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PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO FINANCE

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As a component part of the national market or the international market, the financial market comprises, in its turn, two distinct segments: the bank market and the proper financial market, and the capital market or capitals' market respectively [2].

As part of the capital market, an important place is taken by the derivate financial products or derivatives (e.g. options, forward contracts, futures contracts). These are financial instruments whose value depends on the underlying asset value (stocks, foreign currencies, commodities etc.). The derivatives are used by investors, mainly, in the following purposes: speculation (with the hope of earning a great amount of money with an initial small investment), hedging (covering the risk through buying other contracts which have as a support the same underlying asset [8]) and arbitration (fulfilling income bringing transaction so that the investor should not assume risks or invest capital [1].

The simplest derivative is a forward or futures contract. The forward contract is a sale and purchase contract of a financial asset, commodity or foreign currency, at a certain future date (denoted by T), established through contract and to a price (called strike price and further denoted by E), established at a moment of the transaction end.

At the emission, the premium of a forward contract is zero.

A futures contract has the same features as a forward contract, with a few differences:

- the futures contract is standardized. All futures contracts having the same standardized asset form a sort of futures contracts [3];

1 the price vary daily depending on the demand and supply;

2 it is traded on a secondary market. The options are standardized contracts which give their buyer the right to sell or to buy a certain quantity of underlying asset at a fixed price, at the option expiry date (denoted by T) or before this date.

There are two types of options:

1 buying options (CALL), which give the buyer the right (removing the obligation) to buy the underlying asset at a fixed price at the moment of the transaction end and at a certain established date;

2 selling options (PUT), which give the buyer the right (removing the obligation) to sell the underlying asset at a fixed price at the moment of the transaction end and at a certain established date.

As this is a right and not an obligation, the buyer of the option can choose not to exercise this right and let the option expire. To exchange this right, the buyer of the option pays the option seller a certain amount of money, called premium.

For a CALL option, the payoff function is max(S-E, 0), where S is the underlying asset value. Thus, if the price of the underlying asset, at the T time, is higher than the practicing price, then the option will be exercised. If not, the option will be not exercised.

For a PUT option, the payoff function is max(E-S, 0), the option being exercised only if the price of the underlying asset, at the T time, is lower than the exercise price.

Depending on the moment of her exertion, we distinguish two great categories of options:

1 European options, whose exertion is allowed only at the date of payment;

2 placecountry-region American options, whose exertion is allowed any time before the date of payment.

As time passes, the option value modifies not only because the expiry date comes nearer but also because of the variations of the underlying asset's price. Therefore, the value of an option (denoted by V), at the T expiry moment, depends exclusively on S and T. It is also supposed the same thing for a time moment t<T and we write Vt=V(S, t). If the law of movement for S is known, namely we can write the equation which determines dS, applying Ito's lemma [7], we determine dV_t, where the price increases are denoted by dV_t [8].

One of the most important methods used in finances for evaluating the derivative products consists in using partial differential equations.

Neftci, in [6], shortly presents the logic behind the derivatives evaluating method which led to the use of partial differential equations. A wide accepted model in finances, ever since the moment of its publishing (1973), is the Black_Scholes model of option pricing.

Thus, starting with the stochastic differential equation

$$dS = \mu \cdot Sdt + \sigma \cdot SdB_t, \tag{1}$$

where μ = drift, σ = volatility, B = Brownian motion, building a free risk portfolio and applying Ito's 1_dimensional formula, the Black_Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r\cdot\frac{\partial V}{\partial S}\cdot S-r\cdot V=0), \qquad (2)$$

is obtained, where r is the interest rate.

By adding the final condition (written for a placeEuropean CALL option):

$$Vt=max(S-E,0), \qquad (3)$$

and boundary conditions

$$\lim_{S\to 0}V(S,t)=0$$

and

$$\lim_{S\to\infty}V(S,t)=\infty$$

to a partial differential equation, allows us an accurate determination of the solution. In financial problems, limit conditions show the way in which the solution behaves for $S\to 0$ or $S=0$. Generally, the final conditions consist in payoff function. The second-order partial differential equation written in the general form

$$a\frac{\partial^2 V}{\partial S^2}+b\frac{\partial^2 V}{\partial S\partial t}+c\frac{\partial^2 V}{\partial t^2}+d\frac{\partial V}{\partial S}+e\frac{\partial V}{\partial t}+f=0$$

is of parabolic type if $b^2-4ac=0$. Thus, the Black_Scholes partial differential equation is parabolic and linear (the sum of two solutions of the equation is another solution of the equation). Black and Scholes solved this equation and obtained the solution

$$V(S,t)=SN(d_1)-E\cdot e^{-r\cdot(T-t)}\cdot N(d_2),$$

where $d_1=\frac{\ln\frac{S}{E}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2=d_1-\sigma\sqrt{T-t}$ and $N(d_i)=\int_{-\infty}^{d_i}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}x^2}dx$.

The partial differential equation obtained by Black_Scholes is relevant if a series of hypotheses are fulfilled [1], [6], [8]:

1 returns are log normally distributed;

2 the stock pays no dividends during the option life;

3 no commissions are charged;

4 markets are efficient;

5 the interest rate and volatility remain constant;

6 there are no arbitrage opportunities.

For an placeEuropean PUT option with final condition $V_t = max(E - S, 0)$ and boundary conditions $\lim_{S \to 0} V(S,t) = E \cdot e^{-r(T-t)}$ and $\lim_{S \to \infty} V(S,t) = 0$, the Black_Scholes formula reads

$$V(S,t) = -SN(-d_1) + E \cdot e^{-r \cdot (T-t)} \cdot N(-d_2)$$

with $N(-d) = 1 - N(d)$.

However, there are many situations in which one or more of these hypotheses are not fulfilled, case in which a new partial differential equation is obtained.

If the underlying asset generates constant and continuous share, denoted by δ, it is obtained the equation [1]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) \cdot \frac{\partial V}{\partial S} \cdot S - r \cdot V = 0$$

which has the solutions

$V(S,t) = Se^{-\delta(T-t)} N(d_1) - E \cdot e^{-r \cdot (T-t)} \cdot N(d_2)$ (for placeStateCALL options)

$$V(S,t) = -Se^{-\delta(T-t)} N(-d_1) + E \cdot e^{-r \cdot (T-t)} \cdot N(-d_2) \ (for PUT options)$$

where

$d_1 = \frac{\ln\frac{S}{E}+(r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\frac{S}{E}+(r-\delta-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Numerical example. Consider an European CALL option on a Microsoft stock which has the following specific features[12]: the current price S=28,22, the striking price E=27.5, volatility (%) 19,37, the risk_free interest rate (%) 5.32 and continuous dividend rate 2,1%. If the date of payment is 30 days, it is obtained a value for an option of 1,09.

In Chart 1 there are graphically represented the values of the option calculated by means of the Black_Scholes formula, in the situation in which the current price of the stock has the values given in Table 1, the values of the other specific features remaining constant.

Share price	Option price
25,840	0,1067
25,910	0,1174
25,930	0,1206
25,980	0,1289
26,330	0,2004
26,860	0,3591
27,200	0,4973
27,440	0,6130
28,22	1,0901

Table 1

Chart 1. The graph of the pricing option value.

The options pricing on futures contracts leads to an equation analogous to the equations (2)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r \cdot V = 0,$$

where F = futures prices. The solutions of these equations are

$$V(F,t) = e^{-r(T-t)}[FN(d_1) - E \cdot N(d_2)], \text{ (for placeStateCALL options)}$$

$$V(F,t) = e^{-r(T-t)}[-FN(-d_1) + E \cdot N(-d_2)], \text{ (for PUT options)}$$

where $d_1 = \dfrac{\ln \frac{F}{E} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$. Therefore, solving the partial differential equations and their analysis plays a very important role in derivatives theory.

Notes

1. In concordance with the information taken by the author from the address: www.nasdaq.com.

2. Share prices for the MSFT assets (in the interval 01.09.2006-29.09.2006). See www.nasdaq.com

References

[1] ALTAR, M., *Financial engineering*, A.S.E., Bucharest, 2002. (Romanian)

[2] BUNEA, D., *Integration of the financial and inssurance markets*, PRIMM, year IV, nr. 4/2002. (Romanian)

[3] GHILIC-MICU, B., *Strategies on the capital market*, Ed. Economica, Bucharest, 2002. (Romanian)

[4] HULL, J., *Options, futures and other derivative securities*, Prentice Hall, 1997.

[5] KOZIOL, J. D., *A handbook for professional futures and options traders*, 1987.

[6] NEFTCI, S., N., *An introduction to the mathematics of financial derivatives*, 2nd ed., Academic, New York, 1996.

[7] OKSENDAL, B, *Stochastic differential equations, An introduction with applications*, 6th ed., Springer, New York.

[8] WILMOTT, P., *Derivatives. Financial engineering*, Ed. Economica, Bucharest, 2002. (Romanian)