{pmatrix} =$$

$$p_t^2-p_x^2-p_y^2-p_z^2-(mc)^2=0\ \ iff\ \det\mathrm{K}=0$$

and $\Lambda^*=\mathrm{K}$, where Λ^* is the Hermitian conjugate. Put

$$\Omega=\begin{pmatrix} p_x+i_ep_y & p_t+q & 0 & 0 \\ -(p_t-q) & -(p_x-i_ep_y) & 0 & 0 \\ 0 & 0 & p_x-i_ep_y & p_t+q \\ 0 & 0 & -(p_t-q) & -(p_x+i_ep_y) \end{pmatrix}.$$

where $q=i_hp_z+i_ei_hmc$. Then Dirac's equations become

$$\Omega\begin{pmatrix}\psi_3\\ \psi_4\\ \psi_2\\ \psi_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Leftrightarrow\Omega\varsigma=0$$

Obviously, $\det\Omega=\left(p_t^2-p_x^2-p_y^2-p_z^2-(mc)^2\right)^2=0$. Therefore, it is possible to consider the system of equations as the system with nontrivial solutions. If in (21) ς to represent in the form $\varsigma=u+i_tv+i_hp+i_ei_hq=u+i_ev+i_h\left(p-i_eq\right)\equiv W+i_hQ$, then $\varsigma_x=W_x+i_hQ_x$, $\varsigma_y=W_y+i_hQ_y$. In this representation Q is transformed not in the same way as W, but with complex conjugate equations. Consequently, supposing $\psi_1\equiv\partial_xW,\ \psi_2\equiv\partial_yW,\ \psi_3\equiv\partial_xQ,\ \psi_4\equiv\partial_yQ$, where $\partial_x,\ \partial_y-$ partial derivatives of the corresponding functions, we obtain the system of equations

$$\begin{pmatrix} p_x+i_ep_y & i_h\left(p_t+i_hp_z\right) & 0 & 0 \\ -i_h\left(p_t-i_hp_z\right) & -(p_x-i_ep_y) & 0 & 0 \\ 0 & 0 & p_x-i_ep_y & i_h\left(p_t+i_hp_z\right) \\ 0 & 0 & -i_h\left(p_t-i_hp_z\right) & -(p_x+i_ep_y) \end{pmatrix}\begin{pmatrix}\partial_xW\\ \partial_yW\\ \partial_xQ\\ \partial_yQ\end{pmatrix}=$$

$$\begin{pmatrix}i_emc\varphi_1\\ i_emc\varphi_2\\ i_emc\varphi_3\\ i_emc\varphi_4\end{pmatrix}\Leftrightarrow\begin{pmatrix}\mathrm{M} & 0\\ 0 & \mathrm{N}\end{pmatrix}\begin{pmatrix}\tau\\ \upsilon\end{pmatrix}=\begin{pmatrix}\rho\\ \sigma\end{pmatrix}$$

Let

$$\left(\begin{array}{cc} \mathrm{M} & 0 \\ 0 & \mathrm{N} \end{array}\right)^{+} \tag{29}$$

be the Hermitian conjugate matrix. It is easy to check that

$$\left(\begin{array}{cc} \mathrm{M} & 0 \\ 0 & \mathrm{N} \end{array}\right)^{+} = \left(\begin{array}{cc} \mathrm{N} & 0 \\ 0 & \mathrm{M} \end{array}\right), \left(\begin{array}{cc} \mathrm{M} & 0 \\ 0 & \mathrm{N} \end{array}\right)^{+} \left(\begin{array}{cc} \mathrm{M} & 0 \\ 0 & \mathrm{N} \end{array}\right) = \left(\begin{array}{cc} -(mc)^2 & 0 \\ 0 & -(mc)^2 \end{array}\right).$$

Multiplying equality (29) by the conjugate matrix and using the uniqueness of the corresponding subspaces of the operators M, N, we conclude that $\varphi_1 = \partial_x Q,\ \varphi_2 = \partial_y Q, \varphi_3 = \partial_x W,\ \varphi_4 = \partial_y W$. Taking these into consideration, the system (29) becomes

$$\left\{\begin{array}{l} (p_x + i_e p_y)\,\partial_x W + i_h\,(p_t + i_h p_z)\,\partial_y W = i_e mc \partial_x Q, \\ -i_h\,(p_t - i_h p_z)\,\partial_x W - (p_x - i_e p_y)\,\partial_y W = i_e mc \partial_y Q, \\ (p_x - i_e p_y)\,\partial_x Q + i_h\,(p_t + i_h p_z)\,\partial_y Q = i_e mc \partial_x W, \\ -i_h\,(p_t - i_h p_z)\,\partial_x Q - (p_x + i_e p_y)\,\partial_y Q = i_e mc \partial_y W. \end{array}\right.$$

It is the same form of the Dirac's equations system for the particles with a semispin and with three imaginary units. Suppose that this system describes the particles with integer spins too

$$\left\{\begin{array}{l} \beta\partial_x u + \gamma\partial_y u = -\partial_x p, \\ \beta\partial_x v + \gamma\partial_y v = \partial_x q, \end{array}\right. \quad \left\{\begin{array}{l} \beta\partial_x p + \gamma\partial_y p = -\partial_x u, \\ \beta\partial_x q + \gamma\partial_y q = \partial_x v, \end{array}\right.$$

$$\left\{\begin{array}{l} \beta\partial_x u + \gamma\partial_y u = -\partial_x v, \\ \beta\partial_x p + \gamma\partial_y p = -\partial_x q, \end{array}\right. \quad \left\{\begin{array}{l} \beta\partial_x v + \gamma\partial_y v = \partial_x u, \\ \beta\partial_x q + \gamma\partial_y q = -\partial_x p. \end{array}\right.$$

From these relations the obvious equalities $\partial_x v = \partial_x p,\ \partial_x u = \partial_x q$ follow. Similarly

$$\left\{\begin{array}{l} \alpha\partial_x u + \beta\partial_y u = -\partial_y p, \\ \alpha\partial_x v + \beta\partial_y v = \partial_y q, \end{array}\right. \quad \left\{\begin{array}{l} \alpha\partial_x p + \beta\partial_y p = -\partial_y u, \\ \alpha\partial_x q + \beta\partial_y q = \partial_y v, \end{array}\right.$$

$$\left\{\begin{array}{l} \alpha\partial_x u + \beta\partial_y u = \partial_y v, \\ \alpha\partial_x p + \beta\partial_y p = \partial_y q, \end{array}\right. \quad \left\{\begin{array}{l} \alpha\partial_x v + \beta\partial_y v = -\partial_y u, \\ \alpha\partial_x q + \beta\partial_y q = -\partial_y p, \end{array}\right.$$

whereof we have $\partial_y v = -\partial_y p,\ \partial_y u = -\partial_y q$. The eigenfunctions of the operator A satisfy the following equations system

$$\partial_x u = \beta\partial_x v + \gamma\partial_y v,\ i\,{}_m^2\partial_x v = \beta\partial_x u + \gamma\partial_y u\ .$$

or, equivalently, the system (14)

$$-\partial_y u = \alpha\partial_x v + \beta\partial_y v,\ i_m^2\partial_y v = \alpha\partial_x u + \beta\partial_y u\ .$$

where $i_m^2 = i_e^2$=-1 and $i_m^2 = i_h^2 = 1$. Consequently $\alpha\gamma - \beta^2 = -i_m^2$.

Let us study the problem of the baryon charge conservation (Section 2, point 4). From the Vigner's theorem (Section 1, point 5)and from the statements from the article of Sakurai [26] it follows that the operators which do not form unitary groups can be used to prove conservation law. Therefore, in the representation of the baryons multiplets [19, p. 20], [27, p. 140], instead of the quarks u, d, s, we substitute the operators A, B, C. After suitable computations (physically it can mean – after the strong decay) we get a matrix, by dividing its right upper entry by -2γ, we obtain the number which coincides with the third projection of the baryon and resonance's isotopic spin. On the basis of electric charge conservation laws, strangeness of strong interactions and Gel-Mann-Nashidjimi formulas it is possible to conclude that, in this case, the baryon charge conserves. The abovementioned computations are followed. There are the following completely symmetric states

$$a)\ p = uud + udu + duu = AAB + ABA + BAA = -B = \begin{pmatrix} -\beta & -\gamma \\ \frac{\beta^2-1}{\gamma} & \beta \end{pmatrix};$$

$$b)\ n = udd + dud + ddu = BBA + BAB + ABB = A = \begin{pmatrix} \beta & \gamma \\ \frac{-(\beta^2+1)}{\gamma} & -\beta \end{pmatrix};$$

$$c)\ \Sigma^+ = uus + usu + suu = AAC + ACA + CAA = -(A + C) = \begin{pmatrix} -2\beta & -2\gamma \\ \frac{2\beta^2+1}{\gamma} & 2\beta \end{pmatrix};$$

$$d)\ \Sigma^0 = uds + usd + sud + sdu + dus + dsu = A - B = -2D = \begin{pmatrix} 0 & 0 \\ \frac{-2}{\gamma} & 0 \end{pmatrix};$$

$$e)\ \Sigma^- = dds + dsd + sdd = C + B = 2C + D = \begin{pmatrix} 2\beta & 2\gamma \\ \frac{-(2\beta^2-1)}{\gamma} & -2\beta \end{pmatrix};$$

$$f)\ \Xi^0 = uss + sus + ssu = -C = -\frac{(A+B)}{2} = \begin{pmatrix} -\beta & -\gamma \\ \frac{\beta^2}{\gamma} & \beta \end{pmatrix};$$

$$g)\ \Xi^- = dss + sds + ssd = C = \frac{A+B}{2} = \begin{pmatrix} \beta & \gamma \\ \frac{-\beta^2}{\gamma} & -\beta \end{pmatrix};$$

$$h)\ \Lambda^0 = uds + usd + sud + sdu + dus + dsu = -2D = \begin{pmatrix} 0 & 0 \\ \frac{-2}{\gamma} & 0 \end{pmatrix}.$$

These are baryons, where p is the proton, n is the neutron. Multiplying right upper element of the obtained matrices by $-1/(2\gamma)$ we obtain the third projections of the isotopic spin

$$\begin{aligned} a)\ T_3(p) &= \frac{1}{2},\ b)\ T_3(n) = -\frac{1}{2},\ c)\ T_3(\Sigma^+) = 1,\ d)\ T_3(\Sigma^0) = 0, \\ e)\ T_3(\Sigma^-) &= -1,\ f)\ T_3(\Xi^0) = \frac{1}{2},\ g) T_3(\Xi^-) = -\frac{1}{2},\ h)\ T_3(\Lambda^0) = 0. \end{aligned}$$

In order to obtain the same results for resonances, the concept of quark colors are introduced: R - red, G - green, B - blue. Each quark participating in the multiplet more than once has different colors what allows us to avoid Pauli exclusion principle [27, p. 123 - 125]. So, for the resonances we have (we write the quark colors in brackets)

$$\Delta^0 = sss = s\,(R)\,s\,(G)\,s\,(B) + s\,(G)\,s\,(B)\,s\,(R) + s\,(B)\,s\,(R)\,s\,(G) =$$
$$= CCC + CCC + CCC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$
$$\Delta^{++} = ddd = d\,(R)\,d\,(G)\,d\,(B) + d\,(G)\,d\,(B)\,d\,(R) + d\,(B)\,d\,(R)\,d\,(G) =$$
$$= AAA + AAA + AAA = \begin{pmatrix} -3\beta & -3\gamma \\ \frac{3(\beta^2+1)}{\gamma} & 3\beta \end{pmatrix};$$
$$\Delta^{+} = uuu = u\,(R)\,u\,(G)\,u\,(B) + u\,(G)\,u\,(B)\,u\,(R) + u\,(B)\,u\,(R)\,u\,(G) =$$
$$= BBB + BBB + BBB = \begin{pmatrix} 3\beta & 3\gamma \\ \frac{-3(\beta^2-1)}{\gamma} & -3\beta \end{pmatrix}.$$

Repeating the same procedure as for the baryons, we obtain

$$T_3\left(\Delta^0\right) = 0,\ T_3\left(\Delta^{++}\right) = \frac{3}{2},\ T_3\left(\Delta^{+}\right) = -\frac{3}{2}$$

The obtained results completely coincide with the famous data [19, p. 19], [28, p. 466,467]. Consider the application of imaginary units to quarks color charges. Quarks are fermions. They obey the Pauli principle according to which two particles of a system can not lie in one and the same quantum state. But in many quarks (for example, in nucleons uuu and ddd) two, or even three quarks in the hadron have the same quantum numbers. Therefore, it is necessary to assign an additional quantum number to the quark which is named the *color charge* and which can take three possible values for each quarks' "aroma", i.e. for each quark type. These charges three, conditionally named as color state and characterizing an abstract concept of quark colors, are: *red* (R), *green* (G) and *blue* (B). Corresponding to it, the antiquarks are characterized by *anticolor*. The color is impossible to be directly observed because the nature is *colorless*, the distance 10^{-15} meters corresponding to the beginning of the nucleon's interior structure. Any hadron is *white*, that is it consists of the quarks of different colors neutralizing one another. Each color R, G, B corresponds to the particular combination of color charges. It is supposed that there are three color charges: *red minus green*(RG), *green minus blue* (GB), *blue minus red* (BR). Each color charge can take the values: $+1/2, -1/2, 0$ and any quark color is a combination of these numbers. For example, for red quark red-green color (RG) charge is equal to + 1/2, green-blue (GB) charge is equal to 0 and blue-red (BR) charge is equal to -1/2. For the green quark the color charges are equal to: $(RG) = -1/2, (GB) = +1/2, (BR) = 0$. For the blue quark they are: $(RG) = 0, (GB) = -1/2, (BR) = +1/2 \quad [27, p.134]$. These results can be obtained in the following way. With each quark color we associate the imaginary units i_p, i_e, i_h in this and only in this order. Moreover i_p corresponds to the quark color while the other two imaginary units correspond to the remained quark colors in strict correspondence to their position $R,\ G,\ B$. For example, if the quark is green, then the correspondences are $i_p \to G,\ i_e \to B,\ i_h \to R$ and the color charges are computed as follows

$$(GB) = \frac{(i_p + i_e)\,\overline{(i_p + i_e)}}{2 \cdot 2} = \frac{-\left(i_p^2 + i_p i_e + i_e i_p + i_e^2\right)}{4} = \frac{-\left(0 - i_h^2 - 1\right)}{4} = +\frac{1}{2},$$

$$\begin{aligned}(RG) &= \tfrac{(i_h+i_p)\overline{(i_h+i_p)}}{2\cdot 2} = \tfrac{-\left(i_h^2+i_h i_p+i_p i_h+i_p^2\right)}{4} = \tfrac{-\left(1-i_e^2-0\right)}{4} = -\tfrac{1}{2},\\ (BR) &= \tfrac{(i_e+i_h)\overline{(i_e+i_h)}}{2\cdot 2} = \tfrac{-\left(i_e^2+i_e i_h+i_h i_e+i_h^2\right)}{4} = \tfrac{-\left(\ -1-i_p^2+1\right)}{4} = 0.\end{aligned}$$

For the red quark we have: $i_p \to R,\ i_e \to G,\ i_h \to B$ and $(RG) = \frac{(i_p+i_e)\overline{(i_p+i_e)}}{2\cdot 2} = \frac{1}{2},\ (GB) = \frac{(i_e+i_h)\overline{(i_e+i_h)}}{2\cdot 2} = 0,\ (BR) = \frac{(i_h+i_p)\overline{(i_h+i_p)}}{2\cdot 2} = -\frac{1}{2}$.

For the blue quark we have: $i_p \to B,\ i_e \to R,\ i_h \to G$ and $(RG) = \frac{(i_e+i_h)\overline{(i_e+i_h)}}{2\cdot 2} = 0,\ (GB) = \frac{(i_h+i_p)\overline{(i_h+i_p)}}{2\cdot 2} = -\frac{1}{2},\ (BR) = \frac{(i_p+i_e)\overline{(i_p+i_e)}}{2\cdot 2} = \frac{1}{2}$. These results perfectly agree with the famous data mentioned above.

Consider that the fourth quantum state F is black. The quark colors are arranged in the order: *R, F, G, B* while four color charges are described by: red-black *RF*, black-green *FG*, green-blue *GB*, blue-red *BR*. The computation of these color charges proceed similarly. The imaginary unit i_d corresponds to the black quark. We obtain the following color charge values for the red quark: $RF = -1/4, FG = 0, GB = 0, BR = 1/2$. For the black quark we have: $RF = -1/2, FG = -1/4, GB = 0, BR = 0$. For the green quark we have: $RF = 0, FG = -1/2, GB = -1/4, BR = 0$ while for the blue quark we have: $RF = 0, FG = 0, GB = -1/2, BR = -1/4$.The obtained asymmetry certifies that the states qq and $qqqq$ can not be obtained by means of experiments. Maybe this is related to the problem of Salam, Pati.

Consider the case when the quantity of color states is equal to five and they are arranged in the order: *R, G, F, Y, B*, where Y is the symbol for the yellow color. The imaginary units $i_p,\ j,\ i_d,\ i,\ i_h$ correspond to the red color. Then the quarks' color charges read

Red: $RG = \frac{1}{2},\ GF = 0,\ FY = 0,\ YB = 0,\ BR = -\frac{1}{2}$,

Green: $RG = -\frac{1}{2},\ GF = \frac{1}{2},\ FY = 0,\ YB = 0,\ BR = 0$,

Black: $RG = 0,\ GF = -\frac{1}{2},\ FY = \frac{1}{2},\ YB = 0,\ BR = 0$,

Yellow: $RG = 0,\ GF = 0,\ FY = -\frac{1}{2},\ YB = \frac{1}{2},\ BR = 0$,

Blue: $RG = 0,\ GF = 0,\ FY = 0,\ YB = -\frac{1}{2},\ BR = \frac{1}{2}$.

The obtained results lead us to the conclusion: the state $qqqqq$ is completely possible. The color charges are transmitted when they are involved into the direct contact among each-other otherwise the charges' value is equal to zero.

11. CONCLUSION

1. The Beltrami equations system (BES) in the operator form reads $A\varsigma = i\varsigma$, where A is the matrix depending on the coefficients of the Beltrami equations system and ς is the gradient spinor. This notation of BES allows us to reveal the new side of BES, in particular it is the Cauchy-Riemann equations system.

When the Cauchy-Riemann equations system is dealt with, the matrix $-iA$ is one of the Pauli matrices, σ_y. Then the question arises: which are the systems of equations corresponding to the other Pauli matrices σ_y, σ_z. The eigenvalues of the operator A satisfy the equation $i^4 = 1$, i.e. $i^2 = \pm 1$. The case $i^2 = -1$ is connected with analyticity, while the case $i^2 = 1$, where $i \neq \pm 1$, corresponds to the other representation of the operator A, which is denoted by the operator B (hereinafter operator B). In a special case, the MES of hyperbolical type corresponds to the simplest hyperbolical system: $u_x = v_y, \quad u_y = v_x$ (where u_x, v_y, u_y, v_x are the partial derivatives of the functions $u = u(x, y), v = v(x, y)$ with respect to the corresponding arguments). The corresponding operator is the Pauli matrix σ_x, while the product of A by B, in these simplest cases, is the Pauli matrix σ_z. Moreover, the matrices $A^{2s}$and B^{2s} describe correspondingly the interchange operators in the space of identity particles of fermions and bosons respectively. Thus, the complex analysis complemented with the variables in the space of dual and double numbers describes the basic properties of particles. The operators A and B of MES are applied to describe strongly interactive particles. The operators A and B have the properties $A^4 = E$, $B^2 = E$ (where E is the unit matrix). Hereof we can deduce that all electron shells of chemical elements are located on the Riemann surface of the $g \leq 4$ genus. The first (principal) quantum number of the particle is $n = g + 1 \leq 4$ (where g is the genus of the Riemann surface). Since $A^4 = E$ we conclude that the electron shells of chemical elements are possible to be of genus of the Riemann surface not larger than 4. It is confirmed by the fact that in the electron shells of chemical elements there are no more than $32 = 2 \cdot 4^2$ electrons. It is also possible to describe the baryons decay based on the triplet of all baryons and the properties of the operators A, B, C, where $C = (A + B)/2$. The imaginary units introduced by the author i_e, i_h, i_p are double and dual complex numbers. They describe exactly quarks color and correlations between them and conclude the color charges of the quarks $u,\ d,\ s$. It is also investigated other matrix $D = (B - A)/2$ which has the properties $AD+DA = E, BD+DB = E, CD+DC = E, D^2 = 0$. Taking into account the properties of the stereographic transformation and the properties of MES, it is possible to explain the Fock's puzzle with the period 4. Thus, there are all sound grounds to consider that the matrices A, B, C, D bear the fundamental features to describe the nature in general. They may serve as quarks $u = A,\ d = B,\ s = C,\ b = D$. At the same time, the generalized equations and matrices of Dirac can also be built on the basis of the operators *A, B, AB.*

What is the generalization of the Dirac's matrices? The Dirac equations and matrices are studied in the Minkowskian curved space-time. The grid on the Riemann surface is described by the two arbitrary functions $\beta = \beta(x, y)$, $\gamma = \gamma(x, y)$, where (x, y) are local coordinates of the Riemann surfaces. The Christoffel's symbols of this net are computed. Taking into account all these, it can be proved the existence theorem of mappings of some domains on the surface onto the plane domains. It is here where occur the operators A, B, C, D which describe the above-mentioned physical properties. It is proved that the constructed grid on the Riemann surface represents isothermal coordinates. Consequently, the homeomorphism of the surface domain onto the plane domain is independent of the MES type. It rather corresponds to the ideas of Riemann, Clifford, and Einstein: *everything* from the empty curved space.

In the framework of the developed theory it is very easy to explain the Pauli exclusion principle, Heisenberg's uncertainty principle, mathematical basis of symmetry breakdown for weak interactions etc.

Conclusion: Mixed analyticity, i.e. analytic functions corresponding to the mixed system of the first order differential equations and the fact that $u_x = \beta v_x + \gamma v_y,\ i_m^2 v_x = \beta u_x + \gamma u_y$, where $i_m^2 = 0, \pm 1$ and $\beta = \beta(x, y),\ \gamma = \gamma(x, y),\ (x, y) \in D \subset S$, describe the main laws of nature. Studying this equation in space-time, we get the Maxwell's system of equations on the surface [32]

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \frac{\beta}{\gamma}\left[\left(\frac{\partial \mathbf{\Omega_x}}{\partial y} + \frac{\partial \mathbf{\Omega_x}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial \mathbf{\Omega_y}}{\partial x} + \frac{\partial \mathbf{\Omega_y}}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \mathbf{\Omega_z}}{\partial x} + \frac{\partial \mathbf{\Omega_z}}{\partial y}\right)\mathbf{k}\right] +$$
$$+\frac{1}{\gamma} rot\mathbf{\Omega} = 4M^\circ N^\circ i\mathbf{e},$$

where $\mathbf{\Omega} = E + iH$; $\quad \mathbf{e} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{i}$, $\quad \mathbf{j}$, $\mathbf{k}$ are the basis vectors, $E + iH$ is the voltage of electromagnetic field, $M^\circ,\ N^\circ$ are integrating multipliers of the forms $\gamma dx - (\beta + 1)dy, \quad \gamma dx - (\beta - 1)dy$. In the case $\beta = 0, \quad \gamma = 1$ we get the famous Maxwell's equation, where $\mathbf{\Omega_x},\ \mathbf{\Omega}_y,\ \mathbf{\Omega}_z$ are the projections of $\mathbf{\Omega}$ on the corresponding axis.

Definitely, other authors may have solved some of the above-stated problems in due courses. But in our paper all these problems are solved with a unique method or rather with the method of Minkowskian curved space-time and essentially with the introduced mixed analyticity method. But the mixed "analyticity" is an elliptic-hyperbolic dualism as corpuscular-undulatory dualism in quantum theory.

The following conclusion follows from the abovestated results: the classical Beltrami equations system and MES must play a fundamental role in studying the universe. The author expresses his gratitude to his student Yusupov Hurshidbek for his active assistance in the preparation of this article and for the discussion of some results where he took a direct part in their formation.

The operator form of the Beltrami equations system is realized in [13], [18]. Some quantum mechanical interpretations follow from this operatorial writing. Detailed proof of the Riemann's theorem on existence is given in [14] , [15]. The introduction of imaginary units and their multiplication table is realized in [29]. Applications of our results to quantum mechanical problems and economics are studied in [20], [22], [30], [31]

References

[1] Shulikovskiy, V., I., *Classical differential geometry*, Physical-Mathematical Press, Moscow, 1969.

[2] Shutts, B., *Geometrical methods of mathematical physics*, Mir, Moscow, 1984.

[3] Postnikov, M., M., a) *Lie groups and algebras*, Nauka, Moskow, 1982; b) *Linear algebra and differential geometry*,Nauka, Moscow, 1980.

[4] Lavrentev, M., A., *On one problem of sticking*, Siberia Methematical Journal, **5**, 3(1964), 603-607.

[5] Lavrentev, M., A., Shabat, B., V., *Hydrodynamic problems and their mathematical models*, Nauka, Moscow, 1977.

[6] Kuzmin, A. G., *Non-classical equations of mixed type and their applications to gasdynamics*, LSU Press, Leningrad, 1990.

[7] Suvorov, G., D., *Generalized principle "length and area principle" in mappings theory*, Naukovo Dumka, Kiev, 1985.

[8] Bers, L., *Mathematical problems of subsonic and supersonic gasdynamics*, Literature Press, Moscow, 1961.

[9] Volkovitskiy, L. I., *Quasi-conformal mappimgs*, Lvov University Press, 1954.

[10] Logunov A. A., Mestvirishvili, M. A., *Relativistic theory of gravitation*, Nauka, Moscow, 1989.

[11] Aleksandrov, A. D., Netsvetaev, N. Y., *Geometry*, Nauka, Moscow, 1990.

[12] Penrouz, R., Rindler, V., *Spinors and space-time*, vol.1, Mir, Moscow, 1987.

[13] Zakhirov, M., *Quantum mechanical interpretation of mixed quasi-conformal mappings*, AS of UzSSR, **8**(1990), 9-11.

[14] Zakhirov, M., *Spinors of mixed quasi-conformal mappings*, in Problems of Applied and Computational Mathematics, Collection of Scientific Papers, AS of UzSSR, **89**(1990), 83-94.

[15] Zakhirov, M., *Quantum signatures and mixed mappings*, in Problems of Applied and Computational Mathematics, Collection of Scientific Papers, AS of UzSSR, **99**(1995), 103-115.

[16] Rachevskiy, P. K., *Riemann geometry and tensor analysis*, Nauka, Moscow, 1964.

[17] Dubrovin, B. A., Novikov, S. P., Fomenko, A. T., *Modern geometry*, Nauka, Moscow, 1979.

[18] Zakhirov, M., *Geometrodynamics of ensemble from two particles*, AS of RUz, **2**(1992), 9-11.

[19] Rayder, L., *Quantum theory of field*, Mir, Moscow, 1987.

[20] Islamov, K. A., Zakhirov, M., *Spin structure of Riemannian manifold and its applications*, in Problems of Applied and Computational Mathematics: Collection of Scientific Papers, AS of RUz, **111**(2003), 170-180.

[21] Vigner, E., *Symmetry studies*, Mir, Moscow, 1971.

[22] Zakhirov, M., *MES automorphisms of vector fields and their applications*, Third International Conference "Modern problems of mathematical physics and informational technologies", Tashkent, **1**, 2005, 275-280.

[23] Kantor, I. L., Solodovnikov, A. S., *Hypercomplex numbers*, Nauka, Moscow, 1973.

[24] Boum, A., *Quantum mechanics: bases and applications*, Mir, Moscow, 1990.

[25] Kompaneets, A. S., *Course of theoretical physics*, Prosveshenie, Moscow, 1972.

[26] Sakurai, J. J., Annual Physics, 11(1960).

[27] Blan, D., *Nucleus, particles, nuclear reactors*, Mir, Moscow, 1989.

[28] Yavorskiy, B. M., Detlaf A. A., *Handbook of physics*, Nauka, Moscow, 1981.

[29] Zakhirov, M., *Complex analysis with many imaginary units and quark model of hadrons*, in Problems of applied and computational mathematics, Collection of Scientific Papers, AS of RUz, 107(2000), 161-170.

[30] Zakhirov, M., *Mixed analytic functions in physics of elementary particles* Third International Conference "Modern problems of nuclear physics", Samarkand, 2003, 55-56.

[31] Zakhirov, M., *Econometric and geometric analysis of Cobb-Douglas and CES production functions*, ROMAI J., **1**, 1(2005), 237-242.

[32] Zakhirov, M., *Calibrating transformations in the theory of mixed quasi-conformal mappings*, Third USSR Conference on geometrical theory of physics, Novosibirsk, 1988, p. 40.