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SADDLE-NODE BIFURCATION IN A COMPETING SPECIES MODEL

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Abstract The nature of the saddle-node singularities in a competing species model in the case when four of the six parameters are kept fixed is shown. Some singularities are degenerated, some others not. The local phase portraits around these points are then represented.

1. THE MATHEMATICAL MODEL

One of the models describing the dynamics of two species in competition is the Cauchy problem $x(0) = x_0,\ y(0) = y_0$, for the following system of ordinary differential equations (s.o.d.e.) [4]

$$\begin{cases} \dot{x} &=& r_1 x(1 - x/K_1 - p_{12} y/K_1), \\ \dot{y} &=& r_2 y(1 - y/K_2 - p_{21} x/K_2), \end{cases} \tag{1}$$

where x, y represent the number of individuals of the two species, r_1, r_2 - the growth rates of these species, K_1, K_2 -the carrying capacity of every species, $p_{12} > 0$ - the action of the second population and $p_{21} > 0$ - the action of the first population. In this study we consider r_1, r_2, K_1 and K_2 as fixed, such that in (1) only two parameters occur: p_{12} and p_{21}.

In this paper we are interested only in the saddle-node singularities.

In [3] it is shown that the saddle-node singularities exist at the points $E_1(K_1, 0)$, $E_2(0, K_2)$ and $E_3(\alpha, K_2(K_1 - \alpha)/K_1)$ for $p_{12} = K_1/K_2$, $p_{21} = K_2/K_1$, at $E_1(K_1, 0)$ for $p_{12} \neq K_1/K_2$, $p_{21} = K_2/K_1$ and at $E_2(0, K_2)$ for $p_{12} = K_1/K_2$, $p_{21} \neq K_2/K_1$.

In order to see whether these points are degenerated or nondegenerated singularities we have to derive the normal forms of (1) at $E_i,\ i=1,2,3$ [1].

2. THE NATURE OF THE SADDLE-NODES

Case $\mathbf{p_{12}=K_1/K_2,\ p_{21}=K_2/K_1}$. In this case (1) assumes the form

$$\begin{cases} \dot{x} &= r_1 x\,(1 - x/K_1 - y/K_2)\,, \\ \dot{y} &= r_2 y\,(1 - y/K_2 - x/K_1)\,. \end{cases} \tag{2}$$

Proposition 2.1. *The normal form of (2) at $E_3(\alpha, K_2(K_1-\alpha)/K_1)$ up to second order terms for $\alpha \in [0, K_1]$ is*

$$\begin{cases} \dot{n}_1 &= [(-r_1\alpha - r_2K_1 + r_2\alpha)/K_1]n_1 + (r_2 - r_1)n_1n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{3}$$

and, thus, E_3 is a degenerated saddle-node.

Proof. First, we translate the point E_3 at the origin with the aid of the change $u_1 = x-\alpha,\ u_2 = y - K_2(K_1-\alpha)/K_1$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (2) reads

$$\begin{cases} \dot{u}_1 &= -(r_1\alpha/K_1)u_1 - (r_1\alpha/K_2)u_2 - (r_1/K_1)u_1^2 - (r_1/K_2)u_1u_2, \\ \dot{u}_2 &= -(K_2\gamma/K_1)u_1 - \gamma u_2 - (r_2/K_1)u_1u_2 - (r_2/K_2)u_2^2, \end{cases} \tag{4}$$

where $\gamma = r_2(K_1-\alpha)/K_1$.

The eigenvalues of the matrix defining the system (4) linearized around the origin are $\lambda_1 = (-r_1\alpha - r_2K_1 + r_2\alpha)/K_1,\ \lambda_2 = 0$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1, \gamma K_2/(r_1\alpha))^T$ and $\mathbf{u}_{\lambda_2} = (K_1, -K_2)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & K_1 \\ \gamma K_2/(r_1\alpha) & -K_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (4) achieves the form

$$\begin{cases} \dot{v}_1 &= \lambda_1 v_1 + Pv_1^2 + (r_2 - r_1)v_1v_2, \\ \dot{v}_2 &= Qv_1^2 - (r_2/\alpha)v_1v_2, \end{cases} \tag{5}$$

where $P = -(r_1^2\alpha - r_2^2K_1 + r_2^2\alpha)/(\alpha r_1K_1)$, $Q = -r_2(r_1K_1 - r_1\alpha - r_2K_1 + r_2\alpha)/(\alpha r_1K_1^2)$, and the matrix defining the linear part is diagonal. In order to reduce the second order nonresonant terms in (5) we apply the normal form method [1]. To this aim we determine the transformation $\mathbf{v} = \mathbf{n}+\mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 1.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	P	Q	λ_1	$2\lambda_1$	R	S
1	1	$r_2 - r_1$	$-r_2/\alpha$	0	λ_1	-	T
0	2	0	0	$-\lambda_1$	0	0	-

Table 1.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogeneous vector polynomial in (5), $T = -r_2K_1/[\alpha(-r1\alpha - r_2K_1+r_2\alpha)]$ and $R = -(r_1^2\alpha-r_2^2K_1+r_2^2\alpha)/(\alpha r_1(-r_1\alpha-r_2K_1+r_2\alpha))$. Therefore we find the transformation

$$\begin{cases} v_1 &= n_1 + Rn_1^2, \\ v_2 &= n_2 + Sn_1^2 + Tn_1n_2, \end{cases}$$

carrying (5) into (3). By [2], the equilibrium point E_3 corresponding the dynamical system generated by a s.o.d.e. of the form (3) is a degenerated saddle-node. ∎

Remark 2.1. *For $\alpha = K_1$, the equilibrium point $E_3(\alpha, K_2(K_1 - \alpha)/K_1)$ becomes $E_1(K_1, 0)$. In this case the normal form of (2) at E_1 is*

$$\begin{cases} \dot{n}_1 &= -r_1n_1 + (r_2 - r_1)n_1n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{6}$$

and, thus, E_1 is a degenerated saddle-node.

Remark 2.2. *For $\alpha = 0$, the equilibrium point $E_3(\alpha, K_2(K_1-\alpha)/K_1)$ becomes $E_2(0, K_2)$. In this case the normal form of (2) at E_2 is*

$$\begin{cases} \dot{n}_1 &= -r_2 n_1 + (r_2 - r_1) n_1 n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{7}$$

and, thus, E_2 is a degenerated saddle-node.

Case $\mathbf{p_{12} \neq K_1/K_2,\ p_{21} = K_2/K_1}$. In this case (1) assumes the form

$$\begin{cases} \dot{x} &= r_1 x \left(1 - x/K_1 - p_{12} y/K_1\right), \\ \dot{y} &= r_2 y \left(1 - y/K_2 - x/K_1\right). \end{cases} \tag{8}$$

Proposition 2.2. *The normal form of (8) at $E_1(K_1, 0)$ up to second order terms is*

$$\begin{cases} \dot{n}_1 &= -r_1 n_1 + (p_{12}/K_1)(r_1 - r_2) n_1 n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= r_2 \left(p_{12}/K1 - 1/K_2\right) n_2^2 + O(\mathbf{n}^3), \end{cases} \tag{9}$$

and, thus, E_1 is a nondegenerated saddle-node.

Proof. First, we translate the point E_1 at the origin with the aid of the change $u_1 = x - K_1,\ u_2 = y$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (8) reads

$$\begin{cases} \dot{u}_1 &= -r_1 u_1 - r_1 p_{12} u_2 - (r_1/K_1) u_1^2 - (r_1 p_{12}/K_1) u_1 u_2, \\ \dot{u}_2 &= -(r_2/K_1) u_1 u_2 - (r_2/K_2) u_2^2. \end{cases} \tag{10}$$

The eigenvalues of the matrix defining the linear terms in (10) are $\lambda_1 = -r_1,\ \lambda_2 = 0$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1, 0)^T$ and $\mathbf{u}_{\lambda_2} = (-p_{12}, 1)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (10) achieves the form

$$\begin{cases} \dot{v}_1 &= -r_1 v_1 - (r_1/K_1) v_1^2 + (p_{12}/K_1)/(r_1 - r_2) v_1 v_2 + p_{12} \beta v_2^2, \\ \dot{v}_2 &= -(r_2/K_1) v_1 v_2 + \beta v_2^2, \end{cases} \tag{11}$$

where $\beta = r_2\left(p_{12}/K_1 - 1/K_2\right)$. In (11) the matrix of the linear terms is diagonal. In order to reduce the second order nonresonant terms in (11) we determine the transformation $\mathbf{v} = \mathbf{n} + \mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 2.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	$-r_1/K_1$	0	$-r_1$	$-2r_1$	$1/K_1$	0
1	1	M	$-r_2/K_1$	0	$-r_1$	-	$r_2/(r_1K_1)$
0	2	$p_{12}N$	N	r_1	0	$p_{12}N/r_1$	-

Table 2.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogenous vector polynomial in (11), $M = p_{12}(r_1 - r_2)/K_1$ and $N = r_2\left(p_{12}/K_1 - 1/K_2\right)$. We find the transformation

$$\left\{\begin{array}{lcl} v_1 & = & n_1 + \dfrac{1}{K_1}n_1^2 + \dfrac{p_{12}}{r_1}Nn_2^2, \\ v_2 & = & n_2 + \dfrac{r_2}{r_1K_1}n_1n_2, \end{array}\right.$$

carrying (11) into (9). We have $p_{12} \neq K_1/K_2$, therefore $r_2(p_{12}/K_1 - 1/K_2) \neq 0$. By [2], the equilibrium point E_1 corresponding the dynamical system generated by a s.o.d.e. of the form (9) is a nondegenerated saddle-node. ∎

Case $\mathbf{p_{12} = K_1/K_2}$, $\mathbf{p_{21} \neq K_2/K_1}$. In this case (1) assumes the form

$$\left\{\begin{array}{lcl} \dot{x} & = & r_1x\left(1 - x/K_1 - y/K_2\right), \\ \dot{y} & = & r_2y\left(1 - y/K_2 - p_{21}x/K_2\right). \end{array}\right. \tag{12}$$

Proposition 2.3. *The normal form of (12) at $E_2(0, K_2)$ is*

$$\left\{\begin{array}{lcl} \dot{n}_1 & = & r_1\left(p_{21}/K_2 - 1/K_1\right)n_1^2 + O(\mathbf{n}^3), \\ \dot{n}_2 & = & -r_2n_2 + [p_{21}(r_2 - r_1)/K_2]n_1n_2 + O(\mathbf{n}^3), \end{array}\right. \tag{13}$$

and, thus, E_2 is a nondegenerated saddle-node.

Proof. First, we translate the point E_2 at the origin with the aid of the change $u_1 = x,\ u_2 = y - K_2$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (8) reads

$$\begin{cases} \dot{u}_1 &= -(r_1/K_1)u_1^2 - (r_1/K_2)u_1u_2, \\ \dot{u}_2 &= -r_2p_{21}u_1 - r_2u_2 - (r_2p_{21}/K_2)u_1u_2 - (r_2/K_2)u_2^2. \end{cases} \tag{14}$$

The eigenvalues of the matrix defining the linear terms in (14) are $\lambda_1 = 0$, $\lambda_2 = -r_2$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1, -p_{21})^T$ and $\mathbf{u}_{\lambda_2} = (0,1)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p_{21} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (14) achieves the form

$$\begin{cases} \dot{v}_1 &= Bv_1^2 - (r_1/K_2)v_1v_2, \\ \dot{v}_2 &= -r_2v_2 + p_{21}Bv_1^2 + Cv_1v_2 - (r_2/K_2)v_2^2, \end{cases} \tag{15}$$

where $B = r_1\left(p_{21}/K_2 - 1/K_1\right)$ and $C = p_{21}(r_2 - r_1)/K_2$, and the matrix defining the linear part is diagonal. In order to reduce the second order non-resonant terms in (15) we determine the transformation $\mathbf{v} = \mathbf{n} + \mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 3.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	B	$p_{21}B$	0	r_2	-	$p_{21}B/r_2$
1	1	$-r_1/K_2$	C	$-r_2$	0	$r_1/(r_2K_2)$	-
0	2	0	$-r_2/K_2$	$-2r_2$	$-r_2$	0	$1/K_2$

Table 3.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogenous vector polynomial in (15).

We find the transformation

$$\begin{cases} v_1 &= n_1 + [r_1/(r_2K_2)]n_1n_2, \\ v_2 &= n_2 + (Bp_{21}/r_2)n_1^2 + (1/K_2)n_2^2, \end{cases}$$

carrying (15) into (13). We have $p_{21} \neq K_2/K_1$, therefore $r_1(p_{21}/K_2 - 1/K_1) \neq 0$. By [1], the equilibrium point E_2 corresponding the dynamical system generated by a s.o.d.e. of the form (13) is a nondegenerated saddle-node. ∎

3. THE LOCAL PHASE PORTRAITS AROUND THE SADDLE-NODE SINGULARITIES

In the following we present the local phase portraits around the saddle-node singularities for $r_1 = 0.3,\ r_2 = 0.5,\ K_1 = 40,\ K_2 = 50$.

Namely, in fig. 1 it is shown the local phase portraits around $E_1,\ E_2,\ E_3$ for $p_{12} = K_1/K_2$, $p_{21} = K_2/K_1$ (a); around E_1 for $p_{12} \neq K_1/K_2$, $p_{21} = K_2/K_1$ (b); and around E_2 for $p_{12} = K_1/K_2$, $p_{21} \neq K_2/K_1$ (c).

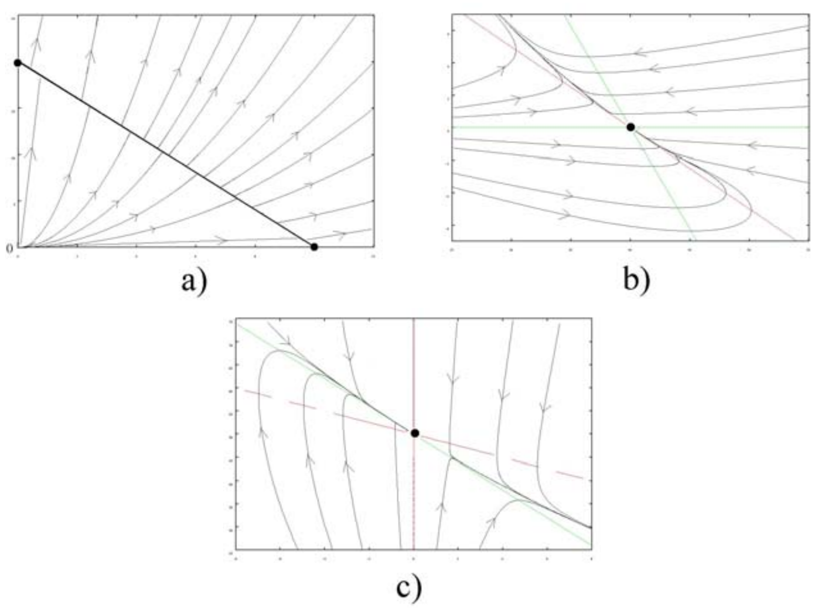

Fig 1. Phase portraits around $E_1,\ E_2,\ E_3$ (a), E_1 (b), E_2 (c).

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