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RECURSIVE UNSOLVABILITY OF A PROBLEM OF EXPRESSIBILITY IN THE LOGIC OF PROVABILITY

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Abstract It is notified the recursive unsolvability of a problem of syntactical expressibility in the Gödel-Löb logic of provability.

In studies lying on the intersection of mathematical logic and its applications an important role is often played by the relation of (functional) expressibility. That is a relation among logical operations (functions) signifying the possibility to obtain some of them from others by means of compositions. In the case of classical logic the expressibility relation was stadied by Post [1-3].

In particular, Post obtained the description of all classes of Boolean functions, closed with respect to expressibility, the so-called *Post classes* [3]. On the basis of the survey of these classes it is not difficult to obtain a not complicate algorithm for determining the expressibility. For any Boolean function f, given by a table or by formula, and for any finite system $\sum$ of such functions, this algorithm enables us to recognize whether the given function f is expressible through $\sum$ by means of superpositions.

At the same time there are some well-known logics represented by means of logical calculus (i.e. systems of axioms and inference rules). For example,

1) the (propositional) intuitionistic logic,

2) the dual intuitionistic logic,

3) the modal logics S4 and S5,

4) the Gödel-Löb logic of provability.

These logics can not be represented, as well as like classical logic by means of some finite system of finite truth tables. But any of them can be formulated by means of a corresponding logical calculus (i.e. syntactically), namely:

I. the intuitionistic calculus of Heyting [4];

II. the dual intuitionistic calculus of Moisil [5,6];

III. the modal systems S4 and S5 of Lewis [7];

IV. the calculus of provability of Gödel-Löb [8-12].

The language of the calculus of provability consists of formulas constructed in the usual way by means of the connectives & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation) and Δ (Gödel provability), and brackets, starting with propositional variables p, q, r (possibly with subscripts).

The calculus of provability is given by the axioms of the classical prepositional calculus and by the two Δ-axioms: $\Delta(\Delta p \supset p) \supset \Delta p$ (axiom of Löb), $\Delta(p \supset q) \supset (\Delta p \supset \Delta q)$, and by the three rules of inference: the substitution rule, the modus ponens rule, and the necessitation rule (permiting the transition from a formula F to ΔF).

The logic of provability of Gödel-Löb (in short, the logic GL) is defined as the set of all formulas derivable in the corresponding calculus.

Two formulas F and G are said to *be equivalent* in a logic L, if their equivalence $(F \sim G) = ((F \supset G)\&(G \supset F))$ is true in L (that is, belongs to the set L).

Now, let us remind the notion of expressibility in syntactical version, proposed by Kuznetsov [13].

Let consider a formula F and a system of formulas Σ. Then formula F is said to be *expressible in the logic L by means of the system* Σ, if F can be obtained from variables and from formulas of Σ by a finite number of applications of the weak rule of substitution $A, B/A[\pi/B]$ and the rule of replacement by equivalents in L A/B if $(A \sim B) \in L$.

By the *problem of expressibility in the logic* L we mean the algorithmic problem requiring the construction of an algorithm which, for every formula F and for every finite system of formulas Σ, enables us to recognize whether F is expressible in L by means of Σ.

Theorem 1. *There is no algorithm solving the problem of expressibility in the Gödel-Löb logic of provability.*

The problem of expressibility in a logic L is said to be recursively solvable if there exists an algorithm solving it in L.

The Theorem 1 is equivalent [14,15] to the following

Theorem 2. *The problem of expressibility in the Gödel-Löb logic of provability is recursive unsolvable.*

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