ROMAI J., **3**, 1(2007), 111–114

A GRADIENT-BASED OPTIMIZATION APPROACH FOR FREE SURFACE VISCOUS FLOWS

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Abstract In this paper, a gradient-based optimization approach to solve the free surface viscous flows is purposed. The main problem to overcome is to compute the gradient itself. The Navier-Stokes equation is used to model the fluid flows and the problem is formulated as an optimization one, then the adjoint problem and the gradient is obtained.

1. INTRODUCTION

The free surface flows of fluids have many application in different industry fields. Among them we mention the ship industry where to improve ship's design knowledge of the water free surface flows are very important. So far many methods have been proposed for this type of problems, many of them related to the basic approach of ideal fluids. However, the viscous flows prove to be an interesting field of research, in order to get a more appropriate mathematical model for the physical problem. Once an optimal shape design method has been chosen to solve the free surface flows problem, the main issue which has to be solved is to compute the gradient for the objective functional.

This aim of this paper is to develop a gradient-based optimization method for the free surface viscous flows. Due to the application we have considered, the water flows in an open environment, it is possible to write the boundary conditions on the free boundary using a simplified approximation. With that approach and using the nonlinear Navier Stokes equation, the problem is formulated into an optimization form. Then the gradient and the adjoint problem is determinated.

2. MATHEMATICAL MODEL AND SOLUTION STRATEGY

The flow of an incompressible and viscous fluid with a free boundary is described by the Navier Stokes equation

$$\begin{aligned} (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - Re^{-1}\nabla\cdot\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) &= -Fr^{-2}\mathbf{j}, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

With $\mathbf{u}$, we denote the velocity field, p is the fluid pressure, Re is the Reynolds number and Fr is the Froude number. We assume that the fluid is defined on a unbounded region D, such that

$$\mathbf{u} = \mathbf{u}_d \tag{1}$$

as $x \to \infty$. Below the fluid domain has a fixed boundary Γ and above a free surface S. The fluid pressure may be splited into a hydrodynamic component φ and a hydrostatic component $Fr^{-2}y$, then it becomes

$$p(x,y) = \varphi(x,y) - Fr^{-2}y. \tag{2}$$

We assume that along the boundary S is true that

$$\mathbf{n}\cdot\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\mathbf{n} = 0. \tag{3}$$

Using (2) and (3) the model for the fluid flow becomes

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi-Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}\right) &= 0, &\quad \mathbf{x}\in D &\qquad (3)\\
\nabla\cdot\mathbf{u} &= 0, &\quad \mathbf{x}\in D &\qquad (4)\\
p &= 0, &\quad \mathbf{x}\in S &\qquad (5)\\
Re^{-1}\mathbf{t}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}\right)\mathbf{n} &= 0, &\quad \mathbf{x}\in S &\qquad (6)\\
\mathbf{u}\cdot\mathbf{n} &= 0, &\quad \mathbf{x}\in S &\qquad (7)\\
\mathbf{u} &= 0, &\quad \mathbf{x}\in \Gamma &\qquad (8)\\
\mathbf{u} &= \mathbf{u}_d, &\quad x\to\infty. &\qquad (9)
\end{aligned}
$$

One method to solve the free surface flows problem is to use a gradient based optimization approach. We assume that the flow domain may be extended beyond its boundaries, a set of domains is then constructed. Let us denote it by $\mathcal{D}$. Indeed, if we define an objective functional $J:\mathcal{D}\to\mathbb{R}$, where

$$
\begin{aligned}
J(S) &= \int_S p^2 dS+\int_D \mathbf{v}\cdot\left((\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi-Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}\right)\right)dD\\
&+ \int_D w\,(\nabla\cdot\mathbf{u})\,dD, \qquad (10)
\end{aligned}
$$

the problem (4) - (10) may be formulated as an optimization one. Then to solve it the main issue is to determine the gradient of the objective functional. Here $\mathbf{v}$ and w are two unknown functions on D such that $\mathbf{v}=0$ on Γ.

We assume that the free boundary is $S=\{(x,y):\ y=\alpha(x)\}$ and we get a new position of S by $(x,\alpha_1(x))=(x,\alpha(x))-\epsilon\mathbf{n}(x,\alpha(x))$. Therefore we compute the gradient of J in $\mathbf{n}$ direction $\mathrm{grad}_{\mathbf{n}}J=\mathrm{grad}J\cdot\mathbf{n}$.

Then we get the gradient of the objective functional (11)

$$
\begin{aligned}
\mathrm{grad}_{\mathbf{n}}J^{*} &= \int_S\{(1-\varphi+Fr^{-2}y)\mathbf{n}\cdot\nabla\varphi-Fr^{-2}\mathbf{n}\cdot\mathbf{j}\\
&\quad -\mathbf{v}\cdot Re^{-1}\left((\nabla(\mathbf{n}\cdot\nabla\mathbf{u}))+(\nabla(\mathbf{n}\cdot\nabla\mathbf{u}))^{T}\right)\mathbf{n}\}dS. \qquad (11)
\end{aligned}
$$

ln (12) there are two variables, $\mathbf{v}$ and w which are unknown. To have a useful formulation of the gradient these variables have to be computed. Based on necessary condition of minimum for J with respect to the flow variables $\mathbf{u}$ and φ we develop a strategy to solve this problem. Then $\mathbf{v}$ and w has to be obtained as solutions of the boundary value problem

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{v}+(\nabla\mathbf{u})\,\mathbf{v}+\nabla w+\nabla\cdot Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^{T}\right) &= 0, \quad \mathbf{x}\in D\\
\nabla\cdot\mathbf{v} &= 0, \quad \mathbf{x}\in D\\
\mathbf{v}\cdot\mathbf{n} &= -p, \; \mathbf{x}\in S\\
Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^{T}\right)\mathbf{n} &= 0, \quad \mathbf{x}\in S\\
\mathbf{v} &= 0, \quad \mathbf{x}\in \Gamma\\
\mathbf{v} &= 0, \quad |x|\to\infty.
\end{aligned}$$

3. CONCLUSIONS

When a gradient based optimization method is used to compute the free surface viscous fluid flows, the main problem is the gradient computation. In this paper we have presented a method to solve it.

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