

## **ASYMPTOTIC STABILITY OF TRIVIAL SOLUTION OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED NONLINEARITY, IN BANACH SPACES**

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This year the mathematical scientific community in Russia celebrated the 150-th birthday of the outstanding mathematician - academician of St-Petersburg Academy Alexandr Mikhaylovich Lyapunov. International Congress of Non-linear Analysis in June 2007 was devoted to this jubilee. On World Congress ICIAM this significant event was marked in reports of the minisymposium organized by our scientific group. Practically during all my life I worked in branching and bifurcation theory of solutions of nonlinear equations, one of initiator of which was Academician Lyapunov (see, for example, [1-2]). Another among his important contribution to the mathematical science was the creation of contemporary stability theory. Before preparing the 150-th Lyapunov jubilee I began a series of articles, trying to over-understand some of this fundamental results in stability theory from the functional analysis point of view. This allows to carry over the Lyapunov approach from ordinary differential equations to functional-differential and integro-partial-differential equations.

At first our attention is payed to the generalization of the Lyapunov theorem

about stability on the first approximation. In a Banach space, a differential equation with linear unbounded operator and nonlinear part depending on time is considered. Non-analytic and analytic cases are studied. In this communication the more general case considered by us is suggested. The nonlinear operator contained in the differential equation can be increasing when the time goes to infinity. This fact essentially extends the application possibilities of our method.

Some separate fragments of this communication were reported, discussed and published on various kinds of international conferences. Note, in particular, International Congress in Steklov Mathematical Institute RAN, devoted to the 100th Birthday of Academician Nikol'sky [6], ISAAC Congress in Sicilia [5], [8], seminar of Academician Il'in on Computation and Cybernetics Faculty of Moscow State Lomonosov University [7], World Congress ICIAM-2007.

In a real or complex Banach space  $X$  let us consider on the semiaxis  $\mathfrak{R}^+ = [0, \infty)$  the Cauchy problem for the differential equation

$$\dot{x} = Ax + R(t, x), \quad x(0) = x_0. \quad (1)$$

We make the following assumptions.

**I.**  $A$  is a closed linear operator, densely defined on  $D(A) \subset X$  with values in  $X$ . It is the generator of a strongly continuous semigroup  $U(t) = \exp(At)$  exponentially decreasing, i.e. there exist  $M > 0$  and  $\alpha > 0$  such that on  $\mathfrak{R}^+$  the inequality  $\|U(t)\| \leq Me^{-\alpha t}$  is valid.

**II.** The nonlinear operator  $R(t, x)$  is continuous by the set of variables  $t \in \mathfrak{R}^+$ ,  $x \in S = \{x \in X \mid \|x\| < p\}$ , and  $R(t, 0) = 0$  for all  $t \in \mathfrak{R}^+$ . Let further exist  $\beta > 0$  and the continuous on  $\mathfrak{R}^+$  function  $C(t) > 0$  such that for all  $t \in \mathfrak{R}^+$ ,  $x_1, x_2 \in S$  the following inequality

$$\|R(t, x_1) - R(t, x_2)\| \leq C(t) \max^{\beta}(\|x_1\|, \|x_2\|) \|x_1 - x_2\|$$

is fulfilled.

For the problem (1) define its classical solution as the function  $x(.) : \mathfrak{R}^+ \rightarrow S$  such that on  $\mathfrak{R}^+$ ,  $x(t) \in D$  is continuously differentiable and its substitution in (1) turns it in identity.

The generalized solution of (1) is understood as a continuous on  $\mathfrak{R}^+$  solution of the integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)R(s, x(s))ds. \tag{2}$$

In addition, the weakened solution of the problem (1) can be considered, when  $x(t)$  is continuous on  $\mathfrak{R}^+$  and continuously differentiable on the semi-axis  $(0, +\infty)$ .

Note that from the condition II the inequality

$$\|R(t, x)\| \leq C(t)\|x\|^{1+\beta}, \forall x \in S, t \in \mathfrak{R}^+. \tag{3}$$

follows. The differential equation from (1) has the trivial classical solution  $x(t) = 0$ .

If it would be established (under some additional conditions), that the problem (1) for all sufficiently small  $\|x_0\|$  has unique small together with small  $\|x_0\|$  solution and  $\|x(t)\| \rightarrow 0, t \rightarrow \infty$ , then by this the asymptotic stability of the trivial solution would be established.

Introduce two complementary restrictions on the function  $C(t)$  in condition II.

**III.** There exist  $\gamma \in (0, \alpha), C^* > 0$  such that  $C(t)e^{-\gamma\beta t} \leq C^*, \forall t \in \mathfrak{R}^+$ .

**III'.**  $\int_0^{+\infty} C(s)e^{-\alpha\beta s}ds = C_1^* < \infty$  (convergent).

Further our aim is the study of solutions to integral equation (2) on  $\mathfrak{R}^+$ , which are exponentially decreasing as  $t \rightarrow +\infty$ . For the achievement of this aim introduce a suitable family of Banach spaces of abstract functions.

**Definition.** For  $\gamma > 0$ , the set of all defined and continuous on  $\mathfrak{R}^+$  abstract functions  $x(t)$  taking the values in  $X$  with the natural operations of their

addition and multiplication on scalars, for which the following norm is finite  $|||x|||_\gamma = \sup_{\mathfrak{R}^+} ||x(t)||e^{\gamma t}$  will be called  $C_\gamma$ .

Note that  $C_\gamma$  is a Banach space.

From this definition it follows that if  $x(t) \in C_\gamma$ , then  $||x(t)|| \leq |||x|||_\gamma e^{-\gamma t}$ .

Note also that if  $\gamma_1 < \gamma_2$  then the space  $C_{\gamma_2}$  is embedded in the space  $C_{\gamma_1}$ .

**Lemma 1.** For  $\gamma \in (0, \alpha)$  define on  $X$  the linear operator in the form  $(Dx_0)(t) = U(t)x_0$ . Then  $D \in L(X, C_\gamma)$ ,  $||D|| \leq M$  and  $Dx_0(t)$  is the generalized solution of Cauchy problem

$$\dot{x} = Ax, \quad x(0) = x_0.$$

**Proof.**  $||e^{\gamma t}U(t)x_0|| \leq M||x_0||$ , and it means  $|||Dx_0|||_\gamma \leq M||x_0||$ .  $\square$

Consider in  $C_\gamma$  the open ball  $S_\gamma = \{x(t) \in C_\gamma; |||x|||_\gamma < p\}$ .

**Lemma 2.** Assume  $\gamma \in (0, \alpha]$ , one of the conditions III or III' be satisfied and  $x(t) \in S_\gamma$ . Define the nonlinear operator  $F$  by formula

$$F(x)(t) = \int_0^t U(t-s)R(s, x(s))ds.$$

Then the abstract function  $F(x)(t)$  belongs to  $C_\gamma$  and there exists an independent on  $x$  constant  $K > 0$  such that  $|||F(x)|||_\alpha \leq K|||x|||_\gamma^{1+\beta}$ .

Proof. From the condition II and inequality (3) it follows the estimate

$$\begin{aligned} ||e^{\gamma t}F(x)(t)|| &\leq Me^{-(\alpha-\gamma)t} \int_0^t e^{\alpha s}C(s)||x(s)e^{\gamma s}||^{1+\beta}e^{-\gamma(1+\beta)s}ds \\ &\leq Me^{-(\alpha-\gamma)t} \int_0^t C(s)e^{(\alpha-\gamma)s}e^{-\gamma\beta s}ds|||x|||_\gamma^{1+\beta}. \end{aligned}$$

For  $\gamma = \alpha$  the passage to supremum on  $\mathfrak{R}^+$  gives  $|||F(x)|||_\alpha \leq MC_1^*|||x|||_\alpha^{1+\beta}$ .

Let be  $\gamma < \alpha$ . Consider the quotient of  $\int_0^t C(s)e^{(\alpha-\gamma)s}e^{-\gamma\beta s}ds$  and  $e^{(\alpha-\gamma)t}$ .

Application of the L'Hopital rule gives the expression

$$(\alpha - \gamma)^{-1} \lim_{t \rightarrow \infty} C(t)e^{\gamma\beta t} \leq (\alpha - \gamma)^{-1}C_1^*.$$

From the proved lemma it follows that the operator  $\Phi(x, x_0) = Dx_0 + F(x)$  is acting in  $C_\gamma$ , mapping every closed ball from  $S_\gamma$  of sufficiently small radius

in a closed ball. The subsequent lemma shows that this operator satisfies the special form of the Lipschitz condition.

**Lemma 3.** *Let  $x_1(t), x_2(t) \in S_\gamma$ . Then the following inequality*

$$\|F(x_1) - F(x_2)\|_\gamma \leq K \max(\|x_1\|_\gamma, \|x_2\|_\gamma) \|x_1 - x_2\|_\gamma$$

*is true.*

The proof is carried out by the scheme of the proof of Lemma 2.

Further, for simplicity of presentation, restrict oneself to the case  $\gamma = \alpha$ .

**Theorem.** *There exist the numbers  $r_* > 0$ ,  $\rho_* > 0$  such that for any  $x_0$ ,  $\|x_0\| \leq \rho_*$  the equation  $x = Dx_0 + F(x)$  has in the ball  $\|x\|_\alpha \leq r_*$  the continuous in the ball  $\|x_0\| \leq \rho_*$  unique solution  $x = x(x_0)$ ,  $x(0) = 0$ .*

The proof of this lemma is contained in the article [7].

Coming back to the integral equation (2) we see that the existence of its continuous on  $\mathfrak{R}^+$  solution is proved and it satisfies the inequality  $\|x(t)\| \leq r_* e^{-\alpha t}$ . If applied to Cauchy problem (1) this means the proof of existence of its exponentially decreasing generalized solution for initial values of sufficiently small norm. This fact means the asymptotic stability of the trivial solution of (1) in the generalized sense.

**Remarks.** Under the assumption of the  $C(t)$  boundedness in the work [7] it is proved that the generalized solution of the problem (1) is its classical solution if  $R(x,t)$  satisfies some special Hölder condition.

We plan to investigate this question in the hypotheses of this paper. More difficult is the generalization on the case when operator  $A$  is dependent on  $t$ . Such situation is closely connected with possibilities of the developments of Lyapunov exponents method and Lyapunov-Floquet theory to the case of unbounded operators in Banach spaces. The author hopes, that all these problems will be resolved in the collaboration at the passage of work on the research grant Romania-Russia.

The research has been supported by the joint grant of the Russian Found of

Fundamental Investigations (RFFI) and the Academy of Sciences of Romania N 07-01-91680\_PA\_a and by the grant RFFI N 05.01.00422a.

## Notes

1. In the printed version of ROMAI Journal, the title of this article appears, by Editor's mistake, as "Asymptotic stability of trivial solution of differential equations with unbounded linearity, in Banach spaces". Since the online edition is much more accessed than the printed one, we make the correction here.

## References

- [1] M. Lyapunov, *Collection of works*, vol 2, Moscow-Leningrad, (1956).
- [2] M.M. Vainberg, V.A. Trenogin, *Branching Theory of solutions of Nonlinear Equations*, Moscow, Nauka (1969), Noordhoff Int. Publ (1974).
- [3] V. A. Trenogin, *Functional analysis*, Moscow.- Phizmathlit, 2002, 2007 (Nauka-1980, 1993, French translation Moscow, Mir, (1980).
- [4] V.A. Trenogin, *First Lyapunov method in analytical case for dynamical system*, Works of 12-th Math. School of Moscow State Social University, Moscow, (2005), 91-97.
- [5] V.A. Trenogin, *First Lyapunov method for the abstract parabolic equations*, Abstracts of ISAAC-2005, University of Catania, 15-16.
- [6] V.A. Trenogin, *Abstract dynamical systems and first Lyapunov method*, Abstracts of International Conference, dedicated to the 100-birthday of academician S.M. Nikolsky, Moscow, Russian Academia Nauk Publ., (2005), 225.
- [7] V.A. Trenogin, *Lyapunov theorem about the stability by linear approximation as the consequence of implicit operators theorem*, Doklady Akademii Nauk, vol 407 (2006), 6, 742-746.
- [8] V.A. Trenogin, *First Lyapunov method for the abstract parabolic equation*, Works of ISAAC-2005, World Scientific,( 2007), Singapoore (in print).
- [9] Trenogin V.A. *Analytical variant of Lyapunov theorem about stability on the first approximation for differential equations in Banach space*, Proc. of Middle-Volga Mathematical Society **3**(2007), 46-52.