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VISCOUS FLOWS DRIVEN BY GRAVITY AND A SURFACE TENSION GRADIENT

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Abstract The lubrication approximation used to investigate the flow of a thin layer is deduced. The starting point for modeling the flow of thin films are the Navier-Stokes equations. The lubrication or reduced Reynolds number approximation to the Navier-Stokes equations has been used to describe a multitude of situations. Our attention has been focussed on the situations in which the surface tension plays an important role.

1. INTRODUCTION

We consider the flow of a thin fluid films where the surface tension is a driving mechanism. In general the introduction of surface tension into standard lubrication theory leads to a fourth-order nonlinear parabolic equation [6], [2]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3 + \left(\frac{\partial^3 h}{\partial x^3} + a\frac{\partial h}{\partial x} + b\right)\right) = 0, \tag{1}$$

where $h = h(x,t)$ is the fluid film height and a, b are constants. For steady situations this equation may be integrated once and a third-order ordinary differential equation is obtained. Appropriate forms of equation (1) have been used to model fluid flows in a number of physical situations such as coating, draining of foams and the movement of contact lenses.

2. PROBLEM FORMULATION

We consider the flow of a thin film on an inclined plane at an angle α to the horizontal plane. Suppose that a Newtonian fluid, of constant density ρ

and the dynamic viscosity μ is undergoing an unsteady flow. With respect to a Cartesian co-ordinates system Oxy as indicated in fig. 1, the velocity has the form $\mathbf{u} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$.

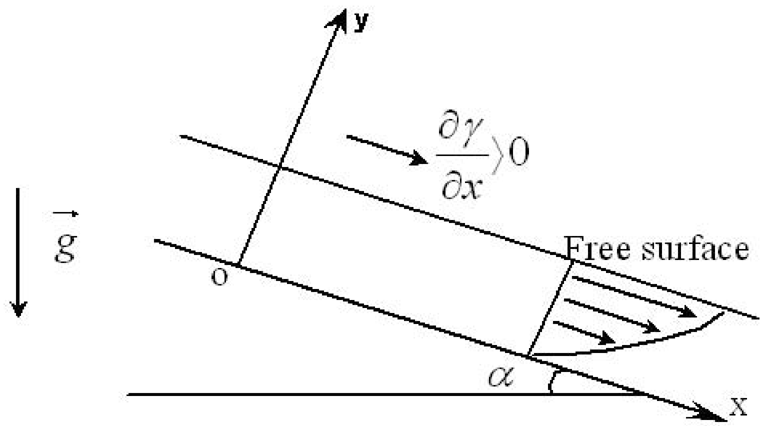

Figure 1

The Navier-Stokes equations are [7]

$$u_t = -\frac{1}{\rho} p_x + \nu \Delta u + g \sin \alpha$$

$$v_t = -\frac{1}{\rho} p_y + \nu \Delta v - g \cos \alpha.$$

Let us non-dimensionalize the length by the length scales (L, h_0) and velocity scales $(U, \delta U)$, where L is a typical length along the film, h_0 a typical film thickness and $\delta = h_0/L << 1$ is the aspect ratio (typical height/typical length scale) [7]. In keeping with the assumption that the surface tension is a mechanism of comparable strength to viscous forces the velocity scale U is chosen to make the capillary number $Ca = \delta^3 \sigma / 3\mu U = 1$. Hence $U = \delta^3 \sigma / 3\mu$, where σ is the value of surface tension (assumed constant although). Note that if the gravity g is dominating, the appropriate scaling is $U = \delta^2 \rho g L^2 / 3\mu$. The vertical velocity v is of order dh/dt and by virtue of $\nabla \cdot \mathbf{u} = 0$ the horizontal velocity u is of order $(L/h) \cdot dh/dt$. Then, we have $\frac{dh}{dt} \sim U\frac{h}{L}$ and so $\frac{d^2h}{dt^2} \sim h\frac{U^2}{L^2}$. Orders of magnitude estimated are the following

$$\frac{\partial v}{\partial t} \sim U^2 \frac{h}{L^2}$$

and

$$\frac{\partial u}{\partial t} \sim L\frac{d}{dt}\left(\frac{v}{h}\right) = 0.$$

These estimates show that the term $\mathbf{u}_t$ may be neglected $(h/L^2 << h/L << 1)$. To first order in δ the Navier-Stokes equations reduce to

$$-p_x + \frac{1}{3}u_{yy} + Bo \sin \alpha = 0, \tag{2}$$

$$-p_y - \delta Bo \cos \alpha = 0. \tag{3}$$

This forms are the thin-film approximation. Here subscripts denote differentiation with respect to the variable. Whilst the continuity equation is unchanged

$$u_x + v_y = 0. \tag{4}$$

The Bond number, $Bo = \delta^2 \rho g L^2 / 3\mu U$, is a ratio of gravity to viscous forces. In order for the gravity terms to be non-negligible either $Bo \sin \alpha$ or $\delta Bo \cos \alpha$ must be $O(1)$. The scaling for fluid pressure p is chosen to balance pressure with viscous forces and is $\delta^2 L/3\mu U$. The appropriate approximate boundary conditions on the free surface $y = h(x,t)$ are [1]

$$\left\{ \begin{array}{rcl} v & = & h_t + u h_x, \\ p & = & -h_{xx}, \\ u_y & = & 0. \end{array} \right. . \tag{5}$$

Here p is the pressure in the fluid. These equations (5) represent the kinematic condition, pressure balancing surface tension and zero shear respectively. The pressure condition is the Laplace-Young equation which reflects the fact that normal stress due to surface tension is proportional to curvature [7]. On the substrate, $y = 0$, the no-slip condition reads

$$u = v = 0. \tag{6}$$

Integrating equation (3) we get an expression for fluid pressure

$$p = -h_{xx} - \delta Bo \sin\alpha \cdot (y - h). \tag{7}$$

Integrating (2) twice and imposing the boundary conditions leads to

$$u = 3(p_x - Bo \sin\alpha)\left(\frac{y^2}{2} - hy\right). \tag{8}$$

This may be used in the continuity equation (4) to determine v. In particular, on the free surface we have

$$v(h) = -\int_0^h u_x dy. \tag{9}$$

This expression together with the kinematic conditions lead to the governing equation for the film height

$$h_t + \frac{\partial}{\partial x}[h^3(h_{xxx} - \delta Bo h_x \cos\alpha + Bo \sin\alpha)] = 0. \tag{10}$$

This is a fourth-order nonlinear degenerate equation. Implicit in the derivation of this equation is the assumption that surface tension and gravity effects are of the same order. A gravity-dominated system would not include the fourth-order surface tension term and the velocity scale, at present based on surface tension, should be changes accordingly.

3. DISCUSSION

Equations of the from (10), apply if the fluid motion is constrained by a no-slip, or similar, condition on one surface, producing a shear flow. If surface tension variations occur, then the zero shear condition becomes $u_y = \frac{\sigma_x}{\delta^2}$, where σ_x is the surface tension gradient [4]. The boundary conditions on the free surface become

$$\begin{cases} v = h_t + uh_x, \\ p = -h_{xx}, \\ u_y = \frac{\sigma_x}{\delta^2}. \end{cases} \quad (11)$$

The continuity equation (4) is unchanged. Then, the evolution equation for the film height is [2]

$$h_t + Ca \cdot \delta^2 \sigma_x \cdot h \cdot h_x + \frac{3}{2\delta} Bo \sin\alpha \cdot h^2 \cdot h_x - \frac{\partial}{\partial x}(h^3 \cdot h_x). \quad (12)$$

This is a nonlinear parabolic equation. A traveling wave substitution $h(x,t) = H(x-t)$ will reduce (12) to a nonlinear dynamic system [5]. For studying this dynamic system we have tried to determine the invariant manifold and the invariant subspaces, corresponding to the linear system. Results show qualitative agreement with experiments. For small time the solution to the linearized problem is found in the form of an infinite series [3].

4. CONCLUDING REMARKS

We deduce the lubrication approximation for the flow of a thin liquid layer down an inclined plane simultaneously driven by a surface tension gradient. We impose the boundary conditions and then we deduce the governing equation for the film height.

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