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# SURVIVAL OPTIMIZATION FOR A DIFFUSION PROCESS

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**Abstract** We consider the problem of optimally controlling a one-dimensional diffusion process in the interval $[-d, +d]$ when there is a reflecting boundary at $-d$ and an absorbing boundary at $+d$. Moreover, the constant $d$ can actually be a random variable. The model can be used to represent the flight of an airplane between ground level and a level at which radar detection is likely. The objective is to maximize the survival time in the continuation region, while taking the quadratic control costs into account.

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## 1. INTRODUCTION

In Lefebvre and Whittle (1988), the authors considered in particular a one-dimensional Brownian motion as a rudimentary model for the flight of an airplane. Let $x(t)$ be the state variable and let $u(t)$ denote the control variable. They assumed that

$$dx(t) = b\,u(t)\,dt + \sigma\,dW(t),$$

where $b \neq 0$ and $\sigma > 0$ are constants, and $W(t)$ is a standard Brownian motion. The objective was to minimize the expected value of the cost function

$$J(x) = \int_0^{\tau(x)} \left[\tfrac{1}{2} q u^2(t) - \lambda\right] dt, \tag{1}$$

where $q$ and $\lambda$ are positive constants, and

$$\tau(x) = \inf\{t > 0 : x(t) = -d \text{ or } d \mid x(0) = x\}$$

with $x \in C := (-d, d)$. That is, $\tau(x)$ is the first time the controlled process $x(t)$ leaves the continuation region $C$, given that it started at $x \in C$. The authors showed that the optimal control $u^*$ is given by $u^* = \frac{\sigma^2}{b} \frac{G'(x)}{G(x)}$, where

$$G(x) := E\left[\exp\{\alpha \lambda T(x)\}\right] \tag{2}$$

with $\alpha = \frac{b^2}{\sigma^2 q}$ $(> 0)$.

In (2), $T(x)$ is the random variable that corresponds to $\tau(x)$ in the case of the *uncontrolled* process

$$dx_1(t) = \sigma\, dW(t).$$

The function $G(x)$ is the moment generating function of $T(x)$. Since $x_1(t)$ is a Wiener process with infinitesimal mean $\mu = 0$ and infinitesimal variance $\sigma^2$, the function $G$ satisfies the ordinary differential equation

$$\frac{\sigma^2}{2} G''(x) = -\alpha \lambda G(x), \tag{3}$$

subject to the boundary conditions $G(\pm d) = 1$. We find that

$$G(x) = \frac{\cos(\gamma x)}{\cos(\gamma d)} \quad \text{for } -d \le x \le d,$$

where $\gamma := \left(\frac{2\alpha\lambda}{\sigma^2}\right)^{1/2}$, and where we assume that $0 < \gamma d < \pi/2$. It follows that

$$u^* = -\frac{\sigma^2 \gamma}{b} \tan(\gamma x) = -\text{sgn}(b) \sqrt{\frac{2\lambda}{q}} \tan(\gamma x) \quad \text{for } -d < x < d.$$

In this model, $x = -d$ represented ground level, while $x = +d$ was a level at which radar detection was likely. Therefore, since $\lambda > 0$, the aim was to maximize the survival time in the continuation region $C$, while taking the quadratic control costs into account. When the parameter $\lambda$ is negative, Whittle (1982) termed this type of problem *LQG homing*.

In Section 2, we will assume that the constant $d$ is actually a random variable $D$ and that the boundary at $x = -d$ is reflecting rather than absorbing. We will compute the optimal control for a particular distribution of $D$. In Section 3, we will treat the case when $\lambda < 0$. Finally, some concluding remarks will be made in Section 4.

## 2. SURVIVAL OPTIMIZATION

If the diffusion process $x_1(t)$ has a reflecting barrier at $x = b_0$, then the function $G(x)$ defined in (2) is such that (see Cox and Miller (1965, p. 231), for instance)

$$G'(x)\big|_{x=b_0} = 0.$$

Hence, if there is a reflecting barrier at $x = -d$, and an absorbing barrier at $x = d$, we must solve the ordinary differential equation (3) subject to

$$G(d) = 1 \quad \text{and} \quad G'(x)\big|_{x=-d} = 0.$$

We easily find that

$$G(x) = \frac{\cos[\gamma(x+d)]}{\cos(2\gamma d)} \quad \text{for } -d \le x \le d,$$

where $\gamma = \sqrt{2\alpha\lambda}/\sigma$, as above.

Now, if $d$ is replaced by the random variable $D$ defined in the interval $(0, \frac{\pi}{4\gamma})$ (so that $2\gamma d \in (0, \frac{\pi}{2})$, as required), we may write that

$$G(x) = E\left[E\left[e^{\alpha\lambda T(x)} \mid D\right]\right] = \int_0^{\frac{\pi}{4\gamma}} \frac{\cos[\gamma(x+y)]}{\cos(2\gamma y)} f_D(y)\, dy. \tag{4}$$

We can obtain an explicit formula for $G(x)$ by choosing, in particular,

$$f_D(d) = k\cos(2\gamma d) \quad \text{for } d \in \left[d_0, \tfrac{\pi}{4\gamma}\right), \tag{5}$$

where $k$ is a normalizing constant and $d_0 \in (0, \frac{\pi}{4\gamma})$. Indeed, we then obtain that

$$\begin{aligned} G(x) &= k \int_{d_0}^{\frac{\pi}{4\gamma}} \cos[\gamma(x+y)]\, dy = \frac{k}{\gamma} \left\{ \sin\left[\gamma\left(x + \frac{\pi}{4\gamma}\right)\right] - \sin\left[\gamma\left(x + d_0\right)\right] \right\} \\ &= \frac{k}{\gamma} \left( \frac{\sqrt{2}}{2} \sin(\gamma x) + \frac{\sqrt{2}}{2} \cos(\gamma x) - \sin\left[\gamma\left(x + d_0\right)\right] \right). \end{aligned}$$

We may state the following proposition.

**Proposition 2.1.** *With the choice in (5) for the distribution of the random variable $D$, when there is a reflecting barrier at $x = -D$ and an absorbing barrier at $x = D$, the optimal control is given by*

$$u^* = \operatorname{sgn}(b) \left(\frac{2\lambda}{q}\right)^{1/2} \frac{\frac{\sqrt{2}}{2} \cos(\gamma x) - \frac{\sqrt{2}}{2} \sin(\gamma x) - \cos\left[\gamma\left(x + d_0\right)\right]}{\frac{\sqrt{2}}{2} \sin(\gamma x) + \frac{\sqrt{2}}{2} \cos(\gamma x) - \sin\left[\gamma\left(x + d_0\right)\right]}$$

*for $-d_0 < x < d_0$.*

*Remarks.* i) The random variable $D$ could of course also be a discrete random variable taking its values in a set contained in the interval $[0, \frac{\pi}{4\gamma})$. For instance, $D$ could be defined on the set $\{\frac{\pi}{6\gamma}, \frac{\pi}{5\gamma}\}$.

ii) If $x(t) = x$, with $x$ such that $|x| \geq d_0$ (but the process has not reached the absorbing barrier yet), we could obtain the optimal control $u^*$ by replacing $f_D(d)$ (in the continuous case) by the conditional density function $f_D(d \mid D > |x|)$ in the calculation of $G(x)$. Moreover, if the process reached the state $x_0 > d_0$ without being absorbed and then returned to the interval $(-d_0, d_0)$, we can use the formula above for the optimal control in the interval $(-x_0, x_0)$. Indeed, the optimizer then knows that the absorbing barrier is not located in the interval $[d_0, x_0]$. Similarly, if the controlled process takes on a value $-x_0$ in the interval $(-\frac{\pi}{4\gamma}, -d_0]$ and returns to $(-d_0, d_0)$, then the optimizer again infers that the value of $D$ is not in the interval $[d_0, x_0]$, and we can now use the formula for $u^*$ in $(-x_0, x_0)$.

iii) Let $b = \sigma = \lambda = 1$ and $q = 2$. Then, $\gamma = 1$ and (with $d_0 = \pi/6$)

$$u^* = \frac{\left(\sqrt{2} - \sqrt{3}\right)\cos(x) + \left(1 - \sqrt{2}\right)\sin(x)}{\left(\sqrt{2} - \sqrt{3}\right)\sin(x) - \left(1 - \sqrt{2}\right)\cos(x)} \quad \text{for } -\pi/6 < x < \pi/6.$$

This function is shown in fig. 1, together with the optimal control when $d = \pi/6$.

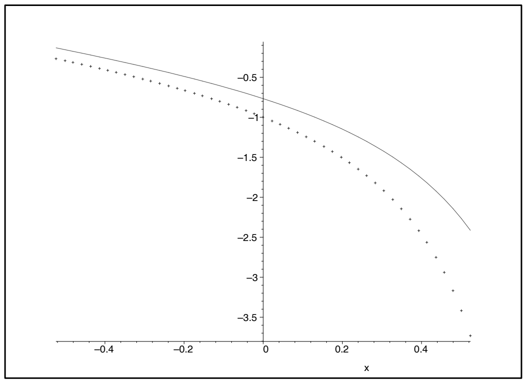


*Fig. 1.* Optimal control when $d = \pi/6$ (dotted line) and when $d$ is random (full line).

iv) The function $G(x)$ being a mathematical expectation, we can make use of the formula in (4). However, the optimal control $u^*$ is *not* a mathematical expectation. Therefore, we could not have obtained it by conditioning on $D$.

v) If we assume that the reflecting barrier is located at $x = 0$ rather than at $x = -d$, we find that

$$G(x) = \frac{\cos(\gamma x)}{\cos(\gamma d)} \quad \text{for } 0 \leq x \leq d.$$

In the case when $d$ is a random variable $D$, we have

$$G(x) = \int_0^\infty \frac{\cos(\gamma x)}{\cos(\gamma y)} f_D(y)\, dy = \cos(\gamma x) \int_0^\infty \frac{f_D(y)}{\cos(\gamma y)}\, dy.$$

We see that the function $G$ is simply multiplied by a constant. Since $u^*$ is proportional to $G'(x)/G(x)$, we deduce that (unfortunately) the optimal control $u^*$ remains the same.

In the next section, we will assume that the parameter $\lambda$ is negative, so that the objective will then be to minimize the time spent in the continuation region $C$.

## 3. LQG HOMING

When $\lambda = -\theta$, with $\theta > 0$, in the cost function (1) and there is a reflecting boundary at $x = -d$, we find that the function $G(x)$ is given by

$$G(x) = \frac{\cosh[\mu(x+d)]}{\cosh(2\mu d)} \quad \text{for } -d \leq x \leq d,$$

where

$$\mu := \frac{1}{\sigma}\sqrt{2\alpha\theta}.$$

*Remarks.* i) We deduce from the function $G(x)$ that

$$u^* \propto \tanh\left[\mu(x+d)\right].$$

Hence, the optimal control does depend on $d$ in that case, while if the barrier at $x = -d$ is absorbing (like in the original problem), then

$$u^* = \frac{\sigma^2}{b}\,\mu\tanh(\mu x),$$

which is independent of $d$.

ii) Notice that there is no constraint on $\mu$, contrary to $\gamma$. That is, the penalty given for survival in $C$ can be as large as we want, whereas we could not give too large a reward for remaining in $C$ in the previous case.

As in the previous problem, we replace $d$ by a random variable $D$, so that

$$G(x) = \int_0^\infty \frac{\cosh[\mu(x+y)]}{\cosh(2\mu y)}\, f_D(y)\, dy.$$

For simplicity, assume that

$$f_D(d) \propto \cosh(2\mu d) \quad \text{for } c_1 \leq d \leq c_2, \tag{6}$$

where $c_1$ and $c_2$ are positive constants. We calculate

$$G(x) \propto \int_{c_1}^{c_2} \cosh[\mu(x+y)]\, dy \propto \sinh[\mu(x+c_2)] - \sinh[\mu(x+c_1)]$$

for $-c_1 \leq x \leq c_1$. A simple calculation then yields the following proposition.

**Proposition 1.** *If $\lambda = -\theta \in (-\infty, 0)$ in (1) and $d$ is a random variable $D$ having the probability density function given in (6), then the optimal control in the case when the barrier at $x = -D$ is reflecting is (see a particular example in fig. 2)*

$$u^* = \frac{\sigma^2}{b}\,\mu\,\frac{\cosh[\mu(x+c_2)] - \cosh[\mu(x+c_1)]}{\sinh[\mu(x+c_2)] - \sinh[\mu(x+c_1)]}$$

*for $-c_1 < x < c_1$.*

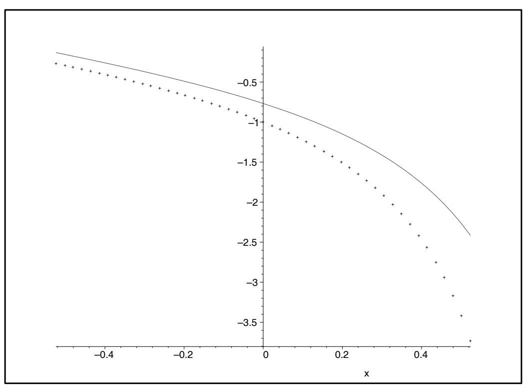


*Fig. 2.* Optimal control when $d$ is deterministic (dotted line) and when $d$ is random (full line).

## 4. CONCLUDING REMARKS

In this note, we have extended the survival optimization or LQG homing problem set up by Whittle (1982), and considered by Lefebvre and Whittle (1988), by assuming that the barriers are located at $x = \pm D$, where $D$ is a random variable rather than a constant.

Going back to the application given in Lefebvre and Whittle (1988), namely that of an airplane trying to avoid hitting the ground at $x = -d$ or being detected by a radar at height $x = d$, to be more realistic we can assume that ground level corresponds to $x = 0$ (and that the height at which radar detection is likely is $x = d$, as before). If the two barriers are absorbing, which again is more realistic, then we find that

$$G(x) = \cos(\gamma x) + k_d \sin(\gamma x) \quad \text{for } 0 \le x \le d,$$

where $k_d$ is a constant (that depends on $d$) given by

$$k_d = \frac{1 - \cos(\gamma d)}{\sin(\gamma d)}$$

(with $0 < \gamma d < \pi/2$). Moreover, the optimal control is

$$u^* = \frac{\sigma^2}{b} \mu \frac{[-\sin(\gamma x) + k_d \cos(\gamma x)]}{\cos(\gamma x) + k_d \sin(\gamma x)}$$

for $0 < x < d$.

Next, if $d$ is replaced by the positive random variable $D$ with range $[d_0, d_1)$, where $d_1 \in (d_0, \infty)$, we have

$$\begin{aligned} G(x) &= \cos(\gamma x) + \left[ \int_{d_0}^{d_1} \frac{1 - \cos(\gamma y)}{\sin(\gamma y)} f_D(y)\, dy \right] \sin(\gamma x) \\ &:= \cos(\gamma x) + k^* \sin(\gamma x). \end{aligned}$$

It follows that

$$u^* = \frac{\sigma^2}{b} \mu \frac{[-\sin(\gamma x) + k^* \cos(\gamma x)]}{\cos(\gamma x) + k^* \sin(\gamma x)}$$

for $x \in (0, d_0)$. Therefore, the constant $k_d$ is simply replaced by the new constant $k^*$, which depends on the distribution of the random variable $D$, in $(0, d_0)$. For $x \in [d_0, d_1)$, we should substitute $f_D(d)$ by $f_D(d \mid D > x)$ in the calculation of $G(x)$.

Similarly, with $\lambda = -\theta \in (-\infty, 0)$, we obtain

$$G(x) = e^{-\mu x} + \kappa_d \left(e^{\mu x} - e^{-\mu x}\right) \quad \text{for } 0 \le x \le d,$$

where

$$\kappa_d := \frac{1}{1 + e^{\mu d}},$$

and

$$u^* = \frac{\sigma^2}{b}\,\mu\,\frac{\kappa_d\,(e^{\mu x} + e^{-\mu x}) - e^{-\mu x}}{\kappa_d\,(e^{\mu x} - e^{-\mu x}) + e^{-\mu x}}.$$

When $d$ becomes the random variable $D \in [d_0, d_1)$, the constant $\kappa_d$ is replaced by the constant

$$\kappa^* := \int_{d_0}^{d_1} \frac{1}{1 + e^{\mu y}}\, f_D(y)\, dy$$

(in the interval $(0, d_0)$).

Finally, to bring the problem closer to reality, we could consider at least a two-dimensional model for the flight of the airplane. However, obtaining an explicit (and exact) expression for the function $G(x)$ is then generally very difficult.

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