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BOUNDEDNESS PROBLEMS FOR JUMPING PETRI NETS

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Abstract The goal of this paper is to extend a decidability result for the classical boundedness problem from the class of finite jumping Petri nets to the class of reduced-computable jumping Petri nets, and also to establish a new one for the generalized boundedness problem.

Keywords: parallel/distributed systems, Petri nets, decidability.

1. INTRODUCTION

A Petri net ([8]) is a mathematical model used for the specification and the analysis of parallel and distributed systems.

Petri nets proved to be a powerful language for system modeling and validation and they are now in widespread use for many different practical and theoretical purposes in various fields of software and hardware development.

One type of problems related to Petri nets is that of finding algorithms which take a Petri net Σ and a property π as input and answer, after a finite number of steps, whether or not Σ satisfies π. For instance, the Karp-Miller graph for Petri nets allows us to decide the boundedness problem (BP), the finiteness reachability set/tree problem (FRSP/FRTP), and the quasi-liveness problem (QLP), or the equivalent problem called the coverability problem (CP) (see [5, 9] for more details).

It is well-known that the behaviour of some distributed systems cannot be adequately modeled by classical Petri nets. Many extensions which increase the computational and expressive power of Petri nets have been thus introduced. One direction has led to various modifications of the firing rule of nets. One of these extension is that of jumping Petri net, introduced in [10]. A jumping Petri net is a classical net Σ equipped with a (recursive) binary relation R on the markings of Σ. The meaning of a pair $(m, m') \in R$ is that the net Σ may "spontaneously jump" from m to m' (this is similar to λ-moves in automata theory).

Previous results (see [10]) showed that the decision problems related to reachability, coverability and quasi-liveness are undecidable for general jumping nets and are decidable only for finite jumping nets, by using the techniques of Karp-Miller coverability graphs in a similar manner as for classical P/T nets ([5]).

In [12] we introduced a larger class of jumping nets than the finite jumping nets, called reduced-computable jumping nets, for which we could define finite Karp-Miller coverability graphs. Based on them, in this paper we will extend a decidability result about the classical boundedness problem to the class of reduced-computable jumping Petri nets, and we shall establish a new one for the generalized boundedness problem.

The paper is organized as follows. Section 2 presents the basic terminology and notation, and also previous results concerning Petri nets and jumping Petri nets. In Section 3, we use the Karp-Miller coverability structures to establish the decidability of some boundedness problems for reduced-computable jumping Petri nets. Finally, in Section 4 we conclude this paper and formulate some open problems.

2. PRELIMINARIES

In this section we will establish the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites for the understanding of this paper (for details the reader is referred to [1, 8, 9, 4]). Mainly, we will follow [4, 10].

2.1. PETRI NETS

A *Place/Transition net*, shortly *P/T-net* or *net*, (finite, with infinite capacities), abbreviated *PTN*, is a 4-tuple $\Sigma = (S, T; F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbb{N}$ is the *weight function* of Σ satisfying $W(x, y) = 0$ iff $(x, y) \notin F$.

A *marking* of a *PTN* Σ is a function $m : S \to \mathbb{N}$; it will be sometimes identified with a $|S|$-dimensional vector. The operations and relations on vectors are defined component-wise. $\mathbb{N}^S$ denotes the set of all markings of Σ. A *marked PTN*, abbreviated *mPTN*, is a pair $\gamma = (\Sigma, m_0)$, where Σ is a *PTN* and m_0, called the *initial marking* of γ, is a marking of Σ.

In the sequel we often use the term "Petri net"(*PN*) or "net" whenever we refer to a *PTN* (*mPTN*) and it is not necessary to specify its type (i.e. marked or unmarked).

Let Σ be a net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbb{N}$ and $\Delta t, \Delta w : S \to \mathbb{Z}$ are defined by $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$ and

$$\Delta w(s) = \begin{cases} 0, & \text{if } w = \lambda, \\ \sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \dots t_n \ (n \geq 1), \end{cases} \quad \text{for all } s \in S.$$

The sequential behaviour of a net Σ is given by the so-called *firing rule*, which consists of

(ER) the *enabling rule*: a transition t is *enabled* at a marking m in Σ (or t is *fireable* from m), abbreviated $m[t\rangle_\Sigma$, iff $t^- \leq m$;

(CR) the *computing rule*: if $m[t\rangle_\Sigma$, then t may *occur* yielding a new marking m', abbreviated $m[t\rangle_\Sigma m'$, defined by $m' = m + \Delta t$.

In fact, for any transition t of Σ we have a binary relation on $\mathbb{N}^S$, denoted by $[t\rangle_\Sigma$ and given by: $m[t\rangle_\Sigma m'$ iff $t^- \leq m$ and $m' = m+\Delta t$. If $t_1, t_2, \ldots, t_n$, $n \geq 1$, are transitions of Σ, the classical product of the relations $[t_1\rangle_\Sigma, \ldots, [t_n\rangle_\Sigma$ will be denoted by $[t_1 t_2 \ldots t_n\rangle_\Sigma$; i.e. $[t_1 t_2 \ldots t_n\rangle_\Sigma = [t_1\rangle_\Sigma \circ \ldots \circ [t_n\rangle_\Sigma$. Moreover, we also consider the relation $[\lambda\rangle_\Sigma$ given by $[\lambda\rangle_\Sigma = \{(m,m) | m \in \mathbb{N}^S\}$.

Let $\gamma = (\Sigma, m_0)$ be a marked Petri net, and $m \in \mathbb{N}^S$. The word $w \in T^*$ is called a *transition sequence* from m in Σ if there exists a marking m' such that $m[w\rangle_\Sigma m'$. Moreover, the marking m' is called *reachable* from m in Σ. We denote by $RS(\Sigma, m) = [m\rangle_\Sigma = \{m' \in \mathbb{N}^S | \exists w \in T^* : m[w\rangle_\Sigma m'\}$ the set of all reachable markings from m in Σ. In the case $m = m_0$, the set $RS(\Sigma, m_0)$ is abbreviated by $RS(\gamma)$ (or $[m_0\rangle_\gamma$) and it is called *the set of all reachable markings* of γ.

We shall assume to be known other notions from P/T-nets, like coverable marking, bounded place, simultaneously unbounded set of places, pseudo-markings etc. For more details about these notions, and about the basic decision problems for P/T-nets, the reachability structures and the Karp-Miller coverability structures for them, and also about the case of P/T-nets with infinite initial markings, the reader is referred to Appendix 4.

2.2. JUMPING PETRI NETS

Jumping Petri nets ([10, 11]) are an extension of classical P/T-nets, which allows them to perform "spontaneous jumps" from one marking to another (this is similar to λ-moves in automata theory).

A *jumping P/T-net*, abbreviated *JPTN*, is a pair $\gamma = (\Sigma, R)$, where Σ is a *PTN* and R, called the *set of (spontaneous) jumps* of γ, is a binary relation on the set of markings of Σ (i.e. $R \subseteq \mathbb{N}^S \times \mathbb{N}^S$). In what follows the set R of jumps of any *JPTN* will be assumed to be *recursive*, that is for any couple of markings (m, m') we can effectively decide whether or not (m, m') is a member of R.

A *marked jumping P/T-net*, abbreviated *mJPTN*, is defined similarly as a *mPTN*, by changing "Σ" into "Σ, R".

Let $\gamma = (\Sigma, R)$ be a *JPTN*. The pairs $(m, m') \in R$ are referred to as *jumps* of γ. If γ has finitely many jumps (i.e. R is finite) then we say that γ is a *finite jumping net*, abbreviated *FJPTN*.

We shall use the term "*jumping net*" (*JN*) ("*finite jumping net*" (*FJN*), resp.) to denote a *JPTN* or a *mJPTN* (a *FJPTN* or a *mFJPTN*, resp.) whenever it is not necessary to specify its type (i.e. marked or unmarked).

Pictorially, a jumping Petri net will be represented as a classical net and, moreover, the relation R will be separately listed.

The behaviour of a jumping net γ is given by the *j-firing rule*, which consists of

(jER) the *j-enabling rule*: a transition t is *j-enabled* at a marking m (in γ), abbreviated $m[t\rangle_{\gamma,j}$, iff there exists a marking m_1 such that $mR^*m_1[t\rangle_\Sigma$ (Σ being the underlying net of γ and R^* the reflexive and transitive closure of R);

(jCR) the *j-computing rule*: if $m[t\rangle_{\gamma,j}$, then the marking m' is *j-produced* by occurring t at the marking m, abbreviated $m[t\rangle_{\gamma,j}m'$, iff there exist markings m_1, m_2 such that $mR^*m_1[t\rangle_\Sigma m_2R^*m'$.

The notions of *transition j-sequence* and *j-reachable marking* are defined similarly as for Petri nets (the relation $[\lambda\rangle_{\gamma,j}$ is defined by $[\lambda\rangle_{\gamma,j} = \{(m, m')|m, m' \in \mathbb{N}^S, mR^*m'\}$). The *set of all j-reachable markings* of a marked jumping net γ is denoted by $RS(\gamma)$ (or by $[m_0\rangle_{\gamma,j}$).

All other notions from P/T-nets (i.e. coverable marking, bounded place, bounded (or safe) net, simultaneously unbounded set of places, pseudo-markings, etc.) are defined for jumping Petri nets similarly as for Petri nets, by considering the notion of *j-reachability* instead of *reachability* from P/T-nets. Also, all the decision problems from P/T-nets, like (RP), (CP), (BP) and (SUbP), are defined for jumping Petri nets similarly as for P/T-nets.

Some jumps of a marked jumping net may be never used. Thus we say that a marked jumping net $\gamma = (\Sigma, R, m_0)$ is *R-reduced* ([10]) if for any jump $(m, m') \in R$ of γ we have $m \neq m'$ and $m \in [m_0\rangle_{\gamma,j}$. The *reduction problem* (RedP) is: Given γ a *JPTN*, is γ R-reduced?

Remark 2.1. *The following decidability results were proved in [10, 4]: i) the problems (RP), (CP), (BP) are undecidable for mJPTN; ii) the problems (RP), (RedP), (CP), (BP) are decidable for mFJPTN.*

Coverability structures for jumping Petri nets The previous positive decidability results from [10] were based on defining Karp-Miller coverability trees only for the subclass of *finite* jumping Petri nets. Therefore, we were interested in extending the class of jumping Petri nets for which we can define finite Karp-Miller coverability structures. Having such a larger class of nets, afterwards we can solve the above decidability problems for it based on these finite coverability structures.

In [12] we succeeded to introduce a class of jumping nets larger than the finite jumping nets, called *reduced-computable jumping nets*, for which we could define finite Karp-Miller coverability structures (trees and graphs), and also minimal coverability structures; moreover, we extended the results about the minimal coverability structures for P/T-nets from [3] to this class of jumping nets.

Let us recall from [12] the definition of reduced-computable jumping nets.

Let $\gamma = (\Sigma, R)$ be an arbitrary jumping net. We associated with γ a finite subset of jumps $R_{\omega-max}$ (which is maximal in a sense specified below, and which can be used instead of the whole set of jumps R to construct the coverability graphs) as follows. We denoted by "ω-jumps" the set

$$R_\omega = \left\{ r \in \mathbb{N}_\omega^{2|S|} - \mathbb{N}^{2|S|} \;;\; \exists \{r_n\}_{n\geq 0} \subseteq R \text{ strictly increasing with } \lim_{n\to\infty} r_n = r \right\}.$$

Let $\overline{R} = R \cup R_\omega$. We defined the *set of ω-maximal jumps* of γ as $R_{\omega-max} = maximal(\overline{R}) = \{r' \in \overline{R} \mid \forall r \in \overline{R} - \{r'\} : r' \not\leq r\}$.

The following are obvious properties of the set of ω-maximal jumps of a jumping net (the proofs are easy and can be found in [12]):

Proposition 2.1. *(1) $R_{\omega-max}$ is finite; (2) $\forall\, r \in \overline{R}, \exists\, r' \in R_{\omega-max}$ such that $r \leq r'$; (3) $\forall\, r \in R_{\omega-max}, \exists\, \{r_n\}_{n\geq 0} \subseteq R$ such that $\lim_{n\to\infty} r_n = r$.*

A marked jumping net is called *reduced-computable jumping net* ([12]), abbreviated *mRCJPTN*, if it is R-reduced and the set $R_{\omega-max}$ is computable.

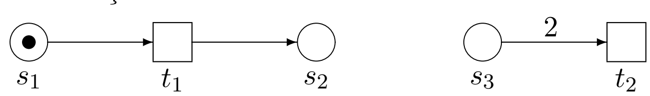
Example 2.1. *Let $\gamma = (\Sigma, R, m_0)$ be the jumping Petri net from Fig. 1; the initial marking is $m_0 = (1,0,0)$ and the set of jumps is $R = \{\ ((0,1,0), (0,0,2\,n)) \mid n \geq 0\}$. Notice that γ is an infinite R-reduced jumping net, and that the only transition sequences in γ are the following ones:*

$$(1,0,0)\ [t_1\rangle_\Sigma\ (0,1,0)\ R\ (0,0,2\,n)\ [t_2\rangle_\Sigma\ (0,0,2\,n-2)\ [t_2\rangle_\Sigma \ldots [t_2\rangle_\Sigma\ (0,0,0),$$

for all $n \geq 0$. Thus, the transition sequence set is $TS(\gamma) = \{t_1 t_2^n \mid n \geq 0\}$, and the reachability set is $RS(\gamma) = \{(1,0,0),(0,1,0)\} \cup \{(0,0,2\,n) \mid n \geq 0\}$. Notice that the reachability set is infinite.

In [12] we also introduced reachability trees and graphs for jumping nets, by a straightforward extension of these structures from classical P/T-nets (i.e. by adding arcs, labelled by "j", for all the jumps of the net).

The reachability graph $\mathcal{RG}(\gamma)$ of the jumping net from example 2.1 is shown in Fig. 2; notice also that it is an infinite graph. This net γ has only one



$$R = \{\ ((0,1,0),(0,0,2\,n)) \mid n \geq 0\}$$

Fig. 1. A jumping Petri net.

"ω-jump", namely $((0,1,0),(0,0,\omega))$, which is also the only ω-maximal jump of the net γ, i.e. $R_{\omega-max} = \{\ ((0,1,0),(0,0,\omega))\ \}$. Thus, γ is a reduced-computable jumping net.

Now let us recall from [12] the definition of coverability trees and graphs generalized for reduced-computable jumping Petri nets.

Let $\gamma = (\Sigma, R, m_0)$ be a *mRCJPTN* with $R \neq \emptyset$. Then the set of ω-maximal jumps is non-empty and finite, i.e. $R_{\omega-max} = \{\ (m_i', m_i'') \mid 1 \leq i \leq n\}$, with $n \geq 1$. Following the same line as in [10], we associated with γ the following P/T-nets: $\gamma_0 = (\Sigma, m_0)$ and $\gamma_i = (\Sigma, m_i'')$, for each $1 \leq i \leq n$, and we defined the notions of *Karp-Miller coverability trees / graphs* of the jumping net γ as being the tuples of the coverability trees / graphs of the P/T-nets $\gamma_0, \gamma_1, \ldots, \gamma_n$:
$\mathcal{KMT}(\gamma) = < \mathcal{KMT}(\gamma_0), \mathcal{KMT}(\gamma_1), \ldots, \mathcal{KMT}(\gamma_n) >$ and
$\mathcal{KMG}(\gamma) = <\ \mathcal{KMG}(\gamma_0),\ \mathcal{KMG}(\gamma_1), \ldots,\ \mathcal{KMG}(\gamma_n)\ >$ respectively. Notice

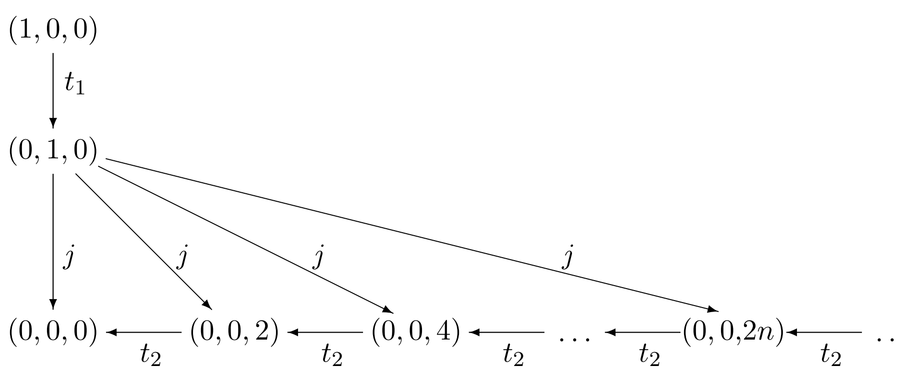


Fig. 2. The reachability graph of γ.

$$\mathcal{KMG}(\gamma_0):\quad \begin{tikzcd} (1,0,0) \arrow[d, "t_1"] \\ (0,1,0) \end{tikzcd} \qquad\qquad \mathcal{KMG}(\gamma_1):\quad \begin{tikzcd} (0,0,\omega) \arrow[loop below, "t_2"] \end{tikzcd}$$

Fig. 3. The Karp-Miller coverability graph of γ.

that it is possible that some of the P/T-nets γ_0,γ_1, $\ldots$,γ_n to have initial markings with ω-components.

Example 2.2. *The P/T-nets associated with the jumping net γ from Example 2.1 are $\gamma_0 = (\Sigma, (1,0,0))$ and $\gamma_1 = (\Sigma, (0,0,\omega))$. Their sets of transition sequences are $TS(\gamma_0) = \{t_1, \lambda\}$ and $TS(\gamma_1) = \{t_2^n \mid n \geq 0\}$, respectively, and their reachability sets are $RS(\gamma_0) = \{(1,0,0),(0,1,0)\}$ and $RS(\gamma_1) = \{(0,0,\omega)\}$ respectively. The Karp-Miller coverability graph of the jumping net γ is $\mathcal{KMG}(\gamma) = < \mathcal{KMG}(\gamma_0), \mathcal{KMG}(\gamma_1) >$, and it is shown in fig. 3.*

3. BOUNDEDNESS PROBLEMS FOR JUMPING PETRI NETS

In this section we shall show how we can use the Karp-Miller coverability graph $\mathcal{KMG}(\gamma)$ to solve some boundedness problems for reduced-computable jumping Petri nets.

First of all, let us give a technical result for P/T-nets, more precisely a property of infinite converging sequences of markings, which we will need later for the proofs of some results.

Proposition 3.1. *Let Σ be a P/T-net and $\{m_n\}_{n\geq 0} \subseteq \mathbb{N}^S$ an infinite sequence of markings. If there exists $\lim_{n\to\infty} m_n = m$, with $m \in \mathbb{N}_\omega^S$, and if $m[w\rangle_\Sigma m'$, with $w \in T^*$, then there exist an integer $n_0 \geq 0$ and an infinite sequence of* *markings* $\{m'_n\}_{n\geq n_0} \subseteq \mathbb{N}^S$ *such that*

$$m_n[w\rangle_\Sigma m'_n, \forall n \geq n_0, \quad and \quad \lim_{n\to\infty} m'_n = m'.$$

Proof. This result is easy to be proved (see the proof in Appendix 4). ∎

We have the following result:

Lemma 3.1. *Let γ be a mRCJPTN, let $s \in S$ be a place and $S' \subseteq S$ a set of places of γ.*
(i) S' is simultaneously unbounded in γ iff $\exists\, 0 \leq i \leq n$ such that S' is simultaneously unbounded in γ_i;
(ii) s is unbounded in γ iff $\exists\, 0 \leq i \leq n$ such that s is unbounded in γ_i,
where $\gamma_0, \gamma_1, \ldots, \gamma_n$ are the P/T-nets associated with the net γ.

Proof. (i) First, we shall prove the direct implication. Let $S' \subseteq S$ be a simultaneously unbounded set of places in γ. This means that

$$\forall k \geq 0, \exists m_k \in RS(\gamma) \text{ such that } \forall s \in S', m_k(s) \geq k. \tag{1}$$

Let us assume by contradiction that S' is not simultaneously unbounded in γ_i, for all $0 \leq i \leq n$. This means that

$$\forall 0 \leq i \leq n, \exists k_i \geq 0 \text{ such that } \forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k_i. \tag{2}$$

Let $k' \in \mathbb{N}$ be an arbitrary integer, satisfying $k' \geq 1 + \max\{k_i | 0 \leq i \leq n\}$; thus, we have $k' > k_i$, for each $0 \leq i \leq n$.
From the relation (1), for $k = k'$, we conclude that

$$\exists m_{k'} \in RS(\gamma) \text{ such that } \forall s \in S', m_{k'}(s) \geq k' \tag{3}$$

Since $m_{k'} \in RS(\gamma)$, we distinguish two cases:
a) $\exists w \in T^*$ such that $m_0[w\rangle_\Sigma m_{k'}$ (i.e. the marking $m_{k'}$ is reachable from m_0 without jumps).

In this case $m_{k'} \in RS(\gamma_0)$ and from relation (3) we have $m_{k'}(s) \geq k' > k_0, \forall s \in S'$. Thus, we can conclude that $\exists m_{k'} \in RS(\gamma_0)$ such that $\forall s \in S', m_{k'}(s) > k_0$, which contradicts (2), for $i = 0$;

b) $\exists w_1, w_2 \in T^*$ and $r = (m', m'') \in R : m_0 \; [w_1\rangle_{\gamma,j} \; m' \; r \; m'' \; [w_2\rangle_\Sigma \; m_{k'}$ (i.e. $m_{k'}$ is reachable from m_0 through jumps, r being the last jump).

By Proposition 2.1(2), we have that $\exists \, 1 \leq i' \leq n$ such that $r \leq r_{i'}$. Thus, $m'' \leq m''_{i'}$ and $m''_{i'}[w_2\rangle_\Sigma m'_{k'}$, where $m'_{k'} = m''_{i'} + \Delta w_2 = m''_{i'} - m'' + m_{k'} \geq m_{k'}$. It follows that $m'_{k'} \in RS(\gamma_{i'})$ and $m'_{k'}(s) \geq m_{k'}(s) \geq k' > k_{i'}, \forall s \in S'$. Therefore, we can conclude that $\exists m'_{k'} \in RS(\gamma_{i'})$ such that $\forall s \in S', m'_{k'}(s) > k_{i'}$, which contradicts (2), for $i = i'$.

Now, we prove the inverse implication. So, assume that there exists $0 \leq i \leq n$ such that S' is simultaneously unbounded in γ_i. Let us assume by contradiction that S' is not simultaneously unbounded in γ. This means that

$$\exists k' \geq 0 \text{ such that } \forall m \in RS(\gamma), \exists s_m \in S' : m(s_m) \leq k' \tag{4}$$

Let $i \in \mathbb{N}$ be arbitrary, satisfying $0 \leq i \leq n$. We distinguish two cases:

a) $r_i \in R_{\omega-max} \cap R$, i.e. the jump $r_i = (m'_i, m''_i)$ does not contain ω-components. Since the net γ is R-reduced, we have that $m''_i \in RS(\gamma)$, and then for each $m \in RS(\gamma_i)$ it follows that $m \in RS(\gamma)$, i.e. $RS(\gamma_i) \subseteq RS(\gamma)$. Therefore, from (4) it follows that

$$\forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k'; \tag{5}$$

b) $r_i \in R_{\omega-max} \cap R_\omega$, i.e. the jump $r_i = (m'_i, m''_i)$ contains ω-components. Thus, proceeding from the definition of the set R_ω, we have

$$\exists \{r_n\}_{n \geq 0} \subseteq R \text{ strictly increasing sequence with } \lim_{n \to \infty} r_n = r_i.$$

Let $r_n = (m'_n, m''_n), \forall n \geq 0$.

Let us consider an arbitrary (pseudo-)marking $m \in RS(\gamma_i)$. So, we have

$$\lim_{n \to \infty} m''_n = m''_i \;\text{ and }\; m''_i[w\rangle_\Sigma m, \text{ with } w \in T^*.$$

Proceeding from Proposition 3.1 it follows that there exists an integer $n_0 \geq 0$ and an infinite sequence of markings $\{m_n\}_{n \geq n_0}$ such that

$$m''_n[w\rangle_\Sigma m_n, \forall n \geq n_0, \quad \text{and} \quad \lim_{n\to\infty} m_n = m.$$

Since the net γ is R-reduced, we have that $m''_n \in RS(\gamma), \forall n \geq 0$, and, therefore, $m_n \in RS(\gamma), \forall n \geq n_0$. Then, from (4) it follows that: $\forall n \geq n_0, \exists s_n \in S'$ such that $m_n(s_n) \leq k'$, and, since from the fact that $\lim_{n\to\infty} m_n = m$ we have

$$\forall n \geq n_0 : \quad \begin{cases} m_n(s) = m(s), & \forall s \in S - \Omega(m) \\ m_n(s) \geq n, & \forall s \in \Omega(m), \end{cases}$$

we can infer, by denoting $n_1 = max\{n_0, k'+1\}$, that $s_n \in S - \Omega(m)$ and $m(s_n) = m_n(s_n) \leq k'$, for all $n \geq n_1$.
Thus, for $n = n_1$ and by denoting $s_m = s_{n_1}$, we have

$$\forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k' \tag{6}$$

Therefore, from relations (5) and (6) for these two possible cases, it follows that we proved the following fact

$$\exists k' \geq 0 : \forall 0 \leq i \leq n, \forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k'.$$

From this fact follows that for each $0 \leq i \leq n$, S' is not simultaneously unbounded in γ_i, which contradicts the hypothesis. Thus, S' is simultaneously unbounded in γ.

The statement (ii) follows easily from (i), by considering the set $S' = \{s\}$, proceeding from the fact that the place s is unbounded in γ' iff the set of places $\{s\}$ is simultaneously unbounded in γ', which holds for P/T-nets (with finite or infinite initial markings), as well as for jumping Petri nets. ∎

Theorem 3.1. *Let γ be a mRCJPTN, and $\mathcal{KMG}(\gamma)$ its Karp-Miller coverability graph.*

(1) A set of places S' is simultaneously unbounded iff there is at least one node (pseudo-marking) m in at least one graph from $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$, for all $s \in S'$.
(2) A place s is unbounded iff there is at least one node (pseudo-marking) m in at least one graph from $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$.

Proof. These statements follow easily from the definition of the Karp-Miller coverability graph $\mathcal{KMG}(\gamma)$, from the previous lemma, and from the similar results for P/T-nets (with finite or infinite initial markings). ∎

Example 3.1. *For the jumping Petri net from Example 2.1 we have that the places s_1 and s_2 are bounded and the place s_3 is unbounded; the trivial set $\{s_3\}$ is the only simultaneously unbounded set of places.*

Theorem 3.1 holds for every finite coverability graph of γ, not only for the Karp-Miller graph, and so to decide the properties listed in the theorem it is sufficient to compute any finite coverability graph, particularly the minimal one.

From Theorem 3.1 and the similar one from P/T-nets [5] we conclude that the following decision problems are solvable by using the Karp-Miller coverability graph for marked reduced-computable jumping Petri nets (or any other finite coverability graph):

Corollary 3.1. *The boundedness problems (SUbP) and (BP) are decidable for mRCJPTN.*

The use of the minimal coverability graph for solving these decision problems is important from the computational point of view because it is, generally speaking, smaller than the Karp-Miller graph.

4. CONCLUSIONS AND FUTURE WORK

In this paper we extended some decidability results from the class of finite jumping Petri nets (*mFJPTN*) to the class of reduced-computable jumping Petri nets (*mRCJPTN*).

More precisely, we have shown some boundedness problems which are decidable by using the Karp-Miller coverability graph for *mRCJPTN*.

An open problem which remains, is to study if there are more eficient algorithms for these decision problems.

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A P/T-nets – some more basic notions and notations

In this appendix we recall the definitions of some more basic notions from P/T-nets, the reachability structures and the Karp-Miller coverability structures, the basic decision problems regarding P/T-nets, and also the case of P/T-nets with infinite initial markings.

Let γ be a Petri net. The marking m is *coverable* in γ if there exists a marking $m' \in [m_0\rangle_\gamma$ such that $m \leq m'$.

A place $s \in S$ is *bounded* (or *safe*) if there exists an integer $k \in \mathbb{N}$ such that we have $m(s) \leq k$, for all $m \in [m_0\rangle_\gamma$. A subset of places $S' \subseteq S$ is *bounded* (or *safe*) if all its places $s \in S'$ are bounded. The net γ is *bounded* (or *safe*) if the set S is bounded.

A subset of places $S' \subseteq S$ is *simultaneously unbounded* if for every integer $k \in \mathbb{N}$ there is a reachable marking $m_k \in [m_0\rangle_\gamma$ such that we have $m_k(s) \geq k$, for all $s \in S'$. The net γ is *simultaneously unbounded* if the set S is simultaneously unbounded. Obviously, if a subset of places S' is simultaneously unbounded then it is also unbounded, but the converse is not always true.

In order to define coverability structures for Petri nets we add to the set of nonnegative integers $\mathbb{N}$ a new symbol, denoted by ω, giving $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$, and extend the operations $+$ and $-$ and the relation $<$ to the set $\mathbb{N}_\omega$ by: a) $n+\omega=\omega+n=\omega$; b) $\omega-n=\omega$; c) $n<\omega$, for all $n\in\mathbb{N}$.

Functions $m: S\rightarrow\mathbb{N}_\omega$ will be called *pseudo-markings*; they will be sometimes identified with $|S|$-dimensional vectors. $\mathbb{N}_\omega^S$ denotes the set of all pseudo-markings. If $m(s)=\omega$, then the component s of m will be called an ω-component; the set of all ω-components of m will be denoted by $\Omega(m)$, i.e. $\Omega(m)=\{s\in S\mid m(s)=\omega\}$. Obviously, any marking is a pseudo-marking. The firing rule is extended to pseudo-markings in the straightforward way: (ER) $m[t\rangle_\Sigma$ iff $t^-\leq m$; (CR) $m[t\rangle_\Sigma m'$ iff $m[t\rangle_\Sigma$ and $m'=m+\Delta t$. The other notions from markings (i.e. transition sequence, reachable marking etc.) are extended similarly to pseudo-markings.

We say that an infinite sequence of pseudo-markings $\{m_n\}_{n\geq 0}$ *converges* to the pseudo-marking m, and we write $\lim_{n\to\infty} m_n=m$, if we have

$$\forall n\geq 0:\begin{cases} m_n(s)=m(s), & \text{for all } s\in S-\Omega(m),\\ m_n(s)\geq n, & \text{for all } s\in\Omega(m).\end{cases}$$

The *reachability tree* of a marked Petri net $\gamma=(\Sigma,m_0)$ is denoted by $\mathcal{RT}(\gamma)$. It is a $(\mathbb{N}^S,T)$-labelled tree (V,E,l_V,l_E) (i.e., a tree with the set of nodes V and the set of arcs E, and the labelling functions $l_V: V\rightarrow\mathbb{N}^S$ and $l_E: E\rightarrow T$), which satisfies the following properties:

(i) the root node is labelled by the initial marking, i.e. $l_V(v_0)=m_0$;

(ii) for each node $v\in V$, the number of direct successors of v in the tree is equal to the number of transitions of the net γ which are fireable from the marking $l_V(v)$;

(iii) for each node $v\in V$ which has successors in the tree and for each transition $t\in T$ which is fireable from the marking $l_V(v)$, there exists an arc $(v,v')\in E$, labelled by $l_E(v,v')=t$, and, moreover, the label of v' is given by $l_V(v')=l_V(v)+\Delta t$.

The *reachability graph* of the net γ is denoted by $\mathcal{RG}(\gamma)$. It is a labelled directed graph (V, T, E), which is obtained from the reachability tree $\mathcal{RT}(\gamma)$ by identifying nodes with the same label; so, the set of nodes V of $\mathcal{RG}(\gamma)$ is the set of the markings which appear in $\mathcal{RT}(\gamma)$ as labels of nodes, i.e. $V = [m_0\rangle_\gamma$, and the set of arcs is given by $\forall m_1, m_2 \in [m_0\rangle_\gamma, \forall t \in T: \quad (m_1, t, m_2) \in E \iff m_1[t\rangle_\gamma m_2$.

The reachability tree/graph of a Petri net could be infinite. That is why people were interested in some refinements of these structures that can produce finite (sub)structures by preserving as much as possible properties.

The first, and well-known, reduced reachability tree/graph was that introduced by Karp and Miller [5]. The Karp-Miller coverability tree of a net γ will be denoted by $\mathcal{KMT}(\gamma)$ and it is a finite $(\mathbb{N}^S, T)$-labelled tree defined by the algorithm given by Karp and Miller in [5]. The Karp-Miller coverability graph of γ will be denoted by $\mathcal{KMG}(\gamma)$ and it is a finite labelled directed graph, obtained from the tree $\mathcal{KMT}(\gamma)$ by identifying nodes with the same label. Later, A. Finkel introduced in [3] the minimal coverability tree and graph. For the general definitions of coverability sets, trees, forests, and graphs for P/T-nets one can see [5, 3].

Some basic decision problems related to P/T-nets are the following:

- (RP) The *Reachability Problem* : Given γ a *mPTN* and m a marking of γ, is m reachable in γ?
- (CP) The *Coverability Problem* : Given γ a *mPTN* and m a marking of γ, is m coverable in γ?
- (BP) The *Boundedness Problem* : Given γ a *mPTN* and s a place of γ, is the place s bounded in γ? We may also ask if: Given γ a *mPTN* and S' a set of places of γ, is the set S' bounded in γ? Or if: Given γ a *mPTN*, is γ a bounded net?

• (SUbP) The *Simultaneously Unboundedness Problem* : Given γ a *mPTN* and S' a set of places, is the set S' simultaneously unbounded in γ? We may also ask if: Given γ a *mPTN*, is the net γ simultaneously unbounded?

Remark 4.1. *It is well-known that: i) the problem (RP) is decidable [7, 6]; ii) the problem (SUbP) is decidable [2]; iii) the problems (CP) and (BP) are decidable by using the Karp-Miller graph [9, 4]; the minimal coverability graph can also be used to solve these problems (see [3]).*

Petri nets with infinite initial markings In [12] we presented the case of P/T-nets with infinite initial markings, and we generalized the notions of coverability structures to them, because we needed them to define coverability structures for jumping Petri nets.

A *marked P/T-net with an infinite initial marking* is a *mPTN* $\gamma = (\Sigma, m_0)$ such that the initial marking has ω-components, i.e. $m_0 \in \mathbb{N}_\omega^S - \mathbb{N}^S$ (or, equivalent, $\Omega(m_0) \neq \emptyset$).

All the notions from P/T-nets with finite initial markings (i.e. firing rule, transition sequence, the set $TS(\gamma, m)$ of transition sequences from a marking, reachable marking, the reachability set $RS(\gamma)$, the reachability tree $\mathcal{RT}(\gamma)$, the reachability graph $\mathcal{RG}(\gamma)$, coverable marking, bounded place, simultaneously unbounded set of places etc.), and all the decision problems (i.e. (RP), (BP), (SUbP) etc.) are defined similarly for P/T-nets with infinite initial markings, with the remark that the initial marking m_0 is actually a pseudo-marking (because m_0 has ω-components), and, consequently, all reachable markings of γ are actually pseudo-markings. Indeed, it is easy to notice that $\Omega(m) = \Omega(m_0), \forall m \in RS(\gamma)$, which means that the ω-components of m_0 are preserved by the firing rule.

Moreover, the coverability structures for P/T-nets with an infinite initial marking are extended from classical nets with finite initial markings, by having the pseudo-markings which appear in these structures extended with ω-components on the set of ω-components of the initial marking. We can use the Karp-Miller graph of a Petri net with an infinite initial marking, to solve the same decision problems as those solved for P/T-nets with finite initial markings. Indeed, we showed the following result [12]:

Theorem 4.1. *Let $\gamma = (\Sigma, m_0)$ be a marked P/T-net (with a finite or an infinite initial marking), and $\mathcal{KMG}(\gamma)$ its Karp-Miller coverability graph.*
(1) A place s is unbounded iff there is at least one node m in $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$;
(2) A set of places S' is simultaneously unbounded iff there is at least one node m in $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$, for all $s \in S'$.

B Some proofs

In this appendix we prove Proposition 3.1 from Section 3.

Proof of proposition 3.1. We proceed by induction on $k = |w|$. If $k = 0$, then the proposition is trivially satisfied ($m' = m$ and we can take $n_0 = 0$ and $m'_n = m_n$).

If $k = 1$, then $w = t$, with $t \in T$, and from $m[t\rangle_\Sigma m'$, we conclude that $m \geq t^-$ and $m' = m + \Delta t$. Since $\lim_{n\to\infty} m_n = m$, it follows that

$$\forall n \geq 0 : \quad \begin{cases} m_n(s) \geq n, & \forall s \in \Omega(m), \\ m_n(s) = m(s), & \forall s \in S - \Omega(m). \end{cases}$$

Let $n_0 = \max\{t^-(s) | s \in \Omega(m)\}$. Then, it follows that

$$\forall n \geq n_0 : \quad \begin{cases} m_n(s) \geq n \geq n_0 \geq t^-(s), & \forall s \in \Omega(m), \\ m_n(s) = m(s) \geq t^-(s), & \forall s \in S - \Omega(m). \end{cases}$$

and, therefore, we have that $m_n \geq t^-$, for all $n \geq n_0$. Thus, we obtain that $m_n[t\rangle_\Sigma m'_n, \forall n \geq n_0$, by taking $m'_n = m_n + \Delta t$. It is easy to verify that $\lim_{n\to\infty} m'_n = m'$.

The induction step. Let $w = w't$ with $|w'| = k$ and $t \in T$. Then, there exists the marking m'' (namely, $m'' = m + \Delta w'$) such that: $m[w'\rangle_\Sigma m''[t\rangle_\Sigma m'$. By applying the induction hypothesis to the sequence of markings $\{m_n\}_{n\geq 0}$ for which $\lim_{n\to\infty} m_n = m$ and $m[w'\rangle_\Sigma m''$, it follows that: there exists an integer $n_1 \geq 0$ and an infinite sequence of markings $\{m''_n\}_{n\geq n_1}$ (namely, $m''_n = m_n + \Delta w', \forall n \geq n_1$) such that

$$m_n[w'\rangle_\Sigma m''_n, \forall n \geq n_1, \quad \text{and} \quad \lim_{n\to\infty} m''_n = m'' \tag{7}$$

Taking the sequence: $\{\tilde{m}''_n\}_{n\geq 0}$, with $\tilde{m}''_n = m''_{n+n_1}, \forall n \geq 0$, we have

$$\lim_{n\to\infty} \tilde{m}''_n = \lim_{n\to\infty} m''_n = m'' \quad \text{and} \quad m''[t\rangle_\Sigma m'.$$

Therefore, by applying to the sequence $\{\tilde{m}''_n\}_{n\geq 0}$ the statement of this proposition (i.e., the step $k = 1$), we conclude that: there exists an integer $n_2 \geq 0$ and an infinite sequence of markings $\{\tilde{m}'_n\}_{n\geq n_2}$ (namely, $\tilde{m}'_n = \tilde{m}''_n + \Delta t, \forall\, n \geq n_2$) such that $\tilde{m}''_n[t\rangle_\Sigma \tilde{m}'_n, \forall\, n \geq n_2$, and $\lim_{n\to\infty} \tilde{m}'_n = m'$.
But $\tilde{m}''_n = m''_{n+n_1}$, and so, by considering $n_0 = n_1 + n_2$ and the sequence of markings $\{m'_n\}_{n\geq n_0}$, with $m'_n = \tilde{m}'_{n-n_1}, \forall\, n \geq n_0$, we obtain

$$m''_n[t\rangle_\Sigma m'_n, \forall\, n \geq n_0, \quad \text{and} \quad \lim_{n\to\infty} m'_n = m' \tag{8}$$

From (7) and (8) we conclude that $m_n[w\rangle_\Sigma m'_n, \forall\, n \geq n_0$, and $\lim_{n\to\infty} m'_n = m'$, which completes the induction. $\square$