

# EXPONENTIAL STABILITY IN MEAN SQUARE OF DISCRETE-TIME TIME-VARYING LINEAR STOCHASTIC SYSTEMS WITH MARKOVIAN SWITCHING

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**Abstract** The problem of the mean square exponential stability for a class of discrete-time time-varying linear stochastic systems subject to Markovian switching is investigated. The case of the linear systems whose coefficients depend both on present state and the previous state of the Markov chain is considered. Three different definitions of the concept of exponential stability in mean square are introduced and it is shown that they are not always equivalent. One definition of the concept of mean square exponential stability is done in terms of the exponential stability of the evolution defined by a sequence of linear positive operators on an ordered Hilbert space. The other two definitions are given in terms of different types of exponential behavior of the trajectories of the considered system. In our approach the Markov chain is not prefixed. The only available information about the Markov chain is the sequence of probability transition matrices and the set of its states. In this way one obtains that if the system is affected by Markovian jumping, the property of exponential stability is independent of the initial distribution of the Markov chain.

The definition expressed in terms of exponential stability of the evolution generated by a sequence of linear positive operators, allows us to characterize the mean square exponential stability based on the existence of some quadratic Lyapunov functions.

The results developed in this paper may be used to derive some procedures for designing stabilizing controllers for the considered class of discrete-time linear stochastic systems in the presence of a delay in the data transmission.

**Keywords:** discrete-time linear stochastic systems, Markov chains, mean square exponential stability, Lyapunov operators, delays in data transmission.

## 1. INTRODUCTION

Stability is one of the main tasks in the analysis and synthesis of a controller in many control problems such as: linear quadratic regulator,  $H_2$  optimal control,  $H_\infty$ -control and other robust control problems (see e.g. [9, 16, 24] for the continuous time case or [1, 4, 5, 15] for the discrete-time case and references therein).

For linear stochastic systems there are various types of stability. However one of the most popular among them is "*exponential stability in mean square*" (ESMS). This is due to the existence of some efficient algorithms to check this property.

In the literature, the problem of ESMS of discrete-time linear stochastic systems with Markovian jumping was intensively investigated. For the readers convenience we refer to [2, 3, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22]. The majority of the works where the problem of ESMS of discrete-time linear systems with Markovian switching is investigated, deals with the so-called time invariant case. That is the case of discrete-time linear systems with the matrix coefficients not depending on time and the Markov chain being homogeneous. There are papers [19, 21, 22] where the matrix coefficients of the system are time dependent and the Markov chain is homogeneous. There are also papers [12, 17] where the matrix coefficients of the discrete time system do not depend on time but the Markov chain is inhomogeneous.

In [8], it was considered the general situation of the discrete-time time-varying linear stochastic systems corrupted by independent random perturbations and by jump Markov perturbations. The general case of discrete-time time-varying linear systems with Markovian switching was investigated in [10].

The aim of the present paper is to extend the results of [10] to the case of discrete-time time-varying linear stochastic systems whose coefficients depend both to the present state  $\eta_t$  and the previous state  $\eta_{t-1}$  of the Markov chain. We assume that the matrix coefficients may depend on time and the Markov chain is not necessarily homogeneous. Such systems arise in connection with the problem of designing of a stabilizing feedback gain in the presence of some delay in transmission of the data either on the channel from the sensors to controller or between controller and actuators. For this class of systems we introduce three different definitions of the concept of exponential stability in mean square. One of these definitions characterizes the concept of exponential stability in mean square in terms of exponential stability of the evolution generated by a suitable sequence of linear positive operators associated with the considered stochastic system. This type of exponential stability in mean square which we shall call "strong exponential stability in mean square" (SESMS) is equivalent to the existence of a quadratic Lyapunov function; meaning that it is equivalent to the solvability of some systems of linear matrix equations or linear matrix inequations.

Other two types of exponential stability in mean square are stated in terms of exponential behavior of the state space trajectories of the considered stochastic systems.

We show that the three definitions of the exponential stability in mean square are not always equivalent. Also, we prove that under some additional assumptions part of types of exponential stability in mean square become equivalent. In the case of discrete-time linear stochastic systems with peri-

odic coefficients all these three types of exponential stability in mean square introduced in the paper become equivalent.

We remark that in our approach the Markov chain is not prefixed. The only available information about the Markov chain consists of the sequence of transition probability matrices  $\{P_t\}_{t \geq 0}$  and the set  $\mathcal{D}$  of its states. The initial distributions of the Markov chain do not play any role in defining and characterizing the exponential stability in mean square.

## 2. THE PROBLEM

### 2.1. DESCRIPTION OF THE SYSTEMS

Let us consider discrete-time linear stochastic systems of the form

$$x(t+1) = A(t, \eta_t, \eta_{t-1})x(t) \quad (1)$$

$t \geq 1, t \in \mathbf{Z}_+$ , where  $x \in \mathbf{R}^n$  and  $\{\eta_t\}_{t \geq 0}$  is a Markov chain defined on a given probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  with the state space the finite set  $\mathcal{D} = \{1, 2, \dots, N\}$  and the sequence of transition probability matrices  $\{P_t\}_{t \geq 0}$ . This means that for  $t \geq 0$ ,  $P_t$  are stochastic matrices of size  $N$ , with the property

$$\mathcal{P}\{\eta_{t+1} = j \mid \mathcal{G}_t\} = p_t(\eta_t, j) \quad (2)$$

for all  $j \in \mathcal{D}$ ,  $t \geq 0$ ,  $t \in \mathbf{Z}_+$ , where  $\mathcal{G}_t = \sigma[\eta_0, \eta_1, \dots, \eta_t]$  is the  $\sigma$ -algebra generated by the random variables  $\eta_s, 0 \leq s \leq t$ . For more details concerning the properties of Markov chains and of the sequences  $\{P_t\}_{t \geq 0}$  of stochastic matrices we refer to [6].

If  $P_t = P$  for all  $t \geq 0$  then the Markov chain is known as a **homogeneous Markov chain**.

In this paper we investigate several aspects of the issue of exponential stability in mean square of the solution  $x = 0$  of the system (1). Our aim is to reveal some difficulties which are due to the fact that the coefficients of the

system are time varying and  $\eta_t$  is an inhomogeneous Markov chain. In order to motivate the study of the problem of exponential stability in mean square for the case of stochastic systems of type (1) we consider the following controlled system

$$x(t+1) = A(t, \eta_t)x(t) + B(t, \eta_t)u(t)y(t) = C(t, \eta_t)x(t), \quad t \geq 0, \quad (3)$$

where  $x(t) \in \mathbf{R}^n$  is the state vector,  $u(t) \in \mathbf{R}^m$  is the vector of the control inputs and  $y(t) \in \mathbf{R}^p$  is the vector of the measurements. The aim is to design a control law of the form

$$u(t) = K(t, \eta_t)y(t) \quad (4)$$

with the property that the trajectories of the corresponding closed-loop system satisfy

$$\lim_{t \rightarrow \infty} E[|x(t)|^2] = 0. \quad (5)$$

In the case when on the channel from the sensor to the controller there exists a delay in transmission of the measurement then, the control (4) is replaced by

$$u(t) = K(t, \eta_t)y(t-1). \quad (6)$$

If the delay occurs on the channel from controller to actuators then instead of (4) we will have

$$u(t) = K(t-1, \eta_{t-1})y(t-1). \quad (7)$$

We have to solve the following two problems:

**P<sub>1</sub>:** *Find a set of sufficient conditions which guarantee the existence of the matrices  $K(t, i) \in \mathbf{R}^{m \times p}$ ,  $i \in \mathcal{D}$ ,  $t \geq 0$ , such that the trajectories of the closed-loop system obtained by coupling (6) with (3) will satisfy a condition of type (5).*

**P<sub>2</sub>:** *Find a set of conditions which guarantee the existence of the matrices  $K(t, i) \in \mathbf{R}^{m \times p}$  with the property that the trajectories of the closed-loop system*

obtained by coupling the control (7) with the system (3) satisfy a condition of type (5).

Coupling a control (6) to the system (3) one obtains the following closed-loop system

$$x(t+1) = A(t, \eta_t)x(t) + B(t, \eta_t)K(t, \eta_t)C(t-1, \eta_{t-1})x(t-1). \quad (8)$$

The closed-loop system obtained combining (7) to (3) is

$$x(t+1) = A(t, \eta_t)x(t) + B(t, \eta_t)K(t-1, \eta_{t-1})C(t-1, \eta_{t-1})x(t-1). \quad (9)$$

Setting  $\xi(t) = (x^T(t), x^T(t-1))^T$  we obtain the following version of (8) and (9)

$$\xi(t+1) = \tilde{A}(t, \eta_t, \eta_{t-1})\xi(t), \quad t \geq 1, \quad (10)$$

$$\xi(t+1) = \hat{A}(t, \eta_t, \eta_{t-1})\xi(t), \quad t \geq 1, \quad (11)$$

respectively, where

$$\begin{aligned} \tilde{A}(t, i, j) &= \begin{pmatrix} A(t, i) & B(t, i)K(t, i)C(t-1, j) \\ I_n & 0 \end{pmatrix}, \\ \hat{A}(t, i, j) &= \begin{pmatrix} A(t, i) & B(t, i)K(t-1, j)C(t-1, j) \\ I_n & 0 \end{pmatrix}. \end{aligned} \quad (12)$$

Hence, the system (10) as well as (11) are both of the form (1).

The condition (5) is equivalent to

$$\lim_{t \rightarrow \infty} E[|\xi(t)|^2] = 0. \quad (13)$$

This motivates the consideration of the problem of stability in mean square of systems of type (1).

Set  $\pi_t = (\pi_t(1), \pi_t(2), \dots, \pi_t(N))$  the distribution of the random variable  $\eta_t$ , that is  $\pi_t(i) = \mathcal{P}\{\eta_t = i\}$ . It can be verified that the sequence  $\{\pi_t\}_{t \geq 0}$  solves

$$\pi_{t+1} = \pi_t P_t, \quad t \geq 0. \quad (14)$$

**Remark 2.1.** a) From (14) it follows that it is possible that  $\pi_t(i) = 0$  for some  $t \geq 1$ ,  $i \in \mathcal{D}$  even if  $\pi_0(j) > 0$ ,  $1 \leq j \leq N$ . This is specific for the discrete-time case.

b) The only available information concerning the system (1) is the set  $\mathcal{D}$  and the sequences  $\{P_t\}_{t \in \mathbf{Z}_+}$ ,  $\{A(t, i, j)\}_{t \geq 1, i, j \in \mathcal{D}}$ . The initial distributions of the Markov chain are not prefixed. Hence, throughout the paper by a Markov chain we will understand any triple  $\{\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D}\}$  where  $\mathcal{D}$  is a fixed set  $\mathcal{D} = \{1, 2, \dots, N\}$ ,  $\{P_t\}_{t \geq 0}$  is a given sequence of stochastic matrices and  $\{\eta_t\}_{t \geq 0}$  is an arbitrary sequence of random variables taking values in  $\mathcal{D}$  and satisfying (2).

Remark 2.1 a) allows us to define the following subsets of the set  $\mathcal{D}$

$$\mathcal{D}_s = \{i \in \mathcal{D} | \pi_s(i) > 0\} \quad (15)$$

for each integer  $s \geq 0$ .

Define  $\Theta(t, s)$  as follows:  $\Theta(t, s) = A(t-1)A(t-2)\dots A(s)$  if  $t \geq s+1$  and  $\Theta(t, s) = I_n$  if  $t = s$ ,  $s \geq 1$ .

Any solution  $x(t)$  of (1) verifies

$$x(t) = \Theta(t, s)x(s)$$

$\Theta(t, s)$  will be called the fundamental (random) matrix solution of (1).

## 2.2. LYAPUNOV TYPE OPERATORS

Let  $\mathfrak{S}_n \subset \mathbf{R}^{n \times n}$  be the linear subspace of symmetric matrices. Set  $\mathfrak{S}_n^N = \mathfrak{S}_n \oplus \mathfrak{S}_n \oplus \dots \oplus \mathfrak{S}_n$ . One can see that  $\mathfrak{S}_n^N$  is a real ordered Hilbert space (see [7], Example 2.5(iii)). The usual inner product on  $\mathfrak{S}_n^N$  is

$$\langle X, Y \rangle = \sum_{i=1}^N \text{Tr}(Y(i)X(i)) \quad (16)$$

for all  $X = (X(1), \dots, X(N))$  and  $Y = (Y(1), \dots, Y(N))$  from  $\mathfrak{S}_n^N$ .

Consider the sequences  $\{A(t, i, j)\}_{t \geq 1}$ ,  $A(t, i, j) \in \mathbf{R}^{n \times n}$ ,  $i, j \in \mathcal{D}$ ,  $\{P_t\}_{t \geq 0}$ ,  $P_t = (p_t(i, j)) \in \mathbf{R}^{N \times N}$ .

Based on these sequences we construct the linear operators  $\Upsilon_t : \mathfrak{S}_n^N \rightarrow \mathfrak{S}_n^N$ ,  $\Upsilon_t S = (\Upsilon_t S(1), \dots, \Upsilon_t S(N))$  with

$$\Upsilon_t S(i) = \sum_{j=1}^N p_{t-1}(j, i) A(t, i, j) S(j) A^T(t, i, j), \quad (17)$$

$t \geq 1, S \in \mathfrak{S}_n^N$ .

If  $A(t, i, j), p_t(i, j)$  are related to the system (1), then the operators  $\Upsilon_t$  are called the Lyapunov type operators associated with the system (1).

By direct calculation based on the definition of the adjoint operator with respect to the inner product (16) one obtains that  $\Upsilon_t^* S = (\Upsilon_t^* S(1), \dots, \Upsilon_t^* S(N))$  with

$$\Upsilon_t^* S(i) = \sum_{j=1}^N p_{t-1}(i, j) A^T(t, j, i) S(j) A(t, j, i) \quad (18)$$

$t \geq 1, S \in \mathfrak{S}_n^N$ .

Let  $R(t, s)$  be the linear evolution operator defined on  $\mathfrak{S}_n^N$  by the sequence  $\{\Upsilon_t\}_{t \geq 1}$ . Hence  $R(t, s) = \Upsilon_{t-1} \Upsilon_{t-2} \dots \Upsilon_s$  if  $t \geq s + 1$  and  $R(t, s) = \mathbb{J}_{\mathfrak{S}_n^N}$  if  $t = s \geq 1$ .

If  $X(t)$  is a solution of discrete-time linear equation on  $\mathfrak{S}_n^N$

$$X_{t+1} = \Upsilon_t X_t, \quad (19)$$

then  $X_t = R(t, s) X_s$  for all  $t \geq s \geq 1$ .

The following result provides a relationship between the operators  $R^*(t, s)$  and the fundamental matrix solution  $\Theta(t, s)$  of the system (1).

**Theorem 2.2.** *Under the considered assumptions the following equality*

$$(R^*(t, s)H)(i) = E[\Theta^T(t, s)H(\eta_{t-1})\Theta(t, s)|\eta_{s-1} = i] \quad (20)$$

holds for all  $H = (H(1), \dots, H(N)) \in \mathfrak{S}_n^N$ ,  $t \geq s \geq 1, i \in \mathcal{D}_{s-1}$ .

**Proof:** see [8, 11].

### 2.3. DEFINITIONS OF MEAN SQUARE EXPONENTIAL STABILITY

Now we are in position to state the concept of exponential stability in mean square of the zero state equilibrium of the system (1).

In this subsection we introduce three different definitions of the concept of exponential stability in mean square.

**Definition 2.3.** We say that the zero state equilibrium of the system (1) is *strongly exponentially stable in mean square* (SESMS) if there exist  $\beta \geq 1, q \in (0, 1)$  such that

$$\|R(t, s)\| \leq \beta q^{t-s} \quad (21)$$

for all  $t \geq s \geq 1$ .

Since  $\mathcal{S}_n^N$  is a finite dimensional linear space, in (21) one can take any norm in  $\mathcal{B}(\mathcal{S}_n^N)$ .

**Definition 2.4.** We say that the zero state equilibrium of the system (1) is *exponentially stable in mean square with conditioning* (ESMS-C) if there exist  $\beta \geq 1, q \in (0, 1)$  such that for any Markov chain  $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$  we have

$$E[|\Theta(t, s)x|^2 | \eta_{s-1} = i] \leq \beta q^{t-s} |x|^2 \quad (22)$$

for all  $t \geq s \geq 1, x \in \mathbf{R}^n, i \in \mathcal{D}_{s-1}$ .

**Definition 2.5.** We say that the zero state equilibrium of the system (1) is *exponentially stable in mean square* (ESMS) if there exist  $\beta \geq 1, q \in (0, 1)$  such that for any Markov chain  $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$  we have

$$E[|\Theta(t, s)x|^2] \leq \beta q^{t-s} |x|^2 \quad (23)$$

for all  $t \geq s \geq 1, x \in \mathbf{R}^n$ .

### 3. EXPONENTIAL STABILITY IN MEAN SQUARE

In this section we establish the relations between the concepts of exponential stability in mean square introduced in Definition 2.3 -Definition 2.5. Firstly we shall show that in the general case of the time-varying system (1) these definitions are not, in general, equivalent. Finally we show that in the case of systems (1) with periodic coefficients these definitions become equivalent.

#### 3.1. THE GENERAL CASE

On the space  $\mathcal{S}_n^N$  one introduces the norm  $|\cdot|_1$  by

$$|X|_1 = \max_{i \in \mathcal{D}} |X(i)| \quad (24)$$

where  $|X(i)|$  is the spectral norm of the matrix  $X(i)$ . Together with the norm  $|\cdot|_2$  induced by the inner product (16), the norm  $|\cdot|_1$  will play an important role in the characterization of the strong exponential stability (SESMS).

If  $T : \mathcal{S}_n^N \rightarrow \mathcal{S}_n^N$  is a linear operator then  $\|T\|_k$  is the operator norm induced by  $|\cdot|_k, k = 1, 2$ .

We recall that (see Proposition 2.2 in [7]) if  $T$  is a linear and positive operator on  $\mathcal{S}_n^N$ , then

$$\|T\|_1 = |TJ|_1 \quad (25)$$

where  $J = (I_n, \dots, I_n) \in \mathcal{S}_n^N$ .

Since  $\|R(t, s)\|$  and  $\|R^*(t, s)\|_1$  are equivalent, from (21) it follows that  $\|R^*(t+1, t)\|_1 \leq \beta_1, t \geq 0$ , hence by using (25) and (18) one obtains

**Corollary 3.1.** *If the zero state equilibrium of the system (1) is SESMS, then*

*$\{\sqrt{p_{t-1}(i, j)}A(t, j, i)\}_{t \geq 1, i, j \in \mathcal{D}}$ , are bounded sequences.*

Now we prove

**Theorem 3.2.** *Under the considered assumptions we have:*

(i) *if the zero state equilibrium of the system (1) is SESMS then it is ESMS-C;*

(ii) *if the zero state equilibrium of the system (1) is ESMS-C then it is ESMS.*

The proof of (i) follows immediately from (20). (ii) follows from the inequality

$$E[|\Theta(t, s)x|^2] \leq \sum_{i \in \mathcal{D}_{s-1}} E[|\Theta(t, s)x|^2 | \eta_{s-1} = i].$$

As we can see in Example 3.5 and Example 3.6 from below the validity of the converse implications from the above theorem is not true in the absence of some additional assumptions.

**Definition 3.3.** We say that a stochastic matrix  $P_t \in \mathbf{R}^{N \times N}$  is a *nondegenerate stochastic matrix* if for any  $j \in \mathcal{D}$  there exists  $i \in \mathcal{D}$  such that  $p_t(i, j) > 0$ .

We remark that if  $P_t, t \geq 0$  are nondegenerate stochastic matrices, then from (14) it follows that  $\pi_t(i) > 0, t \geq 1, i \in \mathcal{D}$  if  $\pi_0(i) > 0$  for all  $i \in \mathcal{D}$ .

Now we have

**Theorem 3.4.** *If for all  $t \geq 0$ ,  $P_t$  are nondegenerate stochastic matrices, then the following are equivalent*

- (i) *the zero state equilibrium of the system (1) is SESMS,*
- (ii) *the zero state equilibrium of the system (1) is ESMS-C,*
- (iii) *there exist a Markov chain  $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$  with  $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$  such that*

$$E[|\Theta(t, s)x|^2 | \eta_{s-1} = i] \leq \beta q^{t-s} |x|^2 \quad (26)$$

for all  $t \geq s \geq 1, i \in \mathcal{D}, x \in \mathbf{R}^n$  where  $\beta \geq 1, q \in (0, 1)$ .

**Proof** (i)  $\rightarrow$  (ii) follows from Theorem 3.2 and (ii)  $\rightarrow$  (iii) is obvious. It remains to prove implication (iii)  $\rightarrow$  (i). Applying Theorem 2.2 for  $H = J =$

$(I_n, \dots, I_n)$  from (26) one obtains that  $x_0^T [R^*(t, s)J](i)x_0 \leq \beta q^{t-s} |x_0|^2$  for all  $t \geq s \geq 1$ ,  $i \in \mathcal{D}$ ,  $x_0 \in \mathbf{R}^n$ .

This allows us to conclude that  $|[R^*(t, s)J](i)| \leq \beta q^{t-s}$  for all  $t \geq s \geq 1$ ,  $i \in \mathcal{D}$ . Based on (24) one gets  $|R^*(t, s)J|_1 \leq \beta q^{t-s}$  for all  $t \geq s$ . Finally from (25) it follows that  $\|R^*(t, s)\|_1 \leq \beta q^{t-s}$  for all  $t \geq s \geq 1$ . Thus the proof is complete, since  $\|R(t, s)\|$  and  $\|R(t, s)^*\|_1$  are equivalent.

The next two examples show that the converse implication in Theorem 3.2 are not always true.

**Example 3.5.** Consider the system (1) in the particular case  $n = 1, N = 2$  described by

$$x(t+1) = a(t, \eta_t, \eta_{t-1})x(t) \quad (27)$$

where  $t \geq 1, a(t, i, 1) = 0, a(t, i, 2) = 2^{\frac{t-1}{2}}, t \geq 1, i \in \{1, 2\}$ . The transition probability matrix is

$$P_t = \begin{pmatrix} 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \\ 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \end{pmatrix}, t \geq 0. \quad (28)$$

We have

$$\sqrt{p_{t-1}(2, 1)}a(t, 1, 2) = \left(1 - \frac{1}{4^t}\right)^{\frac{1}{2}} 2^{\frac{t-1}{2}}.$$

Hence the sequence  $\{\sqrt{p_{t-1}(2, 1)}a(t, 1, 2)\}_{t \geq 1}$  is unbounded. Then via Corollary 3.1, we deduce that the zero state equilibrium of the system (27) is not SESMS. On the other hand one sees that the transition probability matrix (28) is a non-degenerate stochastic matrix. Thus, via Theorem 3.4, we deduce that the zero state equilibrium of (27) cannot be ESMS-C.

We show now that the zero state equilibrium of (27) is ESMS. We write

$$\begin{aligned} E[|\Theta(t, s)x|^2] &= x^2 E[a^2(t-1, \eta_{t-1}, \eta_{t-2})a^2(t-2, \eta_{t-2}, \eta_{t-3}) \dots a^2(s, \eta_s, \eta_{s-1})] = \\ &= x^2 \sum_{i_{t-1}=1}^2 \sum_{i_{t-2}=1}^2 \dots \sum_{i_{s-1}=1}^2 a^2(t-1, i_{t-1}, i_{t-2})a^2(t-2, i_{t-2}, i_{t-3}) \dots a^2(s, i_s, i_{s-1}) \end{aligned}$$

$$\begin{aligned}
 & \mathcal{P}\{\eta_{t-1} = i_{t-1}, \eta_{t-2} = i_{t-2}, \dots, \eta_{s-1} = i_{s-1}\} = \\
 & x^2 \mathcal{P}(\eta_{s-1} = 2) a^2(s, 2, 2) \dots a^2(t-2, 2, 2) p_{s-1}(2, 2) p_s(2, 2) \dots p_{t-3}(2, 2) \\
 & [a^2(t-1, 1, 2) p_{t-2}(2, 1) + a^2(t-1, 2, 2) p_{t-2}(2, 2)] \leq \\
 & x^2 2^{s-1+s+\dots+t-3} \frac{1}{4^{s+s+1+\dots+t-2}} 2^{t-2} = x^2 2^{\frac{(t+s-3)(t-s)}{2}} \frac{1}{2^{(t+s-2)(t-s-1)}}
 \end{aligned}$$

if  $t \geq s+2$ . Finally one obtains that

$$E[|\Theta(t, s)x|^2] \leq x^2 \frac{1}{\sqrt{2^{t^2-s^2-3t-s+4}}} \quad (29)$$

if  $t \geq s+2$ .

For  $t \geq s+2$  we have  $t^2 - 4t - s^2 + 4 = (t-2)^2 - s^2 \geq 0$ . This means that  $t^2 - s^2 - 3t - s + 4 \geq t - s$ . From (29) one obtains that

$$E[|\Theta(t, s)x|^2] \leq x^2 \frac{1}{\sqrt{2^{t-s}}} \quad (30)$$

for all  $t \geq s+2, s \geq 1$ . Further we compute

$$\begin{aligned}
 E[|\Theta(s+1, s)x|^2] &= x^2 E[a^2(s, \eta_s, \eta_{s-1})] = x^2 a^2(s, 1, 2) \mathcal{P}\{\eta_{s-1} = 2, \eta_s = 1\} + \\
 & a^2(s, 2, 2) \mathcal{P}\{\eta_{s-1} = 2, \eta_s = 2\} \leq x^2 2^{s-1} \mathcal{P}\{\eta_{s-1} = 2\} [p_{s-1}(2, 1) + p_{s-1}(2, 2)].
 \end{aligned}$$

Thus we get

$$E[|\Theta(s+1, s)x|^2] \leq x^2 2^{s-1} \mathcal{P}\{\eta_{s-1} = 2\}, \quad (\forall) s \geq 1. \quad (31)$$

Take  $s \geq 2$  and write

$$E[|\Theta(s+1, s)x|^2] \leq x^2 2^{s-1} [\mathcal{P}\{\eta_{s-2} = 1\} p_{s-2}(1, 2) + \mathcal{P}\{\eta_{s-2} = 2\} p_{s-2}(2, 2)] \leq x^2 2^s \frac{1}{4^{s-1}}$$

which leads to

$$E[|\Theta(s+1, s)x|^2] \leq x^2, \quad \forall s \geq 2. \quad (32)$$

Further

$$E[|\Theta(2, 1)x|^2] = E[a^2(1, \eta_1, \eta_0)] x^2 = x^2 [\mathcal{P}\{\eta_0 = 2, \eta_1 = 1\} + \mathcal{P}\{\eta_0 = 2, \eta_1 = 2\}] =$$

$= x^2 \mathcal{P}\{\eta_0 = 2\}$  i.e.

$$E[|\Theta(2, 1)x|^2] \leq x^2. \quad (33)$$

Finally

$$E[|\Theta(s, s)x|^2] = x^2. \quad (34)$$

Combining (30),(32),(33),(34) we conclude that

$$E[|\Theta(t, s)x|^2] \leq \sqrt{2} \frac{1}{\sqrt{2}^{t-s}} x^2$$

for all  $t \geq s \geq 1, x \in \mathbf{R}$  and thus one obtains that the zero state equilibrium of (27) is ESMS.

**Example 3.6.** Consider the system (1) in the particular case  $N = 2, A(t, i, 1) = \mathbf{O} \in \mathbf{R}^{n \times n}, A(t, i, 2) = itI_n, i \in \{1, 2\}, t \geq 1$ . The transition probability matrix is  $P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ . We have

$$\sqrt{p(2, 1)}A(t, 1, 2) = tI_n$$

thus via Corollary 3.1 one obtains that the zero state equilibrium of the considered system cannot be SESMS.

On the other hand we have  $\mathcal{P}\{\eta_t = 2\} = 0$  a.s. for all  $t \geq 1$ . This leads to  $\eta_t = 1$  a.s.,  $t \geq 1$ . Hence  $\Theta(t, s) = 0$  a.s. if  $t \geq \max\{3, s + 1\}, s \geq 1$  for any Markov chain  $(\{\eta_t\}_{t \geq 0}, P, \{1, 2\})$ . This shows that in this particular case the zero state equilibrium of the considered system is both ESMS-C, as well as ESMS.

### 3.2. THE PERIODIC CASE

In this subsection we show that under the periodicity assumption the concepts of exponential stability introduced by Definitions 2.3, 2.4, 2.5 become equivalent.

Firstly we introduce

**Definition 3.7.** We say that the zero state equilibrium of the system (1) is *asymptotically stable in mean square ASMS* if for any Markov chain  $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$  we have

$$\lim_{t \rightarrow \infty} E[|\Theta(t, 1)x|^2] = 0, \forall x \in \mathbf{R}^n.$$

Now we are in position to state

**Theorem 3.8.** *Assume that there exists an integer  $\theta \geq 1$  such that  $A(t + \theta, i, j) = A(t, i, j), i, j \in \mathcal{D}, P_{t+\theta} = P_t, t \geq 0$ . Under these conditions the following are equivalent:*

- (i) *the zero state equilibrium of the system (1) is (SESMS),*
- (ii) *the zero state equilibrium of the system (1) is (ESMS-C),*
- (iii) *the zero state equilibrium of the system (1) is (ESMS),*
- (iv) *the zero state equilibrium of the system (1) is (ASMS),*
- (v) *there exists a Markov chain  $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$  with  $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$  such that*

$$\lim_{t \rightarrow \infty} E[|\Theta(\theta t, 1)x|^2] = 0 \tag{35}$$

for all  $x \in \mathbf{R}^n$ ,

- (vi)  $\rho[R(\theta + 1, 1)] < 1$  where  $\rho[\cdot]$  is the spectral radius.

**Proof.** is similar to the proof of Theorem 3.7 in [11]. It is omitted for shortness.

**Definition 3.9.** We say that the system (1) is in the time invariant case if  $A(t, i, j) = A(i, j)$  for all  $t \geq 1, i, j \in \mathcal{D}$  and  $P_t = P$  for all  $t \geq 0$ .

In this case we have  $\Upsilon_t = \Upsilon$ , for all  $t \geq 1$ . One sees that the system (1) is in the time invariant case if and only if it is periodic with period  $\theta = 1$ . Hence, the equivalences from the above theorem hold in the time invariant case too. In this case, the statement (vi) becomes  $\rho(\Upsilon) < 1$ .

## 4. LYAPUNOV TYPE CRITERIA

In this section we present several conditions for exponential stability in mean square expressed in terms of solvability of some suitable systems of linear matrix equations or linear matrix inequations. The results of this section are special cases of those stated in a more general framework in [7]. That is why we present them here without proofs.

### 4.1. THE GENERAL CASE

In view of Theorem 3.2 it follows that the Lyapunov type criteria are necessary and sufficient conditions for SESMS but they are only sufficient conditions for ESMS. Direct from the above Definition 2.3 and Theorem 3.4 in [7] applied to Lyapunov type operators  $\Upsilon_t$  we obtain

**Theorem 4.1** *Under the considered assumptions the following are equivalent:*

- (i) *the zero state equilibrium of the system (1) is SESMS,*
- (ii) *the system of backward linear equations*

$$X_t(i) = \sum_{j=1}^N p_{t-1}(i, j) A^T(t, j, i) X_{t+1}(j) A(t, j, i) + I_n \quad (36)$$

$t \geq 1$ ,  $i \in \mathcal{D}$  has a bounded solution  $X_t = (X_t(1), \dots, X_t(N))$  with  $X_t(i) \geq I_n$ ,  $t \geq 1$ ,

(iii) *there exists a bounded sequence  $\{Y_t\}_{t \geq 1} \in \mathbb{S}_n^N$  and scalars  $\alpha > 0$ ,  $\delta > 0$  such that*

$$\sum_{j=1}^N p_{t-1}(i, j) A^T(t, j, i) Y_{t+1}(j) A(t, j, i) - Y_t(i) \leq -\alpha I_n \quad (37)$$

$Y_t(i) \geq \delta I_n$ ,  $t \geq 1$ ,  $i \in \mathcal{D}$ .

**Remark 4.2.** Even if the system (36), ((37) respectively) consists of an infinite number of equations (inequations respectively) the criteria derived in

Theorem 4.1 may be useful to obtain sufficient conditions for ESMS in the general time varying case. This can be illustrated by the next simple example.

**Example 4.3** Consider the system (1) in the special case  $n = 1$

$$x(t+1) = a(t, \eta_t, \eta_{t-1})x(t) \quad (38)$$

$t \geq 1$ , where  $a(t, i, j) \in \mathbf{R}$  are such that

$$\sup_{t \geq 1} \sum_{j=1}^N p_{t-1}(i, j) a^2(t, j, i) < 1. \quad (39)$$

Under this condition the zero state equilibrium of (38) is SESMS.

Indeed, if (39) holds, then the corresponding system (37) associated with (38) is fulfilled for  $Y_t(i) = 1, t \geq 1, i \in \mathcal{D}$ .

## 4.2. THE PERIODIC CASE

From Theorem 3.5 in [7] one obtains that if the coefficients of (36) are periodic with period  $\theta$ , the unique bounded and positive solution of (36) is periodic with the same period  $\theta$ . Also, if the system of inequalities (37) has a bounded and uniform positive solution then it has a periodic solution too. This allows us to obtain the following specialized version of Theorem 4.1.

**Theorem 4.4** *Under the assumptions of Theorem 3.8, with  $\theta \geq 2$ , the following are equivalent:*

- (i) *the zero state equilibrium of (1) is ESMS,*
- (ii) *the system of linear matrix equations*

$$X_t(i) = \sum_{j=1}^N p_{t-1}(i, j) A^T(t, j, i) X_{t+1}(j) A(t, j, i) + I_n \quad 1 \leq t \leq \theta - 1$$

$$X_\theta(i) = \sum_{j=1}^N p_{\theta-1}(i, j) A^T(\theta, j, i) X_1(j) A(\theta, j, i) + I_n \quad (40)$$

$i \in \mathcal{D}$  has a solution  $X_t = (X_t(1), \dots, X_t(N))$  with  $X_t(i) > 0$ .

(iii) there exist positive definite matrices  $Y_t(i)$ ,  $1 \leq t \leq \theta$ ,  $i \in \mathcal{D}$ , which solve the following system of LMI's

$$\begin{aligned} \sum_{j=1}^N p_{t-1}(i, j) A^T(t, j, i) Y_{t+1}(j) A(t, j, i) - Y_t(i) < 0, \quad 1 \leq t \leq \theta - 1 \\ \sum_{j=1}^N p_{\theta-1}(i, j) A^T(\theta, j, i) Y_1(j) A(\theta, j, i) - Y_\theta(i) < 0 \quad i \in \mathcal{D}. \end{aligned} \quad (41)$$

It is easy to see that under the conditions of Theorem 3.8 the sequence  $\Upsilon_t$  can be extended in a natural way by periodicity, to the whole set of integers  $\mathbf{Z}$ .

In this case we may use Theorem 3.7 (ii) and Theorem 3.9 from [7] to obtain

**Theorem 4.5.** Under the assumptions of Theorem 3.8 with  $\theta \geq 2$ , the following are equivalent:

- (i) the zero state equilibrium of the system (1) is ESMS,
- (ii) the system of linear matrix equations

$$\begin{aligned} X_{t+1}(i) &= \sum_{j=1}^N p_{t-1}(j, i) A(t, i, j) X_t(j) A^T(t, i, j) + I_n \quad 1 \leq t \leq \theta - 1 \\ X_1(i) &= \sum_{j=1}^N p_{\theta-1}(j, i) A(\theta, i, j) X_\theta(j) A^T(\theta, i, j) + I_n \end{aligned} \quad (42)$$

$i \in \mathcal{D}$  has a solution  $X_t = (X_t(1), \dots, X_t(N))$  such that  $X_t(i) > 0$ ,  $i \in \mathcal{D}$ ,

(iii) there exist positive definite matrices  $Y_t(i)$ ,  $1 \leq t \leq \theta$ ,  $i \in \mathcal{D}$ , which solve the following system of LMI's:

$$\begin{aligned} \sum_{j=1}^N p_{t-1}(j, i) A(t, i, j) Y_t(j) A^T(t, i, j) - Y_{t+1}(i) < 0 \quad 1 \leq t \leq \theta - 1 \\ \sum_{j=1}^N p_{\theta-1}(j, i) A(\theta, i, j) Y_\theta(j) A^T(\theta, i, j) - Y_1(i) < 0 \end{aligned} \quad (43)$$

$i \in \mathcal{D}$ .

**Remark 4.6.** The system of linear equations (40), (42) and the system of linear inequations (41), (43) have  $\hat{n}$  scalar equations (inequations respectively), with  $\hat{n}$  scalar unknowns, where  $\hat{n} = \frac{n(n+1)}{2}N\theta$ .

### 4.3. THE TIME INVARIANT CASE

Using Theorem 3.5 (iii), Theorem 3.7 (iii) and Theorem 3.9 in [7] one obtains the following Lyapunov type criteria for exponential stability in mean square for the system (1) in the time invariant case.

**Corollary 4.7** *If the system is in the time invariant case, the following are equivalent:*

- (i) *the zero state equilibrium of the system (1) is ESMS,*
- (ii) *the system of linear equations*

$$X(i) = \sum_{j=1}^N p(i, j)A^T(j, i)X(j)A(j, i) + I_n \quad (44)$$

$i \in \mathcal{D}$ , has a solution  $X = (X(1), \dots, X(N))$  with  $X(i) > 0, i \in \mathcal{D}$ ,

(iii) *there exist positive definite matrices  $Y(i), i \in \mathcal{D}$ , which solve the following system of LMI's:*

$$\sum_{j=1}^N p(i, j)A^T(j, i)Y(j)A(j, i) - Y(i) < 0 \quad (45)$$

$i \in \mathcal{D}$ .

**Corollary 4.8.** *Under the conditions of Corollary 4.7 the following are equivalent:*

- (i) *the zero state equilibrium of the system (1) is ESMS,*
- (ii) *the system of linear matrix equations*

$$X(i) = \sum_{j=1}^N p(j, i)A(i, j)X(j)A^T(i, j) + I_n \quad (46)$$

$i \in \mathcal{D}$  has a solution  $X = (X(1), \dots, X(N))$  with  $X(i) > 0, i \in \mathcal{D}$ ,

(iii) there exist positive definite matrices  $Y(i), i \in \mathcal{D}$ , which solve the following system of LMI's

$$\sum_{j=1}^N p(j, i) A(i, j) Y(j) A^T(i, j) - Y(i) < 0 \quad (47)$$

$i \in \mathcal{D}$ .

**Remark 4.9.** The system of linear equations (44) and (46) and the system of linear inequations (45) and (47) have  $\tilde{n}$  scalar equations (inequations respectively) with  $\tilde{n}$  scalar unknowns, where  $\tilde{n} = \frac{n(n+1)}{2}N$ .

## 5. APPLICATIONS

In this section we illustrate the applicability of the results concerning the exponential stability in mean square for the systems of type (1) derived in the previous sections to the control problem of the designing of a stabilizing static output feedback in the presence of some delays in the transmission of the data. We restrict our attention to the time-invariant case of the controlled system (3) and of the control law (4). More specific we consider the controlled system

$$x(t+1) = A(t, \eta_t)x(t) + B(t, \eta_t)u(t)y(t) = C(t, \eta_t)x(t), \quad t \geq 0, \quad (48)$$

and the control law

$$u(t) = K(\eta_t)y(t). \quad (49)$$

Based on Theorem 3.8 (for  $\theta = 1$ ) one obtains that (13) is equivalent to exponential stability in mean square. Therefore to find some conditions which guarantee the existence of the feedbacks gains  $K(i)$  with the desired property, we may apply the Lyapunov type criteria derived in Section 4.

The main tools in the derivation of the results in this section are the following well-known lemmas.

**Lemma 5.1** (*The projection lemma*) [23]. Let  $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n \times n}$ ,  $\mathcal{U} \in \mathbf{R}^{m \times n}$ ,  $\mathcal{V} \in \mathbf{R}^{p \times n}$  be given matrices,  $n \geq \max\{m, p\}$ . Let  $\mathcal{U}^\perp, \mathcal{V}^\perp$  be full

column rank matrices such that  $\mathcal{U}\mathcal{U}^\perp = 0$  and  $\mathcal{V}\mathcal{V}^\perp = 0$ . Then the following are equivalent:

(i) the linear matrix inequations

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0$$

with the unknown matrix  $K \in \mathbf{R}^{m \times p}$  is solvable;

(ii)  $(\mathcal{U}^\perp)^T \mathcal{Z} \mathcal{U}^\perp < 0$ ;

$(\mathcal{V}^\perp)^T \mathcal{Z} \mathcal{V}^\perp < 0$ .

**Lemma 5.2.** (Finsler's lemma)[23]. Let  $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n \times n}$ ,  $\mathcal{C} \in \mathbf{R}^{p \times n}$ ,  $n > p$  be given. Take  $\mathcal{C}^\perp$  a full column rank matrix such that  $\mathcal{C}\mathcal{C}^\perp = 0$ . Then the following are equivalent:

(i) there exists a scalar  $\mu$  such that  $\mathcal{Z} + \mu \mathcal{C}^T \mathcal{C} < 0$ ;

(ii)  $(\mathcal{C}^\perp)^T \mathcal{Z} \mathcal{C}^\perp < 0$ .

Combining the above two lemmas one obtains

**Corollary 5.3.** With the previous notations the following statements are equivalent:

(i) the linear inequation

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0 \quad (50)$$

with the unknown  $K \in \mathbf{R}^{m \times p}$  is solvable.

(ii) there exist the scalars  $\mu_1, \mu_2$  such that

$$\mathcal{Z} + \mu_1 \mathcal{U}^T \mathcal{U} < 0$$

$$\mathcal{Z} + \mu_2 \mathcal{V}^T \mathcal{V} < 0.$$

In [23] a parametrization of the whole class of solutions of (50) is given.

Before to state the main results of this section we remark that in (14) written for the special case of (48) and (49) we have the following decomposition

$$\tilde{A}(i, j) = \mathcal{A}_0(i) + \mathcal{B}_0(i)K(i)\mathcal{C}_0(j), \hat{A}(i, j) = \mathcal{A}_0(i) + \mathcal{B}_0(i)K(j)\mathcal{C}_0(j), \quad (51)$$

where  $\mathcal{A}(i) = \begin{pmatrix} A_0(i) & 0 \\ I_n & 0 \end{pmatrix} \in \mathbf{R}^{2n \times 2n}$ ,  $\mathcal{B}_0(i) = \begin{pmatrix} B(i) \\ 0 \end{pmatrix} \in \mathbf{R}^{2n \times m}$ ,  $\mathcal{C}_0(j) = \begin{pmatrix} 0 & C(j) \end{pmatrix} \in \mathbf{R}^{p \times 2n}$ .

Now we have

**Theorem 5.4.** *Assume that there exist the symmetric matrices  $Y(i) \in \mathbf{R}^{2n \times 2n}$  and the scalars  $\mu_1(i)$  and  $\mu_2(i)$ ,  $i \in \mathcal{D}$ , satisfying the following systems of LMI's*

$$\begin{pmatrix} -Y(i) & \Psi_{2i}(Y) & 0 \\ \Psi_{2i}^T(Y) & \Psi_{3i}(Y) & \Psi_{4i}(Y) \\ 0 & \Psi_{4i}^T(Y) & -\mu_1(i)I_p \end{pmatrix} < 0, \quad (52)$$

$$\sum_{j=1}^N p(j, i) \mathcal{A}_0(i) Y(j) \mathcal{A}_0^T(i) - Y(i) + \mu_2(i) \mathcal{B}_0(i) \mathcal{B}_0^T(i) < 0, \quad (53)$$

where

$$\Psi_{2i}(Y) = (\sqrt{p(1, i)} \mathcal{A}_0(i) Y(1), \dots, \sqrt{p(N, i)} \mathcal{A}_0(i) Y(N)),$$

$$\Psi_{3i}(Y) = -\text{diag}(Y(1), \dots, Y(N)),$$

$$\Psi_{4i}^T(Y) = (\sqrt{p(1, i)} \mathcal{C}_0(1) Y(1), \dots, \sqrt{p(N, i)} \mathcal{C}_0(N) Y(N)).$$

Under these conditions there exist stabilizing feedback gains  $K(i) \in \mathbf{R}^{m \times p}$ ,  $i \in \mathcal{D}$  such that the zero state equilibrium of the time-invariant version of the system (10) is ESMS.

Moreover the matrices  $K(i)$  may be obtained as solutions of the following uncoupled LMI's

$$\tilde{\mathcal{Z}}(i) + \tilde{\mathcal{U}}^T(i) K(i) \tilde{\mathcal{V}}(i) + \tilde{\mathcal{V}}^T(i) K^T(i) \mathcal{U}(i) < 0, \quad (54)$$

$i \in \mathcal{D}$ , where

$$\tilde{\mathcal{Z}}(i) = \begin{pmatrix} -\tilde{Y}(i) & \Psi_{2i}(\tilde{Y}) \\ \Psi_{2i}^T(\tilde{Y}) & \Psi_{3i}(\tilde{Y}) \end{pmatrix},$$

$$\tilde{\mathcal{U}}(i) = (\mathcal{B}_0^T(i), 0, \dots, 0) \in \mathbf{R}^{m \times \tilde{n}},$$

$$\tilde{\mathcal{V}}(i) = (0, \sqrt{p(1, i)}\mathcal{C}_0(1)\tilde{Y}(1), \dots, \sqrt{p(N, i)}\mathcal{C}_0(N)\tilde{Y}(N)) \in \mathbf{R}^{p \times \tilde{n}}, \quad (55)$$

where  $\tilde{n} = 2n(N + 1)$ ,  $\tilde{Y} = (\tilde{Y}(1), \dots, \tilde{Y}(N))$  being a solution of (52), (53).

**Proof** is similar to the one of Theorem 6.4 in [11]. It is omitted for shortness.

Let us consider the time-invariant case of control law (7).

In this case we have

**Theorem 5.5.** *Assume that there exist matrices  $Y(i) \in \mathbf{R}^{2n \times 2n}$  and the scalars  $\mu_1(i)$ ,  $\mu_2(i)$  which satisfy the following system of LMI's*

$$\sum_{j=1}^N p(i, j)A_0^T(j)Y(j)A_0(j) - Y(i) + \mu_1(i)\mathcal{C}_0^T(i)\mathcal{C}_0(i) < 0, \quad (56)$$

$$\begin{pmatrix} -Y(i) & \hat{\Gamma}_{2i}(Y) & 0 \\ (\hat{\Gamma}_{2i})^T(Y) & \Gamma_{3i}(Y) & \Gamma_{4i}(Y) \\ 0 & \Gamma_{4i}^T(Y) & -\mu_2(i)I_m \end{pmatrix} < 0, \quad (57)$$

where

$$\hat{\Gamma}_{2i}(Y) = (\sqrt{p(i, 1)}A_0^T(1)Y(1), \dots, \sqrt{p(i, N)}A_0^T(N)Y(N)),$$

$$\Gamma_{3i}(Y) = \Psi_{3i}(Y),$$

$$\Gamma_{4i}^T(Y) = (\sqrt{p(i, 1)}\mathcal{B}_0^T(1)Y(1), \dots, \sqrt{p(i, N)}\mathcal{B}_0^T(N)Y(N)).$$

Under these conditions there exist feedback gains  $K(i) \in \mathbf{R}^{m \times p}$  such that the zero state equilibrium of the corresponding time invariant version of the system (11) is ESMS.

Moreover, if  $(\hat{Y}(i), \hat{\mu}_1(i), \hat{\mu}_2(i), i \in \mathcal{D})$  is a solution of (56), (57) then for each  $i$ ,  $K(i)$  is obtained as a solution of the following LMI

$$\hat{\mathcal{Z}}(i) + \hat{\mathcal{U}}^T(i)K(i)\hat{\mathcal{V}}(i) + \hat{\mathcal{V}}^T(i)K^T(i)\hat{\mathcal{U}}(i) < 0, \quad (58)$$

where

$$\hat{\mathcal{Z}}(i) = \begin{pmatrix} -\hat{Y}(i) & \hat{\Gamma}_{2i}(\hat{Y}) \\ \hat{\Gamma}_{2i}^T(\hat{Y}) & \Gamma_{3i}(\hat{Y}) \end{pmatrix}, \hat{\mathcal{U}}(i) = (\mathcal{C}_0(i), 0, \dots, 0) \in \mathbf{R}^{p \times \tilde{n}}$$

$$\hat{V}(i) = (0, \sqrt{p(i, 1)}\mathcal{B}_0^T(1)\hat{Y}(1), \dots, \sqrt{p(i, N)}\mathcal{B}_0^T(N)\hat{Y}(N)) \in \mathbf{R}^{m \times \bar{n}}. \quad (59)$$

**Proof:** see the proof of Theorem 6.4 in [11].

**Remark 5.6.** Remark that in order to obtain uncoupled LMIs (54) and (58) respectively, we used Corollary 4.8, in the first case and Corollary 4.7 in the second case. We feel that this is a good motivation to deduce stability criteria based on Lyapunov type operators  $\Upsilon_t$  (see Theorem 4.5 and Corollary 4.8) as well as based on its adjoint operators  $\Upsilon_t^*$  (see Theorem 4.4 and Corollary 4.7).

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