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HOMAGE TO PROFESSOR ADELINA GEORGESCU AT THE AGE OF 65

Mitrofan Ciobanu; Anca-Veronica Ion

Professor Adelina Georgescu, the founder and the President of Romanian Society of Applied and Industrial Mathematics - ROMAI, reached, in the spring of 2007, the age of 65. This is a good opportunity for us, her collaborators and friends, to stop a little on our own mathematical and personal paths and to bring once more into light her life and activity up to this moment.

Mitrofan Ciobanu and Adelina Georgescu at Tiraspol State University, Chişinău, 2006

She was born in Drobeta Turnu-Severin on April 25, 1942 in a family of intellectuals (her mother, Maria Berindei, was a teacher of history and geography and her father, Constantin Georgescu, was an advocate). She finished the lyceum with excellent results and in 1960 she began the universitary studies, at the Faculty of Mathematics and Mechanics of the University of Bucharest. In 1965 she successfully defended her graduation thesis of the bachelor degree. After graduation she started to work at the Institute of Fluid Mechanics and Aerospace Constructions of the Romanian Academy. The theory of hydrodynamic stability was the first domain of research of Adelina Georgescu. This theory is mostly known to engineers, meteorologists, hydrologists, physicists. One of the still unsolved great problems of physics of fluids is the problem of the birth of turbulence, which is related to the loss of stability.

Between 1965 and 1990 Professor Georgescu worked at the Institute of Fluid Mechanics and Aerospace Constructions, Bucharest, and at the Institute of Mathematics, Bucharest, alternatively, first as a scientific researcher and then as a main and senior researcher.

In 1970, at the Institute of Mathematics of Romanian Academy, Adelina Georgescu defended her PhD thesis on linear hydrodynamic stability under the supervision of the Academician Caius Iacob, one of the leaders of the Romanian School of Mechanics, and she became a Doctor in Mathematics.

In 1976 Professor A. Georgescu published the first Romanian monograph on hydrodynamic stability. In 1985 Kluwer published its enlarged and improved English version. The 1985 version of the book somehow is a bridge between the classical theory, addressed mainly to engineers, and the pure mathematical ones which looks like an applied functional analysis. This monograph was cited by many authors, since it was widely used as an universitary textbook and by researchers, being a pioneer work in this field.

After 1985 the theory of the hydrodynamic stability evolved very much and the spectrum of scientific interests of Professor A. Georgescu became very broad. It includes the following areas: the stability and the multiparametric bifurcation in fluid dynamics, spectral problems for ODE in the hydrodynamic stability theory, dynamical systems associated with fluid flows, a transition to turbulence as a deterministic chaos, fractals, models of asymptotic approximation in fluid dynamics, synergetics, nonlinear dynamics, numerical analysis, calculus of variations etc.

Besides her research work, she devoted a lot of time to teaching. Since 1974 at the Faculty of Mathematics of Bucharest University she led the courses of the theory of Navier-Stokes equations, Dynamics of rivers, Boundary Layer theory, Turbulence in fluids, Bifurcation theory (the first such course in Romania), Analytical Mechanics, Mechanics of continua, Nonlinear Dynamics, Sinergetics.

We must say that, during the communist regime, the non-proletarian roots of Prof. Adelina Georgescu rised many obstacles in her activity. However, she was not a member of the communist party, even if this would help her in the scientific carrier.

In 1990, as soon as the conditions (after the fall of the communism) allowed the affirmation of personal initiative in science, Professor A. Georgescu became the initiator of the project of a new research institute - Institute of Applied Mathematics of the Romanian Academy. The Institute was created in 1991 and Professor A. Georgescu became its first director. Simultaneously, she initiated the foundation of the Romanian Society of Applied and Industrial

Mathematics - ROMAI. Professor A. Georgescu was President of ROMAI all the time from the moment of the born of this Society up to the present.

In 1997 she completely moved to the didactic activity as a professor of the University of Piteşti and between 1999 and 2005 she was also the Head of the Department of Applied Mathematics of the University of Piteşti.

At the Faculty of Mathematics of Piteşti Professor Adelina Georgescu lead the courses at the 3rd, 4th year and the master level in Analytical Mechanics, Mechanics of continua, Fluid dynamics, Dynamical systems, Bifurcation theory. She also founded and lead the "Victor Vâlcovici" scientific seminar for master students and PhD students. Professor Adelina Georgescu was involved as a supervision of master theses and a supervisor and a referee to PhD Theses, having at this moment 16 PhD's and 7 PhD students. All the time spent at the University of Piteşti she enthusiastically encouraged the work of young researchers in her fields of interest and especially in dynamical systems and bifurcation. New fields as the application of bifurcation theory in the economy and the biology were explored by some of her PhD students with remarkable, internationally recognized results.

The results of Professor's Adelina Georgescu activity were published in 16 books (some of them by well-known publishing houses as Kluwer, Chapman and Hall or Editura Tehnică, Editura Ştiinţifică şi Enciclopedică) and 200 scientific papers in prestigious journals of mathematics. Besides these, around 40 papers were presented at Conferences and published in their Proceedings and she participated to 56 research grants. Among the books we quote: *Hydrodinamic stability theory*, Kluwer, Dordrecht (1985); *The asymptotic treatment of differential equations*, Chapman & Hall, London (1995); *Bifurcation theory. Principles and applications*, Univ. of Pitesti (1999) (with M. Moroianu,

I. Oprea); *FitzHugh-Nagumo model: bifurcation and dynamics*, Kluwer, Dordrecht (2000) (with. C. Rocşoreanu, N. Giurgiţeanu).

A great energy and a lot of work were invested by Professor Adelina Georgescu in gathering the material and writing, with a few collaborators, two editions (2004 and 2006) of a Dictionary of Romanian Mathematicians, a precious tool in making an image of what the Romanian mathematical school represents. This is one of Professor's Adelina Georgescu many acts of patriotism.

As a recognition of her value, Professor Georgescu was a visiting professor at Romanian and foreign universities or research institutes, delivering a large number of conferences, leading doctoral courses and research seminars: University of Craiova; ICEPRONAV, Galaţi; Politecnico di Bari (2000); University of Bari, Dipartimento di Geologia (1998); Istituto per le Applicazioni del Calculo (IAC), Roma (1997, 1996); University of Bari, Dipartimento di Matematica (1991, 1992, 1993, 1994, 1995, 1996, 2004, 2005, 2006, 2007); Instituto Pluridisciplinar, University Complutense, Madrid (1996); Institut für Chemie und Dynamik der Geosphare (ICG-4), KFA, Julich (1999, 1998, 1996, 1995); University of Havre (1994); University of Paris VI (1994, 1995), Istituto per le Ricerche di Matematica Applicata (IRMA), Bari (1992,1994); University of Metz (1991); Polytechnical Institute of Poznan (1991); Institute of Mathematics and its Applications, Minneapolis (1990); Institute für Strömungslehre und Strömungsmechanik, Karlsruhe (1990); University of Catania (2002); University of Lecce (2001, 2002); University of Messina (2001, 2002, 2003, 2004, 2005,2006, 2007); Institute of Mathematics, Belgrad (2003); CREATIS-INSA, Lyon (2003, 2004), University of Patras (2001, 2007).

In addition to her own scientific and didactic activity, Professor A. Georgescu has brought an inestimable contribution to the developing of Romanian applied and industrial mathematics as a talented manager of sciences. She was one of the main organizers of the first (1990) and second (1992) Preparatory Conference for the International Congress of Romanian Mathematics. Since 1993 annually, ROMAI and some universities (of Piteşti, of Oradea, Tiraspol State University from Chişinău, Republic of Moldova) and city halls (of Mioveni) organize the already well-known International Conferences on Applied and Industrial Mathematics - CAIMs, real forums for genuine debates and exchanges of ideas in mathematics and its applications. They enrich the traditional lines in Romanian mathematics of the seventies of the last century with recent topics developed by a large number of professionals, mainly from Romania and Republic of Moldova, ranging from pure mathematics to engineering and high-school mathematics. However, the hard core of CAIMs consists of the few groups of research workers and professors in mathematics, physics and engineering. As the President of ROMAI, Professor Georgescu lead a policy of continuous support towards the mathematical community of the Republic of Moldova. The presence at each edition of CAIM of the mathematicians from the sister country was supported from the ROMAI funds. This is another act of patriotism and it leads to the creation of strong collaboration and friendship relations between the mathematical schools from the two sides of the river Prut. Besides the mathematicians from Romania and Republic of Moldova, mathematicians from countries like Canada, France, Greece, Italy, Russia, Slovakia, Ukraine, Uzbekistan, took part at the successive editions of CAIMs.

From 1997 until 2004, the Proceedings of CAIMs were published mainly in the journal Buletinul Ştiinţific al Universităţii din Piteşti, (Seria Matematica si Informatica) this bringing to this journal a good place in the CNCSIS classification of scientifical journals (namely a category B journal). Professor Adelina Georgescu was practically the Main Editor of this journal in the mentioned period (with the exception of vol. 5). Since 2005, ROMAI began to issue ROMAI Journal that publishes mainly papers presented at CAIM, but also other valuable mathematical papers and Educational ROMAI Journal, which is devoted to preuniversitary mathematics. Professor Georgescu is Editor in Chief of ROMAI Journal. She is also a member of the editorial board of some scientific journals like Int. J. Chaos Theory and Applications, Applied and Numerical Mathematics (e-Journal, Budapest), reviewer for Math. Reviews, Zentralblatt fur Math., and peer-reviewer for Rev. Roum. Math. Pures Appl., Rev. Roum. Phys., Rev. Roum. Sci. Tech.-Mec. Appl.; St. Cerc. Mat.; St. Cerc. Mec. Appl.

As a Professor at University of Piteşti, Professor Georgescu initiated The Series of Applied and Industrial Mathematics of the University of Piteşti, series of books containing 27 titles up to now. Five among her graduated PhD students, are now teaching at this University. The activity devoted to the formation of a school of Dynamical Systems Theory at the University of Piteşti, when there was no similar school in our country, is also an act of patriotism of Professor Adelina Georgescu, as well as the permanent attempt of inducing high universitary standards at this University.

After her first visit to Chişinău in 1990, Professor A.Georgescu filled an important place in the mathematical life of the Republic of Moldova: she became an active participant to many mathematical forums organized in Moldova; she took part in the expert commissions in PhD dissertations; she is a supervisor of PhD and masters theses. Four editions of the CAIM were hold in Republic of Moldova (1995, 1997, 2003, 2006). The Conference CAIM XIV was organized as a Satellite Conference of the International Congress of Mathematicians - 2006. Co-organizeers of this Conference were Moldova State University with its Center for Education and Researches in Mathematics and Computer Sciences, Tiraspol State University, Institute of Mathematics and Computer Sciences of the Academy of Sciences of Moldova, Academy of Economic Studies of Moldova, State Agency of Intellectual Property and Academy of Transports, Computer Sciences and Communications. Her valuable and multilateral contribution to the progress of Mathematics received a high recognition from the Academy of Science of Moldova: Professor A. Georgescu was rewarded with the Diploma of Honour of the Academy. More than that, she received the title of Doctor Honoris Causa of the University of Tiraspol (from Chişinău) in 2006.

Other international signs of recognition of Professor's Adelina Georgescu activity are her election as member of Russian Academy of Nonlinear Sciences and of Accademia Peloritana dei Pericolanti (Messina).

We wish to Professor Adelina Georgescu to complete her prodigious activity with many years of scientific work, of supervising the work of her PhD students, of leading the activity of ROMAI and the editing of ROMAI Journals, and last but not least, to recover the health and joy of life in order to see her grandsons and granddaughters growing and to enjoy their company!

April 2007

Mitrofan Ciobanu, member of the Academy of Science of Moldova, doctor habilitat, Professor, President of the Mathematical Society of the Republic of Moldova.

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VISCOUS FLOWS DRIVEN BY GRAVITY AND A SURFACE TENSION GRADIENT

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Abstract The lubrication approximation used to investigate the flow of a thin layer is deduced. The starting point for modeling the flow of thin films are the Navier-Stokes equations. The lubrication or reduced Reynolds number approximation to the Navier-Stokes equations has been used to describe a multitude of situations. Our attention has been focussed on the situations in which the surface tension plays an important role.

1. INTRODUCTION

We consider the flow of a thin fluid films where the surface tension is a driving mechanism. In general the introduction of surface tension into standard lubrication theory leads to a fourth-order nonlinear parabolic equation [6], [2]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3 + \left(\frac{\partial^3 h}{\partial x^3} + a\frac{\partial h}{\partial x} + b\right)\right) = 0, \qquad (1)$$

where $h = h(x,t)$ is the fluid film height and a, b are constants. For steady situations this equation may be integrated once and a third-order ordinary differential equation is obtained. Appropriate forms of equation (1) have been used to model fluid flows in a number of physical situations such as coating, draining of foams and the movement of contact lenses.

2. PROBLEM FORMULATION

We consider the flow of a thin film on an inclined plane at an angle α to the horizontal plane. Suppose that a Newtonian fluid, of constant density ρ

and the dynamic viscosity μ is undergoing an unsteady flow. With respect to a Cartesian co-ordinates system Oxy as indicated in fig. 1, the velocity has the form $\mathbf{u} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$.

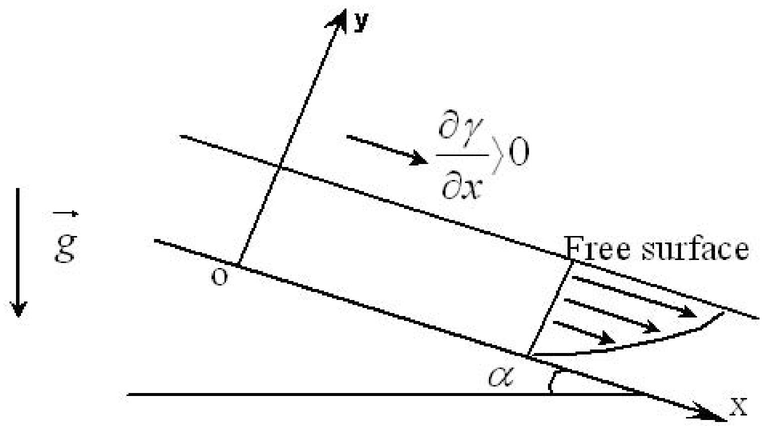

Figure 1

The Navier-Stokes equations are [7]

$$u_t = -\frac{1}{\rho}p_x + \nu\Delta u + g\sin\alpha$$

$$v_t = -\frac{1}{\rho}p_y + \nu\Delta v - g\cos\alpha.$$

Let us non-dimensionalize the length by the length scales (L, h_0) and velocity scales $(U, \delta U)$, where L is a typical length along the film, h_0 a typical film thickness and $\delta = h_0/L << 1$ is the aspect ratio (typical height/typical length scale) [7]. In keeping with the assumption that the surface tension is a mechanism of comparable strength to viscous forces the velocity scale U is chosen to make the capillary number $Ca = \delta^3\sigma/3\mu U = 1$. Hence $U = \delta^3\sigma/3\mu$, where σ is the value of surface tension (assumed constant although). Note that if the gravity g is dominating, the appropriate scaling is $U = \delta^2\rho g L^2/3\mu$. The vertical velocity v is of order dh/dt and by virtue of $\nabla \cdot \mathbf{u} = 0$ the horizontal velocity u is of order $(L/h) \cdot dh/dt$. Then, we have $\frac{dh}{dt} \sim U\frac{h}{L}$ and so $\frac{d^2h}{dt^2} \sim h\frac{U^2}{L^2}$. Orders of magnitude estimated are the following

$$\frac{\partial v}{\partial t} \sim U^2 \frac{h}{L^2}$$

and

$$\frac{\partial u}{\partial t} \sim L\frac{d}{dt}\left(\frac{v}{h}\right) = 0.$$

These estimates show that the term $\mathbf{u}_t$ may be neglected $(h/L^2 << h/L << 1)$. To first order in δ the Navier-Stokes equations reduce to

$$-p_x + \frac{1}{3}u_{yy} + Bo \sin\alpha = 0, \tag{2}$$

$$-p_y - \delta Bo \cos\alpha = 0. \tag{3}$$

This forms are the thin-film approximation. Here subscripts denote differentiation with respect to the variable. Whilst the continuity equation is unchanged

$$u_x + v_y = 0. \tag{4}$$

The Bond number, $Bo = \delta^2 \rho g L^2 / 3\mu U$, is a ratio of gravity to viscous forces. In order for the gravity terms to be non-negligible either $Bo \sin\alpha$ or $\delta Bo \cos\alpha$ must be $O(1)$. The scaling for fluid pressure p is chosen to balance pressure with viscous forces and is $\delta^2 L/3\mu U$. The appropriate approximate boundary conditions on the free surface $y = h(x,t)$ are [1]

$$\begin{cases} v &= h_t + u h_x, \\ p &= -h_{xx}, \\ u_y &= 0. \end{cases} . \tag{5}$$

Here p is the pressure in the fluid. These equations (5) represent the kinematic condition, pressure balancing surface tension and zero shear respectively. The pressure condition is the Laplace-Young equation which reflects the fact that normal stress due to surface tension is proportional to curvature [7]. On the substrate, $y = 0$, the no-slip condition reads

$$u = v = 0. \tag{6}$$

Integrating equation (3) we get an expression for fluid pressure

$$p = -h_{xx} - \delta Bo \sin\alpha \cdot (y - h). \tag{7}$$

Integrating (2) twice and imposing the boundary conditions leads to

$$u = 3(p_x - Bo \sin\alpha)\left(\frac{y^2}{2} - hy\right). \tag{8}$$

This may be used in the continuity equation (4) to determine v. In particular, on the free surface we have

$$v(h) = -\int_0^h u_x dy. \tag{9}$$

This expression together with the kinematic conditions lead to the governing equation for the film height

$$h_t + \frac{\partial}{\partial x}[h^3(h_{xxx} - \delta Boh_x \cos\alpha + Bo \sin\alpha)] = 0. \tag{10}$$

This is a fourth-order nonlinear degenerate equation. Implicit in the derivation of this equation is the assumption that surface tension and gravity effects are of the same order. A gravity-dominated system would not include the fourth-order surface tension term and the velocity scale, at present based on surface tension, should be changes accordingly.

3. DISCUSSION

Equations of the from (10), apply if the fluid motion is constrained by a no-slip, or similar, condition on one surface, producing a shear flow. If surface tension variations occur, then the zero shear condition becomes $u_y = \frac{\sigma_x}{\delta^2}$, where σ_x is the surface tension gradient [4]. The boundary conditions on the free surface become

$$\begin{cases} v = h_t + uh_x, \\ p = -h_{xx}, \\ u_y = \frac{\sigma_x}{\delta^2}. \end{cases} \qquad (11)$$

The continuity equation (4) is unchanged. Then, the evolution equation for the film height is [2]

$$h_t + Ca \cdot \delta^2 \sigma_x \cdot h \cdot h_x + \frac{3}{2\delta} Bo \sin\alpha \cdot h^2 \cdot h_x - \frac{\partial}{\partial x}(h^3 \cdot h_x). \qquad (12)$$

This is a nonlinear parabolic equation. A traveling wave substitution $h(x,t) = H(x-t)$ will reduce (12) to a nonlinear dynamic system [5]. For studying this dynamic system we have tried to determine the invariant manifold and the invariant subspaces, corresponding to the linear system. Results show qualitative agreement with experiments. For small time the solution to the linearized problem is found in the form of an infinite series [3].

4. CONCLUDING REMARKS

We deduce the lubrication approximation for the flow of a thin liquid layer down an inclined plane simultaneously driven by a surface tension gradient. We impose the boundary conditions and then we deduce the governing equation for the film height.

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A CLASS OF COMMUTATIVE ALGEBRAS AND THEIR APPLICATIONS IN LIE TRIPLE SYSTEM THEORY

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Abstract It is especially important to solve the problem of representing the ternary composition of a Lie triple system by means of an appropriate binary composition defined on its ground space. In this paper, some necessary conditions for the existence of a binary algebra whose standardly associated h-system be a given Lie triple system are exhibited. They were used to classify, up to an isomorphism, some low-dimensional special Lie triple systems. Some examples show us that this problem does not always have a positive answer.

Keywords: Lie triple system, homogeneous system, LT-algebra.

2000 MSC: 17Dxx, 18G60.

1. INTRODUCTION

In [21] J. M. Osborn proved that any commutative (nonassociative) algebra $A(\cdot)$ with unity element, over a field of characteristic not 2 or 3, which satisfies an identity of degree ≤ 4 not implied by the commutativity law, must satisfy at least one of the following three identities: (i) $(x^2 \cdot x) \cdot x = x^2 \cdot x^2$, (ii) $2(yx \cdot x) \cdot x + y \cdot x^3 = 3(yx^2) \cdot x$, (iii) $2(y^2 \cdot x) \cdot x - 2(yx \cdot y) \cdot x - 2(yx \cdot x) \cdot y + 2(x^2 y) \cdot y - y^2 \cdot x^2 + yx \cdot yx = 0$.

The algebras satisfying the identity (ii) were studied in [20]. The relationships between the algebras satisfying (ii) and the Jordan algebras were clarified by Osborn [20]. The identity (ii) is implied by Jordan identity. The class of such algebras is strictly larger than the class of Jordan algebras and it is not included in the class of power-associative algebras. Further, any simple algebra satisfying (ii) is necessarily a Jordan algebra. Since these algebras are closely connected with Lie triple systems we name them *LT-algebras*.

Lie triple systems (briefly, LTSs) have been introduced by N. Jacobson [12], in 1948, as being the abstract algebraic structure describing the subspaces of an associative algebra which are closed relatively to the superpositions $[[\cdot,\cdot],\cdot]$ of the Lie bracket associated, in the standard way, with the composition of the algebra. On the other hand, in 1958 [28], K. Yamaguti introduced the notion of *homogeneous system* (shortly, h-*system* or HS) as being the algebraic structure induced by the multiplication of a Lie algebra L by a vector subspace of L which is a complement to a Lie subalgebra of L. The term *homogeneous system* is motivated by the remark that every tangent space of a homogeneous space can be naturally organized as an h-system. In [3], it was achieved a construction which associates with every binary algebra $A(\cdot)$ a h-system structure defined on its ground space; this h-system, denoted by $\mathcal{H}_A$, gives the opportunity to identify a set of generators spanning the Lie algebra generated by the left/right multiplications of $A(\cdot)$ [4]. Since any LTS is a particular h-system it becomes important to know whether it is or it is not just the h-system associated with a certain binary algebra defined on its own ground space. Of course, the whole study for such a class of particular LTSs, the so called *special LTSs* (see Definition 2.3), can be reduced to that of the appropriately associated binary algebras, the so called *LT-algebras* (see Definition 5.1). Unfortunately, as it will be shown later, not any LTS can be obtained from a binary algebra by means of the typical construction given in [3]. On the other hand, every LTS has a universal enveloping which is an LT-algebra [8].

In this paper we give some necessary conditions for an LTS to be the h-system associated with a binary algebra. The obtained results are used to classify, up to an isomorphism, the 2-dimensional special LTSs and some 3-dimensional special LTSs.

2. PRELIMINARIES

LIE **triple systems**. Recall that the LIE triple systems were firstly noted by CARTAN in his famous PhD thesis [9] but, as algebraic structures, they were introduced by JACOBSON [12].

Definition 2.1. *A* LIE ***triple system*** *(shortly, LTS) over a field K is a K-vector space T endowed with a 3-multilinear mapping $[\cdot,\cdot,\cdot] : T \times T \times T \to T$ satisfying the following axioms:*

$$(LTS1) \qquad [x,x,y] = 0, \ \ \forall x,y \in T,$$

$$(LTS2) \qquad [x,y,z] + [y,z,x] + [z,x,y] = 0, \ \ \forall x,y,z \in T,$$

$$(LTS3) \qquad [x,y,[u,v,w]] = [[x,y,u],v,w] + [u,[x,y,v],w] + [u,v,[x,y,w]], \quad \forall x,y,u,v,w \in T.$$

*Any LTS $(T,[\cdot,\cdot,\cdot])$ for which $[x,y,z] = 0$, for all $x,y,z \in T$ is called the **null** LTS on T.*

For every $x,y \in T$ we can define the endomorphism $D_{(x,y)} : T \to T$ by $D_{(x,y)}(z) = [x,y,z]$, for all $z \in T$. The set $\mathbb{D} = \{D_{(x,y)} | x,y \in T\}$ spans a vector subspace $\mathcal{D}_T$ of *End A*, i.e., $\mathcal{D}_T = Span_K(\mathbb{D})$. The axiom (LTS3) assures us that the usual bracket of two endomorphisms endows $\mathcal{D}_T$ with a LIE algebra structure, i.e., $\mathcal{D}_T$ is a LIE subalgebra of $g\ell\ (T)$. Moreover, $\mathcal{D}_T$ is a LIE subalgebra of *Der T* and it is called the *algebra of inner derivations* of T. Consequently, it becomes clear that the possibility of using the LIE algebra representation theory in the study of LTSs has arisen.

Examples of LTSs. 1°. Any 1-dimensional LTS is necessarily a null LTS. Further, any K-vector space T, endowed with the null ternary composition becomes a null LTS.

2°. In [7] the real two-dimensional LTSs were classified. It was proved that for any real two-dimensional LTS, there exists a basis such that its ternary multiplication has one of the following six multiplication tables:

$$(i)\begin{cases}[a,b,a]=0\\ [a,b,b]=0\end{cases}\quad (ii)\begin{cases}[a,b,a]=0\\ [a,b,b]=a\end{cases}\quad (iii)\begin{cases}[a,b,a]=0\\ [a,b,b]=-a.\end{cases}$$
$$(iv)\begin{cases}[a,b,a]=b\\ [a,b,b]=a\end{cases}\quad (v)\begin{cases}[a,b,a]=b\\ [a,b,b]=-a\end{cases}\quad (vi)\begin{cases}[a,b,a]=-b\\ [a,b,b]=a.\end{cases}$$

In [18] it was defined the radical of a LTS, the semisimple LTS, and it was proved a Levi-type Theorem for LTSs. E. L. Stitzinger [27] proved that the derivation algebra *Der* T of a LTS T over a field of characteristic 0 acts completely reducibly on T if and only if the radical $\mathcal{R}(T)$ of T is contained in the center of T.

Derivations of an algebra. Let $A(\cdot)$ be a binary algebra and let $DerA$ be its derivation algebra. For associative algebras, Lie algebras, Leibniz algebras the corresponding derivation algebras are well known. Another important class of algebras for which exists an important information about their derivation algebras consists in the so-called NN-*algebras* (i.e. real algebras without niplotents). For such an algebra $A(\cdot)$, it was proved that the Lie subalgebra $\mathcal{D}$ of $DerA$, consisting in all commuting derivations, has $dim\ \mathcal{D} \leq \left[\frac{1}{2}(dimA-1)\right]$ (see Theorem 2, [5]). Here we are interested in the particular cases when dim $A=2$ or 3; this time $dim\ \mathcal{D}\in\{0,1\}$ (see Theorem 1, [5]).

Homogeneous system associated with a commutative algebra. For the definition of a homogeneous system (shortly, *h-system* or HS) it can be used [30], [3], [4]. With any non necessarily associative algebra in a standard (functorial) way (see [4]) a HS can be associated. The HSs associated, in this standard way, with commutative algebras are characterized by a set of specific axioms, simpler than that of a generic HS; such an HS will be called a *commutative* HS.

Let us consider a binary commutative algebra $A(\cdot)$. A commutative homogeneous system $\mathcal{H}_A$ on A, in the YAMAGUTI's sense, is the algebraic structure defined on A by using the following multilinear compositions [3]

$$\begin{cases} [a,b,c] = D_{(a,b)}(c), \;\; \forall a,b,c \in A, \\ [a_1,a_2,...,a_k,a_{k+1}] = D_{(a_1,a_2,...,a_k)}(a_{k+1}), \\ \forall a_1,a_2,...,a_k,a_{k+1} \in A, \;\; \forall k \geq 3, \end{cases} \tag{1}$$

where

$$\begin{cases} D_{(a,b)} = [L_a, L_b], \\ D_{(a_1,a_2,...,a_k)} = [D_{(a_1,a_2,...,a_{k-1})}, L_{a_k}] - L_{[a_1,a_2,...,a_k]} \end{cases} \tag{2}$$

(here $[f,g] = f \circ g - g \circ f$). These compositions satisfy the axioms listed in the following definition.

Definition 2.2. *A* ***commutative homogeneous system*** *(briefly, commutative h-system or* ***CHS****) over a field K is a K-vector space A together a countable set of multilinear compositions* $[.,...,.]$:$\underbrace{A \times A \times ... \times A}_{n \;\; times} \to A$ *satisfying the axioms:*

(h.s.1) $[x,x,x_1,\cdots,x_k] = 0, \quad \forall k \in \mathbb{N}^\star, \; \forall x,x_1,\cdots,x_k \in A,$

(h.s.2) $[x,y,z] + [y,z,x] + [z,x,y] = 0, \quad \forall x,y,z \in A,$

(h.s.3) $[x,y,z,w] + [y,z,x,w] + [z,x,y,w] = 0, \quad \forall x,y,z,w \in A,$

(h.s.4) $[x_1,\cdots,x_k,y,z] - [x_1,\cdots,x_k,z,y] = 0, \;\; \forall x_1,\cdots,x_k,y,z \in A,$

(h.s.5) $\big[D_{(x_1,\cdots,x_k)}, D_{(y_1,y_2)}\big] = D_{([x_1,\cdots,x_k,y_1],y_2)} - D_{([x_1,\cdots,x_k,y_2],y_1)} +$

. $+ D_{(x_1,\cdots,x_k,y_1,y_2)} - D_{(x_1,\cdots,x_k,y_2,y_1)},$

. $\forall k = 2,3,\cdots, \;\; \forall x_1,\cdots,x_k,y_1,y_2 \in A$

(h.s.6) $\big[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell,y_{\ell+1})}\big] = \widetilde{\mathcal{D}}\big(\big[D_{(x_1,\cdots,x_k)}, D_{(y_1,\cdots,y_\ell)}\big], y_{\ell+1}\big) -$

$$. -\big[D_{(x_1,\cdots,x_k,y_{\ell+1})}, D_{(y_1,\cdots,y_\ell)}\big] - D_{(x_1,\cdots,x_k,[y_1,\cdots,y_\ell,y_{\ell+1}])} + D_{(y_1,\cdots,y_\ell,[x_1,\cdots,x_k,y_{\ell+1}])},$$
$$. \qquad \forall \ell, k = 2, 3, \cdots, \ \forall x_1, \cdots, x_k, y_1, \cdots, y_{\ell+1} \in A$$

($\widetilde{\mathcal{D}}$ *is the bilinear mapping* $\widetilde{\mathcal{D}} : \mathcal{D} \times A \to \mathcal{D}$ *defined by* $\widetilde{\mathcal{D}}\big(D_{(x_1,\cdots,x_n)}, y\big) = D_{(x_1,\cdots x_n,y)}$).

The axioms (h.s.5) and (h.s.6) assure us that the family of endomorphisms $\mathcal{D}_A = Span_K\{D_{(x_1,\cdots,x_n)} | \forall x_1, \cdots, x_n \in A, n \geq 2\}$, endowed with the usual LIE bracket of endomorphisms, is just a LIE subalgebra of $g\ell(A)$.

It was proved (see Theorem 6, [4]) that the LIE algebra $\mathcal{L}_A$ generated by all the left multiplications of $A(\cdot)$ is isomorphic to the LIE algebra $\mathcal{A}_L$ associated canonically (by YAMAGUTI's functor [28]) with the h-system $\mathcal{H}_A$ earlier defined on A by (1) and (2).

Remark that $\mathcal{H}_A$ becomes an LTS in the case when $[x, y, z, u] = 0$, for all $x, y, z, u \in A$, i.e. when every $D_{(x,y)}$ is necessarily a derivation both for $A(\cdot)$ and $\mathcal{H}_A$. This time, the ternary composition on $\mathcal{H}_A$ is defined by

$$[x, y, z] = x \cdot (y \cdot z) - y \cdot (x \cdot z), \ \ \forall x, y, z \in A.$$

Obviously, if $A(\cdot)$ is an associative and commutative algebra, then $\mathcal{H}_A$ is the null LTS.

Definition 2.3. *Any LTS which is the HS associated with a binary algebra is called a* ***special*** *LTS (shortly, SLTS).*

3. PROBLEM SETTING

Let $A(\cdot)$ be a commutative algebra over the field K and let $\mathcal{H}_A$ be the YAMAGUTI's type homogeneous system (shortly, h-system or hs) associated with it as before. It was already remarked that every LTS is a particular h-system. In this framework, the following problem arises.

Problem. *Let* $(T, [\cdot,\cdot,\cdot])$ *be a LTS on the vector space* T *over the field* K. *Does exists or not a binary (commutative)* K*-algebra* $T(\cdot)$ *whose associated h-system* $\mathcal{H}_T$ *(via* (1) *and (2)) be just* $(T, [\cdot,\cdot,\cdot])$?

Actually, our problem is the same with finding a procedure to recognize if a given LTS is or is not a SLTS.

4. NECESSARY CONDITIONS

Let us suppose that $T(\cdot)$ is a solution of the above posed problem. Then, $T(\cdot)$ must be necessarily a commutative algebra and, moreover, it must satisfy the identity [3]

$$[x, y, z, w] = 0 \quad \forall x, y, z, w \in T. \tag{3}$$

Obviously, (3) is equivalent to the identity

$$\begin{aligned} x\cdot(y\cdot zw) - y\cdot(x\cdot zw) - z\cdot(x\cdot yw) + z\cdot(y\cdot xw) = \\ = (x\cdot yz)\cdot w - (y\cdot xz)\cdot w, \qquad \forall x, y, z, w \in T. \end{aligned} \tag{4}$$

Setting $x = z = w$ in (4) yields

$$2(yx\cdot x)\cdot x + y\cdot x^3 = 3yx^2\cdot x, \quad \forall x, y \in T, \tag{5}$$

i.e. the left multiplications must satisfy the equation

$$2L_x^3 + L_{x^3} - 3L_x \circ L_{x^2} = 0, \qquad \forall x \in T. \tag{6}$$

Consequently, the before defined problem could have as its solution a binary commutative operation that necessarily satisfies the identity (5).

Definition 4.1. *Any binary algebra* $T(\cdot)$ *whose elements satisfy the axiom* (5) (*or, equivalently,* (6)) *is called a LT-**algebra**.*

It follows that the problem of classification up to an isomorphism of special LTSs is equivalent to the problem of classification of LT-algebras up to an isomorphism.

Examples of LT-algebras. 1. Any 1-dimensional algebra is a LT-algebra.

2. Any associative and commutative algebra is a LT-algebra.

3. Any Jordan algebra is a LT-algebra.

4. The two-dimensional commutative algebra $T(\cdot)$, having in basis $B = \{a, b\}$ the multiplication table

$$a^2 = 0, \qquad a \cdot b = a, \qquad b^2 = a + 2b,$$

is a LT-algebra whose associated Lts is just (*ii*) Section 2. Further, the two-dimensional commutative algebra $T(\cdot)$, having in basis $B = \{a, b\}$ the multiplication table

$$a^2 = 0, \qquad a \cdot b = -a, \qquad b^2 = a - 2b,$$

is also a LT-algebra whose associated Lts is (*iii*) Section 2.

Remark. It can be easily proved that the two LT-algebras in **4.** are the only nonisomorphic (nonassociative) real two-dimensional algebras. That is why, in what follows we are interested in the study of real n-dimensional LT-algebras with $n > 2$.

5. FINITE-DIMENSIONAL LT-ALGEBRAS

In the following $T(\cdot)$ is a n-dimensional LT-algebra, with $n > 2$. Also, we shall denote by $\mathcal{I}(T)$ the set of all nonzero idempotents of $T(\cdot)$, and by $\mathcal{N}(T)$ the set of all nonzero nilpotents of $T(\cdot)$.

In accordance with the Theorem 1 [14], $T(\cdot)$ has necessarily (at least) either an idempotent element or a nilpotent one, i.e. $\mathcal{I}(T) \cup \mathcal{N}(T) \neq \varnothing$. Every idempotent which is not an identity helps us to obtain a vector direct sum decomposition of the ground space of algebra (similar to Peirce decomposition of a power-associative algebra). That is why it is important to know whether a finite dimensional LT-algebra contains a (unique) ideal which is maximal with respect to the property of not containing an idempotent. Therefore, it is natural to analyze the following two complimentary situations:

Case 1. $T(\cdot)$ has an idempotent element which is not a unity element,

Case 2. $T(\cdot)$ has no idempotent element as in 1.

In its turn, the Case 2 is covered by

Case 2a. $T(\cdot)$ has a nilpotent element (with or without a unity element),

Case 2b. $T(\cdot)$ has a unity element but it has no nilpotent or (other) idempotent element.

Case 1

Let e be an idempotent which is not a unity element. Following Osborn [20], it follows that T has the following vector direct sum decomposition

$$T = T_e(0) \oplus T_e(1/2) \oplus T_e(1), \qquad (7)$$

where $T_e(\lambda) = \{x \in T | e \cdot x = \lambda x\}$ with $\lambda \in \{0, \frac{1}{2}, 1\}$. In what follows, we shall use the more convenient notation $T_\lambda = T_e(\lambda)$, i.e., $T = T_0 \oplus T_{\frac{1}{2}} \oplus T_1$.

Lemma 5.1. (*Osborn* [20].) *For a fixed idempotent e the following relations, connecting T_λ's ($\lambda \in \{0, \frac{1}{2}, 1\}$), hold*

$$\begin{cases} T_0^2 \subset T_0, \ \ T_0 T_{\frac{1}{2}} \subset T_{\frac{1}{2}}, \ \ T_0 T_1 = \{0\}, \\ T_{\frac{1}{2}}^2 \subset T_0 + T_1, \ \ T_{\frac{1}{2}} T_1 \subset T_{\frac{1}{2}} \\ T_1^2 \subset T_1. \end{cases} \qquad (8)$$

Consequently, $T_0(\cdot), T_1(\cdot)$ and $\widetilde{T}(\cdot) = T_0 \oplus T_1(\cdot)$ are subalgebras of $T(\cdot)$; actually, $T_0(\cdot)$, $T_1(\cdot)$ are ideals in $\widetilde{T}(\cdot)$ (not necessarily in $T(\cdot)$).

Taking into account Lemma 5.1, it follows that $(T_0, [\cdot,\cdot,\cdot])$, $(T_{\frac{1}{2}}[\cdot,\cdot,\cdot])$, $(T_1, [\cdot,\cdot,\cdot])$ are sub-LTS's of $(T, [\cdot,\cdot,\cdot])$, even if $T_{\frac{1}{2}}$ is not necessarily a subalgebra of $T(\cdot)$.

Recall that, for such LT-algebra, Osborn proved the identity

$$(L_e - kI)[x, y, z] + (j - i)xy \cdot z = 0, \quad \forall x \in T_i, y \in T_j, z \in T_k \qquad (9)$$

(this is the Osborn's identity (8) of [20]). Setting i, j and k equal to $0, \frac{1}{2}, 1$ in all possible ways, we shall obtain some structural properties of $(T, [\cdot,\cdot,\cdot])$ in connection with its sub-LTS's before mentioned. According to the 27 possibilities occurring in this setting, we get the following results.

Lemma 5.2. . *The ternary operation of the LTS T satisfies the relations*

1°. $[x_0, y_0, z_0] \in T_0$

2°. $[x_{\frac{1}{2}}, y_{\frac{1}{2}}, z_{\frac{1}{2}}] \in T_{\frac{1}{2}}$

3°. $[x_1, y_1, z_1] \in T_1$

4°. $[x_0, y_0, z_{\frac{1}{2}}] \in T_{\frac{1}{2}}$

5°. $[x_0, y_{\frac{1}{2}}, z_0] = -x_0 y_{\frac{1}{2}} \cdot z_0 \in T_{\frac{1}{2}}$

6°. $[x_{\frac{1}{2}}, y_0, z_0] = x_{\frac{1}{2}} y_0 \cdot z_0 \in T_{\frac{1}{2}}$

7°. $[x_0, y_0, z_1] = 0$

8°. $[x_0, y_1, z_0] = 0$

9°. $[x_1, y_0, z_0] = 0$

10°. $[x_0, y_{\frac{1}{2}}, z_{\frac{1}{2}}]_0 = (x_0 y_{\frac{1}{2}} \cdot z_{\frac{1}{2}})_0$
$[x_0, y_{\frac{1}{2}}, z_{\frac{1}{2}}]_1 = -(x_0 y_{\frac{1}{2}} \cdot z_{\frac{1}{2}})_1$

11°. $[x_{\frac{1}{2}}, y_0, z_{\frac{1}{2}}]_0 = -(x_{\frac{1}{2}} y_0 \cdot z_{\frac{1}{2}})_0$
$[x_{\frac{1}{2}}, y_0, z_{\frac{1}{2}}]_1 = (x_{\frac{1}{2}} y_0 \cdot z_{\frac{1}{2}})_1$

12°. $[x_{\frac{1}{2}}, y_{\frac{1}{2}}, z_0] \in T_0$

13°. $[x_0, y_1, z_1] = 0$

14°. $[x_1, y_0, z_1] = 0$

15°. $[x_1, y_1, z_0] = 0$

16°. $[x_1, y_{\frac{1}{2}}, z_{\frac{1}{2}}]_0 = -(x_1 y_{\frac{1}{2}} \cdot z_{\frac{1}{2}})_0$
$[x_1, y_{\frac{1}{2}}, z_{\frac{1}{2}}]_1 = (x_1 y_{\frac{1}{2}} \cdot z_{\frac{1}{2}})_1$

17°. $[x_{\frac{1}{2}}, y_1, z_{\frac{1}{2}}]_0 = (x_{\frac{1}{2}} y_1 \cdot z_{\frac{1}{2}})_0$
$[x_{\frac{1}{2}}, y_1, z_{\frac{1}{2}}]_1 = -(x_{\frac{1}{2}} y_1 \cdot z_{\frac{1}{2}})_1$

18°. $[x_{\frac{1}{2}}, y_{\frac{1}{2}}, z_1] \in T_1$

19°. $[x_{\frac{1}{2}}, y_1, z_1] = x_{\frac{1}{2}} y_1 \cdot z_1 \in T_{\frac{1}{2}}$

20°. $[x_1, y_{\frac{1}{2}}, z_1] = -x_1 y_{\frac{1}{2}} \cdot z_1 \in T_{\frac{1}{2}}$

21°. $[x_1, y_1, z_{\frac{1}{2}}] \in T_{\frac{1}{2}}$

22°. $[x_0, y_{\frac{1}{2}}, z_1] = x_0 y_{\frac{1}{2}} \cdot z_1 \in T_{\frac{1}{2}}$

23°. $[x_{\frac{1}{2}}, y_1, z_0] = -x_{\frac{1}{2}} y_1 \cdot z_0 \in T_{\frac{1}{2}}$

24° $[x_1, y_0, z_{\frac{1}{2}}] \in T_{\frac{1}{2}}$

25° $[x_{\frac{1}{2}}, y_0, z_1] = -y_0 \cdot x_{\frac{1}{2}} z_1 \in T_{\frac{1}{2}}$

26°. $[x_1, y_{\frac{1}{2}}, z_0] = x_1 \cdot y_{\frac{1}{2}} z_0 \in T_{\frac{1}{2}}$

27°. $[x_0, y_1, z_{\frac{1}{2}}] \in T_{\frac{1}{2}}$,

where the subscript i for w_i means $w_i \in T_i$ with $i \in \{0, \frac{1}{2}, 1\}$.

Remark that $1° - 4°$, $7° - 9°$, $13° - 15°$, $21°$, $24°$, $27°$ are straightforward consequences of Lemma 5.1, while $19°, 6°, 16^\circ_2, 10^\circ_1, 18°, 12°, 22°$ are, respectively, equivalent to the identities listed in Lemma 2 [20]. Moreover, the following equivalences hold: $5° \Leftrightarrow 6°$, $16° \Leftrightarrow 17°$ and $19° \Leftrightarrow 20°$.

Corollary 5.1. . *The idempotent e necessarily satisfies the following identities:*

$$
\begin{array}{ll}
[x_0,y_0,e]=[x_0,e,y_0]=[e,x_0,y_0]=0, & [x_1,e,y_0]=[e,x_1,y_0]=0,\\
[x_1,y_1,e]=[x_1,e,y_1]=[e,x_1,y_1]=0, & [x_1,e,y_{\frac12}]=[e,x_1,y_{\frac12}]=0,\\
[x_0,y_1,e]=[x_0,e,y_1]=0, & [x_{\frac12},y_{\frac12},e]=0,\\
[x_1,y_0,e]=[e,x_0,y_1]=0, & [x_0,e,y_{\frac12}]=0\\
 & [e,x_0,y_{\frac12}]=0.
\end{array}
$$

Corollary 5.2. . *The following identities*

$$
\begin{aligned}
x_0y_{\frac12}&=2[x_0,y_{\frac12},e],\\
x_{\frac12}y_1&=2[x_{\frac12},y_1,e],\\
x_{\frac12}y_{\frac12}&=2\overline{[x_{\frac12},e,y_{\frac12}]},\\
x_0y_1&=0,
\end{aligned}
$$

hold (where, for any $v=v_0+v_1\in T_0+T_1$, the conjugate element $\overline{v}\in T_0+T_1$ is defined by $\overline{v}=v_0-v_1$).

Consequently, if the idempotent e is identified, then it suffices to solve the stated problem for the LTSs $(T_0,[\cdot,\cdot,\cdot])$ and $(T_1,[\cdot,\cdot,\cdot])$, only.

If $e=e_0+e_{\frac12}+e_1$, then

$$
\begin{array}{lll}
e\cdot e_0=0 & \Rightarrow & e_0^2=0,\;\; e_0e_{\frac12}=0\\
e\cdot e_{\frac12}=\frac12 e_{\frac12} & \Rightarrow & e_{\frac11}^2=0,\;\; e_0e_{\frac12}+e_1e_{\frac12}=\frac12 e_{\frac12}\\
e\cdot e_1=e_1 & \Rightarrow & e_{\frac12}\cdot e_1=0,\;\; e_1^2=e_1.
\end{array}
$$

By equating $e^2=e$ we get $e=e_1\in T_1$. Moreover, e is unity element for subalgebra T_1.

Corollary 5.3. .$T_{\frac12}$ *is a T_0T_1-bimodule.*

In order to prove this assertion it suffices to remark that 22° implies the identity

$$x_0y_{\frac12}\cdot z_1=x_0\cdot y_{\frac12}z_1.$$

Let us consider the LTS $T([\cdot,\cdot,\cdot])$ and define the endomorphisms $D_{(x,y)}$ of T by

$$D_{(x,y)}(z) = [x, y, z], \qquad \forall x, y, z \in T. \tag{10}$$

Then, the axiom T_3 (from the definition of a LTS) becomes

$$\left[D_{(x,y)}, D_{(z,w)}\right] = D_{([x,y,z],w)} + D_{(z,[x,y,w])}, \qquad \forall x, y, z, w \in T. \tag{11}$$

Consequently, $\mathcal{D}_T = Span_K\{D_{(x,y)} | \forall x, y \in T\}$ becomes a Lie algebra over K relatively to the usual bracket of two endomorphisms. Further, let us suppose that the LTS T is just $\mathcal{H}_T$ for the binary commutative algebra $T(\cdot)$. As it has already been remarked, in this case $\mathcal{D}_T$ must be a Lie algebra of derivations of $T(\cdot)$ so that the solution of our problem implies to solve the problem of constructing commutative algebras with a given Lie algebra of derivations. It is suitable to denote

$$D_{ij} = Span_K\{D_{(x,y)} | \forall x \in T_i, y \in T_j\}$$

where $i, j \in \{0, \frac{1}{2}, 1\}$. Since $D_{(x,y)} = -D_{(y,x)}$ it follows that

$$\mathcal{D}_T = \mathcal{D}_{00} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}} + \mathcal{D}_{11} + \mathcal{D}_{01} + \mathcal{D}_{0\frac{1}{2}} + \mathcal{D}_{\frac{1}{2}1}$$

(a vector space sum).

Using Lemma 5.2, one gets the following result.

Lemma 5.3. . *The following assertions:*

$$\begin{array}{llll}
1^\circ. & \mathcal{D}_{00}(T_0) \subset T_0, & \mathcal{D}_{00}(T_{\frac{1}{2}}) \subset T_{\frac{1}{2}}, & \mathcal{D}_{00}(T_1) = \{0\}, \\
2^\circ. & \mathcal{D}_{\frac{1}{2}\frac{1}{2}}(T_0) \subset T_0, & \mathcal{D}_{\frac{1}{2}\frac{1}{2}}(T_{\frac{1}{2}}) \subset T_{\frac{1}{2}}, & \mathcal{D}_{\frac{1}{2}\frac{1}{2}}(T_1) \subset T_1, \\
3^\circ. & \mathcal{D}_{11}(T_0) = \{0\}, & \mathcal{D}_{11}(T_{\frac{1}{2}}) \subset T_{\frac{1}{2}}, & \mathcal{D}_{11}(T_1) \subset T_1, \\
4^\circ. & \mathcal{D}_{01}(T_0) = \{0\}, & \mathcal{D}_{01}(T_{\frac{1}{2}}) \subset T_{\frac{1}{2}}, & \mathcal{D}_{01}(T_1) = \{0\}, \\
5^\circ. & \mathcal{D}_{0\frac{1}{2}}(T_0) \subset T_{\frac{1}{2}}, & \mathcal{D}_{0\frac{1}{2}}(T_{\frac{1}{2}}) \subset T_0 + T_1, & \mathcal{D}_{0\frac{1}{2}}(T_1) \subset T_{\frac{1}{2}}, \\
6^\circ. & \mathcal{D}_{\frac{1}{2}1}(T_0) \subset T_{\frac{1}{2}}, & \mathcal{D}_{\frac{1}{2}1}(T_{\frac{1}{2}}) \subset T_0 + T_1, & \mathcal{D}_{\frac{1}{2}1}(T_1) \subset T_{\frac{1}{2}}
\end{array}$$

hold.

Consequently, $\mathcal{D}_{00}$, $\mathcal{D}_{\frac{1}{2}\frac{1}{2}}$, $\mathcal{D}_{11}$ and $\mathcal{D}_{01}$ respect the vector direct sum decomposition of T. From (11) and Lemma 5.2 it follows

Proposition 5.1. . *For the* LIE *algebra* $\mathcal{D}_T$ *of a LTS* T *whose ternary operation is de fined by* $[x, y, z] = x \cdot yz - y \cdot xz$, *for all* $x, y, z \in T$, *the following statements:*

1°. $[\mathcal{D}_{00}, \mathcal{D}_{00}] \subset \mathcal{D}_{00}, [\mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{\frac{1}{2}\frac{1}{2}}] \subset \mathcal{D}_{\frac{1}{2}\frac{1}{2}}, [\mathcal{D}_{11}, \mathcal{D}_{11}] \subset \mathcal{D}_{11}, [\mathcal{D}_{01}, \mathcal{D}_{01}] = 0,$
$[\mathcal{D}_{0\frac{1}{2}}, \mathcal{D}_{0\frac{1}{2}}] \subset \mathcal{D}_{00} + \mathcal{D}_{01} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}}, [\mathcal{D}_{\frac{1}{2}1}, \mathcal{D}_{\frac{1}{2}1}] \subset \mathcal{D}_{01} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}},$

2°. $[\mathcal{D}_{00}, \mathcal{D}_{\frac{1}{2}\frac{1}{2}}] \subset \mathcal{D}_{\frac{1}{2}\frac{1}{2}}, [\mathcal{D}_{00}, \mathcal{D}_{11}] = 0, [\mathcal{D}_{00}, \mathcal{D}_{01}] \subset \mathcal{D}_{01},$
$[\mathcal{D}_{00}, \mathcal{D}_{0\frac{1}{2}}] \subset \mathcal{D}_{0\frac{1}{2}}, [\mathcal{D}_{00}, \mathcal{D}_{\frac{1}{2}1}] \subset \mathcal{D}_{\frac{1}{2}1},$

3°. $[\mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{11}] \subset \mathcal{D}_{11}, [\mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{01}] \subset \mathcal{D}_{01}, [\mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{0\frac{1}{2}}] \subset \mathcal{D}_{0\frac{1}{2}},$
$[\mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{1\frac{1}{2}}] \subset \mathcal{D}_{\frac{1}{2}1},$

4°. $[\mathcal{D}_{11}, \mathcal{D}_{01}] \subset \mathcal{D}_{01}, [\mathcal{D}_{11}, \mathcal{D}_{0\frac{1}{2}}] \subset \mathcal{D}_{0\frac{1}{2}}, [\mathcal{D}_{11}, \mathcal{D}_{1\frac{1}{2}}] \subset \mathcal{D}_{\frac{1}{2}1},$

5°. $[\mathcal{D}_{01}, \mathcal{D}_{0\frac{1}{2}}] \subset \mathcal{D}_{0\frac{1}{2}}, [\mathcal{D}_{01}, \mathcal{D}_{\frac{1}{2}1}] \subset \mathcal{D}_{\frac{1}{2}1},$

6°. $[\mathcal{D}_{0\frac{1}{2}}, \mathcal{D}_{\frac{1}{2}1}] \subset \mathcal{D}_{01} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}}.$

hold.

Consequently, $\mathcal{D}_{00}, \mathcal{D}_{\frac{1}{2}\frac{1}{2}}, \mathcal{D}_{11}, \mathcal{D}_{01}, \mathcal{D}_{00} \oplus \mathcal{D}_{11}, \mathcal{D}_{00} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}} + \mathcal{D}_{11}, \mathcal{D}_{00} + \mathcal{D}_{11} + \mathcal{D}_{01}, \mathcal{D}_{01} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}}$ and $\mathcal{D}_{00} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}} + \mathcal{D}_{11} + \mathcal{D}_{01}$ (vector space sums) are LIE subalgebras of $\mathcal{D}_T$. All these subalgebras keep invariant both subspaces $\mathcal{D}_{0\frac{1}{2}}$ and $\mathcal{D}_{\frac{1}{2}1}$. Actually, there exists the following ascending chain of LIE (simple) subalgebras

$$\begin{gathered}\mathcal{D}_{01} \subset \mathcal{D}_{01} + \mathcal{D}_{00} \subset \mathcal{D}_{01} + \mathcal{D}_{00} + \mathcal{D}_{11} \subset \\ \subset \mathcal{D}_{01} + \mathcal{D}_{00} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}} \subset \\ \subset \mathcal{D}_{01} + \mathcal{D}_{00} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}} + \mathcal{D}_{0\frac{1}{2}} \subset \mathcal{D}_T.\end{gathered} \tag{12}$$

$\mathcal{D}_{01}$ is an Abelian ideal of $\mathcal{D}_{01} + \mathcal{D}_{00} + \mathcal{D}_{11} + \mathcal{D}_{\frac{1}{2}\frac{1}{2}}$ and $\mathcal{D}_{01} + \mathcal{D}_{00}$ is an ideal of $\mathcal{D}_{01} + \mathcal{D}_{00} + \mathcal{D}_{11}$; further $\mathcal{D}_T$ is a simple LIE algebra. The LIE algebra $\mathcal{D}_T$

and its LIE subalgebras act naturally on T as endomorphism algebras. Their adjoint representations have the before presented properties.

On another hand, it must be remarked that (3) assures us that $\mathcal{D}_T$ is a derivation algebra for $T(\cdot)$. Moreover, $5°, 19°$ Lemma 5.1 are equivalent to the existence of a set of derivations for $T_0(\cdot)$ and $T_1(\cdot)$. Actually, we get the next result.

Proposition 5.2. . *The following identities*

$$y_{\frac{1}{2}} \cdot x_0 z_0 = y_{\frac{1}{2}} x_0 \cdot z_0 + x_0 \cdot y_{\frac{1}{2}} z_0,$$
$$y_{\frac{1}{2}} \cdot x_1 z_1 = y_{\frac{1}{2}} x_1 \cdot z_1 + x_1 \cdot y_{\frac{1}{2}} z_1$$

hold.

These Propositions ensure us that every left multiplication $L_{y_{\frac{1}{2}}}$ is a derivation both for $T_0(\cdot)$ and for $T_1(\cdot)$. Consequently, it suffices to solve the stated problem for the LTSs $(T_0, [\cdot,\cdot,\cdot])$ and $(T_1, [\cdot,\cdot,\cdot])$; their dimensions are less than $dim\ T$, while their derivation algebras have dimensions at least equal either to $dim\ \mathcal{D}_{00} + dim\ T_{\frac{1}{2}}$ or $dim\ \mathcal{D}_{11} + dim\ T_{\frac{1}{2}}$, respectively.

As it was already remarked in [20], the presence of an orthogonal idempotent to e can be considered, too. Indeed, if for example, the LT-algebra has a unity ($\neq e$), then $e' = 1 - e$ is an idempotent orthogonal to e.

We apply the before presented results to 3-dimensional LT-algebras, in order to classify them up to an isomorphism.

6. 3-DIMENSIONAL LT-ALGEBRAS WITH AN IDEMPOTENT WHICH IS NOT A UNITY ELEMENT

Let $A(\cdot)$ be a 3-dimensional LT-algebra having the idempotent element e. L_e has three simple eigenvalues, namely $s_1 = 0, s_2 = \frac{1}{2}, s_3 = 1$, what implies

$A = A_0 \oplus A_{\frac{1}{2}} \oplus A_1$, where A_0 and A_1 are 1-dimensional LT-algebras. If the basis $B = (e_0, e_{\frac{1}{2}}, e_1 = e)$ consists of the eigenvectors corresponding to s_1, s_2, s_3, respectively, then the multiplication table of the commutative algebra $A(\cdot)$ must have one of the following two forms

$$I. \quad \begin{array}{lll} e_0^2 = 0 & e_0 e_{\frac{1}{2}} = \alpha e_{\frac{1}{2}} & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = \beta e_0 + \gamma e_1 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

or, the form

$$II. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = \alpha e_{\frac{1}{2}} & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = \beta e_0 + \gamma e_1 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1. \end{array}$$

I. By imposing to $L_{e_{\frac{1}{2}}}$ be a derivation both for A_0 and A_1 (see Proposition 5.8) it follows $\alpha = 0$. Moreover, $L_{e_0}, L_{e_{\frac{1}{2}}}, L_{e_1}$ satisfy (6) iff $\gamma = 0$. The obtained algebra is isomorphic, in accordance with the values of β, with one of the three algebras listed below

$$1^\circ. \quad \begin{array}{lll} e_0^2 = 0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = e_0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

$$2^\circ. \quad \begin{array}{lll} e_0^2 = 0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = 0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1. \end{array}$$

$$3^\circ. \quad \begin{array}{lll} e_0^2 = 0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = -e_0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

The algebras 1° and 3° are isomorphic. On other hand, 1° and 2° are not isomorphic, because $\mathcal{N}(A) = \mathbb{R}^* e_0$ for 1°, while $\mathcal{N}(A) = \mathbb{R}^* e_0 \cup \mathbb{R}^* e_{\frac{1}{2}}$ for 2°.

II. By imposing to $L_{e_{\frac{1}{2}}}$ to be a derivation both for A_0 and A_1 (see Proposition 5.8) it follows $\alpha = 0$ or $\alpha = \frac{1}{2}$. As before, we get the following six algebras

$$4^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = e_0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

$$5^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = 0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1. \end{array}$$

$$6^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = 0 & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = -e_0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

$$7^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = \frac{1}{2} e_{\frac{1}{2}} & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = e_0 + e_1 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1, \end{array}$$

$$8^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = \frac{1}{2} e_{\frac{1}{2}} & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = 0 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1. \end{array}$$

$$9^\circ. \quad \begin{array}{lll} e_0^2 = e_0 & e_0 e_{\frac{1}{2}} = \frac{1}{2} e_{\frac{1}{2}} & e_0 e_1 = 0 \\ e_{\frac{1}{2}} e_{\frac{1}{2}} = -e_0 - e_1 & e_{\frac{1}{2}} e_1 = \frac{1}{2} e_{\frac{1}{2}} & e_1^2 = e_1. \end{array}$$

It this list, only algebras 5° and 8° are LT-algebras.

In order to prove that LT-algebras $1^\circ, 2^\circ, 5^\circ$ and 8° are nonisomorphic to each other, it suffices to compare their sets of idempotents and nilpotents. We get:

1° has the family of idempotents $\{y^2 e_0 + y e_{\frac{1}{2}} + e_1 | y \in \mathbb{R}\}$, and the family of nilpotents $\mathbb{R}e_0$;

2° has the family of idempotents $\{y e_{\frac{1}{2}} + e_1 | y \in \mathbb{R}\}$, and the family of nilpotents $\mathbb{R}e_0 \cup \mathbb{R}e_{\frac{1}{2}}$;

5° has the family of idempotents $\mathcal{I}(A) = \{e_0, e_1, e_0 + e_1, e_0 + y e_{\frac{1}{2}} + e_1 | y \in \mathbb{R}\}$, and the family of nilpotents $\mathcal{N}(A) = \mathbb{R}e_{\frac{1}{2}}$;

8° has the family of idempotents $\mathcal{I}(A) = \{e_0, e_1, e_0 + e_1\}$ and the family of nilpotents $\mathcal{N}(A) = \mathbb{R}e_{\frac{1}{2}}$; moreover, $u = e_0 + e_1$ is identity element.

Consequently, the LT-algebras $1^\circ, 2^\circ, 5^\circ$ and 8° are not isomorphic to each other. Actually, it was proved the following result.

Proposition 6.1. *Any 3-dimensional real LT-algebra, having an idempotent which is not a unity element, is isomorphic with one of the following four nonisomorphic LT-algebras*: $1°, 2°, 5°$ *and* $8°$.

Case 2

We can now pass to approach our problem in the cases when the solution is an algebra either with an identity element or with a nilpotent element, only.

Case 2a

Let us suppose that $T(\cdot)$ has no idempotent element. Then, Theorem 1 [14] assures us that there exists a nilpotent element n in T. Setting $x = n$ in (5), it yields $L_n^3 = 0$, i.e. the left multiplication $B = L_n$ is a nilpotent operator. It follows that there exists a basis $\mathcal{B}$ in T such that the corresponding matrix of B has a quasi-diagonal form whose diagonal consists of blocks of the form

$$[0], \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

only. Indeed, in the case $dim\ T > 2$, the basis $\mathcal{B}$ can be structured on three levels as below

$$\begin{aligned} \mathcal{B} = \ & \{e_1, e_2, ..., e_{p_1}, \\ & Be_1, Be_2, ..., Be_{p_1}, e_{p_1+1}, ..., e_{p_2}, \\ & B^2e_1, B^2e_2, ..., B^2e_{p_1}, Be_{p_1+1}, ..., Be_{p_2}, e_{p_2+1}, ..., e_{p_3} = n\}, \end{aligned}$$

where the first line contains elements of height 3 (i.e. elements $x \in T$ such that $B^3x = 0, B^2x \neq 0$), the second line contains elements of height 2 and the third line contains elements of height 1. This means that lines 2 and 3 together form a basis in $ker\ B^2$ while the third line gives a basis in $ker\ B$. Such a basis gives the vector sum decomposition $T = T_1 \oplus T_2 \oplus T_3$ where T_i is the subspace spanned by the i-line of $\mathcal{B}$. Unfortunately, a basis like $\mathcal{B}$ is not uniquely defined, so that the decomposition just presented is not necessarily unique, although $T_3 = kerB$ and $T_2 \oplus T_3 = kerB^2$ are uniquely determined vector subspaces. In what follows, the elements of T_i will be appropriately labelled with the index i. Setting $x = x_3, y = y_3, z = z_3, w = n$ in (4) it follows that

$$n \cdot [x_3, y_3, z_3] = 0 \Leftrightarrow [x_3, y_3, z_3] \in T_3$$

i.e. $(T_3, [\cdot,\cdot,\cdot])$ is sub-LTS of $(T, [\cdot,\cdot,\cdot])$. By setting $x = x_3, y = y_3, z = n, w = z_i$ $(i = 1, 2, 3)$ in (4) we obtain

$$[x_3, y_3, z_1 \cdot n] = n \cdot [x_3, y_3, z_1], [x_3, y_3, z_2 \cdot n] = n \cdot [x_3, y_3, z_2],$$
$$[x_3, y_3, z_3 \cdot n] = n \cdot [x_3, y_3, z_3] = 0,$$

i.e. $[x_3, y_3, z \cdot n] = n \cdot [x_3, y_3, z], \forall z \in T$ and $[D_{(x_3,y_3)}, B] = 0, \forall x_3, y_3 \in T$.

Consequently, $\mathcal{D}_3 = Span_K\{D_{(x,y)} | x, y \in T_3\}$ is a Lie subalgebra of $\mathcal{D}$ keeping invariant the subspace T_3. $\mathcal{D}_3$ commutes with B. However, T_3 is not necessarily a subalgebra of $T(\cdot)$.

Example of a 3-dimensional LT-algebra. We look for a 3-dimensional LT-algebra $A(\cdot)$, with or without identity, which has at least a nilpotent n. Then, there exists a basis, with elements ordered by their heights, $B = (e_1 = B^2 e_3 = n, e_2 = Be_3, e_3)$ such that B has one of the following matrices

$$(i) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (ii) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Case (i). A straightforward computation shows us that, in this Case, there exists no LT-algebra without identity.

Case (ii). After a suitable change of basis, one gets the following three LT-algebras

$$10^\circ. \quad \begin{array}{lll} e_1^2 = e_2 & e_2^2 = 0 & e_3^2 = 0 \\ e_1 \cdot e_2 = e_3 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = 0, \end{array}$$

$$11^\circ. \qquad \begin{array}{lll} e_1^2 = e_3 & e_2^2 = -e_3 & e_3^2 = 0 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = 0, \end{array}$$

$$12^\circ. \qquad \begin{array}{lll} e_1^2 = e_2 + e_3 & e_2^2 = 0 & e_3^2 = 0 \\ e_1 \cdot e_2 = e_3 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = 0. \end{array}$$

These algebras are associative and they are among the algebras listed by Scorza in [24].

Case (iii). This time we have to consider two complementary cases:

(iii_1) there exists a subalgebra complementary to the $\mathbb{R}n$, i.e., $A = \mathbb{R}n \oplus A_1$, with A_1 a subalgebra,

(iii_2) does not exist such an subalgebra.

Case (iii$_1$). Since A_1 is a real two-dimensional LT-algebra, the classification of algebras A is obtained when the classification in two-dimensional case (see the last section) is given. To this end, it suffices to consider an appropriate basis in A. Actually, $\mathbb{R}n$ and A_1 must be ideals of A (so that the classification of such algebras is well known by means of Proposition 7.1).

Case (iii$_2$). In a basis $B = (e_1 = n, e_2, e_3)$ the multiplication table for A has the form

$$\textbf{Table 1} \qquad \begin{array}{lll} e_1^2 = 0 & e_2^2 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 & e_3^2 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, \end{array}$$

where $\alpha_1^2 + \beta_1^2 + \gamma_1^2 \neq 0$.

Proposition 6.2. *Algebra $A(\cdot)$, with the multiplication table given in **Table 1**, has at least an element e for which the corresponding left multiplication L_e has real numbers as its eigenvalues.*

More exactly, there exists a basis $B = (e_1, e_2, e_3,)$ such that the multiplication table of the algebra $A(\cdot)$ has necessarily the form

$$13^\circ. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = \alpha_1 e_1 + \alpha_2 e_2 & e_3^2 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3. \end{array}$$

This result can be used to obtain some classifying results for 3-dimensional LT-algebras.

Case 2b

Let us suppose that $T(\cdot)$ has a unity element 1 but it has neither an idempotent element nor a nilpotent one. In this case the unity 1 is a common eigenvector for all elements of $\mathcal{D}$ corresponding to the eigenvalue 0. Then, taking into account the results from [1], [2], in case $K = \mathbb{R}$, every derivation $D_{(x,y)} \in \mathcal{D}$ has either 0 or imaginary complex numbers as eigenvalues. Some important results were proved in [1], [2], [5] on derivation algebras of real algebras without nilpotents of order two (the so called NN-algebras). Using the representation theory of Lie algebras, Benkart&Osborn [1], [2] exhibited real division algebras having certain Lie algebras as derivation algebras. This analysis was extended in [5] to real algebras without nilpotents of order two. Recall that, according to Theorem 1 [5], any real 3-dimensional algebra without nilpotent elements has $dim\ DerT \in \{0, 1\}$.

Any LT-algebra $A(\cdot)$ which is an NN-algebra but is not associative has at least a nonzero derivation D. Then, according to the results from [5], there exists a basis $B = (e_1, e_2, e_3)$ in A such that

$$D(e_1) = 0, \quad D(e_2) = -e_3, \quad D(e_3) = e_2,$$

and $A = kerD \oplus ImD$.

Consequently, the multiplication table of the algebra in basis B has the form

$$14^\circ.\quad \begin{array}{lll} e_1^2 = e_1 & e_2^2 = \varepsilon e_1 & e_3^2 = \varepsilon e_1 \\ e_1 \cdot e_2 = b e_2 + c e_3 & e_1 \cdot e_3 = -c e_2 + b e_3 & e_2 \cdot e_3 = 0. \end{array}$$

where $\varepsilon = \pm 1$, $b^2 + c^2 \neq 0$, or $\varepsilon = 1$, $b = c = 0$. The algebra must have identity element, that imposes $b = 1$ and $c = 0$. If moreover, $\varepsilon = 1$ then the corresponding algebra is necessarily a LT-algebra. Similarly, the algebra obtained from 14° for $b = 1$, $c = 0$ and $\varepsilon = -1$ is also a LT-algebra.

7. LOW-DIMENSIONAL LT-ALGEBRAS

Two-dimensional LT-algebras. In Section 2, it was presented the classification of real two-dimensional LTSs. It was already remarked that the null LTS (i) is a SLTS associated with any real two dimensional commutative and associative algebra. Recall that, there exist four such nonisomorphic algebras, namely the null algebra, and the real two-dimensional (commutative and associative) algebras of double and dual or complex numbers respectively. A straightforward computation allows us to prove that LTS LTS (ii) can be obtained, by means of the standard construction, from any 2-dimensional algebra $A_{\theta,\beta}(\cdot)$ of the form

$$a^2 = 0, a \cdot b = \beta a, b^2 = \theta a + 2\beta b, \quad \theta \in \mathbb{R}, \ \ \beta^2 = 1.$$

Any algebra $A_{\theta,1}$ is isomorphic with $A_{1,1}$, and any algebra $A_{\theta,-1}$ is isomorphic with $A_{1,-1}$; the corresponding isomorphisms are $\{a' = \theta a,\ b' = a + b\}$ and $\{a' = \theta a,\ b' = -a + b\}$, respectively. The algebras $A_{1,1}$ and $A_{1,-1}$ are not isomorphic. Consequently, there exist two nonisomorphic real two-dimensional LT-algebras: $A_{1,1}$ and $A_{1,-1}$. Moreover, they are not JORDAN algebras.

The real two-dimensional LTSs (iii)-(vi) are not SLTSs.

The following classification result was obtained.

Proposition 7.1. *Any real two-dimensional LT-algebra is isomorphic with one of the following six algebras: null algebra, algebra of double numbers, algebra of dual numbers, algebra of complex numbers,* $A_{1,1}$ *and* $A_{1,-1}$*. Their associated LTSs are (i)(for the first four algebras) and (ii)(for* $A_{1,1}$ *and* $A_{1,-1}$*), respectively.*

Three-dimensional LT-algebras. Each of the nonisomorphic real three-dimensional commutative and associative algebras identified by G. Scorza in [24] is a LT-algebra.

In order to obtain some classifying results for real three-dimensional LT-algebras, we shall use the result of Proposition 6.2. Let $B = (e_1, e_2, e_3)$ be a basis in LT-algebra $A(\cdot)$, such that L_{e_2} has the eigenvalues $s_1 = 0$, $s_2 = \alpha_2$, $s_3 = \beta_3$. We have to consider the following five cases:

1°. $\alpha_2 \cdot \beta_3 \neq 0$, $\alpha_2 \neq \beta_3$,

2°. $\alpha_2 = \beta_3 \neq 0$,

3°. $\alpha_2 \neq 0$, $\beta_3 = 0$,

4°. $\alpha_2 = 0$, $\beta_3 \neq 0$,

5°. $\alpha_2 = 0$, $\beta_3 = 0$.

Case 1°. This algebra is necessarily associative.

Case 2°. and *Case* 3°. This time, there is no LT-algebra.

Case 4°. There exists a basis such that the multiplication table of the algebra is

$$15^\circ. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = e_1 & e_3^2 = \gamma_1 e_1 + e_2 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = e_1, \end{array}$$

$$16^\circ. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = e_1 & e_3^2 = \gamma_1 e_1 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = e_1. \end{array}$$

The former is a LT-algebra which is not associative if $\gamma_2 \neq 0$. The second algebra is a LT-algebra if and only if $\gamma_2 = 0$ and it is an associative algebra.

Case 5°. There exists a basis $B = (e_1, e_2, e_3)$ in $A(\cdot)$ such that the matrix of L_{e_2} is similar to one of the following matrices

$$(i)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (ii)\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii)\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the case (i) there exists a basis such that the multiplication table of the algebra is

$$17°. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = e_1 & e_3^2 = \gamma_1 e_1 + 2\gamma_3 e_3 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = e_2. \end{array}$$

In the case (ii) there exists a basis such that the multiplication table of the algebra is one of the following two algebras

$$18°. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = 0 & e_3^2 = \gamma_1 e_1 + \gamma_2 e_2 + e_3 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = e_2. \end{array}$$

$$19°. \quad \begin{array}{lll} e_1^2 = 0 & e_2^2 = 0 & e_3^2 = \gamma_1 e_1 + \gamma_2 e_2 + 2e_3 \\ e_1 \cdot e_2 = 0 & e_1 \cdot e_3 = 0 & e_2 \cdot e_3 = e_2. \end{array}$$

Finally, in the case (iii) the algebra is necessarily associative.

With every exhibited LT-algebra, the corresponding LTS must be constructed. Also, it must be proved if these LTSs are or are not isomorphic to each other.

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ON ORDERED N-GROUPOIDS AND SPECTRAL COMPACTIFICATIONS

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Abstract The ideal spaces of n-l-groupoids is applied to the construction of the spectral compactifications of topological spaces.

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1. INTRODUCTION

In the following every space is considered to be a T_0-space. A pair (Y, f) is a g-compactification of a space X if $f : X \to Y$ is a continuous mapping, Y is a compact space and $f(X)$ is a dense subspace of Y.

Let X be a space with the topology T. Let $T_c = \{U \in T : U$ is a compact subset$\}$ and T_h be the topology on X generated by the subbase $T \cup T_c$. By hX we denote the set X endowed with the topology T_h.

A space Z is called a spectral space if T_c is an open base of Z and hZ is a compact space. If Z is a spectral space, then hZ is a Hausdorff zero-dimensional space.

A mapping $\varphi : Z \to Y$ is a spectral mapping if φ is continuous and $\varphi^{-1}(U)$ is a compact subset of Z provided U is an open compact subset of Y, i.e. $\varphi : hZ \to hY$ is continuous too.

A g-compactification (Y, f) of X is a spectral g-compactification if the set $f(X)$ is dense in the space hY.

Denote by $SC(X)$ the family of all spectral g-compactifications of a space X. Let $(Y, f), (Z, g) \in SC(X)$ We consider that $(Y, f) \leq (Z, g)$ if there exists a spectral mapping $\varphi : Z \to Y$ such that $f = \varphi \circ g$. The mapping φ is unique. If $(Y, f) \leq (Z, g)$ and $(Z, g) \leq (Y, f)$, then there exists a unique homeomorphism $\varphi : Z \to Y$ such that φ and φ^{-1} are spectral mappings and $\varphi(g(X)) = f(X)$. In this case we identify the g-compactification $(Y, f), (Z, g)$ and consider that $(Y, f) = (Z, g)$.

In this article we study the class of the family of spectral g-compactifications. We mention that $SC(X)$ is a set. Moreover, $SC(X)$ is a complete lattice with the maximal element (sX, e_X) and the minimal element (mX, m_X) is a singleton space.

2. SPECTRUM OF THE *L*-*N*-GROUPOID

Fix a natural number $n \geq 2$.

A set G is an l-n-groupoid or an ordered groupoid of the rank n if the following conditions hold (see[3]):

- G is a lattice with a minimal element 0 and a maximal element $1 \neq 0$;

- for every $x_1, x_2, ..., x_n \in G$ it is determined an element $[x_1, x_2, ..., x_n] \in G$;

- if $0 \in \{x_1, x_2, ..., x_n\}$, then $[x_1, x_2, ..., x_n] = 0$;

- if $i \leq n$ and $x_j = 1$ for any $j \neq i$, then $[x_1, x_2, ..., x_n] = x_i$;

- if $x_1 \leq y_1, x_2 \leq y_2, ..., x_n \leq y_n$, then $[x_1, x_2, ..., x_n] \leq [y_1, y_2, ..., y_n]$.

Let G be an l-n-groupoid.

A set $H \subseteq G$ is called $\vee$-open provided:

- if $x \leq y$ and $x \in H$, then $y \in H$;

- if $A \subseteq G$, there exists the maximum $\vee A$ and $\vee A \in H$, then $\vee B \in H$ for some finite subset $B \subseteq A$.

The family $B(G)$ of all $\vee$-open subsets of G is an open subbase of the ordered topology T_l on G.

A set $I \subseteq G$ is called an ideal of G provided:

- if $x_1, x_2, ..., x_n \in G$ and $I \cap \{x_1, x_2, ..., x_n\} \neq \emptyset$, then $[x_1, x_2, ..., x_n] \in I$;
- if $x, y \in G \setminus I$, then $x \wedge y \notin I$;
- if $x \leq y$ and $x \in I$, then $y \in I$;
- $0 \in I$.

An ideal I is simple if: $[x_1, x_2, ..., x_n] \in I$ if and only if $I \cap \{x_1, x_2, ..., x_n\} \neq \emptyset$; $I \neq G$; $x \wedge y$ implies that $I \cap \{x, y\} \neq \emptyset$.

Denote by $S(G)$ the space of all simple ideals of G. For every $x \in G$ we put $h(x) = \{I \in S(G) : x \notin I\}$ and $r(x) = S(G) \setminus h(x)$. On the space $S(G)$ consider the T_0-topology T_s with the closed subbase $\{r(x) : x \in G\}$ and the Hausdorff zero-dimensional topology T_h with the open subbase $\{r(x) : x \in G\} \cup \{h(x) : x \in G\}$. We say that T_s is the Stone-Zariski topology on $S(G)$ and T_h is the Hochster modification of the topology T_s (see [1,2]).

If I is an ideal, P is a simple ideal of G and $I \subseteq P$, then there exists a minimal simple ideal Q such that $I \subseteq Q \subseteq P$.

For any ideal (set) I we put $h(I) = \{P \in S(G) : I \subsetneqq P\} = \cup\{h(x) : x \in I\}$. Then $h(I) \in T_s$.

Lemma 2.1. *If $U \in T_s$, then $U = h(I)$ for some ideal $I \subseteq G$.*

Proof. Let $I = \{x \in G : h(x) \subseteq U\}$. If $x, y \in G$, then $h(x) \subseteq h(y)$ if and only if $x \leq y$. Thus $x \wedge y \in I$ if and only if $\{x, y\} \cap I \neq \emptyset$. Moreover if $\{x_1, x_2, ..., x_n\} \cap I \neq \emptyset$, then $[x_1, x_2, ..., x_n] \in I$. Thus I is an ideal and $h(I) = U$.

Let $A \subseteq S(G)$. Denote by $s\text{-}clA$ the closure of the set A in $(S(G), T_s)$.

Lemma 2.2. *$s\text{-}cl\{I\} = S(G) \setminus h(I)$ for every $I \in S(G)$.*

Proof. By definition, $s\text{-}clA = \cap\{r(x) : x \in G, A \subseteq r(x)\}$. Thus $s\text{-}cl\{I\} = \cap\{r(x) : x \in G, I \subseteq r(x)\} = \cap \cup \{S(G) \setminus h(I) : I \notin h(x)\} = S(G) \setminus h(I)$.

Lemma 2.3. *For every $a \in G$ the set $h(a)$ is a compact subset of the space $(S(G), T_s)$.*

Proof. Suppose that $h(a) \subseteq \cup\{h(x) : x \in A \subseteq G\}$. Then $\cup\{h(x) : x \in A\} = h(I)$ for some ideal $I \supseteq A$. Thus $r(a) \supseteq S(G)\backslash h(A)$. There exists a finite subset $B \subseteq A$ such that $r(a) \supseteq S(G)\backslash h(B)$. Therefore $h(a) \subseteq \cup\{h(x) : x \in B\}$. The Alexander Theorem of compactness complete the proof.

A subset F of a space X is reducible if there exist two closed subsets A and B of X such that $F = A \cup B$, $F\backslash A \neq \emptyset$ and $F\backslash B \neq \emptyset$.

Lemma 2.4. *If F is an irreducible subset of the space $(S(G), T_s)$, then $F = s\text{-}cl\{P\}$ for some $P \in S(G)$.*

Proof. Obvious.

Corollary 2.5. *$(S(G), T_s)$ is a spectral space.*

Corollary 2.6. *$(S(G), T_h)$ is a compact Hausdorff space.*

3. SPECTRAL COMPACTIFICATIONS OF SPACES

Fix $n \geq 2$ and an l-n-groupoid E with the following properties:

- E is a complete lattice;
- T_l is a T_0-topology on E;
- (E, T_l) is a topological groupoid;
- the lattice operations $\vee, \wedge : E^2 \to E$ are continuous in the topology T_l.

On E we consider the topology T_l. For every space X consider the space $C(X, E)$ of all continuous mappings of X in E into the topology T_l.

Proposition 3.1. *$C(X, E)$ is an l-n-groupoid.*

Proof. Obvious.

Corollary 3.2. *$(S(C(X, E)), T_s)$ is a spectral space.*

Proof. Fix a space X and a sublattice $L \subseteq C(X, E)$. Then $(S(L), T_s)$ is a spectral space.

For every $x \in X$ we put $I(x, L) = \{f \in L : f(x) = 0\}$. Then $I(x, L) \in S(L)$. Thus the mapping $v_L : X \to S(L)$, where $v_L(x) = I(x, L)$, is constructed.

Corollary 3.3. *$(S(L), v_L)$ is a spectral g-compactification of the space X.*

Corollary 3.4. *$(S(C(X,E)), v)$ is the maximal spectral compactification of the space X.*

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A NOTE ON SUBCLASSES OF UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

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Abstract The object of this paper is to derive some inclusion relations regarding a new class denoted by $S_n^m(\lambda, \alpha)$ using the generalized Sălăgean operator.

Keywords: univalent, Sălăgean operator, differential subordination.

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1. INTRODUCTION

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \in \mathbb{N}^* = \{1, 2, \dots\}, \tag{1}$$

which are analytic and univalent in the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : \ |z| < 1\} \tag{2}$$

with $\mathcal{A}_1 = \mathcal{A}$.

F.M. Al-Oboudi in [1] defined, for a function in $\mathcal{A}_n$, the following differential operator

$$D^0 f(z) = f(z) \tag{3}$$

$$D_\lambda^1 f(z) = D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) \tag{4}$$

$$D_\lambda^m f(z) = D_\lambda (D_\lambda^{m-1} f(z)), \quad \lambda > 0. \tag{5}$$

When $\lambda = 1$, we get the Sălăgean operator [6].

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(u) \subset g(U)$.

We shall use the following lemmas to prove our results.

Lemma 1.1. *[3] Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* Re $\gamma > 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma 1.2. *[4] Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z),$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + \cdots \in \mathcal{H}[q(0), n]$$

and

$$p(z) + \alpha z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

2. MAIN RESULTS

Definition 2.1. *Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}^*$. We say that the function f is in the class $S_n^m(\lambda, \alpha)$, $\lambda > 0$, $\alpha \in [0, 1)$, $m \in \mathbb{N}$, if f satisfies the condition*

$$\text{Re } [D_\lambda^m f(z)]' > \alpha, \quad z \in U. \tag{6}$$

Remark 2.1. *The class $S_1^m(\lambda, \alpha) \equiv S^m(\lambda, \alpha)$ was studied in [2] and the class $S_1^m(1, \alpha)$ was studied in [5].*

Theorem 2.1. *If $\alpha \in [0, 1)$, $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$ then*

$$S_n^{m+1}(\lambda, \alpha) \subset S_n^m(\lambda, \delta), \tag{7}$$

where

$$\delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1 - \alpha)\frac{1}{n\lambda}\beta\left(\frac{1}{\lambda n}\right), \tag{8}$$

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt \tag{9}$$

is the Beta function.

Proof. Let $f \in S_n^{m+1}(\lambda, \alpha)$. By using the properties of the operator D_λ^m, we get

$$D_\lambda^{m+1} f(z) = (1 - \lambda) D_\lambda^m f(z) + \lambda z (D_\lambda^m f(z))' \tag{10}$$

If we denote

$$p(z) = (D_\lambda^m f(z))', \tag{11}$$

where $p(z) = 1 + p_n z^n + \ldots$, $p(z) \in \mathcal{H}[1, n]$, then after a short computation we get

$$(D_\lambda^{m+1} f(z))' = p(z) + \lambda z p'(z), \quad z \in U. \tag{12}$$

Since $f \in S_n^{m+1}(\lambda, \alpha)$, from Definition 2.1 one obtains

$$\text{Re } (D_\lambda^{m+1} f(z))' > \alpha, \quad z \in U.$$

Using (12) we get

$$\text{Re }(p(z) + \lambda z p'(z)) > \alpha,$$

which is equivalent to

$$p(z) + \lambda z p'(z) \prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z). \tag{13}$$

Making use of Lemma 1.1 we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{1}{n\lambda z^{1/\lambda n}} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{(1/\lambda n) - 1} dt.$$

The function q is convex and is the best $(1, n)$-dominant.

Since

$$(D_\lambda^m f(z))' \prec 2\alpha - 1 + 2(1 - \alpha) \frac{1}{n\lambda} \cdot \frac{1}{z^{1/\lambda n}} \int_0^z \frac{t^{(1/\lambda n) - 1}}{t + 1} dt$$

it follows that

$$\text{Re }(D_\lambda^m f(z))' > q(1) = \delta, \tag{14}$$

where

$$\delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1 - \alpha) \frac{1}{n\lambda} \beta\left(\frac{1}{\lambda n}\right), \tag{15}$$

$$\beta\left(\frac{1}{\lambda n}\right) = \int_0^1 \frac{t^{(1/\lambda n) - 1}}{t + 1} dt. \tag{16}$$

From (14) we deduce that $f \in S_n^m(\lambda, \alpha, \delta)$ and the proof of the theorem is complete. ∎

Making use of Lemma 1.2 we now prove the following theorems.

Theorem 2.2. *Let $q(z)$ be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + n\lambda z q'(z), \quad \lambda > 0. \tag{17}$$

If $f \in \mathcal{A}_n$ *and satisfies the differential subordination*

$$(D_\lambda^{m+1} f(z))' \prec h(z), \tag{18}$$

then

$$(D_\lambda^m f(z))' \prec q(z) \tag{19}$$

and the result is sharp.

Proof. From (12) and (18) one obtains

$$p(z) + \lambda z p'(z) \prec q(z) + n\lambda z q'(z) \equiv h(z).$$

Then, by Lemma 1.2, we get

$$p(z) \prec q(z),$$

or

$$(D_\lambda^m f(z))' \prec q(z), \quad z \in U$$

and this result is sharp. ∎

Theorem 2.3. *Let* q *be a convex function with* $q(0) = 1$ *and let* h *be a function of the form*

$$h(z) = q(z) + nzq'(z), \quad \lambda > 0, \ z \in U. \tag{20}$$

If $f \in \mathcal{A}_n$ *satisfies the differential subordination*

$$(D_\lambda^m f(z))' \prec h(z), \quad z \in U, \tag{21}$$

then

$$\frac{D_\lambda^m f(z)}{z} \prec q(z) \tag{22}$$

and this result is sharp.

Proof. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{z}, \quad z \in U,$$

then we obtain

$$(D_\lambda^m f(z))' = p(z) + zp'(z), \quad z \in U.$$

The subordination (21) becomes

$$p(z) + zp'(z) \prec q(z) + nzq'(z)$$

and, by Lemma 1.2, we have (22). The result is sharp. ∎

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WEAK PARAMETRICAL COMPLETENESS IN ACQUAINT NON-CHAIN EXTENSION OF INTUITIONISTIC LOGIC

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Abstract The criteria of weak parametrical completeness in the classical propositional logic and in the 6-element extension of intuitionistic logic are shown.

Formulas are constructed from the symbols of variables $p, q, \ldots$ by means of the operators of set $\Omega = \{\&, \vee, \supset, \neg\}$ and parentheses. The symbol $(A \sim B)$ denotes the formula $((A \supset B)\&(B \supset A))$. Following [1], a formula F is said to be parametrically expressible in the logic L by means of a system Σ of formulas, if there exist numbers l and s, variables $\pi, \pi_1, \ldots, \pi_l$ not occurring in F, pairis of formulas A_i, B_i $(i = 1, \ldots, s)$ that are expressible in L in the terms of Σ, and formulas $D_1, \ldots, D_l$ that do not contain the variables $\pi, \pi_1, \ldots, \pi_l$, such that the relations

$L \vdash (F \sim \pi) \supset (A_1 \sim B_1)\& \ldots \&(A_s \sim B_s)[d_1/D_1] \ldots [d_l/D_l]$,

$L \vdash (A_1 \sim B_1)\& \ldots \&(A_s \sim B_s) \supset (F \sim \pi)$

take place.

A system Σ of formulas is said to be parametrically complete in a logic L if all formulas of the language of L are parametrically expressible in L by means of Σ. By a Pseudo-Boolean algebra [2] (p.b.a.) we mean a system $\langle M; \Omega \rangle$, which is a lattice $\langle M; \&, \vee \rangle$ with relative pseudo-complement $\supset$ and pseudo-complement $\neg$. The set of all formulas true on algebra $\mathfrak{A}$ constitute the logic of $\mathfrak{A}$, denoted by $L\mathfrak{A}$. Let us consider a p.b.a. $Z_2 + Z_5 = \langle\{0, \tau, \rho, \sigma, \omega, 1\}; \Omega\rangle$, where $0 < \tau < \rho < \omega < 1$, $0 < \tau < \sigma < \omega < 1$, and ρ and σ are incomparable.

Let us define the operations $f_1(0) = f_1(\tau) = 0, f_1(1) = 1$; $f_2(0) = 0, f_2(\tau) = f_2(\omega) = \tau, f_2(1) = 1$; $f_3(0) = 0, f_3(\tau) = 1, f_3(\omega) = \tau, f_3(1) = 1$; $f_4(0) = 0, f_4(\tau) = \tau, f_4(\rho) = \sigma, f_4(\sigma) = \rho, f_4(\omega) = f_4(1) = 1$; $f_5(0) = 0, f_5(\tau) = f_5(\rho) = f_5(\sigma) = \tau, f_5(\omega) = f_5(1) = 1$; $f_6(0) = 0, f_6(\tau) = \tau, f_6(\rho) = f_6(\sigma) = f_6(\omega) = f_6(1) = 1$; $f_7(0) = 0, f_7(\tau) = 1, f_7(\rho) = f_7(\sigma) = \tau, f_7(\omega) = f_7(1) = 1$.

Following A. V. Kuznetsov [1], we say that a formula $F(p_1, \ldots p_n)$ *preserves the predicate* $R(x_1, \ldots x_m)$ on the algebra $\mathfrak{A}$ if, for any elements $\alpha_{ij} \in \mathfrak{A}$ $(i = 1, \ldots, m; j = 1, \ldots, n)$, the truth of propositions $R[\alpha_{11}, \alpha_{21}, \ldots, \alpha_{m1}], \ldots, R[\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{mn}]$ implies $R[F[\alpha_{11}, \ldots, \alpha_{1n}], \ldots, F[\alpha_{m1}, \ldots, \alpha_{mn}]]$.

The next criterion of parametrical completeness in the $L(Z_2 + Z_5)$ logic was published in [3].

Theorem 1. *In order that a system Σ of formulas be parametrically complete in the logic $L(Z_2 + Z_5)$ it is necessary and sufficient that Σ be parametrically complete in the classical logic and for every $i = 1, \ldots, 7$ there exist a formula of Σ, does not preserving the predicate $y = f_i(x)$.*

A system Σ is said to be weakly parametrically complete in L if the system, consisting of the functions belonging to Σ and the functions constants 0 and 1, is parametrically complete in L.

Let consider a Boolean algebra $Z_2 = \langle\{0, 1\}; \Omega\rangle$.

The next criterion of weak parametrical completeness in the LZ_2 classical logic follows from criterion of parametrical completeness in LZ_2 [1, 3].

Proposition 1. *A system Σ of formulas is weakly parametrically complete in the classical logic LZ_2 iff there exists the formulas of Σ that do not preserve the predicates $(x\&y) = z$, $(x \vee y) = z$, $((x \sim y) \sim z) = u$ on the algebra Z_2.*

Theorem 2. *In order that a system Σ of formulas be weakly parametrically complete in the logic $L(Z_2+Z_5)$ it is necessary and sufficient that Σ be weakly parametrically complete in the classical logic and for any $i = 1, \dots, 7$ there exist a formula of Σ, does not preserving the predicate $y = f_i(x)$.*

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ON EXTENDED LEWIS CONFORMAL MAPPING IN HYDRODYNAMICS

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Abstract The ship hull forms have been described by the well-known classic Lewis transformation [9], and by an extended-Lewis transformation with three parameters, as given by Athanassoulis and Loukakis [1], with practical applicability for any types of ships. We already have presented [2], an algorithmic method solving directly the problems that appear in naval architecture domain concerning the contour of ship's cross-section. In this paper we present how we may extend the Lewis transformation to obtain the contour of the ship's cross section of different types of ships.

1. INTRODUCTION

It was in 1949 that Ursell published his potential theory for determining the hydrodynamic coefficients of semicircular cross sections, oscillating in deep water in the frequency domain. Using this, for the first time a rough estimation could be made of the motions of a ship in regular waves at zero forward speed.

Shortly after that Tasai, Grim, Gerritsma and many other scientists used various already existing conformal mapping techniques (to transform ship-like cross section to a semicircle) together with Urssel's theory, in such a way that the motion in regular waves of more realistic hull forms could be calculated too. Most popular was the 2-parameter Lewis conformal mapping technique.

It is necessary to approximate the ship's shape by continuous functions, in order to get some practical results. A method, which has imposed itself during the last few years, is that of multi-parameter conformal mapping, with good results also in the case of extreme bulbous forms.

The advantage of conformal mapping is that the velocity potential of the fluid around an arbitrary shape of a cross section in a complex plane can be derived from the more convenient circular section in another complex plane. In this manner, hydrodynamic problems can be solved directly by using the coefficients of the mapping function.

The general transformation formula is given by:

$$f(Z) = \mu_s \sum_{k=0}^{n} a_{2k-1} Z^{-2k+1}, \tag{1}$$

with $f(Z) = z, z = x + iy$ is the plane of the ship's cross section, $Z = ie^{\alpha}e^{-i\varphi}$ is the plane of the unit circle, $\mu_s = 1$, a_{2k-1} are the conformal mapping coefficients ($k = 1, ...n$), n is the number of parameters.

Therefore we can write

$$x + iy = \mu_s \sum_{k=0}^{n} a_{2k-1} (ie^{\alpha}e^{-i\varphi})^{-(2k-1)}, \tag{2}$$

$$x + iy = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} e^{-(2k-1)\alpha} [icos(2k-1)\varphi - sin(2k-1)\varphi]. \tag{3}$$

From the relation between the coordinates in the z - plane (the ship's cross section) and the variables in the Z - plane (the circular cross section), it follows

$$x = -\mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} e^{-(2k-1)\alpha} sin(2k-1)\varphi, \tag{4}$$

$$y = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} e^{-(2k-1)\alpha} cos(2k-1)\varphi. \tag{5}$$

Now by using conformal mapping approximations, the contour of the ship's cross section, follows from putting $\alpha = 0$ in (4) and (5). We get

$$x_o = -\mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} sin(2k-1)\varphi,$$

$$y_0 = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} cos(2k-1)\varphi.$$

The breadth on the waterline of the approximate ship's cross section is defined by

$$B_0 = 2\mu_s\beta, \text{ with } \beta = \sum_{k=0}^{n} a_{2k-1},$$

and the draft is defined by

$$D_0 = 2\mu(s)\delta, \text{ with } \delta = \sum_{k=0}^{n} (-1)^k a_{2k-1}.$$

The breadth on the waterline is obtained for $\varphi = \pi/2$, that means

$$x_{\pi/2} = -\mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} sin(2k-1)\pi/2,$$

hence

$$x_{\pi/2} = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1}, \text{ and } B_0 = 2x_{\pi/2}.$$

The scale factor is $\mu_s = B_o/2\beta$ and the draft is obtained for $\varphi = 0$:

$$y_0 = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} cos(2k-1)0, \text{ hence } y_0 = \mu_s \sum_{k=0}^{n} (-1)^k a_{2k-1} \text{ and } D_0 = y_0$$

with $\mu_s = D_0/\delta$.

2. EXTENDED LEWIS CONFORMAL MAPPING

We can obtain better approximations of the cross sectional hull form by taking into account also the first order moments of half the cross section about the x-and y-axes. These two additions to the Lewis formulation were proposed by Reed and Nowacki [11] and have been simplified by Athanassoulis and Loukakis [1] by taking into account the vertical position of the centroid of the cross section. This has been done by extending the Lewis transformation from n=2 to n=3 in the general transformation formula.

The three-parameter extended Lewis transformation of a cross section is defined by

$$z = f(Z) = \mu_s a_{-1} Z + \mu_s a_1 Z^{-1} + \mu_s a_3 Z^{-3} + \mu_s a_5 Z^{-5}, \tag{6}$$

where $a_{-1} = 1$, μ_s is the scale factor and the conformal mapping coefficients a_1, a_3, a_5 are called Lewis coefficients. Then, for $z = x + iy$ and $Z = ie^{\alpha}e^{-i\varphi}$, that is $Z = ie^{\alpha}[cos(-\varphi) + isin(-\varphi]$, we have

$$x = \mu_s(e^{\alpha} sin\varphi + a_1 e^{-\alpha} sin\varphi - a_3 e^{-3\alpha} sin3\varphi + a_5 e^{-5\alpha} sin5\varphi)$$

and

$$y = \mu_s(e^{\alpha} cos\varphi - a_1 e^{-\alpha} cos\varphi + a_3 e^{-3\alpha} cos3\varphi - a_5 e^{-5\alpha} cos5\varphi).$$

For $\alpha = 0$ we obtain the contour of the so- called extended Lewis form

$$x_0 = \mu_s(sin\varphi + a_1 sin\varphi - a_3 sin3\varphi + a_5 sin5\varphi),$$

and

$$y_0 = \mu_s(cos\varphi - a_1 cos\varphi + a_3 cos3\varphi - a_5 cos5\varphi)$$

where the scale factor μ_s is

$$\mu_s = B_s/2(1 + a_1 + a_3 + a_5) \text{ or } \mu_s = D_s/(1 - a_1 + a_3 - a_5),$$

in which B_s is the sectional breadth on the waterline and D_s is the sectional draught. The half breadth to draft ratio H_0 is given by

$$H_0 = \frac{B_s}{2D_s} = (1 + a_1 + a_3 + a_5)/(1 - a_1 + a_3 - a_5).$$

An integration of the extended Lewis form delivers the sectional area coefficient:

$$\sigma_s = A_s/B_s D_s = \pi/4 \cdot (1 - a_1^2 - 3a_3^2 - 5a_5^2)/[(1 + a_3)^2 - (a_1 + a_5)^2]$$

in which A_s is the area of the cross section, $A_s = \pi/2 \cdot \mu_s^2(1 - a_1^2 - 3a_3^2 - 5a_5^2)$ and $B_s D_s = 2[(1 + a_3)^2 - (a_1 + a_5)^2]$.

Now the coefficients a_1, a_3, a_5 and the scale factor μ_s will be determined in such a manner that the sectional breadth, the draft and the area of the approximate cross section and of the actual cross section are identical. We have already presented [2] a typical and realistic form. More precisely we have considered a dry bulk carrier of 55.000 tone deadweight capacity and "Mircea" school ship. That application was made in Java language and created both a text file and a graphical chart. A more complex expression has been obtained by Athanassoulis and Loukakis [1] for the relative distance of the centroid to the keel point

$$k = \frac{KB}{D_s} = 1 - \frac{\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} A_{ijk} a_{2i-1} a_{2j-1} a_{2k-1}}{H_0 \sigma_s \sum_{i=0}^{3} a_3^{2i-1}}$$

in which

$$A_{ijk} = \frac{1}{4}\left\{\frac{1-2k}{3-2(i+j+k)} - \frac{1-2k}{1-2(i-j+k)} + \frac{1-2k}{1-(i+j-k)} + \frac{1-2k}{1-2(-i+j+k)}\right\}.$$

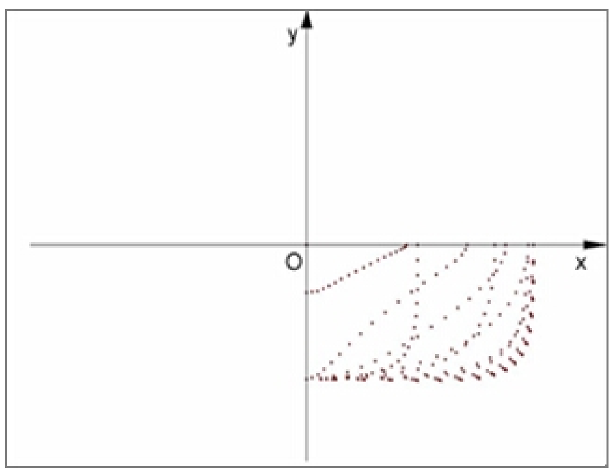

Fig. 1. The contour of the ship's cross section of the dry bulk carrier.

Figs. 1 and 2 were obtained from the previous method with a software package specially developed for this purpose [2].

The graphical representation of the points shows the contour of the ship's cross section of the dry bulk carrier and "Mircea" school ship.

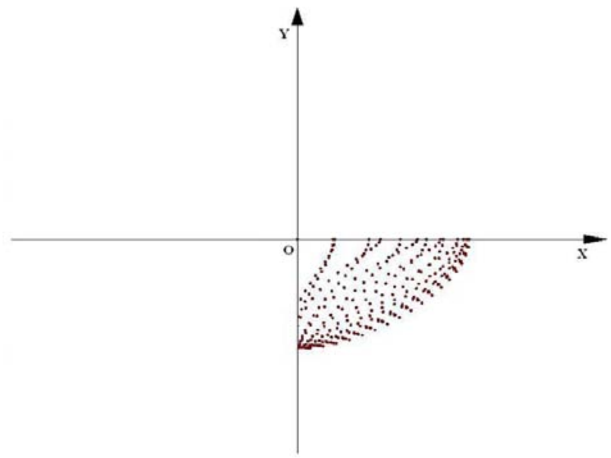

Fig. 2. The contour of the ship's cross section of the "Mircea" school ship.

3. CONCLUSIONS

This is a mathematical solution in order to obtain the contour of the ship's cross section of different types of ships, using conformal mapping approximations.

The advantage of conformal mapping is that the velocity potential of the fluid around an arbitrary shape of a cross section in a complex plane can be derived from the more convenient circular section in another complex plane.

In this manner hydrodynamic problems can be solved directly with the coefficients of the mapping function.

In the future we hope to obtain much better graphical representation by considering three or more coefficients.

Acknowledgements

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TENDENCIES IN VERIFYING OBJECT-ORIENTED SOFTWARE

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Abstract

The goal of this paper is to investigate the main approaches to the concept of object-oriented systems verification, namely:

- the adaptation of the classic technique to the context of new technologies of analysis, design and programming (object-oriented)

- applying the specific characteristics of the context of the new technologies (analysis, design and programming) in order to get new techniques.

Some considerations regarding the checking approaches to the aspect-oriented programming (AOP) whose defining characteristic is the moment, depending on the woven program, are also taken into account, when the systems verification is completed.

1. PRELIMINARY

Systems verification was developed at the same time with the projection and programming techniques which became in a short time a concept.

The complexity of systems verification determined a new checking technique based on new data. Throughout the development of a project steps were established (analysis, projection and programming) and for all these the methodology was settled. Among these methodologies the object oriented software has the capability to resolve systems complexity (the main problem of the new systems).

When resolving complexity [12] object-oriented software has some benefits like the abstract, encapsulation, the heritage which ensures the capacity of a new reality dimension report, seen from a certain point of view, but also two important facilities like reusing the code and the libraries.

The paper is structured as follows: we define the concept of systems verification and its classic techniques. Then we examine the verification techniques that are used in the context of object-oriented systems, namely those developed after the classic techniques of system verification - model checking - and the techniques specific to the object-oriented context - checking the consistency of class of collaboration diagrams.

Close attention is paid to the checking approaches to the aspect-oriented programming (AOP) whose defining characteristic is the moment, depending on the woven program, when the systems verification is completed.

In the first approach that we develop the aspects are considered as independent components which can be woven together with the other software system components. In this case the verification is completed before the software system starts its functionalities.

In the last part of the paper we consider another approach that is based on the use of classic techniques (model checking) to verify aspect-oriented programming system after the woven program, that is it verifies the programming code (Java) with familiar instruments (JPF/Java).

The techniques of system verification are exemplified by checking the deadlock freedom propriety.

2. THE CONCEPT OF SYSTEMS VERIFICATION

The concept of systems verification correctness was developed after applying (within program systems) formal language and methodologies, which rely on well-developed mathematical procedures. This enabled the verification of the correctness of these specifications.

In the formal language of temporal logistic [1], formalization permitted the right investigation of fundamental system program properties:

- safety proprieties (deadlock freedom, partial correctness, global invariant, generalized invariant);

- liveness proprieties (total correctness and termination, accessibility property);

The development of programming systems and their increasing complexity necessitate an adaptation of the systems verification or a new technique.

The paper [8] presents a verification of the relevant properties for the program systems like "deadlock freedom" property in the case of concurrent systems.

Another important technique is "model checking" which means the systematic verification of a given property, in all the phases the system is passing through [5].

3. SYSTEMS VERIFICATION OF OBJECT ORIENTED SOFTWARE

In object oriented software context the first method of systems verification consists in adapting the classic methods mentioned in the previous section.

At this point a good example is checking the context of the deadlock freedom properties approached in [8].

Another example is the adaptation of the model checking technique to the context of object oriented software, discussed in [17].

The second method, specific to the context, is the diagrams consistency checking [15] which could start with the first steps of analysis and projection.

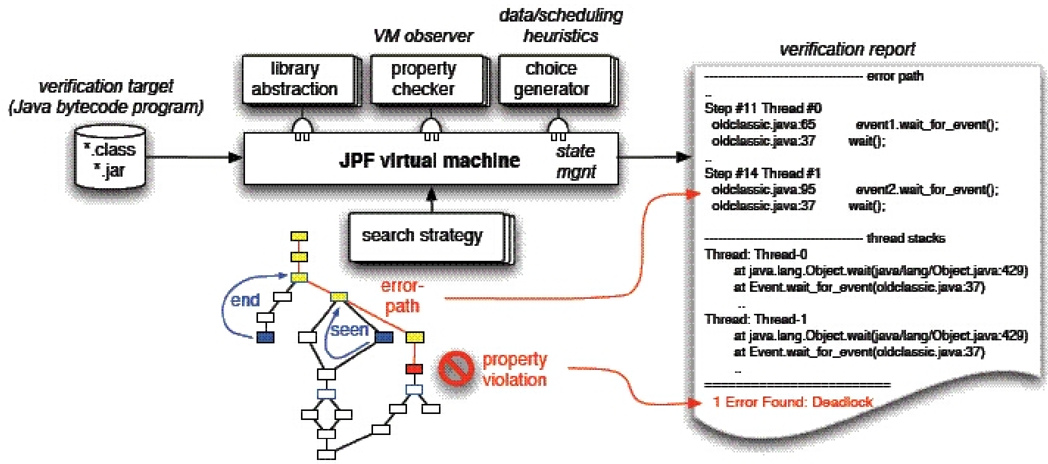


Fig.1. JPF usage.

This method includes the verification of some properties by type of preconditions, post conditions or invariant states.

The implementation of these perspectives on the system verification in the case of Java object-oriented software is done through the Java Path Finder instrument, which is in progress.

The way of using this tool is shown in [17] from where we choose the suggestive fig.1 in which the "deadlock freedom" property is shown.

4. SYSTEMS VERIFICATION OF ASPECT-ORIENTED SOFTWARE

The perspectives on the systems verification of aspect-oriented software are defined by the moment, according to the woven process, in which the way of achieving the verification is foreseen.

For example, it takes the generic problem producer/consumer (illustrated in fig. 2) defined as in [1]:

-the producer - for each going through the curl - generates an object which is stored in a limited buffer (which cannot contain more than N such objects)

-the consumer : from time to time it drags out such an object from the buffer and eats it

-stocking up the object in the buffer (on being made) means that it is not full and dragging out the same objects from the buffer (on consumption) means that it is not empty.

For this problem we will analyze the section with the buffer functioning, whose description is illustrated in fig. 3 by means of object-oriented software.

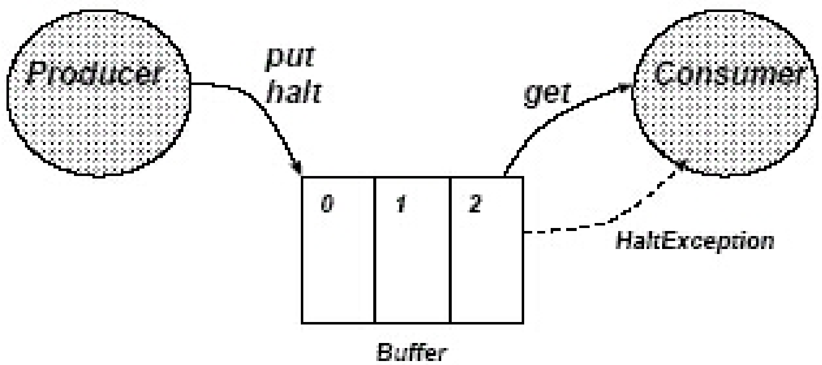


Fig.2.

The generic problem of producer/consumer.

4.1. THE VERIFICATION PRIOR TO THE WOVEN PROCESS

Within an aspect-oriented program verification aspects are considered as independent components which can be woven together with the other entities of an object oriented system. Its verification can be achieved before the system accomplishing its functions.

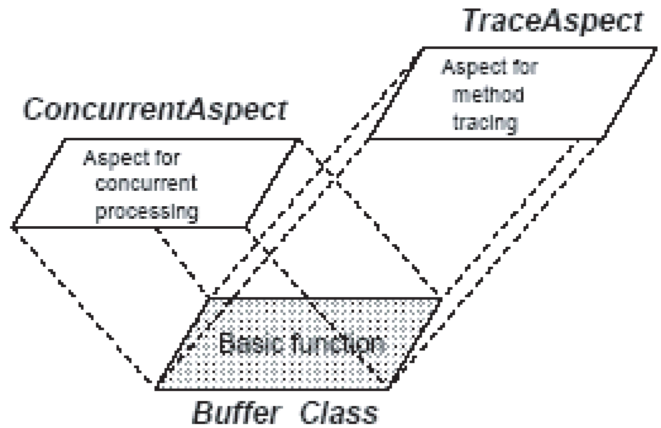


Fig.3.

Buffer description in AOP.

In the paper [8] the following methodology is suggested (the approach is recommended for safety properties verification defining the aspect oriented programs).

1. For the beginning, safety proprieties are selected, which are relevant for the soft system (such as deadlock freedom, liveness of state).

To illustrate this method [8] analyzes the deadlock freedom property in the case of a hypothetical concurrent system rendered in aspect oriented language (AspectJ - shown in fig. 4).

Similarly, the deadlock freedom propriety is selected for analysis in the case of the generic problem producer/consumer (illustrated in fig. 2).

2. The formal verification technique is selected according to the above established properties.

In the case of deadlock freedom property such a technique can be model checking.

3. The aspects which encapsulate the code, relevant to the context of established properties are identified.

```
public class User1 {
  public void removeUseless(File f){
    if(f.isUseless()){
      Directory d = f.getDir();
      d.removeFile(f);
      System.out.println("Removed: " + f.getName());
    }
  }
}
```

(a)

```
public class User2 {
  public void updateDir(Directory d){
    Enumeration e = d.getFiles();
    while(e.hasMoreElements()){
      File f = (File)e.nextElement();
        f.update();
        System.out.println("Updated: " +
                          f.getDir().getName()
                          + f.getName());
        }
    }
}
```

(b)

Fig.4.

Hypothetical concurrent system.

For the example of a hypothetical concurrent system the relevant aspects are those which encapsulate the synchronized policy - shown in fig. 5.

For the example of the generic problem producer/consumer the relevant aspects are those which encapsulate the buffer whose description, by means of aspect-oriented method, is shown in fig. 3.

4. Subsequently, the formal verification technique is used; this was selected for the identified aspects.

In the case of the aspects which encapsulate the synchronized procedure as in the example of hypothetical concurrent system, modeled in PROMELA, SPIN was used for the verification process.

```
aspect ConcurrencyAspect{

 /* pointcut declarations */
 pointcut file(File f):
   (instanceof(f) && executions(void File.update()))
     || executions(void User1.removeUseless(f));
 pointcut dir(Directory d):
   (instanceof(d)
    && executions(boolean Directory.removeFile(File))
     || executions(void User2.updateDir(d));

 /* code to be woven when pointcuts occur */
 static around(File f) returns void: file(f){
      synchronized(f){
        System.out.println("locked: "  + f.getName());
        proceed(f);
        System.out.println("unlocked: " + f.getName());
      }
 }
 static around(Directory d) returns boolean: dir(d){
      synchronized(d){
        boolean b;
        System.out.println("locked: " + d.getName());
        b = proceed(d);
        System.out.println("unlocked: " + d.getName());
        return b;
      }
 }
}
```

Fig.5.

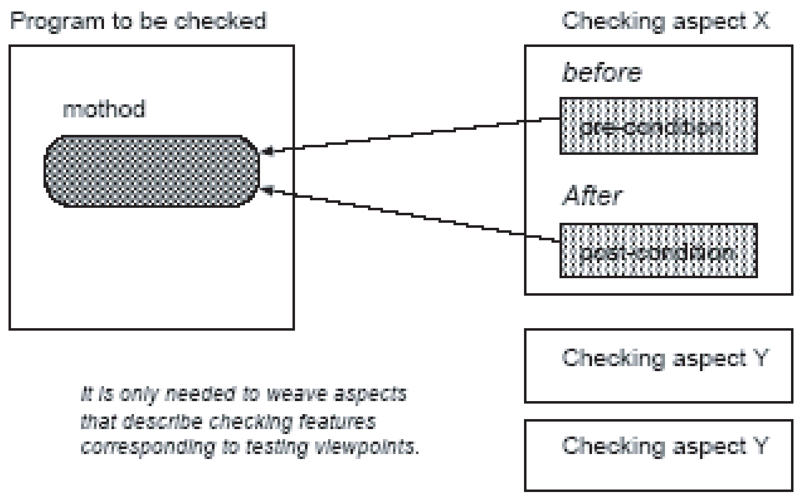
Aspects which encapsulate synchronized policy.

Similarly, one can approach the buffer described by means of aspect-oriented method as in the example of the generic problem producer/consumer.

4.2. THE SUBSEQUENT VERIFICATION OF THE WOVEN PROCESS

Another approach is focused on using the model checking techniques to verify the aspect-oriented programs after the woven process, shown in the paper [11] and to verify the code which has resulted (Java) with the already known instruments (JPF/Java) respectively.

In the paper [11] both a procedure and a working flow are presented - shown in fig. 6.




```
aspect Sample {
   // specify a join point
   pointcut fooPointcut(Foo o):
     call(public void bar()) && target(o);
   before(Foo o): fooPointcut(o){
     // specify a pre-condition
   }
   after(Foo o): fooPointcut(o){
     // specify a post-condition
   }
}
```

Fig.6.

The procedure and the working flow for verification whit the help of JPF.

This approach is exemplified in the paper regarding the generic problem producer/consumer (described in figs. 2 and 3) by the model checking verification procedure, illustrated in fig. 7.

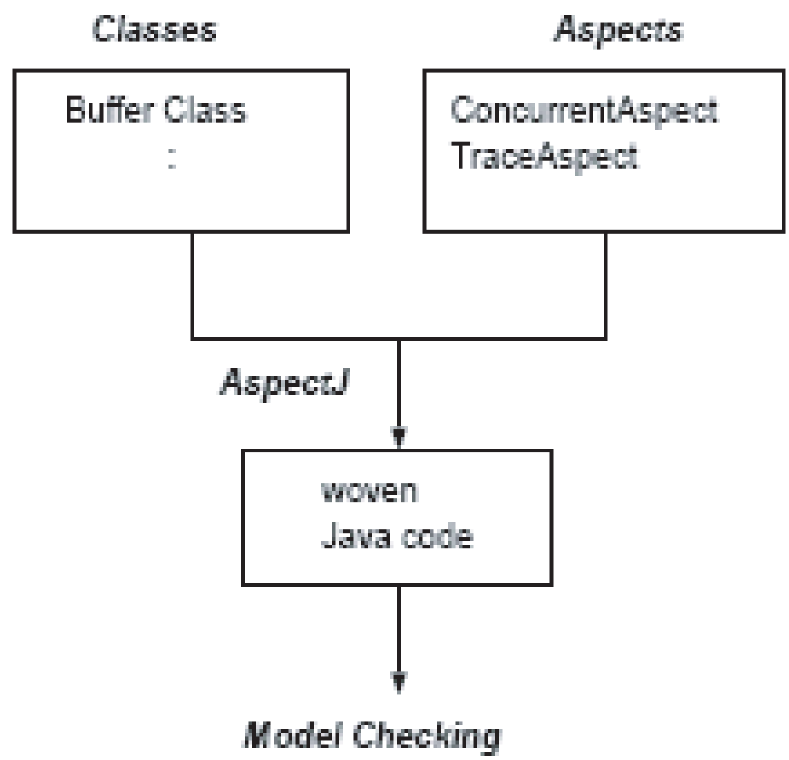


Fig.7.

Model checking procedure for the generic problem producer/consumer.

5. CONCLUSIONS

The recent results regarding the concept of programs systems verifications have led to two main tendencies.

The first tendency - the adaptation of the classic technique to the context of new technologies of analysis, design and programming (object-oriented).

The second tendency - applying the specific characteristics of the context of the new technologies (analysis, design and programming) in order to get new techniques.

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EXPONENTIAL STABILITY IN MEAN SQUARE OF DISCRETE-TIME TIME-VARYING LINEAR STOCHASTIC SYSTEMS WITH MARKOVIAN SWITCHING

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Abstract The problem of the mean square exponential stability for a class of discrete-time time-varying linear stochastic systems subject to Markovian switching is investigated. The case of the linear systems whose coefficients depend both on present state and the previous state of the Markov chain is considered. Three different definitions of the concept of exponential stability in mean square are introduced and it is shown that they are not always equivalent. One definition of the concept of mean square exponential stability is done in terms of the exponential stability of the evolution defined by a sequence of linear positive operators on an ordered Hilbert space. The other two definitions are given in terms of different types of exponential behavior of the trajectories of the considered system. In our approach the Markov chain is not prefixed. The only available information about the Markov chain is the sequence of probability transition matrices and the set of its states. In this way one obtains that if the system is affected by Markovian jumping, the property of exponential stability is independent of the initial distribution of the Markov chain.

The definition expressed in terms of exponential stability of the evolution generated by a sequence of linear positive operators, allows us to characterize the mean square exponential stability based on the existence of some quadratic Lyapunov functions.

The results developed in this paper may be used to derive some procedures for designing stabilizing controllers for the considered class of discrete-time linear stochastic systems in the presence of a delay in the data transmission.

Keywords: discrete-time linear stochastic systems, Markov chains, mean square exponential stability, Lyapunov operators, delays in data transmission.

1. INTRODUCTION

Stability is one of the main tasks in the analysis and synthesis of a controller in many control problems such as: linear quadratic regulator, H_2 optimal control, H_∞-control and other robust control problems (see e.g. [9, 16, 24] for the continuous time case or [1, 4, 5, 15] for the discrete-time case and references therein).

For linear stochastic systems there are various types of stability. However one of the most popular among them is "***exponential stability in mean square***" (ESMS). This is due to the existence of some efficient algorithms to check this property.

In the literature, the problem of ESMS of discrete-time linear stochastic systems with Markovian jumping was intensively investigated. For the readers convenience we refer to [2, 3, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22]. The majority of the works where the problem of ESMS of discrete-time linear systems with Markovian switching is investigated, deals with the so-called time invariant case. That is the case of discrete-time linear systems with the matrix coefficients not depending on time and the Markov chain being homogeneous. There are papers [19, 21, 22] where the matrix coefficients of the system are time dependent and the Markov chain is homogeneous. There are also papers [12, 17] where the matrix coefficients of the discrete time system do not depend on time but the Markov chain is inhomogeneous.

In [8], it was considered the general situation of the discrete-time time-varying linear stochastic systems corrupted by independent random perturbations and by jump Markov perturbations. The general case of discrete-time time-varying linear systems with Markovian switching was investigated in [10].

The aim of the present paper is to extend the results of [10] to the case of discrete-time time-varying linear stochastic systems whose coefficients depend both to the present state η_t and the previous state η_{t-1} of the Markov chain. We assume that the matrix coefficients may depend on time and the Markov chain is not necessarily homogeneous. Such systems arise in connection with the problem of designing of a stabilizing feedback gain in the presence of some delay in transmission of the date either on the channel from the sensors to controller or between controller and actuators. For this class of systems we introduce three different definitions of the concept of exponential stability in mean square. One of these definitions characterizes the concept of exponential stability in mean square in terms of exponential stability of the evolution generated by a suitable sequence of linear positive operators associated with the considered stochastic system. This type of exponential stability in mean square which we shall called "strong exponential stability in mean square" (SESMS) is equivalent to the existence of a quadratic Lyapunov function; meaning that it is equivalent to the solvability of some systems of linear matrix equations or linear matrix inequations.

Other two types of exponential stability in mean square are stated in terms of exponential behavior of the state space trajectories of the considered stochastic systems.

We show that the three definitions of the exponential stability in mean square are not always equivalent. Also, we prove that under some additional assumptions part of types of exponential stability in mean square become equivalent. In the case of discrete-time linear stochastic systems with periodic coefficients all these three types of exponential stability in mean square introduced in the paper become equivalent.

We remark that in our approach the Markov chain is not prefixed. The only available information about the Markov chain consists of the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$ and the set $\mathcal{D}$ of its states. The initial distributions of the Markov chain do not play any role in defining and characterizing the exponential stability in mean square.

2. THE PROBLEM

2.1. DESCRIPTION OF THE SYSTEMS

Let us consider discrete-time linear stochastic systems of the form

$$x(t+1) = A(t, \eta_t, \eta_{t-1})x(t) \tag{1}$$

$t \geq 1$, $t \in \mathbf{Z}_+$, where $x \in \mathbf{R^n}$ and $\{\eta_t\}_{t\geq 0}$ is a Markov chain defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the state space the finite set $\mathcal{D} = \{1, 2, ..., N\}$ and the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$. This means that for $t \geq 0$, P_t are stochastic matrices of size N, with the property

$$\mathcal{P}\{\eta_{t+1} = j \mid \mathcal{G}_t\} = p_t(\eta_t, j) \tag{2}$$

for all $j \in \mathcal{D}$, $t \geq 0$, $t \in \mathbf{Z}_+$, where $\mathcal{G}_t = \sigma[\eta_0, \eta_1, ..., \eta_t]$ is the σ-algebra generated by the random variables $\eta_s, 0 \leq s \leq t$. For more details concerning the properties of Markov chains and of the sequences $\{P_t\}_{t\geq 0}$ of stochastic matrices we refer to [6].

If $P_t = P$ for all $t \geq 0$ then the Markov chain is known as a ***homogeneous Markov chain***.

In this paper we investigate several aspects of the issue of exponential stability in mean square of the solution $x = 0$ of the system (1). Our aim is to reveal some difficulties which are due to the fact that the coefficients of the system are time varying and η_t is an inhomogeneous Markov chain. In order to motivate the study of the problem of exponential stability in mean square for the case of stochastic systems of type (1) we consider the following controlled system

$$x(t+1) = A(t,\eta_t)x(t) + B(t,\eta_t)u(t)y(t) = C(t,\eta_t)x(t), \quad t \geq 0, \tag{3}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the vector of the control inputs and $y(t) \in \mathbf{R}^p$ is the vector of the measurements. The aim is to design a control law of the form

$$u(t) = K(t,\eta_t)y(t) \tag{4}$$

with the property that the trajectories of the corresponding closed-loop system satisfy

$$\lim_{t\to\infty} E[|x(t)|^2] = 0. \tag{5}$$

In the case when on the channel from the sensor to the controller there exists a delay in transmission of the measurement then, the control (4) is replaced by

$$u(t) = K(t,\eta_t)y(t-1). \tag{6}$$

If the delay occurs on the channel from controller to actuators then instead of (4) we will have

$$u(t) = K(t-1,\eta_{t-1})y(t-1). \tag{7}$$

We have to solve the following two problems:

$\mathbf{P_1}$: *Find a set of sufficient conditions which guarantee the existence of the matrices* $K(t,i) \in \mathbf{R}^{m\times p}, i \in \mathcal{D}, t \geq 0$, *such that the trajectories of the closed-loop system obtained by coupling (6) with (3) will satisfy a condition of type (5).*

$\mathbf{P_2}$: *Find a set of conditions which guarantee the existence of the matrices* $K(t,i) \in \mathbf{R}^{m\times p}$ *with the property that the trajectories of the closed-loop system* *obtained by coupling the control (7) with the system (3) satisfy a condition of type (5).*

Coupling a control (6) to the system (3) one obtains the following closed-loop system

$$x(t+1) = A(t,\eta_t)x(t) + B(t,\eta_t)K(t,\eta_t)C(t-1,\eta_{t-1})x(t-1). \tag{8}$$

The closed-loop system obtained combining (7) to (3) is

$$x(t+1) = A(t,\eta_t)x(t) + B(t,\eta_t)K(t-1,\eta_{t-1})C(t-1,\eta_{t-1})x(t-1). \tag{9}$$

Setting $\xi(t) = (x^T(t), x^T(t-1))^T$ we obtain the following version of (8) and (9)

$$\xi(t+1) = \tilde{A}(t,\eta_t,\eta_{t-1})\xi(t), t \geq 1, \tag{10}$$

$$\xi(t+1) = \hat{A}(t,\eta_t,\eta_{t-1})\xi(t), \quad t \geq 1, \tag{11}$$

respectively, where

$$\tilde{A}(t,i,j) = \begin{pmatrix} A(t,i) & B(t,i)K(t,i)C(t-1,j) \\ I_n & 0 \end{pmatrix},$$

$$\hat{A}(t,i,j) = \begin{pmatrix} A(t,i) & B(t,i)K(t-1,j)C(t-1,j) \\ I_n & 0 \end{pmatrix}. \tag{12}$$

Hence, the system (10) as well as (11) are both of the form (1).

The condition (5) is equivalent to

$$\lim_{t\to\infty} E[|\xi(t)|^2] = 0. \tag{13}$$

This motivates the consideration of the problem of stability in mean square of systems of type (1).

Set $\pi_t = (\pi_t(1), \pi_t(2), ..., \pi_t(N))$ the distribution of the random variable η_t, that is $\pi_t(i) = \mathcal{P}\{\eta_t = i\}$. It can be verified that the sequence $\{\pi_t\}_{t\geq 0}$ solves

$$\pi_{t+1} = \pi_t P_t, \quad t \geq 0. \tag{14}$$

Remark 2.1. a) From (14) it follows that it is possible that $\pi_t(i) = 0$ for some $t \geq 1$, $i \in \mathcal{D}$ even if $\pi_0(j) > 0$, $1 \leq j \leq N$. This is specific for the discrete-time case.

b) The only available information concerning the system (1) is the set $\mathcal{D}$ and the sequences $\{P_t\}_{t \in \mathbf{Z}_+}$, $\{A(t,i,j)\}_{t \geq 1}, i, j \in \mathcal{D}$. The initial distributions of the Markov chain are not prefixed. Hence, throughout the paper by a Markov chain we will understand any triple $\{\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D}\}$ where $\mathcal{D}$ is a fixed set $\mathcal{D} = \{1, 2, ..., N\}$, $\{P_t\}_{t \geq 0}$ is a given sequence of stochastic matrices and $\{\eta_t\}_{t \geq 0}$ is an arbitrary sequence of random variables taking values in $\mathcal{D}$ and satisfying (2).

Remark 2.1 a) allows us to define the following subsets of the set $\mathcal{D}$

$$\mathcal{D}_s = \{i \in \mathcal{D} | \pi_s(i) > 0\} \tag{15}$$

for each integer $s \geq 0$.

Define $\Theta(t,s)$ as follows: $\Theta(t,s) = A(t-1)A(t-2)...A(s)$ if $t \geq s+1$ and $\Theta(t,s) = I_n$ if $t = s, s \geq 1$.

Any solution $x(t)$ of (1) verifies

$$x(t) = \Theta(t,s)x(s)$$

$\Theta(t,s)$ will be called the fundamental (random) matrix solution of (1).

2.2. LYAPUNOV TYPE OPERATORS

Let $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$ be the linear subspace of symmetric matrices. Set $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus ... \oplus \mathcal{S}_n$. One can see that $\mathcal{S}_n^N$ is a real ordered Hilbert space (see [7], Example 2.5(iii)). The usual inner product on $\mathcal{S}_n^N$ is

$$\langle X, Y \rangle = \sum_{i=1}^{N} Tr(Y(i)X(i)) \tag{16}$$

for all $X = (X(1), ..., X(N))$ and $Y = (Y(1), ..., Y(N))$ from $\mathcal{S}_n^N$.

Consider the sequences $\{A(t,i,j)\}_{t\geq 1}, A(t,i,j) \in \mathbf{R}^{n\times n}$, $i,j \in \mathcal{D}$, $\{P_t\}_{t\geq 0}, P_t = (p_t(i,j)) \in \mathbf{R}^{N\times N}$.

Based on these sequences we construct the linear operators $\Upsilon_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\Upsilon_t S = (\Upsilon_t S(1), ..., \Upsilon_t S(N))$ with

$$\Upsilon_t S(i) = \sum_{j=1}^{N} p_{t-1}(j,i) A(t,i,j) S(j) A^T(t,i,j), \tag{17}$$

$t \geq 1, S \in \mathcal{S}_n^N$.

If $A(t,i,j), p_t(i,j)$ are related to the system (1), then the operators Υ_t are called the Lyapunov type operators associated with the system (1).

By direct calculation based on the definition of the adjoint operator with respect to the inner product (16) one obtains that $\Upsilon_t^* S = (\Upsilon_t^* S(1), ..., \Upsilon_t^* S(N))$ with

$$\Upsilon_t^* S(i) = \sum_{j=1}^{N} p_{t-1}(i,j) A^T(t,j,i) S(j) A(t,j,i) \tag{18}$$

$t \geq 1, S \in \mathcal{S}_n^N$.

Let $R(t,s)$ be the linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\Upsilon_t\}_{t\geq 1}$. Hence $R(t,s) = \Upsilon_{t-1}\Upsilon_{t-2}...\Upsilon_s$ if $t \geq s+1$ and $R(t,s) = \mathcal{I}_{\mathcal{S}_n^N}$ if $t = s \geq 1$.

If $X(t)$ is a solution of discrete-time linear equation on $\mathcal{S}_n^N$

$$X_{t+1} = \Upsilon_t X_t, \tag{19}$$

then $X_t = R(t,s)X_s$ for all $t \geq s \geq 1$.

The following result provides a relationship between the operators $R^*(t,s)$ and the fundamental matrix solution $\Theta(t,s)$ of the system (1).

Theorem 2.2. *Under the considered assumptions the following equality*

$$(R^*(t,s)H)(i) = E[\Theta^T(t,s) H(\eta_{t-1}) \Theta(t,s) | \eta_{s-1} = i] \tag{20}$$

holds for all $H = (H(1), ..., H(N)) \in \mathcal{S}_n^N, t \geq s \geq 1, i \in \mathcal{D}_{s-1}$.

Proof: see [8, 11].

2.3. DEFINITIONS OF MEAN SQUARE EXPONENTIAL STABILITY

Now we are in position to state the concept of exponential stability in mean square of the zero state equilibrium of the system (1).

In this subsection we introduce three different definitions of the concept of exponential stability in mean square.

Definition 2.3. We say that the zero state equilibrium of the system (1) is *strongly exponentially stable in mean square* (SESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that

$$\|R(t,s)\| \leq \beta q^{t-s} \tag{21}$$

for all $t \geq s \geq 1$.

Since $\mathcal{S}_n^N$ is a finite dimensional linear space, in (21) one can take any norm in $\mathcal{B}(\mathcal{S}_n^N)$.

Definition 2.4. We say that the zero state equilibrium of the system (1) is *exponentially stable in mean square with conditioning* (ESMS-C) if there exist $\beta \geq 1, q \in (0,1)$ such that for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ we have

$$E[|\Theta(t,s)x|^2 | \eta_{s-1} = i] \leq \beta q^{t-s}|x|^2 \tag{22}$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n, i \in \mathcal{D}_{s-1}$.

Definition 2.5. We say that the zero state equilibrium of the system (1) is *exponentially stable in mean square* (ESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ we have

$$E[|\Theta(t,s)x|^2] \leq \beta q^{t-s}|x|^2 \tag{23}$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n$.

3. EXPONENTIAL STABILITY IN MEAN SQUARE

In this section we establish the relations between the concepts of exponential stability in mean square introduced in Definition 2.3 -Definition 2.5. Firstly we shall show that in the general case of the time-varying system (1) these definitions are not, in general, equivalent. Finally we show that in the case of systems (1) with periodic coefficients these definitions become equivalent.

3.1. THE GENERAL CASE

On the space $\mathcal{S}_n^N$ one introduces the norm $|\cdot|_1$ by

$$|X|_1 = max_{i\in\mathcal{D}}|X(i)| \tag{24}$$

where $|X(i)|$ is the spectral norm of the matrix $X(i)$. Together with the norm $|\cdot|_2$ induced by the inner product (16), the norm $|\cdot|_1$ will play an important role in the characterization of the strong exponential stability (SESMS).

If $T : \mathcal{S}_n^N \to \mathcal{S}_n^N$ is a linear operator then $\|T\|_k$ is the operator norm induced by $|\cdot|_k, k = 1, 2$.

We recall that (see Proposition 2.2 in [7]) if T is a linear and positive operator on $\mathcal{S}_n^N$, then

$$\|T\|_1 = |TJ|_1 \tag{25}$$

where $J = (I_n, ..., I_n) \in \mathcal{S}_n^N$.

Since $\|R(t,s)\|$ and $\|R^*(t,s)\|_1$ are equivalent, from (21) it follows that $\|R^*(t+1,t)\|_1 \le \beta_1, t \ge 0$, hence by using (25) and (18) one obtains

Corollary 3.1. *If the zero state equilibrium of the system (1) is SESMS, then*

$\{\sqrt{p_{t-1}(i,j)}A(t,j,i)\}_{t\ge 1}, i, j \in \mathcal{D}$, *are bounded sequences.*

Now we prove

Theorem 3.2. *Under the considered assumptions we have:*

(i) *if the zero state equilibrium of the system (1) is SESMS then it is ESMS-C;*

(ii) *if the zero state equilibrium of the system (1) is ESMS-C then it is ESMS.*

The proof of (i) follows immediately from (20). (ii) follows from the inequality

$$E[|\Theta(t,s)x|^2] \leq \sum_{i \in \mathcal{D}_{s-1}} E[|\Theta(t,s)x|^2 | \eta_{s-1} = i].$$

As we can see in Example 3.5 and Example 3.6 from below the validity of the converse implications from the above theorem is not true in the absence of some additional assumptions.

Definition 3.3. We say that a stochastic matrix $P_t \in \mathbf{R}^{N \times N}$ is a *nondegenerate stochastic matrix* if for any $j \in \mathcal{D}$ there exists $i \in \mathcal{D}$ such that $p_t(i,j) > 0$.

We remark that if $P_t, t \geq 0$ are nondegenerate stochastic matrices, then from (14) it follows that $\pi_t(i) > 0, t \geq 1, i \in \mathcal{D}$ if $\pi_0(i) > 0$ for all $i \in \mathcal{D}$.

Now we have

Theorem 3.4. *If for all $t \geq 0$, P_t are nondegenerate stochastic matrices, then the following are equivalent*

(i) *the zero state equilibrium of the system (1) is SESMS,*

(ii) *the zero state equilibrium of the system (1) is ESMS-C,*

(iii) *there exist a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$ with $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$ such that*

$$E[|\Theta(t,s)x|^2 | \eta_{s-1} = i] \leq \beta q^{t-s} |x|^2 \tag{26}$$

for all $t \geq s \geq 1, i \in \mathcal{D}, x \in \mathbf{R}^n$ where $\beta \geq 1, q \in (0,1)$.

Proof (i) $\to$ (ii) follows from Theorem 3.2 and (ii) $\to$ (iii) is obvious. It remains to prove implication (iii) $\to$ (i). Applying Theorem 2.2 for $H = J =$

$(I_n, ..., I_n)$ from (26) one obtains that $x_0^T[R^*(t,s)J](i)x_0 \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s \geq 1,\ i \in \mathcal{D},\ x_0 \in \mathbf{R}^n$.

This allows us to conclude that $|[R^*(t,s)J](i)| \leq \beta q^{t-s}$ for all $t \geq s \geq 1,\ i \in \mathcal{D}$. Based on (24) one gets $|R^*(t,s)J|_1 \leq \beta q^{t-s}$ for all $t \geq s$. Finally from (25) it follows that $\|R^*(t,s)\|_1 \leq \beta q^(t-s)$ for all $t \geq s \geq 1$. Thus the proof is complete, since $\|R(t,s)\|$ and $\|R(t,s)^*\|_1$ are equivalent.

The next two examples show that the converse implication in Theorem 3.2 are not always true.

Example 3.5. Consider the system (1) in the particular case $n = 1, N = 2$ described by

$$x(t+1) = a(t, \eta_t, \eta_{t-1})x(t) \tag{27}$$

where $t \geq 1, a(t,i,1) = 0, a(t,i,2) = 2^{\frac{t-1}{2}}, t \geq 1, i \in \{1,2\}$. The transition probability matrix is

$$P_t = \begin{pmatrix} 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \\ 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \end{pmatrix}, t \geq 0. \tag{28}$$

We have

$$\sqrt{p_{t-1}(2,1)}a(t,1,2) = (1 - \frac{1}{4^t})^{\frac{1}{2}} 2^{\frac{t-1}{2}}.$$

Hence the sequence $\{\sqrt{p_{t-1}(2,1)}a(t,1,2)\}_{t\geq 1}$ is unbounded. Then via Corollary 3.1, we deduce that the zero state equilibrium of the system (27) is not SESMS. On the other hand one sees that the transition probability matrix (28) is a non-degenerate stochastic matrix. Thus, via Theorem 3.4, we deduce that the zero state equilibrium of (27) cannot be ESMS-C.

We show now that the zero state equilibrium of (27) is ESMS. We write

$$E[|\Theta(t,s)x|^2] = x^2 E[a^2(t-1,\eta_{t-1},\eta_{t-2})a^2(t-2,\eta_{t-2},\eta_{t-3})...a^2(s,\eta_s,\eta_{s-1})] =$$

$$x^2 \sum_{i_{t-1}=1}^{2} \sum_{i_{t-2}=1}^{2} ... \sum_{i_{s-1}=1}^{2} a^2(t-1, i_{t-1}, i_{t-2})a^2(t-2, i_{t-2}, i_{t-3})...a^2(s, i_s, i_{s-1})$$

$$\mathcal{P}\{\eta_{t-1} = i_{t-1}, \eta_{t-2} = i_{t-2}, ..., \eta_{s-1} = i_{s-1}\} =$$

$$x^2\mathcal{P}(\eta_{s-1} = 2)a^2(s,2,2)...a^2(t-2,2,2)p_{s-1}(2,2)p_s(2,2)...p_{t-3}(2,2)$$

$$[a^2(t-1,1,2)p_{t-2}(2,1) + a^2(t-1,2,2)p_{t-2}(2,2)] \leq$$

$$x^2 2^{s-1+s+...+t-3}\frac{1}{4^{s+s+1+...t-2}}2^{t-2} = x^2 2^{\frac{(t+s-3)(t-s)}{2}}\frac{1}{2^{(t+s-2)(t-s-1)}}$$

if $t \geq s+2$. Finally one obtains that

$$E[|\Theta(t,s)x|^2] \leq x^2\frac{1}{\sqrt{2}^{t^2-s^2-3t-s+4}} \tag{29}$$

if $t \geq s+2$.

For $t \geq s+2$ we have $t^2 - 4t - s^2 + 4 = (t-2)^2 - s^2 \geq 0$. This means that $t^2 - s^2 - 3t - s + 4 \geq t - s$. From (29) one obtains that

$$E[|\Theta(t,s)x|^2] \leq x^2\frac{1}{\sqrt{2}^{t-s}} \tag{30}$$

for all $t \geq s+2, s \geq 1$. Further we compute

$$E[|\Theta(s+1,s)x|^2] = x^2E[a^2(s,\eta_s,\eta_{s-1})] = x^2a^2(s,1,2)\mathcal{P}\{\eta_{s-1} = 2, \eta_s = 1\}+$$

$$a^2(s,2,2)\mathcal{P}\{\eta_{s-1} = 2, \eta_s = 2\} \leq x^2 2^{s-1}\mathcal{P}\{\eta_{s-1} = 2\}[p_{s-1}(2,1) + p_{s-1}(2,2)].$$

Thus we get

$$E[|\Theta(s+1,s)x|^2] \leq x^2 2^{s-1}\mathcal{P}\{\eta_{s-1} = 2\}, \quad (\forall)s \geq 1. \tag{31}$$

Take $s \geq 2$ and write

$$E[|\Theta(s+1,s)x|^2] \leq x^2 2^{s-1}[\mathcal{P}\{\eta_{s-2} = 1\}p_{s-2}(1,2)+\mathcal{P}\{\eta_{s-2} = 2\}p_{s-2}(2,2)] \leq x^2 2^s\frac{1}{4^{s-1}}$$

which leads to

$$E[|\Theta(s+1,s)x|^2] \leq x^2, \quad \forall s \geq 2. \tag{32}$$

Further

$$E[|\Theta(2,1)x|^2] = E[a^2(1,\eta_1,\eta_0)]x^2 = x^2[\mathcal{P}\{\eta_0 = 2, \eta_1 = 1\}+\mathcal{P}\{\eta_0 = 2, \eta_1 = 2\}] =$$

$= x^2\mathcal{P}\{\eta_0 = 2\}$ *i.e.*

$$E[|\Theta(2,1)x|^2] \leq x^2. \tag{33}$$

Finally

$$E[|\Theta(s,s)x|^2] = x^2. \tag{34}$$

Combining (30),(32),(33),(34) we conclude that

$$E[|\Theta(t,s)x|^2] \leq \sqrt{2}\frac{1}{\sqrt{2}^{t-s}}x^2$$

for all $t \geq s \geq 1, x \in \mathbf{R}$ and thus one obtains that the zero state equilibrium of (27) is ESMS.

Example 3.6. Consider the system (1) in the particular case $N = 2, A(t,i,1) = \mathbf{O} \in \mathbf{R}^{n\times n}, A(t,i,2) = itI_n, i \in \{1,2\}, t \geq 1$. The transition probability matrix is $P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. We have

$$\sqrt{p(2,1)}A(t,1,2) = tI_n$$

thus via Corollary 3.1 one obtains that the zero state equilibrium of the considered system cannot be SESMS.

On the other hand we have $\mathcal{P}\{\eta_t = 2\} = 0$ a.s. for all $t \geq 1$. This leads to $\eta_t = 1 a.s., t \geq 1$. Hence $\Theta(t,s) = 0 a.s.$ if $t \geq \max\{3, s+1\}, s \geq 1$ for any Markov chain $(\{\eta_t\}_{t\geq 0}, P, \{1,2\})$. This shows that in this particular case the zero state equilibrium of the considered system is both ESMS-C, as well as ESMS.

3.2. THE PERIODIC CASE

In this subsection we show that under the periodicity assumption the concepts of exponential stability introduced by Definitions 2.3, 2.4, 2.5 become equivalent.

Firstly we introduce

Definition 3.7. We say that the zero state equilibrium of the system (1) is *asymptotically stable in mean square* ASMS if for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ we have

$$\lim_{t\to\infty} E[|\Theta(t,1)x|^2] = 0, \forall x \in \mathbf{R}^n.$$

Now we are in position to state

Theorem 3.8. *Assume that there exists an integer $\theta \geq 1$ such that $A(t+\theta, i, j) = A(t,i,j), i, j \in \mathcal{D}, P_{t+\theta} = P_t, t \geq 0$. Under these conditions the following are equivalent:*

(i) the zero state equilibrium of the system (1) is (SESMS),

(ii) the zero state equilibrium of the system (1) is (ESMS-C),

(iii) the zero state equilibrium of the system (1) is (ESMS),

(iv) the zero state equilibrium of the system (1) is (ASMS),

(v) there exists a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ with $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$ such that

$$\lim_{t\to\infty} E[|\Theta(\theta t,1)x|^2] = 0 \tag{35}$$

for all $x \in \mathbf{R}^n$,

(vi) $\rho[R(\theta+1,1)] < 1$ where $\rho[\cdot]$ is the spectral radius.

Proof. is similar to the proof of Theorem 3.7 in [11]. It is omitted for shortness.

Definition 3.9. We say that the system (1) is in the time invariant case if $A(t,i,j) = A(i,j)$ for all $t \geq 1$, $i, j \in \mathcal{D}$ and $P_t = P$ for all $t \geq 0$.

In this case we have $\Upsilon_t = \Upsilon$, for all $t \geq 1$. One sees that the system (1) is in the time invariant case if and only if it is periodic with period $\theta = 1$. Hence, the equivalences from the above theorem hold in the time invariant case too. In this case, the statement (vi) becomes $\rho(\Upsilon) < 1$.

4. LYAPUNOV TYPE CRITERIA

In this section we present several conditions for exponential stability in mean square expressed in terms of solvability of some suitable systems of linear matrix equations or linear matrix inequations. The results of this section are special cases of those stated in a more general framework in [7]. That is why we present them here without proofs.

4.1. THE GENERAL CASE

In view of Theorem 3.2 it follows that the Lyapunov type criteria are necessary and sufficient conditions for SESMS but they are only sufficient conditions for ESMS. Direct from the above Definition 2.3 and Theorem 3.4 in [7] applied to Lyapunov type operators Υ_t we obtain

Theorem 4.1 *Under the considered assumptions the following are equivalent:*

(i) *the zero state equilibrium of the system (1) is SESMS,*

(ii) *the system of backward linear equations*

$$X_t(i) = \sum_{j=1}^{N} p_{t-1}(i,j)A^T(t,j,i)X_{t+1}(j)A(t,j,i) + I_n \tag{36}$$

$t \geq 1$, $i \in \mathcal{D}$ *has a bounded solution* $X_t = (X_t(1), ..., X_t(N))$ *with* $X_t(i) \geq I_n, t \geq 1$,

(iii) *there exists a bounded sequence* $\{Y_t\}_{t\geq 1} \in \mathcal{S}_n^N$ *and scalars* $\alpha > 0$, $\delta > 0$ *such that*

$$\sum_{j=1}^{N} p_{t-1}(i,j)A^T(t,j,i)Y_{t+1}(j)A(t,j,i) - Y_t(i) \leq -\alpha I_n \tag{37}$$

$Y_t(i) \geq \delta I_n, t \geq 1$, $i \in \mathcal{D}$.

Remark 4.2. Even if the system (36), ((37)respectively) consists of an infinite number of equations (inequations respectively) the criteria derived in

Theorem 4.1 may be useful to obtain sufficient conditions for ESMS in the general time varying case. This can be illustrated by the next simple example.

Example 4.3 Consider the system (1) in the special case $n = 1$

$$x(t+1) = a(t, \eta_t, \eta_{t-1})x(t) \tag{38}$$

$t \geq 1$, where $a(t, i, j) \in \mathbf{R}$ are such that

$$\sup_{t \geq 1} \sum_{j=1}^{N} p_{t-1}(i, j)a^2(t, j, i) < 1. \tag{39}$$

Under this condition the zero state equilibrium of (38) is SESMS.

Indeed, if (39) holds, then the corresponding system (37) associated with (38) is fulfilled for $Y_t(i) = 1, t \geq 1, i \in \mathcal{D}$.

4.2. THE PERIODIC CASE

From Theorem 3.5 in [7] one obtains that if the coefficients of (36) are periodic with period θ, the unique bounded and positive solution of (36) is periodic with the same period θ. Also, if the system of inequalities (37) has a bounded and uniform positive solution then it has a periodic solution too. This allows us to obtain the following specialized version of Theorem 4.1.

Theorem 4.4 *Under the assumptions of Theorem 3.8, with $\theta \geq 2$, the following are equivalent:*

(i) the zero state equilibrium of (1) is ESMS,

(ii) the system of linear matrix equations

$$X_t(i) = \sum_{j=1}^{N} p_{t-1}(i, j)A^T(t, j, i)X_{t+1}(j)A(t, j, i) + I_n \qquad 1 \leq t \leq \theta - 1$$

$$X_\theta(i) = \sum_{j=1}^{N} p_{\theta-1}(i, j)A^T(\theta, j, i)X_1(j)A(\theta, j, i) + I_n \tag{40}$$

$i \in \mathcal{D}$ has a solution $X_t = (X_t(1), ..., X_t(N))$ with $X_t(i) > 0$.

(iii) there exist positive definite matrices $Y_t(i), 1 \le t \le \theta, i \in \mathcal{D}$*, which solve the following system of LMI's*

$$\sum_{j=1}^{N} p_{t-1}(i,j)A^T(t,j,i)Y_{t+1}(j)A(t,j,i) - Y_t(i) < 0, \qquad 1 \le t \le \theta - 1$$

$$\sum_{j=1}^{N} p_{\theta-1}(i,j)A^T(\theta,j,i)Y_1(j)A(\theta,j,i) - Y_\theta(i) < 0 \quad i \in \mathcal{D}. \tag{41}$$

It is easy to see that under the conditions of Theorem 3.8 the sequence Υ_t *can be extended in a natural way by periodicity, to the whole set of integers* $\mathbf{Z}$.

In this case we may use Theorem 3.7 (ii) and Theorem 3.9 from [7] to obtain

Theorem 4.5. *Under the assumptions of Theorem 3.8 with* $\theta \ge 2$*, the following are equivalent:*

(i) the zero state equilibrium of the system (1) is ESMS,

(ii) the system of linear matrix equations

$$X_{t+1}(i) = \sum_{j=1}^{N} p_{t-1}(j,i)A(t,i,j)X_t(j)A^T(t,i,j) + I_n \qquad 1 \le t \le \theta - 1$$

$$X_1(i) = \sum_{j=1}^{N} p_{\theta-1}(j,i)A(\theta,i,j)X_\theta(j)A^T(\theta,i,j) + I_n \tag{42}$$

$i \in \mathcal{D}$ *has a solution* $X_t = (X_t(1), ..., X_t(N))$ *such that* $X_t(i) > 0, i \in \mathcal{D}$,

(iii) there exist positive definite matrices $Y_t(i), 1 \le t \le \theta, i \in \mathcal{D}$*, which solve the following system of LMI's:*

$$\sum_{j=1}^{N} p_{t-1}(j,i)A(t,i,j)Y_t(j)A^T(t,i,j) - Y_{t+1}(i) < 0 \qquad 1 \le t \le \theta - 1$$

$$\sum_{j=1}^{N} p_{\theta-1}(j,i)A(\theta,i,j)Y_\theta(j)A^T(\theta,i,j) - Y_1(i) < 0 \tag{43}$$

$i \in \mathcal{D}$.

Remark 4.6. The system of linear equations (40), (42) and the system of linear inequations (41), (43) have $\hat{n}$ scalar equations (inequations respectively), with $\hat{n}$ scalar unknowns, where $\hat{n} = \frac{n(n+1)}{2} N\theta$.

4.3. THE TIME INVARIANT CASE

Using Theorem 3.5 (*iii*), Theorem 3.7 (*iii*) and Theorem 3.9 in [7] one obtains the following Lyapunov type criteria for exponential stability in mean square for the system (1) in the time invariant case.

Corollary 4.7 *If the system is in the time invariant case, the following are equivalent:*

(i) *the zero state equilibrium of the system (1) is ESMS,*

(ii) *the system of linear equations*

$$X(i) = \sum_{j=1}^{N} p(i,j) A^T(j,i) X(j) A(j,i) + I_n \tag{44}$$

$i \in \mathcal{D}$, *has a solution* $X = (X(1), ..., X(N))$ *with* $X(i) > 0, i \in \mathcal{D}$,

(iii) *there exist positive definite matrices* $Y(i), i \in \mathcal{D}$, *which solve the following system of LMI's*:

$$\sum_{j=1}^{N} p(i,j) A^T(j,i) Y(j) A(j,i) - Y(i) < 0 \tag{45}$$

$i \in \mathcal{D}$.

Corollary 4.8. *Under the conditions of Corollary 4.7 the following are equivalent:*

(i) *the zero state equilibrium of the system (1) is ESMS,*

(ii) *the system of linear matrix equations*

$$X(i) = \sum_{j=1}^{N} p(j,i) A(i,j) X(j) A^T(i,j) + I_n \tag{46}$$

$i \in \mathcal{D}$ *has a solution* $X = (X(1), ..., X(N))$ *with* $X(i) > 0, i \in \mathcal{D}$, *(iii) there exist positive definite matrices $Y(i), i \in \mathcal{D}$, which solve the following system of LMI's*

$$\sum_{j=1}^{N} p(j,i)A(i,j)Y(j)A^T(i,j) - Y(i) < 0 \tag{47}$$

$i \in \mathcal{D}$.

Remark 4.9. The system of linear equations (44) and (46) and the system of linear inequations (45) and (47) have $\check{n}$ scalar equations (inequations respectively) with $\check{n}$ scalar unknowns, where $\check{n} = \frac{n(n+1)}{2}N$.

5. APPLICATIONS

In this section we illustrate the applicability of the results concerning the exponential stability in mean square for the systems of type (1) derived in the previous sections to the control problem of the designing of a stabilizing static output feedback in the presence of some delays in the transmission of the date. We restrict our attention to the time-invariant case of the controlled system (3) and of the control law (4). More specific we consider the controlled system

$$x(t+1) = A(t,\eta_t)x(t) + B(t,\eta_t)u(t)y(t) = C(t,\eta_t)x(t), \quad t \geq 0, \tag{48}$$

and the control law

$$u(t) = K(\eta_t)y(t). \tag{49}$$

Based on Theorem 3.8 (for $\theta = 1$) one obtains that (13) is equivalent to exponential stability in mean square. Therefore to find some conditions which guarantee the existence of the feedbacks gains $K(i)$ with the desired property, we may apply the Lyapunov type criteria derived in Section 4.

The main tools in the derivation of the results in this section are the following well-known lemmas.

Lemma 5.1 *(The projection lemma) [23]. Let $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n\times n}, \mathcal{U} \in \mathbf{R}^{m\times n}, \mathcal{V} \in \mathbf{R}^{p\times n}$ be given matrices, $n \geq \max\{m,p\}$. Let $\mathcal{U}^\perp, \mathcal{V}^\perp$ be full* *column rank matrices such that* $\mathcal{U}\mathcal{U}^{\perp} = 0$ *and* $\mathcal{V}\mathcal{V}^{\perp} = 0$. *Then the following are equivalent:*

(i) *the linear matrix inequations*

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0$$

with the unknown matrix $K \in \mathbf{R}^{m \times p}$ *is solvable*;

(ii) $(\mathcal{U}^{\perp})^T \mathcal{Z} \mathcal{U}^{\perp} < 0$;

$(\mathcal{V}^{\perp})^T \mathcal{Z} \mathcal{V}^{\perp|} < 0$.

Lemma 5.2. *(Finsler's lemma)[23]. Let* $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n \times n}, \mathcal{C} \in \mathbf{R}^{p \times n}, n > p$ *be given. Take* $\mathcal{C}^{\perp}$ *a full column rank matrix such that* $\mathcal{C}\mathcal{C}^{\perp} = 0$. *Then the following are equivalent:*

(i) *there exists a scalar* μ *such that* $\mathcal{Z} + \mu \mathcal{C}^T \mathcal{C} < 0$;

(ii) $(\mathcal{C}^{\perp})^T \mathcal{Z} \mathcal{C}^{\perp} < 0$.

Combining the above two lemmas one obtains

Corollary 5.3. *With the previous notations the following statements are equivalent:*

(i) *the linear inequation*

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0 \tag{50}$$

with the unknown $K \in \mathbf{R}^{m \times p}$ *is solvable.*

(ii) *there exist the scalars* μ_1, μ_2 *such that*

$$\mathcal{Z} + \mu_1 \mathcal{U}^T \mathcal{U} < 0$$

$$\mathcal{Z} + \mu_2 \mathcal{V}^T \mathcal{V} < 0.$$

In [23] a parametrization of the whole class of solutions of (50) is given.

Before to state the main results of this section we remark that in (14) written for the special case of (48) and (49) we have the following decomposition

$$\tilde{A}(i,j) = \mathcal{A}_0(i) + \mathcal{B}_0(i) K(i) \mathcal{C}_0(j), \hat{A}(i,j) = \mathcal{A}_0(i) + \mathcal{B}_0(i) K(j) \mathcal{C}_0(j), \tag{51}$$

where $\mathcal{A}(i) = \begin{pmatrix} A_0(i) & 0 \\ I_n & 0 \end{pmatrix} \in \mathbf{R}^{2n\times 2n}, \mathcal{B}_0(i) = \begin{pmatrix} B(i) \\ 0 \end{pmatrix} \in \mathbf{R}^{2n\times m}, \mathcal{C}_0(j) = (0 \quad C(j)) \in \mathbf{R}^{p\times 2n}$.

Now we have

Theorem 5.4. *Assume that there exist the symmetric matrices* $Y(i) \in \mathbf{R}^{2n\times 2n}$ *and the scalars* $\mu_1(i)$ *and* $\mu_2(i), i \in \mathcal{D}$, *satisfying the following systems of LMI's*

$$\begin{pmatrix} -Y(i) & \Psi_{2i}(Y) & 0 \\ \Psi_{2i}^T(Y) & \Psi_{3i}(Y) & \Psi_{4i}(Y) \\ 0 & \Psi_{4i}^T(Y) & -\mu_1(i)I_p \end{pmatrix} < 0, \tag{52}$$

$$\sum_{j=1}^{N} p(j,i)\mathcal{A}_0(i)Y(j)\mathcal{A}_0^T(i) - Y(i) + \mu_2(i)\mathcal{B}_0(i)\mathcal{B}_0^T(i) < 0, \tag{53}$$

where

$$\Psi_{2i}(Y) = (\sqrt{p(1,i)}\mathcal{A}_0(i)Y(1), ..., \sqrt{p(N,i)}\mathcal{A}_0(i)Y(N)),$$

$$\Psi_{3i}(Y) = -diag(Y(1), ..., Y(N)),$$

$$\Psi_{4i}^T(Y) = (\sqrt{p(1,i)}\mathcal{C}_0(1)Y(1), ..., \sqrt{p(N,i)}\mathcal{C}_0(N)Y(N)).$$

Under these conditions there exist stabilizing feedback gains $K(i) \in \mathbf{R}^{m\times p}, i \in \mathcal{D}$ *such that the zero state equilibrium of the time-invariant version of the system (10) is ESMS.*

Moreover the matrices $K(i)$ *may be obtained as solutions of the following uncoupled LMI's*

$$\tilde{\mathcal{Z}}(i) + \tilde{\mathcal{U}}^T(i)K(i)\tilde{\mathcal{V}}(i) + \tilde{\mathcal{V}}^T(i)K^T(i)\mathcal{U}(i) < 0, \tag{54}$$

$i \in \mathcal{D}$, *where*

$$\tilde{\mathcal{Z}}(i) = \begin{pmatrix} -\tilde{Y}(i) & \Psi_{2i}(\tilde{Y}) \\ \Psi_{2i}^T(\tilde{Y}) & \Psi_{3i}(\tilde{Y}) \end{pmatrix},$$

$$\tilde{\mathcal{U}}(i) = (\mathcal{B}_0^T(i), 0, ..., 0) \in \mathbf{R}^{m\times\tilde{n}},$$

$$\tilde{\mathcal{V}}(i) = (0, \sqrt{p(1,i)}\mathcal{C}_0(1)\tilde{Y}(1), ..., \sqrt{p(N,i)}\mathcal{C}_0(N)\tilde{Y}(N)) \in \mathbf{R}^{p\times\tilde{n}}, \tag{55}$$

where $\tilde{n} = 2n(N+1), \tilde{Y} = (\tilde{Y}(1), ..., \tilde{Y}(N))$ *being a solution of (52), (53).*

Proof is similar to the one of Theorem 6.4 in [11]. It is omitted for shortness.

Let us consider the time-invariant case of control law (7).

In this case we have

Theorem 5.5. *Assume that there exist matrices* $Y(i) \in \mathbf{R}^{2n\times 2n}$ *and the scalars* $\mu_1(i), \quad \mu_2(i)$ *which satisfy the following system of LMI's*

$$\sum_{j=1}^{N} p(i,j)\mathcal{A}_0^T(j)Y(j)\mathcal{A}_0(j) - Y(i) + \mu_1(i)\mathcal{C}_0^T(i)\mathcal{C}_0(i) < 0, \tag{56}$$

$$\begin{pmatrix} -Y(i) & \hat{\Gamma}_{2i}(Y) & 0 \\ (\hat{\Gamma}_{2i})^T(Y) & \Gamma_{3i}(Y) & \Gamma_{4i}(Y) \\ 0 & \Gamma_{41}^T(Y) & -\mu_2(i)I_m \end{pmatrix} < 0, \tag{57}$$

where

$$\hat{\Gamma}_{2i}(Y) = (\sqrt{p(i,1)}\mathcal{A}_0^T(1)Y(1), ..., \sqrt{p(i,N)}\mathcal{A}_0^T(N)Y(N)),$$

$$\Gamma_{3i}(Y) = \Psi_{3i}(Y),$$

$$\Gamma_{4i}^T(Y) = (\sqrt{p(i,1)}\mathcal{B}_0^T(1)Y(1), ..., \sqrt{p(i,N)}\mathcal{B}_0(N)Y(N)).$$

Under these conditions there exist feedback gains $K(i) \in \mathbf{R}^{m\times p}$ *such that the zero state equilibrium of the corresponding time invariant version of the system (11) is ESMS.*

Moreover, if $(\hat{Y}(i), \hat{\mu}_1(i), \hat{\mu}_2(i), i \in \mathcal{D})$ *is a solution of (56), (57) then for each* $i, K(i)$ *is obtained as a solution of the following LMI*

$$\hat{\mathcal{Z}}(i) + \hat{\mathcal{U}}^T(i)K(i)\hat{\mathcal{V}}(i) + \hat{\mathcal{V}}^T(i)K^T(i)\hat{\mathcal{U}}(i) < 0, \tag{58}$$

where

$$\hat{\mathcal{Z}}(i) = \begin{pmatrix} -\hat{Y}(i) & \hat{\Gamma}_{2i}(\hat{Y}) \\ \hat{\Gamma}_{2i}^T(\hat{Y}) & \Gamma_{3i}(\hat{Y}) \end{pmatrix}, \hat{\mathcal{U}}(i) = (\mathcal{C}_0(i), 0, ..., 0) \in \mathbf{R}^{p\times\tilde{n}}$$

$$\hat{\mathcal{V}}(i) = (0, \sqrt{p(i,1)}\mathcal{B}_0^T(1)\hat{Y}(1), ..., \sqrt{p(i,N)}\mathcal{B}_0^T(N)\hat{Y}(N)) \in \mathbf{R}^{m\times\tilde{n}}. \quad (59)$$

Proof: see the proof of Theorem 6.4 in [11].

Remark 5.6. Remark that in order to obtain uncoupled LMIs (54) and (58) respectively, we used Corollary 4.8, in the first case and Corollary 4.7 in the second case. We feel that this is a good motivation to deduce stability criteria based on Lyapunov type operators Υ_t (see Theorem 4.5 and Corollary 4.8) as well as based on its adjoint operators Υ_t^* (see Theorem 4.4 and Corollary 4.7).

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SADDLE-NODE BIFURCATION IN A COMPETING SPECIES MODEL

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Abstract The nature of the saddle-node singularities in a competing species model in the case when four of the six parameters are kept fixed is shown. Some singularities are degenerated, some others not. The local phase portraits around these points are then represented.

1. THE MATHEMATICAL MODEL

One of the models describing the dynamics of two species in competition is the Cauchy problem $x(0) = x_0,\ y(0) = y_0$, for the following system of ordinary differential equations (s.o.d.e.) [4]

$$\begin{cases} \dot{x} &= r_1 x(1 - x/K_1 - p_{12} y/K_1), \\ \dot{y} &= r_2 y(1 - y/K_2 - p_{21} x/K_2), \end{cases} \tag{1}$$

where x, y represent the number of individuals of the two species, r_1, r_2 - the growth rates of these species, K_1, K_2 -the carrying capacity of every species, $p_{12} > 0$ - the action of the second population and $p_{21} > 0$ - the action of the first population. In this study we consider r_1, r_2, K_1 and K_2 as fixed, such that in (1) only two parameters occur: p_{12} and p_{21}.

In this paper we are interested only in the saddle-node singularities.

In [3] it is shown that the saddle-node singularities exist at the points $E_1(K_1, 0)$, $E_2(0, K_2)$ and $E_3(\alpha, K_2(K_1 - \alpha)/K_1)$ for $p_{12} = K_1/K_2$, $p_{21} = K_2/K_1$, at $E_1(K_1, 0)$ for $p_{12} \neq K_1/K_2$, $p_{21} = K_2/K_1$ and at $E_2(0, K_2)$ for $p_{12} = K_1/K_2$, $p_{21} \neq K_2/K_1$.

In order to see whether these points are degenerated or nondegenerated singularities we have to derive the normal forms of (1) at $E_i,\ i = 1, 2, 3$ [1].

2. THE NATURE OF THE SADDLE-NODES

Case $\mathbf{p_{12} = K_1/K_2,\ p_{21} = K_2/K_1}$. In this case (1) assumes the form

$$\begin{cases} \dot{x} &= r_1 x \left(1 - x/K_1 - y/K_2\right), \\ \dot{y} &= r_2 y \left(1 - y/K_2 - x/K_1\right). \end{cases} \tag{2}$$

Proposition 2.1. *The normal form of (2) at $E_3(\alpha, K_2(K_1-\alpha)/K_1)$ up to second order terms for $\alpha \in [0, K_1]$ is*

$$\begin{cases} \dot{n}_1 &= [(-r_1\alpha - r_2K_1 + r_2\alpha)/K_1]n_1 + (r_2 - r_1)n_1n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{3}$$

and, thus, E_3 is a degenerated saddle-node.

Proof. First, we translate the point E_3 at the origin with the aid of the change $u_1 = x - \alpha,\ u_2 = y - K_2(K_1 - \alpha)/K_1$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (2) reads

$$\begin{cases} \dot{u}_1 &= -(r_1\alpha/K_1)u_1 - (r_1\alpha/K_2)u_2 - (r_1/K_1)u_1^2 - (r_1/K_2)u_1u_2, \\ \dot{u}_2 &= -(K_2\gamma/K_1)u_1 - \gamma u_2 - (r_2/K_1)u_1u_2 - (r_2/K_2)u_2^2, \end{cases} \tag{4}$$

where $\gamma = r_2(K_1 - \alpha)/K_1$.

The eigenvalues of the matrix defining the system (4) linearized around the origin are $\lambda_1 = (-r_1\alpha - r_2K_1 + r_2\alpha)/K_1,\ \lambda_2 = 0$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1, \gamma K_2/(r_1\alpha))^T$ and $\mathbf{u}_{\lambda_2} = (K_1, -K_2)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & K_1 \\ \gamma K_2/(r_1\alpha) & -K_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (4) achieves the form

$$\begin{cases} \dot{v}_1 &= \lambda_1 v_1 + Pv_1^2 + (r_2 - r_1)v_1v_2, \\ \dot{v}_2 &= Qv_1^2 - (r_2/\alpha)v_1v_2, \end{cases} \tag{5}$$

where $P = -(r_1^2\alpha - r_2^2K_1 + r_2^2\alpha)/(\alpha r_1 K_1)$, $Q = -r_2(r_1K_1 - r_1\alpha - r_2K_1 + r_2\alpha)/(\alpha r_1 K_1^2)$, and the matrix defining the linear part is diagonal. In order to reduce the second order nonresonant terms in (5) we apply the normal form method [1]. To this aim we determine the transformation $\mathbf{v} = \mathbf{n} + \mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 1.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	P	Q	λ_1	$2\lambda_1$	R	S
1	1	$r_2 - r_1$	$-r_2/\alpha$	0	λ_1	-	T
0	2	0	0	$-\lambda_1$	0	0	-

Table 1.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogeneous vector polynomial in (5), $T = -r_2K_1/[\alpha(-r1\alpha - r_2K_1 + r_2\alpha)]$ and $R = -(r_1^2\alpha - r_2^2K_1 + r_2^2\alpha)/(\alpha r_1(-r_1\alpha - r_2K_1 + r_2\alpha))$. Therefore we find the transformation

$$\begin{cases} v_1 &= n_1 + Rn_1^2, \\ v_2 &= n_2 + Sn_1^2 + Tn_1n_2, \end{cases}$$

carrying (5) into (3). By [2], the equilibrium point E_3 corresponding the dynamical system generated by a s.o.d.e. of the form (3) is a degenerated saddle-node. ∎

Remark 2.1. *For $\alpha = K_1$, the equilibrium point $E_3(\alpha, K_2(K_1 - \alpha)/K_1)$ becomes $E_1(K_1, 0)$. In this case the normal form of (2) at E_1 is*

$$\begin{cases} \dot{n}_1 &= -r_1n_1 + (r_2 - r_1)n_1n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{6}$$

and, thus, E_1 is a degenerated saddle-node.

Remark 2.2. *For $\alpha = 0$, the equilibrium point $E_3(\alpha, K_2(K_1-\alpha)/K_1)$ becomes $E_2(0, K_2)$. In this case the normal form of (2) at E_2 is*

$$\begin{cases} \dot{n}_1 &= -r_2 n_1 + (r_2 - r_1) n_1 n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= O(\mathbf{n}^3), \end{cases} \tag{7}$$

and, thus, E_2 is a degenerated saddle-node.

Case $\mathbf{p_{12} \neq K_1/K_2,\ p_{21} = K_2/K_1}$. In this case (1) assumes the form

$$\begin{cases} \dot{x} &= r_1 x \left(1 - x/K_1 - p_{12} y/K_1\right), \\ \dot{y} &= r_2 y \left(1 - y/K_2 - x/K_1\right). \end{cases} \tag{8}$$

Proposition 2.2. *The normal form of (8) at $E_1(K_1, 0)$ up to second order terms is*

$$\begin{cases} \dot{n}_1 &= -r_1 n_1 + (p_{12}/K_1)(r_1 - r_2) n_1 n_2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= r_2 \left(p_{12}/K1 - 1/K_2\right) n_2^2 + O(\mathbf{n}^3), \end{cases} \tag{9}$$

and, thus, E_1 is a nondegenerated saddle-node.

Proof. First, we translate the point E_1 at the origin with the aid of the change $u_1 = x - K_1,\ u_2 = y$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (8) reads

$$\begin{cases} \dot{u}_1 &= -r_1 u_1 - r_1 p_{12} u_2 - (r_1/K_1) u_1^2 - (r_1 p_{12}/K_1) u_1 u_2, \\ \dot{u}_2 &= -(r_2/K_1) u_1 u_2 - (r_2/K_2) u_2^2. \end{cases} \tag{10}$$

The eigenvalues of the matrix defining the linear terms in (10) are $\lambda_1 = -r_1,\ \lambda_2 = 0$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1,0)^T$ and $\mathbf{u}_{\lambda_2} = (-p_{12}, 1)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (10) achieves the form

$$\begin{cases} \dot{v}_1 &= -r_1 v_1 - (r_1/K_1) v_1^2 + (p_{12}/K_1)/(r_1 - r_2) v_1 v_2 + p_{12} \beta v_2^2, \\ \dot{v}_2 &= -(r_2/K_1) v_1 v_2 + \beta v_2^2, \end{cases} \tag{11}$$

where $\beta = r_2\left(p_{12}/K_1 - 1/K_2\right)$. In (11) the matrix of the linear terms is diagonal. In order to reduce the second order nonresonant terms in (11) we determine the transformation $\mathbf{v} = \mathbf{n}+\mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 2.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	$-r_1/K_1$	0	$-r_1$	$-2r_1$	$1/K_1$	0
1	1	M	$-r_2/K_1$	0	$-r_1$	-	$r_2/(r_1K_1)$
0	2	$p_{12}N$	N	r_1	0	$p_{12}N/r_1$	-

Table 2.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogenous vector polynomial in (11), $M = p_{12}(r_1 - r_2)/K_1$ and $N = r_2\left(p_{12}/K_1 - 1/K_2\right)$. We find the transformation

$$\begin{cases} v_1 &= n_1 + \dfrac{1}{K_1}n_1^2 + \dfrac{p_{12}}{r_1}Nn_2^2, \\ v_2 &= n_2 + \dfrac{r_2}{r_1K_1}n_1n_2, \end{cases}$$

carrying (11) into (9). We have $p_{12} \neq K_1/K_2$, therefore $r_2(p_{12}/K_1 - 1/K_2) \neq 0$. By [2], the equilibrium point E_1 corresponding the dynamical system generated by a s.o.d.e. of the form (9) is a nondegenerated saddle-node. ∎

Case $\mathbf{p_{12} = K_1/K_2}$, $\mathbf{p_{21} \neq K_2/K_1}$. In this case (1) assumes the form

$$\begin{cases} \dot{x} &= r_1x\left(1 - x/K_1 - y/K_2\right), \\ \dot{y} &= r_2y\left(1 - y/K_2 - p_{21}x/K_2\right). \end{cases} \tag{12}$$

Proposition 2.3. *The normal form of (12) at $E_2(0, K_2)$ is*

$$\begin{cases} \dot{n}_1 &= r_1\left(p_{21}/K_2 - 1/K_1\right)n_1^2 + O(\mathbf{n}^3), \\ \dot{n}_2 &= -r_2n_2 + [p_{21}(r_2 - r_1)/K_2]n_1n_2 + O(\mathbf{n}^3), \end{cases} \tag{13}$$

and, thus, E_2 is a nondegenerated saddle-node.

Proof. First, we translate the point E_2 at the origin with the aid of the change $u_1 = x,\ u_2 = y - K_2$. Let $\mathbf{u} = (u_1, u_2)^T$. Then, in $\mathbf{u}$, (8) reads

$$\begin{cases} \dot{u}_1 &= -(r_1/K_1)u_1^2 - (r_1/K_2)u_1u_2, \\ \dot{u}_2 &= -r_2p_{21}u_1 - r_2u_2 - (r_2p_{21}/K_2)u_1u_2 - (r_2/K_2)u_2^2. \end{cases} \tag{14}$$

The eigenvalues of the matrix defining the linear terms in (14) are $\lambda_1 = 0$, $\lambda_2 = -r_2$ and the corresponding eigenvectors read $\mathbf{u}_{\lambda_1} = (1, -p_{21})^T$ and $\mathbf{u}_{\lambda_2} = (0,1)^T$. Thus, with the change of coordinates $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p_{21} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (14) achieves the form

$$\begin{cases} \dot{v}_1 &= Bv_1^2 - (r_1/K_2)v_1v_2, \\ \dot{v}_2 &= -r_2v_2 + p_{21}Bv_1^2 + Cv_1v_2 - (r_2/K_2)v_2^2, \end{cases} \tag{15}$$

where $B = r_1\left(p_{21}/K_2 - 1/K_1\right)$ and $C = p_{21}(r_2 - r_1)/K_2$, and the matrix defining the linear part is diagonal. In order to reduce the second order non-resonant terms in (15) we determine the transformation $\mathbf{v} = \mathbf{n} + \mathbf{h}(\mathbf{n})$, where $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{n} = (n_1, n_2)^T$, suggested by the Table 3.

m_1	m_2	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$h_{\mathbf{m},1}$	$h_{\mathbf{m},2}$
2	0	B	$p_{21}B$	0	r_2	-	$p_{21}B/r_2$
1	1	$-r_1/K_2$	C	$-r_2$	0	$r_1/(r_2K_2)$	-
0	2	0	$-r_2/K_2$	$-2r_2$	$-r_2$	0	$1/K_2$

Table 3.

Here $\Lambda_{\mathbf{m},1}$, $\Lambda_{\mathbf{m},2}$ are the eigenvalues of the associated Lie operator, $X_{\mathbf{m}}$ is a second order homogenous vector polynomial in (15).

We find the transformation

$$\begin{cases} v_1 &= n_1 + [r_1/(r_2K_2)]n_1n_2, \\ v_2 &= n_2 + (Bp_{21}/r_2)n_1^2 + (1/K_2)n_2^2, \end{cases}$$

carrying (15) into (13). We have $p_{21} \neq K_2/K_1$, therefore $r_1(p_{21}/K_2 - 1/K_1) \neq 0$. By [1], the equilibrium point E_2 corresponding the dynamical system generated by a s.o.d.e. of the form (13) is a nondegenerated saddle-node. ∎

3. THE LOCAL PHASE PORTRAITS AROUND THE SADDLE-NODE SINGULARITIES

In the following we present the local phase portraits around the saddle-node singularities for $r_1 = 0.3,\ r_2 = 0.5,\ K_1 = 40,\ K_2 = 50$.

Namely, in fig. 1 it is shown the local phase portraits around $E_1,\ E_2,\ E_3$ for $p_{12} = K_1/K_2$, $p_{21} = K_2/K_1$ (a); around E_1 for $p_{12} \neq K_1/K_2$, $p_{21} = K_2/K_1$ (b); and around E_2 for $p_{12} = K_1/K_2$, $p_{21} \neq K_2/K_1$ (c).

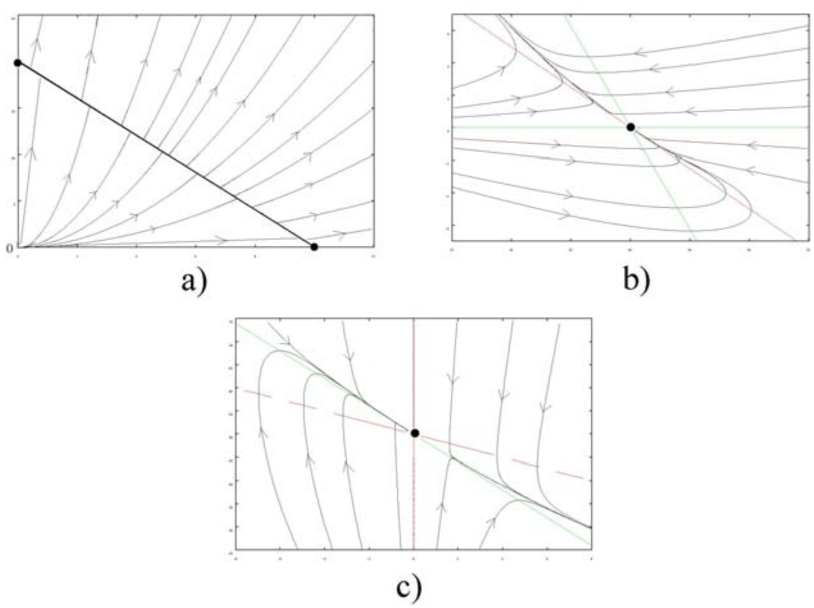

Fig 1. Phase portraits around $E_1,\ E_2,\ E_3$ (a), E_1 (b), E_2 (c).

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A GRADIENT-BASED OPTIMIZATION APPROACH FOR FREE SURFACE VISCOUS FLOWS

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Abstract In this paper, a gradient-based optimization approach to solve the free surface viscous flows is purposed. The main problem to overcome is to compute the gradient itself. The Navier-Stokes equation is used to model the fluid flows and the problem is formulated as an optimization one, then the adjoint problem and the gradient is obtained.

1. INTRODUCTION

The free surface flows of fluids have many application in different industry fields. Among them we mention the ship industry where to improve ship's design knowledge of the water free surface flows are very important. So far many methods have been proposed for this type of problems, many of them related to the basic approach of ideal fluids. However, the viscous flows prove to be an interesting field of research, in order to get a more appropriate mathematical model for the physical problem. Once an optimal shape design method has been chosen to solve the free surface flows problem, the main issue which has to be solved is to compute the gradient for the objective functional.

This aim of this paper is to develop a gradient-based optimization method for the free surface viscous flows. Due to the application we have considered, the water flows in an open environment, it is possible to write the boundary conditions on the free boundary using a simplified approximation. With that approach and using the nonlinear Navier Stokes equation, the problem is formulated into an optimization form. Then the gradient and the adjoint problem is determinated.

2. MATHEMATICAL MODEL AND SOLUTION STRATEGY

The flow of an incompressible and viscous fluid with a free boundary is described by the Navier Stokes equation

$$\begin{aligned} (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - Re^{-1}\nabla\cdot\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) &= -Fr^{-2}\mathbf{j}, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned}$$

With $\mathbf{u}$, we denote the velocity field, p is the fluid pressure, Re is the Reynolds number and Fr is the Froude number. We assume that the fluid is defined on a unbounded region D, such that

$$\mathbf{u} = \mathbf{u}_d \tag{1}$$

as $x \to \infty$. Below the fluid domain has a fixed boundary Γ and above a free surface S. The fluid pressure may be splited into a hydrodynamic component φ and a hydrostatic component $Fr^{-2}y$, then it becomes

$$p(x,y) = \varphi(x,y) - Fr^{-2}y. \tag{2}$$

We assume that along the boundary S is true that

$$\mathbf{n}\cdot\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\mathbf{n} = 0. \tag{3}$$

Using (2) and (3) the model for the fluid flow becomes

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi-Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right) &= 0, & \mathbf{x}\in D \quad (3)\\
\nabla\cdot\mathbf{u} &= 0, & \mathbf{x}\in D \quad (4)\\
p &= 0, & \mathbf{x}\in S \quad (5)\\
Re^{-1}\mathbf{t}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\mathbf{n} &= 0, & \mathbf{x}\in S \quad (6)\\
\mathbf{u}\cdot\mathbf{n} &= 0, & \mathbf{x}\in S \quad (7)\\
\mathbf{u} &= 0, & \mathbf{x}\in \Gamma \quad (8)\\
\mathbf{u} &= \mathbf{u}_d, & x\to\infty. \quad (9)
\end{aligned}
$$

One method to solve the free surface flows problem is to use a gradient based optimization approach. We assume that the flow domain may be extended beyond its boundaries, a set of domains is then constructed. Let us denote it by $\mathcal{D}$. Indeed, if we define an objective functional $J:\mathcal{D}\to\mathbb{R}$, where

$$
\begin{aligned}
J(S) &= \int_S p^2 dS + \int_D \mathbf{v}\cdot\left((\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi-Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\right)dD\\
&+ \int_D w\,(\nabla\cdot\mathbf{u})\,dD, \qquad (10)
\end{aligned}
$$

the problem (4) - (10) may be formulated as an optimization one. Then to solve it the main issue is to determine the gradient of the objective functional. Here $\mathbf{v}$ and w are two unknown functions on D such that $\mathbf{v}=0$ on Γ.

We assume that the free boundary is $S=\{(x,y):\ y=\alpha(x)\}$ and we get a new position of S by $(x,\alpha_1(x))=(x,\alpha(x))-\epsilon\mathbf{n}(x,\alpha(x))$. Therefore we compute the gradient of J in $\mathbf{n}$ direction $\mathrm{grad}_{\mathbf{n}}J=\mathrm{grad}J\cdot\mathbf{n}$.

Then we get the gradient of the objective functional (11)

$$
\begin{aligned}
\mathrm{grad}_{\mathbf{n}}J^* &= \int_S\{(1-\varphi+Fr^{-2}y)\mathbf{n}\cdot\nabla\varphi-Fr^{-2}\mathbf{n}\cdot\mathbf{j}\\
&-\mathbf{v}\cdot Re^{-1}\left((\nabla(\mathbf{n}\cdot\nabla\mathbf{u}))+(\nabla(\mathbf{n}\cdot\nabla\mathbf{u}))^T\right)\mathbf{n}\}dS. \qquad (11)
\end{aligned}
$$

ln (12) there are two variables, $\mathbf{v}$ and w which are unknown. To have a useful formulation of the gradient these variables have to be computed. Based on necessary condition of minimum for J with respect to the flow variables $\mathbf{u}$ and φ we develop a strategy to solve this problem. Then $\mathbf{v}$ and w has to be obtained as solutions of the boundary value problem

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{v}+(\nabla\mathbf{u})\,\mathbf{v}+\nabla w+\nabla\cdot Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right) &= 0, \quad \mathbf{x}\in D\\
\nabla\cdot\mathbf{v} &= 0, \quad \mathbf{x}\in D\\
\mathbf{v}\cdot\mathbf{n} &= -p, \ \mathbf{x}\in S\\
Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right)\mathbf{n} &= 0, \quad \mathbf{x}\in S\\
\mathbf{v} &= 0, \quad \mathbf{x}\in\Gamma\\
\mathbf{v} &= 0, \quad |x|\to\infty.
\end{aligned}$$

3. CONCLUSIONS

When a gradient based optimization method is used to compute the free surface viscous fluid flows, the main problem is the gradient computation. In this paper we have presented a method to solve it.

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COMPUTATIONAL TECHNIQUES FOR TIME DEPENDENT NON-NEWTONIAN FLUID FLOW

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Abstract In this paper a numerical algorithm for computing a class of non-Newtonian fluid flows is investigated. The method is based on the finite volume technique and on a time step adaptive backward differentiation formula. A solver for computing the evolutionary 1D flow of a pseudo-plastic fluid is presented. The numerical results obtained indicate a fast and robust code. Some basic components of a 2D solver as the discrete gradient of a discrete scalar field, the discrete divergence of a discrete vector field, the discrete Hodge decomposition formula etc, defined on a nonuniform rectangular grid, are also presented.

1. INTRODUCTION

In this paper we study a numerical approximation of the solution of the incompressible Navier-Stokes type equations for generalized Newtonian fluid

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0, \\ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla p + \nabla \cdot \boldsymbol{\sigma}, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity vector field, p is the hydrodynamic pressure field and $\boldsymbol{\sigma}$ the extra stress tensor field.

The constitutive equation that relates the stress with the strain rate tensor can be viewed as a generalization of that of a Newtonian fluid, namely,

$$\sigma_{ab} = 2\nu(\dot{\gamma}) d_{ab}. \tag{2}$$

The strain rate tensor d_{ab} is given by $d_{ab} = \frac{1}{2}\left(\partial_a u_b + \partial_b u_a\right)$ and $\dot{\gamma} = \left[2\mathrm{tr}\left(d^2\right)\right]^{1/2}, \nu = \frac{\mu}{\rho}$. The Newtonian fluid corresponds to $\nu = constant$.

To have an appropriate image on the nature of the equation that we shall deal with, we revised here some generalized Newtonian model for pseudo-plastic fluids. Such fluids are characterized by the fact that the apparent viscosity is a decreasing function with respect to shear rate.

In all models μ_0 and $\mu_\infty(\ \mu_0 > \mu_\infty)$ are the asymptotic apparent viscosities as $\dot{\gamma} \to 0$ and ∞ respectively, and $\Lambda \geq 0$ is a material constant with dimension of time.

• *Powell-Eyring model* is an old three parameters model for the suspensions of polymer in solvents and polymer melts with low elasticity, [6], [12], [15]

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{\sinh^{-1}\Lambda\dot{\gamma}}{\Lambda\dot{\gamma}}\,. \tag{3}$$

• *Yeleswarapu model*is proposed in [22] for the constitutive behavior of the blood,

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{1 + \ln(1 + \Lambda\dot{\gamma})}{1 + \Lambda\dot{\gamma}}. \tag{4}$$

• *Cross model* is a four parameters model, [13]

$$\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (\Lambda\dot{\gamma})^m}. \tag{5}$$

• *Carreau-Yasuda model* in its general form is a five parameters model, [7], [21]

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + (\Lambda\dot{\gamma})^a\right)^{(n-1)/a}, \quad 0 < n < 1. \tag{6}$$

We note that in all preceding models the constitutive function $\nu(\cdot)$ is a continuous, bounded and monotonous function with respect to its arguments.

In writing down a numerical algorithm for the non-stationary incompressible Navier-Stokes type equations for generalized Newtonian fluids, three main difficulties occur, namely: (i) the incompressibility constraint on the velocity field, (ii) the presence of the nonlinear advective term and (iii) the nonlinear dependence of the viscosity on the shear rate.

The first two are common to the Newtonian fluid and there exists an extensive literature devoted to the subject. In the context of finite volume method, the method which we will work with, the first primitive-variable numerical method, developed by Harlow and Welch [17], attempts to enforce the incompressibility constraint by deriving a Poisson equation for the pressure, taking the divergence of the momentum balance equation. Such kind of methods need an artificial boundary condition for the pressure. Chorin [11] developed a practical numerical method based on a discrete form of the Hodge decomposition. We present the method in the next section. This method, known as the projection method, computes an intermediate vector field that is then projected onto divergence free fields to recover the velocity. Kim and Moin [18] proposed a method for advancing velocity field in two fractional steps. They use a version of Chorin's algorithm replacing the treatment of nonlinear advective term by the second order explicit Adams-Bashford scheme and the implicit second-order Crank-Nicholson method for viscous term. The divergent-free velocity is updated by using a projection function that solves a Poisson equation. Bell, Colella and Glaz, [3] developed a second order accurate method in time and space. The method is a kind of Crank-Nicholson method for time stepping and uses intermediate values for velocity field and pressure field.

When one deals with non-Newtonian fluids the nonlinearity of the viscosity rises a new problem in obtaining a discrete form a Navier-Stokes type equations. The new issue is the development of an appropriate discrete form of the action of the stress tensor on the boundary of the volume-control. Andreianov, Boyer and Hubert [2] develop a method for solving p-Laplacian problem for rectangular grid. They use primal and dual mesh and define the flux on the boundary of the control volume. The discrete gradient of the unknown function is defined on the dual mesh and the norm of the gradient on the boundary of the control volume is evaluated as quadratic bilinear function on the gradient defined on the dual mesh.

We develop a numerical method based on: the projection method introduced by Bell et al., the discretization of the velocity field and pressure field on a staggered mesh and a similar method of Andreianov at al. regarding the discretization of the gradient of the velocity field.

The outline of the paper is as follows. In Section 2 we establish the semi-discrete, space discrete coordinates and time continuum variable, form of the equation (1) and we present some general concepts concerning the space discretization and related notions like admissible mesh, primal and dual mesh, the discretization of the derivative operators etc. In Section 3 we present a code to solve a 1D model. We will prove the existence of a maximal principle property of the ODE model. As an application we consider the evolutionary 1D Couette flow of a pseudo-plastic fluid.

2. SPATIAL DISCRETIZATION AND DISCRETE FUNCTIONAL ANALYSIS

Let Ω be the space domain of the flow and $\{\Omega_i\}_{i\in\mathfrak{I}}$ a partition of it. The finite volume method (FVM) deals with the integral form of the equation rather than local form. For each control volume Ω_i the integral form of the balance momentum equation reads

$$\partial_t \int_{\Omega_i} \boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}x + \int_{\partial\Omega_i} \boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s + \int_{\Omega_i} \nabla p \mathrm{d}x = \int_{\partial\Omega_i} \boldsymbol{\sigma}\cdot\boldsymbol{n}\mathrm{d}s. \tag{7}$$

On each control volume Ω_i we approximate the velocity field $\boldsymbol{u}$ by a constant value $\boldsymbol{u}_i$ and we consider a set of discrete values $\{p_a\}_{a\in\mathcal{J}}$ that constitute the approximation of the pressure field. Next one needs to define the discrete form of the advective flux, the flux of the stress tensor and the gradient of the pressure,

$$\begin{aligned} \mathcal{F}_i(\{\boldsymbol{u}\}) &\approx \textstyle\int_{\partial\Omega_i} \boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s, \\ \mathrm{Grad}_i(\{p\}) &\approx \textstyle\int_{\Omega_i} \nabla p\mathrm{d}x, \\ \mathcal{S}_i(\{\boldsymbol{u}\}) &\approx \textstyle\int_{\partial\Omega_i} \boldsymbol{\sigma}\cdot\boldsymbol{n}\mathrm{d}s. \end{aligned} \tag{8}$$

One obtains the discrete form of the balance of the momentum, a system of ordinary differential equations

$$m(\Omega_i)\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} + \mathcal{F}_i(\{\boldsymbol{u}\}) + \mathrm{Grad}_i(\{p\}) = \mathcal{S}_i(\{\boldsymbol{u}\}.) \tag{9}$$

As the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ must satisfy the continuity equation in (1) a set of discrete constraints on the discrete values $\{\boldsymbol{u}\}$ must exist. The discrete constraints are obtained from the integral form of the continuity equation. Let $\{\widetilde{\Omega}_a\}_{a\in\mathcal{J}}$ be another partition of the domain Ω. For each $\widetilde{\Omega_a}$ we impose to the velocity field to satisfy the integral equation

$$\int_{\partial\widetilde{\Omega}_a} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s = 0.$$

We introduce the discrete divergence as an approximation of the flux of the velocity through the boundary of the control volume $\widetilde{\Omega}_a$,

$$\mathrm{Div}_a(\{\boldsymbol{u}\}) \approx \int_{\partial\widetilde{\Omega}_a} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s. \tag{10}$$

Then we supplement the ODE system (9) by a system of algebraic equation

$$\mathrm{Div}_a(\{\boldsymbol{u}\}) = 0 \tag{11}$$

The problem is to find the values $\{\boldsymbol{u}_i\}_{i\in\mathcal{I}}$, $\{p_a\}_{a\in\mathcal{J}}$ that satisfy the differential algebraic system of equations (DAE)

$$\begin{aligned} m(\Omega_i)\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} + \mathcal{F}_i(\{\boldsymbol{u}\}) + \mathrm{Grad}_i(\{p\}) - \mathcal{S}_i(\{\boldsymbol{u}\}) = 0, \quad i\in\mathcal{I} \\ \mathrm{Div}_a(\{\boldsymbol{u}\}) = 0, \quad a\in\mathcal{J} \end{aligned} \tag{12}$$

The main issue is to define the discrete gradient of the scalar function and the discrete divergence of the vector function in a manner such that the solution of the DAE (12) is uniquely determined by the initial data of the discrete velocity $\{\boldsymbol{u}_i\}_{i\in\mathcal{I}}$. This is the object of the next section.

2.1. ADMISSIBLE DISCRETIZATION

A *mesh* is a discretization of a domain into small domains of simple geometric shapes, such as segments in 1D, triangles or quadrangles in dimension two or tetrahedrons in 3D. A *structured* mesh is one in which all interior vertices are topological alike. An *unstructured* mesh is one in which vertices may have arbitrarily varying local neighborhoods. In the 2D case a structured mesh typically uses quadrilaterals, while an unstructured mesh uses triangles.

An admissible *primal mesh* $\mathcal{T} = \{\Omega_{\mathcal{J}}, \boldsymbol{r}_{\mathcal{J}}\}$ of Ω is a partition $\Omega_{\mathcal{J}}$ of Ω into closed polygonal domains ω_i and a net of the inner knots $\boldsymbol{r}_i$ that satisfies

$$\left\| \begin{array}{l} (1)\ \overline{\cup_{i\in I}\omega_i} = \overline{\Omega}, \\ (2)\ \forall i \neq j \in I \text{ and } \overline{\omega}_i \cap \overline{\omega}_j \neq \Phi, \text{either } \mathcal{H}_{n-1}(\overline{\omega}_i \cap \overline{\omega}_j) = 0 \text{ or} \\ \qquad \sigma_{ij} := \overline{\omega}_i \cap \overline{\omega}_j \text{ is a common } (n-1) - \text{face of } \omega_i \text{ and } \omega_j, \\ (3)\ \boldsymbol{r}_i \in \omega_i. \end{array} \right.$$

Here $\mathcal{H}_{n-1}$ is the $n-1$ –dimensional Hausdorff measure.

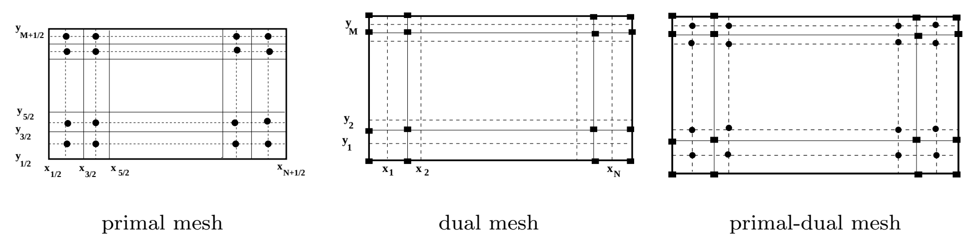


Fig. 1. Rectangular structured mesh.

A mesh which satisfies the property (2) is called a *matching mesh*.

• **Rectangular structured meshes.** Let Ω be the rectangle $[0, a] \times [0, b]$ and $0 = x_{1/2} < x_{1+1/2} < \cdots x_{N+1/2} = a$, $0 = y_{1/2} < y_{1+1/2} < \cdots y_{M+1/2} = b$ be two partitions of the intervals $[0, a]$ and $[0, b]$ respectively.

One defines a structured primal mesh by:

- control volumes; $\omega_{i,j} = \left[x_{i-1/2}, x_{i+1/2}\right] \times \left[y_{j-1/2}, y_{j+1/2}\right], i = \overline{1,N},\ j = \overline{1,M}$;
- inner knots; $\boldsymbol{r}_{i,j} = (x_i, y_j),\ x_i = \dfrac{x_{i-1/2} + x_{i+1/2}}{2}, y_j = \dfrac{y_{j-1/2} + y_{j+1/2}}{2}, i = \overline{1,N},\ j = \overline{1,M}$.

A *dual mesh* $\widetilde{\mathfrak{T}} = \left\{\widetilde{\Omega}_{\mathfrak{J}}, \widetilde{\boldsymbol{r}}_{\mathfrak{J}}\right\}$ is another partition of $\widetilde{\Omega}_{\mathfrak{J}}$ of Ω into closed polygonal domains $\widetilde{\omega}_a$ and a net of vertices $\widetilde{\boldsymbol{r}}_{\mathfrak{J}}$ which satisfies

(1) $\widetilde{\boldsymbol{r}}_a \in \widetilde{\omega}_a, \forall a \in \mathfrak{J}$,

(2) $\forall a \in \mathfrak{J}\ \widetilde{\boldsymbol{r}}_a$is a vertex of $\mathfrak{T}$,

(3) the set $\boldsymbol{r}_{\mathfrak{J}}$ is contained in the set of vertices of $\widetilde{\mathfrak{T}}$.

A dual mesh for rectangular grid is given by:

- $\widetilde{\omega}_{i+1/2,j+1/2} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$;
- $\widetilde{\boldsymbol{r}}_{i+1/2,j+1/2} = \left(x_{i+1/2}, y_{j+1/2}\right)$,

where $i = \overline{0,N},\ j = \overline{0,M}$; $x_0 \equiv 0, y_0 \equiv 0, x_{N+1} \equiv a, y_{M+1} \equiv b$.

2.2. DISCRETE DERIVATIVE OPERATORS. NONUNIFORM RECTANGULAR MESH

In the sequel we deal with a rectangular mesh $(\mathfrak{T}, \widetilde{\mathfrak{T}})$ as in the above example on the rectangle Ω. Unlike in the continuum case, where the derivative operators are precisely defined, there is no canonical way to define the discrete space derivatives of a discrete field.

We denote by $H_{\widetilde{\mathfrak{T}}}(\Omega)$ the space of functions which are piecewise constant on each volume $\widetilde{\omega}_a \in \widetilde{\Omega}_{\mathfrak{J}}$, $\boldsymbol{H}_{\mathfrak{T}}(\Omega)$ the space of two dimensional functions which are piecewise constant on each volume $\omega_i \in \Omega_{\mathfrak{I}}$.

For all $\phi \in H_{\widetilde{\mathfrak{T}}}(\Omega)$ and for all $\widetilde{\omega}_a \in \widetilde{\Omega}_{\mathfrak{J}}$ we denote by ϕ_a the constant value of ϕ in $\widetilde{\omega}_a$. Similarly, $\boldsymbol{u}_i$ stands for the constant value of a vector function $\boldsymbol{u} \in \boldsymbol{H}_{\mathfrak{T}}(\Omega)$ in ω_i.

Discrete divergence. We define a discrete divergence operator

$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})} : \boldsymbol{H}_{\mathcal{T}}(\Omega) \to H_{\widetilde{\mathcal{T}}}(\Omega)$, by

$$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u})_a = \int\limits_{\partial\widetilde{\omega}_a} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s. \tag{13}$$

Discrete gradient. A discrete gradient operator $\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \to \boldsymbol{H}_{\mathcal{T}}(\Omega)$ is defined by

$$\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi)_i = \int\limits_{\partial\omega_i} \phi\,\boldsymbol{n}\mathrm{d}s. \tag{14}$$

Discrete "rotation". The "rotation" of a scalar field is a vector field and any vector field that result as the rotation of a scalar is divergent free vector field. The discrete counterpart of that results can be obtained by a proper definition of a discrete rotation. We define

$\mathbf{Rot}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \longrightarrow \boldsymbol{H}_{\mathcal{T}}(\Omega)$ by

$$\mathbf{Rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi)_i = \frac{1}{\mathrm{m}(\omega_i)}\int\limits_{\partial\omega_i} \phi\mathrm{d}\boldsymbol{r}. \tag{15}$$

Using the discrete divergence operator and discrete gradient operator one defines the discrete Laplace operator by

$$\triangle_{(\mathcal{T},\widetilde{\mathcal{T}})} = \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}. \tag{16}$$

For an irregular rectangular mesh the discrete operators take the following forms. Frequently we will use an index $a, b, ...$ as a short notation for some two indices $(i+1/2, j+1/2)$. Also, an index like $a+(0,-1)$ equals the $(i+1/2, j-1/2)$.

$$\begin{aligned}\mathrm{Div}(\boldsymbol{u})_{i+1/2,j+1/2} \quad &= \begin{bmatrix} u \\ v \end{bmatrix}_{i,j} \cdot \begin{bmatrix} -\dfrac{\triangle y_j}{2} \\ -\dfrac{\triangle x_i}{2} \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}_{i+1,j} \cdot \begin{bmatrix} \dfrac{\triangle y_j}{2} \\ -\dfrac{\triangle x_{i+1}}{2} \end{bmatrix} + \\ &+ \begin{bmatrix} u \\ v \end{bmatrix}_{i+1,j+1} \cdot \begin{bmatrix} \dfrac{\triangle y_{j+1}}{2} \\ \dfrac{\triangle x_{i+1}}{2} \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}_{i,j+1} \cdot \begin{bmatrix} -\dfrac{\triangle y_{j+1}}{2} \\ \dfrac{\triangle x_i}{2} \end{bmatrix}\end{aligned} \tag{17}$$

where $\triangle x_i = x_{i+1/2} - x_{i-1/2}$ and $\triangle y_j = y_{j+1/2} - y_{j-1/2}$. The above formula is equal true for $i = \overline{0,N}$, $j = \overline{0,M}$ with the conventions: $v_{i,0} = v_{i,M+1} = 0, u_{0,j} = u_{N+1,j} = 0$ and $\triangle x_0 = \triangle y_0 = \triangle x_{N+1} = \triangle y_{M+1} = 0$, for example

$$\mathrm{Div}(\boldsymbol{u})_{1/2,1/2} = \frac{1}{2}\left(v_{1,1}\triangle x_1 + u_{1,1})\triangle y_1\right).$$

For any $i \in \{1,.,.,.,N\}$, $j \in \{1,.,.,.,M\}$ and $a = (i+1/2, j+1/2)$ the discrete gradient is given by

$$\mathbf{Grad}(\phi)_{i,j} = \begin{bmatrix} \left(\phi_{a+(0,0)} - \phi_{a+(-1,0)} - \phi_{a+(-1,-1)} + \phi_{a+(0,-1)}\right)\dfrac{\triangle y_j}{2} \\ \left(\phi_{a+(0,0)} + \phi_{a+(-1,0)} - \phi_{a+(-1,-1)} - \phi_{a+(0,-1)}\right)\dfrac{\triangle x_i}{2} \end{bmatrix} \tag{18}$$

and discrete rotation is given by

$$\begin{aligned} \mathbf{Rot}(\psi)_{i,j} = \quad & \psi_{a+(0,0)}\begin{bmatrix} \dfrac{-1}{2\triangle y_j} \\ \dfrac{1}{2\triangle x_i} \end{bmatrix} + \psi_{a+(-1,0)}\begin{bmatrix} \dfrac{-1}{2\triangle y_j} \\ \dfrac{-1}{2\triangle x_i} \end{bmatrix} + \\ & + \psi_{a+(-1,-1)}\begin{bmatrix} \dfrac{1}{2\triangle y_j} \\ \dfrac{-1}{2\triangle x_i} \end{bmatrix} + \psi_{a+(0,-1)}\begin{bmatrix} \dfrac{1}{2\triangle y_j} \\ \dfrac{1}{2\triangle x_i} \end{bmatrix} \end{aligned} \tag{19}$$

the r.h.s in (19) can be rewritten as

$$\begin{bmatrix} -\dfrac{\psi_{a+(0,0)} + \psi_{a+(-1,0)} - \psi_{a+(-1,-1)} - \psi_{a+(0,-1)}}{2\triangle y_j} \\ \dfrac{\psi_{a+(0,0)} - \psi_{a+(-1,0)} - \psi_{a+(-1,-1)} + \psi_{a+(0,-1)}}{2\triangle x_i} \end{bmatrix} \tag{20}$$

On the space $\boldsymbol{H}_{\mathfrak{T}}(\Omega)$ we define the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ by

$$\langle\langle \boldsymbol{u}, \boldsymbol{v} \rangle\rangle = \sum_{i=1,N;j=1,M} \boldsymbol{u}_{i,j} \cdot \boldsymbol{v}_{i,j} \tag{21}$$

and on the space $H_{\widetilde{\mathfrak{T}}}(\Omega)$ we define the scalar product $\langle\cdot,\cdot\rangle$ by

$$\langle \phi, \psi \rangle = \sum_{i=0,N;j=0,M} \phi_{i+1/2,j+1/2} \cdot \psi_{i+1/2,j+1/2}. \tag{22}$$

In the following lemma we prove certain properties of the discrete derivative operators.

Lemma 2.1. *Let* $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$ *be the primal-dual irregular structured mesh,* $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ *and* $H_{\widetilde{\mathcal{T}}}(\Omega)$ *the space of the discrete vector fields and the space of discrete scalar fields associated with it, respectively.*

Let the discrete divergence be defined by (17)*, the discrete gradient be defined by* (18)*, the discrete rotation be defined by* (19)*. Then:*
(a1) *discrete Stokes formula. For any* $\boldsymbol{u} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ *and any* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ *there exists a discrete integration by parts formula*

$$\left\langle \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}), \phi \right\rangle + \left\langle\left\langle \boldsymbol{u}, \mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi) \right\rangle\right\rangle = 0; \tag{23}$$

(a2) *for any* $\psi \in H_{\widetilde{\mathcal{T}}}(\Omega)$*,* $\psi|_{\partial\Omega} = 0$ *one has*

$$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})} \mathbf{Rot}_{(\mathcal{T},\widetilde{\mathcal{T}})} \psi = 0; \tag{24}$$

(a3) $\dim\left(\mathrm{Ker}\left(\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}\right)\right) = 2.$

Proof. The first two assertions can be proved by direct calculations. To calculate the algebraic dimension of the kernel of the discrete gradient we note that, for any $a = (i + 1/2, j + 1/2)$,

$$\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi)_{i,j} = 0$$

implies

$$\phi_{a+(0,0)} = \phi_{a+(-1,-1)},$$
$$\phi_{a+(-1,0)} = \phi_{a+(0,-1)}.$$

The general solution is determined by two constants, fig. 2.

2.3. DISCRETE HODGE DECOMPOSITION FORMULA

The Hodge decomposition formula asserts that, [19], for any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ there exists two scalar functions ϕ and ψ and a vector function $\boldsymbol{u}$ such that

$$\boldsymbol{w} = \boldsymbol{u} + \nabla\phi + \nabla\psi, \tag{25}$$

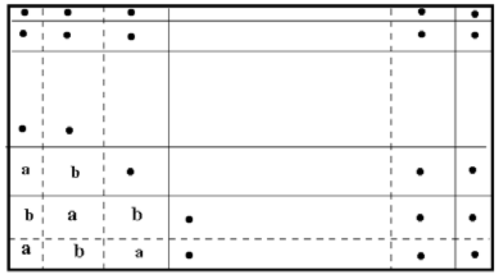


Fig. 2. The graphic of a function in the kernel of the discrete gradient operator.

with

$$\operatorname{div} \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \triangle\phi = 0, \quad \psi|_{\partial\Omega} = 0 \tag{26}$$

Moreover, the three functions occuring in the decomposition are orthogonal

$$< \boldsymbol{u}, \nabla\psi >=< \boldsymbol{u}, \nabla\phi >=< \nabla\psi, \nabla\phi >= 0.$$

The orthogonal projection operator P_H from $\boldsymbol{L}^2(\Omega)$ to the space $\boldsymbol{H} = \{\boldsymbol{u}; \operatorname{div} \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega}\}$ is a continuous mapping into $\boldsymbol{L}^2(\Omega)$.

There exists a discrete version of the Hodge decomposition formula for structured rectangular grid.

Let $\boldsymbol{G}$ be the image of the discrete gradient, i.e

$$\boldsymbol{G} = \operatorname{Im}\left(\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}\right) \subset \boldsymbol{H}_{\mathcal{T}}(\Omega)$$

and $\boldsymbol{G}^{\perp}$ its orthogonal complement in $\boldsymbol{H}_{\mathcal{T}}(\Omega)$. By definition one has

$$\boldsymbol{H}_{\mathcal{T}}(\Omega) = \boldsymbol{G}^{\perp} \oplus \boldsymbol{G}. \tag{27}$$

Next we prove that a discrete vector field $\boldsymbol{u}$ belongs to $\boldsymbol{G}^{\perp}$ iff $\operatorname{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0$ and also we define a basis of $\boldsymbol{G}^{\perp}$. To this aim we firstly define a basis $\{\Psi^a\}_{a\in\mathcal{J}}$ for the space $H_{\widetilde{\mathcal{T}}}(\Omega)$. Each function is defined by

$$\Psi^a(x) = \begin{cases} 1, & \text{if } x \in \widetilde{w}^a \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

where the index a stands for the pair indices $(i+1/2, j+1/2)$ with $i = 0, N; j = 0, M$. Denote by $\mathrm{Int}(\mathcal{J})$ the set of indices of the interior vertices points $\boldsymbol{r}_a$, that is $i = 1, N-1; j = 1, M-1$.

Proposition 2.1. *Let* $(\mathcal{T}, \widetilde{\mathcal{T}})$ *be the rectangular primal-dual mesh. Then* (a1)

$$\boldsymbol{u} \in \boldsymbol{G}^{\perp} \iff \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0,$$

(a2)

$$\dim(\boldsymbol{G}^{\perp}) = (N-1) \times (M-1),$$

(a3) *Using the basis* $\{\Psi^a\}_{a\in\mathcal{J}}$ *we define the discrete vector field* $\boldsymbol{\mathcal{U}}^a \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ *by*

$$\boldsymbol{\mathcal{U}}^a = \mathbf{Rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\Psi^a) \tag{29}$$

Each vector function $\boldsymbol{\mathcal{U}}^a$ *satisfies*

$$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{\mathcal{U}}^{\boldsymbol{a}}) = 0 \tag{30}$$

and the set $\{\boldsymbol{\mathcal{U}}^a; a \in \mathrm{Int}(\mathcal{J})\}$ *defines a basis for* $\boldsymbol{G}^{\perp}$.

Proof. (a1) Using the discrete Stokes formula (23), one can show that if $\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0$, then $\boldsymbol{u} \in \boldsymbol{G}^{\perp}$. Conversely, let $\boldsymbol{u} \in \boldsymbol{G}^{\perp}$ and let Ψ^a be an arbitrary element of the basis of the space $H_{\widetilde{\mathcal{T}}}(\Omega)$. Using again the discrete Stokes formula one obtains

$$\left\langle \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}), \Psi^a \right\rangle = -\left\langle\left\langle \boldsymbol{u}, \mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\Psi^a) \right\rangle\right\rangle,$$

which shows that $\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u})$ is orthogonal to the space $H_{\widetilde{\mathcal{T}}}(\Omega)$, hence $\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0$.

(a2) From the rank-nullity theorem it follows that

$$\dim(\boldsymbol{G}) = \dim(H_{\widetilde{\mathcal{T}}}(\Omega)) - \dim(\mathrm{Ker}(\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})})).$$

As $\dim(H_{\widetilde{\mathcal{T}}}(\Omega)) = (N+1) \times (M+1)$ and $\dim(\mathrm{Ker}(\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})})) = 2$, see (2.1), it follows that $\dim(\boldsymbol{G}) = (N+1) \times (M+1) - 2$. Using the orthogonal decomposition (27) and $\dim(H_{\widetilde{\mathcal{T}}}(\Omega)) = 2 \times N \times M$ one obtains (a2).

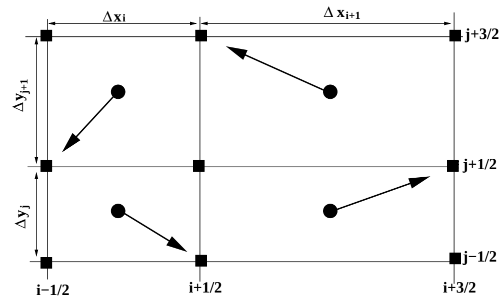


Fig. 3. The support and the graph of the function $\boldsymbol{\mathcal{U}}^{i+1/2,j+1/2}$.

(a3) The fact that each $\mathcal{U}^{i+1/2,j+1/2}$ is a divergent free discrete vector field follows from lemma (2.1). From the definition one can show that for $a = (i+1/2, j+1/2)$ the only nonzero values of the $\mathcal{U}^{i+1/2,j+1/2}$ are

$$\boldsymbol{\mathcal{U}}^a_{i,j} = \begin{bmatrix} \dfrac{1}{2\triangle y_j} \\ -\dfrac{1}{2\triangle x_i} \end{bmatrix}, \ \boldsymbol{\mathcal{U}}^a_{i+1,j} = \begin{bmatrix} \dfrac{1}{2\triangle y_j} \\ \dfrac{1}{2\triangle x_{i+1}} \end{bmatrix},$$

$$\boldsymbol{\mathcal{U}}^a_{i+1,j+1} = \begin{bmatrix} -\dfrac{1}{2\triangle y_{j+1}} \\ \dfrac{1}{2\triangle x_{i+1}} \end{bmatrix}, \boldsymbol{\mathcal{U}}^a_{i,j+1} = \begin{bmatrix} -\dfrac{1}{2\triangle y_{j+1}} \\ -\dfrac{1}{2\triangle x_i} \end{bmatrix},$$

which implies that the supports of two different functions $\boldsymbol{\mathcal{U}}^a$ and $\boldsymbol{\mathcal{U}}^b$ do not overlap, hence the family $\{\boldsymbol{\mathcal{U}}^a; a \in \mathrm{Int}(\mathcal{J})\}$ is linearly independent. Since the number of the functions equals the dimension of the $\boldsymbol{G}^\perp$ one concludes that the family constitute a basis of $\boldsymbol{G}^\perp$.

We will prove the discrete Hodge decomposition for the structured rectangular grid.

Proposition 2.2 (Discrete Hodge formula)**.** *Let* $(\mathcal{T}, \widetilde{\mathcal{T}})$ *be the rectangular primal-dual mesh. Let the discrete divergence be defined by* (17)*, and the discrete gradient be defined by* (18)*. Then for any* $\boldsymbol{w} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ *there exists an element* $\boldsymbol{u} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ *and a scalar function* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ *such that*

$$\boldsymbol{w} = \boldsymbol{u} + \mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\phi), \quad \text{with } \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0. \tag{31}$$

Proof. We search for a divergence free vector $\boldsymbol{u}$ as a linear combination

$$\boldsymbol{u} = \sum_{a \in \mathcal{J}} \alpha_a \boldsymbol{\mathcal{U}}^a. \tag{32}$$

Inserting it in the decomposition, taking the scalar product with the basis elements and using the discrete Stokes formula for the determination of the unknowns $\{\alpha_a\}_{a \in \mathcal{J}}$, one obtains a linear algebraic system of equations

$$\left\langle\left\langle \boldsymbol{w}, \boldsymbol{\mathcal{U}}^b \right\rangle\right\rangle = \sum_{a \in \mathcal{J}} \alpha_a \left\langle\left\langle \boldsymbol{\mathcal{U}}^a, \boldsymbol{\mathcal{U}}^b \right\rangle\right\rangle. \tag{33}$$

This matrix of the system is the Gram matrix of a linearly independent family, hence there exists a unique solution $\boldsymbol{u}$.

Since $\langle\langle \boldsymbol{w} - \boldsymbol{u}, \mathcal{U}^a \rangle\rangle = 0$ for any basis function it follows that $\boldsymbol{w} - \boldsymbol{u}$ is orthogonal to $\boldsymbol{G}^{\perp}$, which implies that $\boldsymbol{w} - \boldsymbol{u} \in \boldsymbol{G}$. Hence there exists $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that

$$\boldsymbol{w} - \boldsymbol{u} = \mathbf{Grad}_{(\mathcal{T}, \widetilde{\mathcal{T}})}(\phi). \tag{34}$$

Once one solves the system (33) the projection on the divergent free discrete vector field space is given by

$$\mathcal{P}(\boldsymbol{w})_{i,j} = \begin{bmatrix} \dfrac{\alpha_{a+(0,0)} + \alpha_{a+(-1,0)} - \alpha_{a+(-1,-1)} - \alpha_{a+(0,-1)}}{2\triangle y_j} \\ \dfrac{-\alpha_{a+(0,0)} + \alpha_{a+(-1,0)} + \alpha_{a+(-1,-1)} - \alpha_{a+(0,-1)}}{2\triangle x_i}. \end{bmatrix} \tag{35}$$

Note that the projection $\boldsymbol{u}$ is always a free divergent vector field independently how exactly is solved the equation (33).

3. 1D COUETTE FLOW. ODE APPROXIMATION AND NUMERICAL SIMULATION

A very simple case to test the response of the numerical model to the numerical approximation of the viscosity is the 1D flow. A nontrivial example, and rich in applications, is the evolutionary Couette flow. The fluid flows through a flat channel of thickness h between two fixed horizontal plates. The length

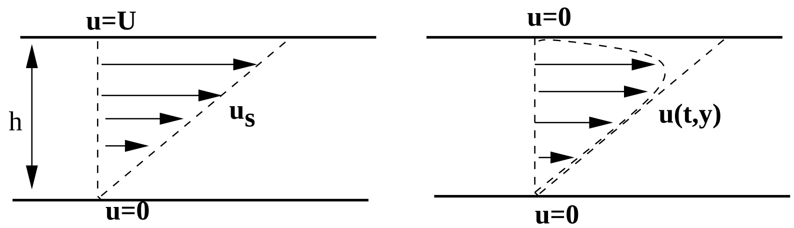


Fig. 4. Couette 1D flow. Steady solution (left) and evolutionary profile of the velocity from non-slip boundary condition (right).

L and the width W of the channel are much greater than its thickness h, so we may assume fully developed flow and neglect edge effects. Take the x direction along of the flow and parallel to the plates, and y direction perpendicular to the plates.

Assume that there is no flow in the direction perpendicular to the plate and all field variables do not depend on the x variable. According to the incompressibility constraint the x component of the velocity, u is depending only on the space variable y and on the time variable t. The momentum balance equations read

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\nu\left(\left|\frac{\partial u}{\partial y}\right|\right), \frac{\partial u}{\partial y}\right), \\ 0 &= -\frac{\partial p}{\partial y}. \end{aligned} \tag{36}$$

From the last equation one can conclude that the pressure field is a constant function across the channel. Hence we deal with the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\nu\left(\left|\frac{\partial u}{\partial y}\right|\right)\frac{\partial u}{\partial y}\right). \tag{37}$$

The stationary solution of the equation (37) is given by

$$u_s(y) = \frac{y}{h}U, \tag{38}$$

where U is the velocity of the upper plate. In our test model we assume that at $t = 0^+$ the upper plate stops moving. So, the boundary condition are

$$u(t,0) = u(t,h) = 0, \quad t > 0 \tag{39}$$

and the initial data are

$$u(0,y) = u_s(y), \quad 0 < y < h. \tag{40}$$

3.1. ODE APPROXIMATION

Let $0 = x_{1/2} < x_{1+1/2} < \cdots x_{N+1/2} = h$ be the knots of the partition of the interval $[0,h]$, $\omega_i = \left[x_{i-1/2}, x_{i+1/2}\right]$ be a control volume and $x_i = (x_{i-1/2} + x_{i+1/2})/2$ be the center of ω_i, $i = \overline{1,N}$.

The discrete form of the equation (37) is

$$m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = \nu(u_{i+1}, u_i)\frac{u_{i+1} - u_i}{d_{i+1,i}} - \nu(u_i, u_{i-1})\frac{u_i - u_{i-1}}{d_{i,i-1}}, \quad i = 1, ., ., ., N \tag{41}$$

where m_i is the length of the volume control ω_i, $d_{i,j} = |x_i - x_j|$ and $\nu(u_{i+1}, u_i)$ is a discrete approximation of the constitutive function $\nu\left(\left|\frac{\partial u}{\partial y}\right|\right)$ supposed to be a continuous function. A very simple choice for the numerical viscosity is

$$\nu(u_{i+1}, u_i) = \nu\left(\left|\frac{u_{i+1} - u_i}{d_{i+1,i}}\right|\right). \tag{42}$$

In the r.h.s of the ODE (41), the quantities u_0 and u_{N+1} are equal to the boundary data. So, according to the boundary data (39), we set

$$u_{N+1}(t) = 0, u_0(t) = 0. \tag{43}$$

The initial conditions are given by

$$u_i(0) = u_s(x_i). \tag{44}$$

To resume, the ODE approximation of the PDE (37) with the boundary conditions (39) and initial condition (40) is given by (41), (42), (43) and (44).

The solution of the ODE model has two important properties , namely the maximal principle property and the monotony of the "kinetic energy". The maximal principle is equally true for general boundary data and initial condition. Assume that there exist two constants α and β such that

$$\alpha < u_i(0), u_0, u_{N+1} < \beta, i = 1, \cdots, N$$

Proposition 3.1. *Let $(0, T)$ be the maximal interval of the time of the existence of the solution of the ODE approximation. Then*

$$\alpha < u_i(t) < \beta, t \in (0, T), i = 1, \cdots, N \tag{45}$$

If $u_0 = u_{N+1} = 0$ then

$$\sum_i m_i u_i^2(t_1) <= \sum_i m_i u_i^2(t_2),\ t_1 < t_2. \tag{46}$$

Proof. In order to prove that the solution $\{u(t)\}$ remains in the N-dimensional rectangle $\Omega = [\alpha, \beta]^N$ we will show that any face of the rectangle is an entry face for the trajectories of the ODE. Consider a face $u_i = \alpha$ and assume there exists a moment of time t^* such that $\{u(t)\} \in \Omega$ for $t < t^*$ and for $t = t^*$ $u_i = \alpha$ and remainder of the components still stay in Ω. We have

$$m_i \left.\frac{\mathrm{d}u_i}{\mathrm{d}t}\right|_{t=t^*} = \nu(u_{i+1}, \alpha)\frac{u_{i+1} - \alpha}{d_{i+1,i}} - \nu(\alpha, u_{i-1})\frac{\alpha - u_{i-1}}{d_{i,i-1}} \geq 0,$$

therefore $u_i \geq \alpha$. In order to prove the second assertion we multiply each $i-$ equation (41) by u_i,

$$\sum_i m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} u_i = \sum_i \left(\nu(u_{i+1}, u_i)\frac{u_{i+1} - u_i}{d_{i+1,i}} - \nu(u_i, u_{i-1})\frac{u_i - u_{i-1}}{d_{i,i-1}}\right) u_i$$

and then sum up. After some manipulation we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i \frac{u_i^2}{2} + \sum_{i=0}^{N} \nu(u_{i+1}, u_i)\frac{(u_{i+1} - u_i)^2}{d_{i+1,i}} = 0,$$

which yields (46).

In order to integrate the ODE (41) we use an implicit multi-step method [5]. Let $\{t_{n-k}, t_{n-k+1}, ..., t_n\}$ be a sequence of moments of time and denote $u^m = u(t_m) \in \mathbb{R}^N$. Supposing that one knows the values $\{u^{n-k}, u^{n-k+1}, ..., u^n\}$ the values u^{n+1} at the next moment of time t_{n+1} is calculated as follows. Define a predictor polynomial $\omega^P(t)$ and a corrector polynomial $\omega^C(t)$. The predictor polynomial interpolates the values $\{u^{n-k}, u^{n-k+1}, ..., u^n\}$ at moments of time $\{t_{n-k}, t_{n-k+1}, ..., t_n\}$, Lagrange interpolation,

$$\omega^P(t) = \sum_{j=0}^{k} q_j(t) u^{n-j}. \tag{47}$$

For each $j = \overline{0,k}$ the polynomial $q_j(t)$ is given by

$$q_j(t) = \prod_{i=0, i\neq j}^{k} \frac{t - t_{n-i}}{t_{n-j} - t_{n-i}}.$$

The corrector polynomial $\omega^C(t)$ interpolates the unknowns u^{n+1} and the values of $\omega^P(t)$ at the moments of time t_{n+1} and $\{t_{n+1} - j\triangle t_n; j = \overline{1,k}\}$, respectively. The unknowns u^{n+1} are determined by imposing to the corrector polynomial $\omega^C(t)$ to satisfies the ODE. Then it follows a system of algebraic equations

$$\begin{aligned} m_i\left(\frac{a}{\triangle t_{n+1}} u_i^{n+1} - w_i^P(t_{n+1})\right) = \quad & \nu_{i+1/2}(u^{n+1}) \frac{u_{i+1}^{n+1} - u_i^{n+1}}{d_{i+1,i}} - \\ & -\nu_{i-1/2}(u^{n+1}) \frac{u_i^{n+1} - u_{i-1}^{n+1}}{d_{i,i-1}}, \end{aligned} \tag{48}$$

where

$$\nu_{i+1/2}(v) = \nu(v_{i+1}, v_i),$$

a is a constant specific to the order of the method and $\triangle t_{n+1} = t_{n+1} - t_n$ is the time step. The term $w^P(t_{n+1})$ in the l.h.s of (48) is a known quantity as a function of $\{u^n, ..., u^{n-k}\}$, namely

$$w_i^P(t_{n+1}) = -\dot{\omega}_i^P(t_{n+1}) + \frac{a}{\triangle t_{n+1}} \omega_i^P(t_{n+1}) = \sum_{j=0}^{k} \beta_j^{n+1} q_j(t_{n+1}) u^{n-j}. \tag{49}$$

where

$$\beta_j^{n+1} = -\sum_{l=0, l\neq j}^{k} \frac{1}{t_{n+1} - t_{n-l}} + \frac{a}{\triangle t_{n+1}}.$$

The implicit Euler method can be also described by (48) and (49). In this case the constant $a = 1$ and

$$w_i^P(t_{n+1}) = \frac{1}{\triangle t_{n+1}} u_i^n. \tag{50}$$

In the case of a Newtonian fluid one deals with a linear system of algebraic equations that can be solved. In the case of a non-Newtonian fluid we are facing with a nonlinear system and we must develop a nonlinear solver. We build up an iterative algorithm to solve the nonlinear equations (48) that provided good results. Our algorithm reads as

$$\begin{aligned} m_i\left(\frac{a}{\triangle t_{n+1}} u_i^{n+1,k} - w_i^P(t_{n+1})\right) = \; & \nu_{i+1/2}(u^{n+1,k-1}) \frac{u_{i+1}^{n+1,k} - u_i^{n+1,k}}{d_{i+1,i}} - \\ & -\nu_{i-1/2}(u^{n+1,k-1}) \frac{u_i^{n+1,k} - u_{i-1}^{n+1,k}}{d_{i,i-1}}. \end{aligned} \tag{51}$$

As the initial guess in the algorithm we use

$$u^{n+1,0} = u^n. \tag{52}$$

The iterative process is considered successful if for a given tolerance ε the residue $\mathcal{R}$

$$\begin{aligned} \mathcal{R}_i(\boldsymbol{u}) = \; & m_i\left(\frac{a}{\triangle t_{n+1}} u_i - w_i^P(t_{n+1})\right) - \nu_{i+1/2}(u) \frac{u_{i+1} - u_i}{d_{i+1,i}} - \\ & -\nu_{i-1/2}(u) \frac{u_i - u_{i-1}}{d_{i,i-1}} \end{aligned} \tag{53}$$

satisfies

$$\left|\left|\mathcal{R}\left(\boldsymbol{u}^{n+1,k}\right)\right|\right|_\infty \leq \varepsilon \tag{54}$$

in a maximum $LMAX$ iterations.

Proposition 3.2. (a) *For any time step $\triangle t_{n+1}$ there exists a solution of the equations* (48)*. The solution satisfies the inequality*

$$||u^{n+1}|| \leq \frac{M}{m} \frac{\triangle t_{n+1}}{a} ||w^P||.$$

(b) *For each iteration step l there exists a unique solution $u^{n+1,l}$ of the equation (51) which satisfies*

$$\frac{\triangle t_{n+1}}{a}\inf_j w_j^P \leq u_i^{n+1,l} \leq \frac{\triangle t_{n+1}}{a}\sup_j w_j^P.$$

Proof. (a) In order to prove the existence of a solution of the nonlinear equation (48) we use an application of the Browder fixed-point theorem that asserts that if there exists a positive real constant η such that $(\mathcal{R}(u),u) \geq 0$ for any $||u|| = \eta$ and the function $\mathcal{R}(u)$ is a continuous function on the sphere $||u|| \leq \eta$ then there exists a solution of the equation

$$\mathcal{R}_i(u) = 0$$

on that sphere, [23]. One has

$$(\mathcal{R}(u),u) = \frac{a}{\triangle t_{n+1}}\,||u||_2^2 - \left\langle u, w^P \right\rangle + \sum_i \nu_{i+1/2}(u)\frac{(u_{i+1}-u_i)^2}{d_{i+1,i}}$$

and from that and Cauchy-Schwartz inequality it follows

$$(\mathcal{R}(u),u) \geq \frac{a}{\triangle t_{n+1}}\,||u||_2^2 - \left\langle u, w^P \right\rangle \geq ||u||_2 \left(\frac{a}{\triangle t_{n+1}} m\,||u|| - M\,\left|\left|w^P\right|\right|\right)$$

(b) The existence and uniqueness of the solution of the linear system follows from the fact that the matrix of system is nonsingular. In order to prove the boundedness of a solution $u^{n+1,l}$ we analyze an index i_0 for which $u_{i_0}^{n+1,l} \leq u_i^{n+1,l}; \forall i$. We have

$$m_{i_0}\left(\frac{a}{\triangle t_{n+1}} u_{i_0}^{n+1,k} - w_{i_0}^P(t_{n+1})\right) \geq 0,$$

(the r.h.s of i_0–equation in (51) is a positive number), hence

$$u_i^{n+1,l} \geq u_{i_0}^{n+1,l} \geq \frac{\triangle t_{n+1}}{a} w_{i_0}^P \geq \frac{\triangle t_{n+1}}{a}\inf_i w_i^P$$

The upper bound case can be proved in the same manner.

3.2. NUMERICAL RESULTS

The core of the code used for numerical integration of the ODE (48) consists in the iterative algorithm (51) and the time step marching strategy to control the local error and to keep the time step size at a maximum value. As the first test we compare the numerical results obtained by our method with the exact solution for a Newtonian fluid. For a Newtonian fluid the iterative process (51) needs only a iteration.

The analytical solution of the problem (37) with the boundary conditions (39) and the initial data (40) is given by

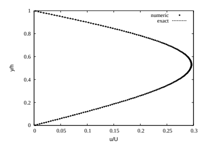
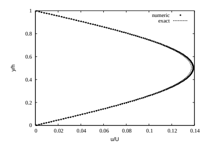
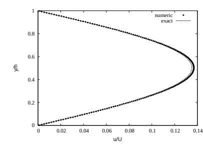
$$u_{exact}(t,y) = U\frac{2}{\pi}\sum_j \frac{1}{j}\sin\frac{j\pi}{h}(h-y)\mathrm{e}^{-\frac{j^2\pi^2}{h^2}\nu t}/. \tag{55}$$

The fig. 5 summarizes the essential facts about our numerical code.

The next aim is to test the nonlinear solver. For that we consider the Carreau-Yasuda model and we analyze the response of the code to some external data entries, like residual error ε, local time error TOL, maximal admissible time step $\triangle t_{max}$. The parameters in the constitutive model are $\nu_{inf} = 1.57 \times 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$, $\nu_0 = 15.7 \times 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$, $\Lambda = 0.11\mathrm{s}$, $n = 0.392$ and $a = 0.644$, human blood [16], [10].

Remark 1. As we can see in Table 1 the low tolerance in local time error implies high number of iterations in the iterative process and large residual error in the solution of nonlinear equations destroys the performance of the time integration scheme.

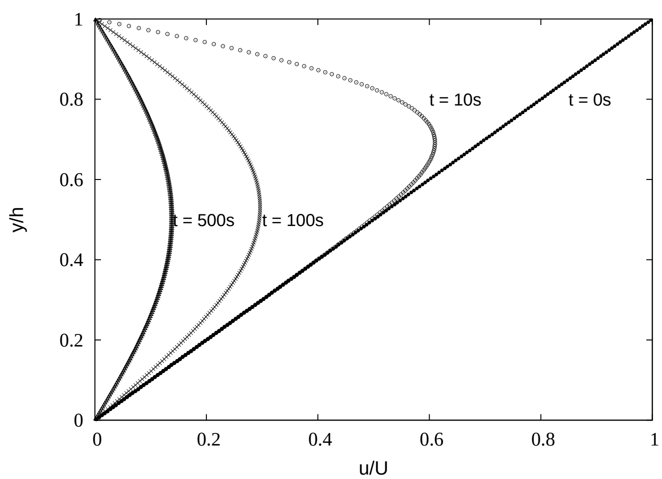
Remark 2. In the paper [9] the authors studied the problem of cessing of Couette flow of a Bingham plastic fluid. By numerical simulation they showed that the velocity decays to zero in a finite time, see also the references in that paper about the existence of finite stopping time. In the case of the pseudo-plastic fluid our numerical simulation shows that its velocity is greater than the velocity of a Newtonian fluid, hence there is no finite stowing time.

$t = 100\text{s}$	$t = 500\text{s}$	$t = 1000\text{s}$
$e_\infty = 0.5\%$ 	$e_\infty = 0.4\%$ 	$e_\infty = 0.4\%$ 

Comparison of analytical solution with the numerical solution

$$e_\infty = \sup_i \frac{|u_i(t) - u_{exact}(t, x_i)|}{||u_{exact}(t,x)||_\infty} \times 100$$

BDF of the fix order 4, $\triangle t_{max} = 2.\text{s}$, $TOL = 10^{-3} \div 10^{-7}$

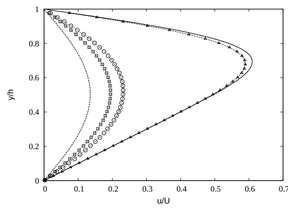
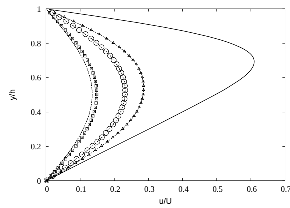
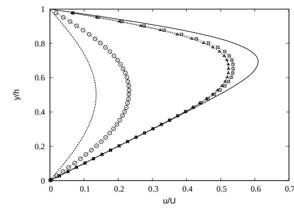


Evolution of the velocity. Numerical solution.

Fig. 5. Newtonian Fluid with $\nu = 15.7 \times 10^{-6}$. The evaluation was made on an uniform distributed net wit 200 internal knots. The distance be between walls was taken $h = 0.1\text{m}$ and the relative velocity of the wall was $U = 0.4\text{ms}^{-1}$.

Table 1 Response of the code to the data entry TOL and ε. $NREJECTED$ numbers of time step rejected, local error is bigger than tolerance TOL, $NSTEP$ number of time step accepted local error is smaller than tolerance TOL, $MAXITER$ the greater number of iterations in the iterative process encountered on the time integration interval $(0, T)$, $T = 100$s. All calculations were made on a uniform distributed net of 200 internal knots.

	$NREJECTED$	$NSTEP$	$MAXITER$
$\varepsilon = 10^{-5}$	374	2739	2
$\varepsilon = 10^{-7}$	190	1590	3
$\varepsilon = 10^{-9}$	8	609	5
$\varepsilon = 10^{-11}$	FAILED		
$TOL = 10^{-7}$, $\triangle t_{max} = 2.5$s			
$TOL = 10^{-3}$	0	136	7
$TOL = 10^{-5}$	0	199	6
$TOL = 10^{-7}$	8	196	5
$TOL = 10^{-8}$	FAILED		
$\varepsilon = 10^{-9}$, $\triangle t_{max} = 2.5$s			

$a = 0.1, 0.644, 0.9$	$n = 0.1, 0.392, 0.9$	$\Lambda = 0.11, 10.0, 14.1$
		

The reference parameters: $a = 0.644$, $n = 0.392$, $\Lambda = 0.11\mathrm{s}$

Fig. 6. The response of the numerical model to the variation of the parameters in Carreau-Yasuda model. The line draws the profiles of the velocity of the Newtonian fluid with viscosity ν_0 (left plot) and ν_∞ (right plot). The line-point draws the profiles of the velocity of pseudo-plastic fluids for several values of the parameters; the smallest value corresponds to the left plot.

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SURVIVAL OPTIMIZATION FOR A DIFFUSION PROCESS

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Abstract We consider the problem of optimally controlling a one-dimensional diffusion process in the interval $[-d, +d]$ when there is a reflecting boundary at $-d$ and an absorbing boundary at $+d$. Moreover, the constant d can actually be a random variable. The model can be used to represent the flight of an airplane between ground level and a level at which radar detection is likely. The objective is to maximize the survival time in the continuation region, while taking the quadratic control costs into account.

Keywords: LQG homing, Brownian motion, hitting time.

2000 MSC: 93E20.

1. INTRODUCTION

In Lefebvre and Whittle (1988), the authors considered in particular a one-dimensional Brownian motion as a rudimentary model for the flight of an airplane. Let $x(t)$ be the state variable and let $u(t)$ denote the control variable. They assumed that

$$dx(t) = b\,u(t)\,dt + \sigma\,dW(t),$$

where $b \neq 0$ and $\sigma > 0$ are constants, and $W(t)$ is a standard Brownian motion. The objective was to minimize the expected value of the cost function

$$J(x) = \int_0^{\tau(x)} \left[\tfrac{1}{2} q u^2(t) - \lambda\right] dt, \tag{1}$$

where q and λ are positive constants, and

$$\tau(x) = \inf\{t > 0 : x(t) = -d \text{ or } d \mid x(0) = x\}$$

with $x \in C := (-d, d)$. That is, $\tau(x)$ is the first time the controlled process $x(t)$ leaves the continuation region C, given that it started at $x \in C$. The authors showed that the optimal control u^* is given by $u^* = \frac{\sigma^2}{b} \frac{G'(x)}{G(x)}$, where

$$G(x) := E\left[\exp\{\alpha \lambda T(x)\}\right] \tag{2}$$

with $\alpha = \frac{b^2}{\sigma^2 q}$ (> 0).

In (2), $T(x)$ is the random variable that corresponds to $\tau(x)$ in the case of the *uncontrolled* process

$$dx_1(t) = \sigma \, dW(t).$$

The function $G(x)$ is the moment generating function of $T(x)$. Since $x_1(t)$ is a Wiener process with infinitesimal mean $\mu = 0$ and infinitesimal variance σ^2, the function G satisfies the ordinary differential equation

$$\frac{\sigma^2}{2} G''(x) = -\alpha \lambda G(x), \tag{3}$$

subject to the boundary conditions $G(\pm d) = 1$. We find that

$$G(x) = \frac{\cos(\gamma x)}{\cos(\gamma d)} \quad \text{for } -d \le x \le d,$$

where $\gamma := \left(\frac{2\alpha\lambda}{\sigma^2}\right)^{1/2}$, and where we assume that $0 < \gamma d < \pi/2$. It follows that

$$u^* = -\frac{\sigma^2 \gamma}{b} \tan(\gamma x) = -\text{sgn}(b) \sqrt{\frac{2\lambda}{q}} \tan(\gamma x) \quad \text{for } -d < x < d.$$

In this model, $x = -d$ represented ground level, while $x = +d$ was a level at which radar detection was likely. Therefore, since $\lambda > 0$, the aim was to maximize the survival time in the continuation region C, while taking the quadratic control costs into account. When the parameter λ is negative, Whittle (1982) termed this type of problem *LQG homing*.

In Section 2, we will assume that the constant d is actually a random variable D and that the boundary at $x = -d$ is reflecting rather than absorbing. We will compute the optimal control for a particular distribution of D. In Section 3, we will treat the case when $\lambda < 0$. Finally, some concluding remarks will be made in Section 4.

2. SURVIVAL OPTIMIZATION

If the diffusion process $x_1(t)$ has a reflecting barrier at $x = b_0$, then the function $G(x)$ defined in (2) is such that (see Cox and Miller (1965, p. 231), for instance)

$$G'(x)\big|_{x=b_0} = 0.$$

Hence, if there is a reflecting barrier at $x = -d$, and an absorbing barrier at $x = d$, we must solve the ordinary differential equation (3) subject to

$$G(d) = 1 \quad \text{and} \quad G'(x)\big|_{x=-d} = 0.$$

We easily find that

$$G(x) = \frac{\cos[\gamma(x+d)]}{\cos(2\gamma d)} \quad \text{for } -d \leq x \leq d,$$

where $\gamma = \sqrt{2\alpha\lambda}/\sigma$, as above.

Now, if d is replaced by the random variable D defined in the interval $(0, \frac{\pi}{4\gamma})$ (so that $2\gamma d \in (0, \frac{\pi}{2})$, as required), we may write that

$$G(x) = E\left[E\left[e^{\alpha\lambda T(x)} \mid D\right]\right] = \int_0^{\frac{\pi}{4\gamma}} \frac{\cos[\gamma(x+y)]}{\cos(2\gamma y)} f_D(y)\, dy. \tag{4}$$

We can obtain an explicit formula for $G(x)$ by choosing, in particular,

$$f_D(d) = k\cos(2\gamma d) \quad \text{for } d \in \left[d_0, \frac{\pi}{4\gamma}\right), \tag{5}$$

where k is a normalizing constant and $d_0 \in (0, \frac{\pi}{4\gamma})$. Indeed, we then obtain that

$$\begin{aligned} G(x) &= k \int_{d_0}^{\frac{\pi}{4\gamma}} \cos[\gamma(x+y)]\, dy = \frac{k}{\gamma} \left\{ \sin\left[\gamma\left(x + \frac{\pi}{4\gamma}\right)\right] - \sin\left[\gamma\left(x + d_0\right)\right] \right\} \\ &= \frac{k}{\gamma} \left(\frac{\sqrt{2}}{2} \sin(\gamma x) + \frac{\sqrt{2}}{2} \cos(\gamma x) - \sin\left[\gamma\left(x + d_0\right)\right] \right). \end{aligned}$$

We may state the following proposition.

Proposition 2.1. *With the choice in (5) for the distribution of the random variable D, when there is a reflecting barrier at $x = -D$ and an absorbing barrier at $x = D$, the optimal control is given by*

$$u^* = \mathrm{sgn}(b) \left(\frac{2\lambda}{q}\right)^{1/2} \frac{\frac{\sqrt{2}}{2}\cos(\gamma x) - \frac{\sqrt{2}}{2}\sin(\gamma x) - \cos\left[\gamma\left(x + d_0\right)\right]}{\frac{\sqrt{2}}{2}\sin(\gamma x) + \frac{\sqrt{2}}{2}\cos(\gamma x) - \sin\left[\gamma\left(x + d_0\right)\right]}$$

for $-d_0 < x < d_0$.

Remarks. i) The random variable D could of course also be a discrete random variable taking its values in a set contained in the interval $[0, \frac{\pi}{4\gamma})$. For instance, D could be defined on the set $\{\frac{\pi}{6\gamma}, \frac{\pi}{5\gamma}\}$.

ii) If $x(t) = x$, with x such that $|x| \geq d_0$ (but the process has not reached the absorbing barrier yet), we could obtain the optimal control u^* by replacing $f_D(d)$ (in the continuous case) by the conditional density function $f_D(d \mid D > |x|)$ in the calculation of $G(x)$. Moreover, if the process reached the state $x_0 > d_0$ without being absorbed and then returned to the interval $(-d_0, d_0)$, we can use the formula above for the optimal control in the interval $(-x_0, x_0)$. Indeed, the optimizer then knows that the absorbing barrier is not located in the interval $[d_0, x_0]$. Similarly, if the controlled process takes on a value $-x_0$ in the interval $(-\frac{\pi}{4\gamma}, -d_0]$ and returns to $(-d_0, d_0)$, then the optimizer again infers that the value of D is not in the interval $[d_0, x_0]$, and we can now use the formula for u^* in $(-x_0, x_0)$.

iii) Let $b = \sigma = \lambda = 1$ and $q = 2$. Then, $\gamma = 1$ and (with $d_0 = \pi/6$)

$$u^* = \frac{(\sqrt{2}-\sqrt{3})\cos(x) + (1-\sqrt{2})\sin(x)}{(\sqrt{2}-\sqrt{3})\sin(x) - (1-\sqrt{2})\cos(x)} \quad \text{for } -\pi/6 < x < \pi/6.$$

This function is shown in fig. 1, together with the optimal control when $d = \pi/6$.

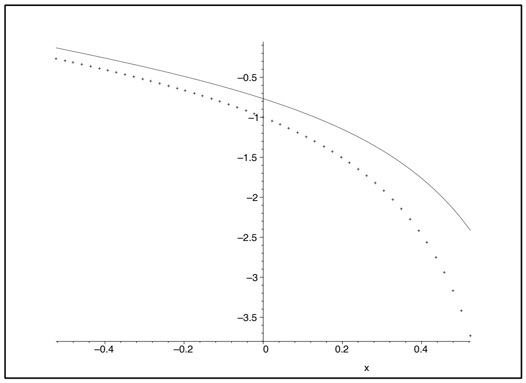

Fig. 1. Optimal control when $d = \pi/6$ (dotted line) and when d is random (full line).

iv) The function $G(x)$ being a mathematical expectation, we can make use of the formula in (4). However, the optimal control u^* is *not* a mathematical expectation. Therefore, we could not have obtained it by conditioning on D.

v) If we assume that the reflecting barrier is located at $x = 0$ rather than at $x = -d$, we find that

$$G(x) = \frac{\cos(\gamma x)}{\cos(\gamma d)} \quad \text{for } 0 \le x \le d.$$

In the case when d is a random variable D, we have

$$G(x) = \int_0^\infty \frac{\cos(\gamma x)}{\cos(\gamma y)} f_D(y)\, dy = \cos(\gamma x) \int_0^\infty \frac{f_D(y)}{\cos(\gamma y)}\, dy.$$

We see that the function G is simply multiplied by a constant. Since u^* is proportional to $G'(x)/G(x)$, we deduce that (unfortunately) the optimal control u^* remains the same.

In the next section, we will assume that the parameter λ is negative, so that the objective will then be to minimize the time spent in the continuation region C.

3. LQG HOMING

When $\lambda = -\theta$, with $\theta > 0$, in the cost function (1) and there is a reflecting boundary at $x = -d$, we find that the function $G(x)$ is given by

$$G(x) = \frac{\cosh[\mu(x+d)]}{\cosh(2\mu d)} \quad \text{for } -d \leq x \leq d,$$

where

$$\mu := \frac{1}{\sigma}\sqrt{2\alpha\theta}.$$

Remarks. i) We deduce from the function $G(x)$ that

$$u^* \propto \tanh\left[\mu(x+d)\right].$$

Hence, the optimal control does depend on d in that case, while if the barrier at $x = -d$ is absorbing (like in the original problem), then

$$u^* = \frac{\sigma^2}{b}\mu \tanh(\mu x),$$

which is independent of d.

ii) Notice that there is no constraint on μ, contrary to γ. That is, the penalty given for survival in C can be as large as we want, whereas we could not give too large a reward for remaining in C in the previous case.

As in the previous problem, we replace d by a random variable D, so that

$$G(x) = \int_0^\infty \frac{\cosh[\mu(x+y)]}{\cosh(2\mu y)} f_D(y)\,dy.$$

For simplicity, assume that

$$f_D(d) \propto \cosh(2\mu d) \quad \text{for } c_1 \leq d \leq c_2, \tag{6}$$

where c_1 and c_2 are positive constants. We calculate

$$G(x) \propto \int_{c_1}^{c_2} \cosh[\mu(x+y)]\, dy \propto \sinh[\mu(x+c_2)] - \sinh[\mu(x+c_1)]$$

for $-c_1 \leq x \leq c_1$. A simple calculation then yields the following proposition.

Proposition 1. *If $\lambda = -\theta \in (-\infty, 0)$ in (1) and d is a random variable D having the probability density function given in (6), then the optimal control in the case when the barrier at $x = -D$ is reflecting is (see a particular example in fig. 2)*

$$u^* = \frac{\sigma^2}{b}\mu \frac{\cosh[\mu(x+c_2)] - \cosh[\mu(x+c_1)]}{\sinh[\mu(x+c_2)] - \sinh[\mu(x+c_1)]}$$

for $-c_1 < x < c_1$.

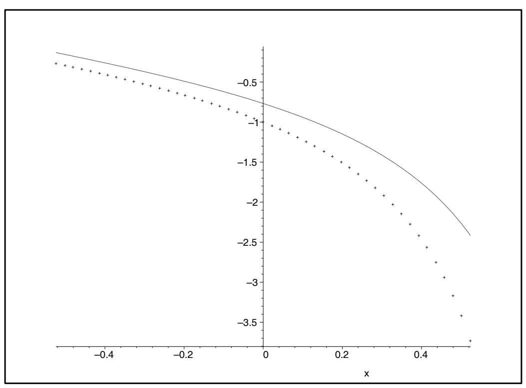

Fig. 2. Optimal control when d is deterministic (dotted line) and when d is random (full line).

4. CONCLUDING REMARKS

In this note, we have extended the survival optimization or LQG homing problem set up by Whittle (1982), and considered by Lefebvre and Whittle (1988), by assuming that the barriers are located at $x = \pm D$, where D is a random variable rather than a constant.

Going back to the application given in Lefebvre and Whittle (1988), namely that of an airplane trying to avoid hitting the ground at $x = -d$ or being detected by a radar at height $x = d$, to be more realistic we can assume that ground level corresponds to $x = 0$ (and that the height at which radar detection is likely is $x = d$, as before). If the two barriers are absorbing, which again is more realistic, then we find that

$$G(x) = \cos(\gamma x) + k_d \sin(\gamma x) \quad \text{for } 0 \le x \le d,$$

where k_d is a constant (that depends on d) given by

$$k_d = \frac{1 - \cos(\gamma d)}{\sin(\gamma d)}$$

(with $0 < \gamma d < \pi/2$). Moreover, the optimal control is

$$u^* = \frac{\sigma^2}{b} \mu \frac{[-\sin(\gamma x) + k_d \cos(\gamma x)]}{\cos(\gamma x) + k_d \sin(\gamma x)}$$

for $0 < x < d$.

Next, if d is replaced by the positive random variable D with range $[d_0, d_1)$, where $d_1 \in (d_0, \infty)$, we have

$$\begin{aligned} G(x) &= \cos(\gamma x) + \left[\int_{d_0}^{d_1} \frac{1 - \cos(\gamma y)}{\sin(\gamma y)} f_D(y)\, dy \right] \sin(\gamma x) \\ &:= \cos(\gamma x) + k^* \sin(\gamma x). \end{aligned}$$

It follows that

$$u^* = \frac{\sigma^2}{b} \mu \frac{[-\sin(\gamma x) + k^* \cos(\gamma x)]}{\cos(\gamma x) + k^* \sin(\gamma x)}$$

for $x \in (0, d_0)$. Therefore, the constant k_d is simply replaced by the new constant k^*, which depends on the distribution of the random variable D, in $(0, d_0)$. For $x \in [d_0, d_1)$, we should substitute $f_D(d)$ by $f_D(d \mid D > x)$ in the calculation of $G(x)$.

Similarly, with $\lambda = -\theta \in (-\infty, 0)$, we obtain

$$G(x) = e^{-\mu x} + \kappa_d \left(e^{\mu x} - e^{-\mu x}\right) \quad \text{for } 0 \le x \le d,$$

where

$$\kappa_d := \frac{1}{1 + e^{\mu d}},$$

and

$$u^* = \frac{\sigma^2}{b} \mu \frac{\kappa_d \left(e^{\mu x} + e^{-\mu x}\right) - e^{-\mu x}}{\kappa_d \left(e^{\mu x} - e^{-\mu x}\right) + e^{-\mu x}}.$$

When d becomes the random variable $D \in [d_0, d_1)$, the constant κ_d is replaced by the constant

$$\kappa^* := \int_{d_0}^{d_1} \frac{1}{1 + e^{\mu y}} f_D(y)\, dy$$

(in the interval $(0, d_0)$).

Finally, to bring the problem closer to reality, we could consider at least a two-dimensional model for the flight of the airplane. However, obtaining an explicit (and exact) expression for the function $G(x)$ is then generally very difficult.

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BIFURCATION AND SYMMETRY IN DIFFERENTIAL EQUATIONS NON-RESOLVED WITH RESPECT TO DERIVATIVE

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Abstract For the differential equations in Banach spaces nonresolved under derivative on the base of general theorem about group symmetry inheritance by relevant A. M. Lyapounov and E. Schmidt branching equations the possibilities of the reduction (dimension lowering) of the potential type branching equations are studied. As corollaries the results about general form of branching equations with rotational symmetries for evolution equation with degenerate Fredholm operator at the derivative are established. The closed results for stationary bifurcation [12]–[14] are quoted and used.

1. THE STATEMENT OF THE PROBLEM

The Lyapounov–Schmidt method in nonstationary bifurcation was suggested in the V. I. Yudovich works [1], [2] and some later in the articles [5], [6] of V. A. Trenogin. Then it was developed for the case of group symmetry presence for convection problems in hydrodynamics [3] and also for the evolution equations in Banach spaces with degenerate operator at the derivative [9], [10]. Here it is applied under group symmetry conditions to differential equations in Banach spaces non-resolved under derivative.

In real Banach spaces E_1 and E_2 the general problem of dynamic branching

$$F(p,x,\varepsilon)=0, \quad p=\frac{dx}{dt}, \quad F(0,x_0,\varepsilon)\equiv 0, \tag{1}$$

$$A=A_{x_0}=F'_p(0,x_0,0), \qquad B=B_{x_0}=-F'_{x_0}(0,x_0,0)$$

is studied, where A_{x_0} and B_{x_0} are Fredholm operators. We define $N(A) = span\{\phi_i\}_1^m$ – the zero subspace (kernel) of the operator A_{x_0}, $N^*(A_{x_0}) = span\{\widehat{\psi}_i\}_1^m$ – the subspace of defect functionals, $\{\vartheta_j\}_1^m \in E_1$, $\langle\phi_i, \vartheta_j\rangle = \delta_{ij}$ and $\{\zeta_j^{(1)}\}_1^m \in E_2$, $\langle\zeta_i^{(1)}, \widehat{\psi}_j^{(1)}\rangle = \delta_{ij}$ are relevant biorthogonal systems. Elements of complete $B-$ and B^*-Jordan sets [7]

$$\begin{aligned}&\{\phi_i^{(j)}\},\ j = 1,\dots,q_i,\ i = 1,\dots,m;\ \{\widehat{\psi}_k^{(l)}\},\ l = 1,\dots,q_k, k = 1,\dots,m,\\ &A\phi_i^{(\rho)} = B\phi_i^{(\rho-1)},\ \langle\phi_i^{(\rho)}, \vartheta_j\rangle = 0, \rho = 2,\dots,q_i, i, j = 1,\dots,m;\\ &A^*\widehat{\psi}_k^{(l)} = B^*\widehat{\psi}_k^{(l-1)},\ \langle\zeta_r^{(1)}, \widehat{\psi}_k^{(l)}\rangle = 0,\ l = 1,\dots,q_k,\ r, k = 1,\dots,m\end{aligned}$$

can be chosen [7], [8] satisfying the following biorthogonality relations

$$\langle\zeta_i^{(j)}, \widehat{\psi}_k^{(l)}\rangle = \delta_{ik}\delta_{jl}, \quad j(l) = 1,\dots,q_i(q_k),$$
$$\vartheta_k^{(l)} = B^*\widehat{\psi}_k^{(q_k+1-l)}, \quad \zeta_i^{(j)} = B\phi_i^{(q_i+1-j)}, \quad i, k = 1,\dots,m.$$

These relations generate the projectors on the root-subspace $E_1^{k_A} = K(A; B) = span\{\phi_i^{(j)}\},\ j = 1,\dots,q_i,\ i = 1,\dots,m$ and $E_{2,k_A} = span\{\zeta_i^{(j)}\},\ j = 1,\dots,q_1,\ i = 1,\dots,m$

$$\mathbf{p} = \sum_{i=1}^{m}\sum_{j=1}^{q_i}\langle\cdot, \vartheta_i^{(j)}\rangle\phi_i^{(j)} = \langle\cdot, \vartheta\rangle\phi :\ E_1 \to E_1^{k_A} = K(A, B),$$
$$\mathbf{q} = \sum_{i=1}^{m}\sum_{j=1}^{q_i}\langle\cdot, \widehat{\psi}_i^{(j)}\rangle\zeta_i^{(j)} = \langle\cdot, \widehat{\psi}\rangle\zeta :\ E_2 \to E_{2,k_A} = span\{\zeta_i^{(j)},\}$$

$k_A = \sum_{i=1}^m q_i = dim\ K(A; B)$ is the root-number. By virtue of smoothness property of the operator F the equation (1) can be written in the form

$$A_{x_0}\frac{dx}{dt} = B_{x_0}(x - x_0) - R(x_0, \frac{dx}{dt}, x - x_0, \varepsilon), \tag{2}$$
$$R(x_0, p, x - x_0, \varepsilon) = F(p, x, \varepsilon) - A_{x_0}p + B_{x_0}(x - x_0),\ p = \frac{dx}{dt} = \frac{d(x - x_0)}{dt}.$$

Let the A-spectrum $\sigma_A(B)$ of the operator B be splitting on two parts: $\sigma_A^-(B)$ lying strictly in the left complex half-plane and $\sigma_A^0(B)$ lying on the imaginary axis containing a finite number of the nonzero points $\pm i\beta$ of finite multiplicity eigenvalues decomposed on the disjoint subclasses $\pm i\alpha_r$, $\alpha_r = k_r\alpha$, $r = 1, \dots, \nu$ (k_r are natural numbers without nontrivial common divisors).

Let $\pm i\alpha_r \in \sigma_A(B)$, $r = 1, \dots, \nu$, be n_r–multiple eigenvalues with relevant eigenelements $u_{rj} = u_{1rj} \pm iu_{2rj}$, i.e.

$$Bu_{rj} = i\alpha_r Au_{rj} = ik_r\alpha_r Au_{rj}, B\bar{u}_{rj} = -i\alpha_r A\bar{u}_{rj} = -ik_r\alpha A\bar{u}_{rj}, j = 1, \dots, n_r,$$

and $v_{rj} = v_{1rj} \pm iv_{2rj}$ be eigenelements of adjoint operator, i.e.

$$B^*v_{rj} = -i\alpha_r A^*v_{rj} = -ik_r\alpha A^*v_{rj}, B^*\bar{v}_{rj} = i\alpha_r A^*\bar{v}_{rj} = ik_r\alpha A^*\bar{v}_{rj}, j = 1, \dots, n_r$$

with generalized Jordan chains (GJCh $\equiv$ A-JCh) of the lengths p_{rj}. This means the existence of the elements u^k_{rj}, $\bar{u}^{(k)}_{rj}$ and v^k_{rj}, $\bar{v}^{(k)}_{rj}$ such that

$$u^{(1)}_{rj} = u_{rj}, v^{(1)}_{rj} = v_{rj},$$

$$(B - ik_r\alpha A)u^{(k)}_{rj} = Au^{(k-1)}_{rj}, (B + ik_r\alpha A)\bar{u}^{(k)}_{rj} = -A\bar{u}^{(k-1)}_{rj},$$

$$(B^* + ik_r\alpha A^*)v^{(k)}_{rj} = -A^*v^{(k-1)}_{rj}, (B^* - ik_r\alpha A^*)\bar{b}^{(k)}_{rj} = A^*\bar{v}^{(k-1)}_{rj},$$

where, by virtue of GJChs biorthogonality lemma [7], $\langle Au^{(k)}_{rj}, v^{p_{\rho s}+1-l}_{\rho s}\rangle = \delta_{l\rho}\delta_{js}\delta_{kl}$ can be taken into account.

For every α its own branching equation (BEq) is constructing [9], [10] by the introduction of A. Poincaré substitution $t = \frac{\tau}{\alpha+\mu}$, $x(t) = y(\tau)$. Then the problem about $\frac{2\pi}{\alpha+\mu}$–periodic solution of (2) can be reduced to the determination of 2π–periodic solutions of the following equation

$$\mathfrak{B}_{x_0}y = \mu\mathcal{C}y + R(x_0, (\alpha+\mu)\frac{dy}{d\tau}, \varepsilon) \equiv \mathcal{R}(x_0, \frac{dy}{d\tau}, y, \mu, \varepsilon), \tag{3}$$

$$(\mathfrak{B}_{x_0}y)(\tau) \equiv B_{x_0}y(\tau) - \alpha A_{x_0}\frac{dy}{d\tau}, \quad (\mathcal{C}y)(\tau) \equiv A_{x_0}\frac{dy}{d\tau}.$$

Further the operator $(\mathfrak{B}y)(\tau)$ is supposed to be Fredholm. The operators $(\mathfrak{B}y)(\tau)$ and $(\mathcal{C}y)(\tau)$ map the space Y of 2π–periodic continuously differentiable functions of τ with values in $\mathcal{E}_1 = E_1\dot{+}iE_1$ into the space Z of 2π–periodic continuous functions of τ with values in $\mathcal{E}_2 = E_2\dot{+}iE_2$. Here the functionals of the special form are used

$$\langle\langle y, f\rangle\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle y(\tau), f(\tau)\rangle d\tau \quad y \in Y,\ f \in Y^* \ (y \in Z,\ f \in Z^*).$$

Zero-subspaces $N(\mathfrak{B})$ and $N(\mathfrak{B}^*)$ of the operators $\mathfrak{B}$ and $\mathfrak{B}^*$ are $2n$–dimensional, $n = n_1 + \ldots + n_\nu$

$$N(\mathfrak{B}) = N(\mathfrak{B}_{x_0}) = span\{\varphi_{rj}^{(1)} = \varphi_{rj}(x_0,\tau) = u_{rj}(x_0)e^{ik_r\tau};\ \bar{\varphi}_{rj}^{(1)} = \bar{u}_{rj}(x_0)e^{-ik_r\tau},\}$$
$$N(\mathfrak{B}^*) = N(\mathfrak{B}^*_{x_0}) = span\{\psi_{rj}^{(1)} = \psi_{rj}(x_0,\tau) = v_{rj}(x_0)e^{ik_r\tau};\ \bar{\psi}_{rj}^{(1)} = \bar{v}_{rj}(x_0)e^{-ik_r\tau}\},$$

$j = 1,\ldots,n_r,\ r = 1,\ldots,\nu$. To their basis elements, A–Jordan, A^*–Jordan chains $\varphi_{rj}^{(k)} = u_{rj}^{(k)}(x_0)e^{ik_r\tau}$, $\psi_{rj}^{(k)} = v_{rj}^{(k)}(x_0)e^{ik_r\tau}$, $k = 1,\ldots,p_{rj}$ of the lengths p_{rj} are corresponding, which also can be chosen satisfying the biorthogonality relations [9], [10]

$$\langle\langle\varphi_{rj}^k, \gamma_{\rho s}^{(l)}\rangle\rangle = \delta_{r\rho}\delta_{js}\delta_{kl}, \langle\langle z_{rj}^k, \psi_{\rho s}^{(l)}\rangle\rangle = \delta_{r\rho}\delta_{js}\delta_{kl}, k(l) = 1,\ldots,p_{rj}(p_{\rho s}),$$
$$\gamma_{\rho s}^{(l)} = A^*\psi_{\rho s}^{(p_s+1-l)}, z_{rj}^{(k)} = A\varphi_{rj}^{(p_{rj}+1-k)}, j(s) = 1,\ldots,n_r(n_\rho),\ r,\rho = 1,\ldots,\nu.$$

In order to construct the equivalent to (3) BEq A. Lyapounov and E. Schmidt approaches are applied [1]– [10]. The first uses the restriction $\widehat{\mathfrak{B}}$ to the complement of $N(\mathfrak{B})$ subspace; the second one – the generalized E. Schmidt lemma. Introduce the projectors

$$\mathbb{P}_{x_0} = \mathbf{P}_{x_0} + \overline{\mathbf{P}}_{x_0} = \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}\left[\langle\langle\cdot, \gamma_{rj}^{(1)}\rangle\rangle\varphi_{rj}^{(1)} + \langle\langle\cdot, \bar{\gamma}_{ij}^{(1)}\rangle\rangle\bar{\varphi}_{rj}^{(1)}\right],$$

$$\mathbf{P}_{x_0} : Y \to N(\mathfrak{B}) = Y^{2n},$$

$$\mathbb{Q}_{x_0} = \mathbf{Q}_{x_0} + \overline{\mathbf{Q}}_{x_0} = \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}\left[\langle\langle\cdot, \psi_{rj}^{(1)}\rangle\rangle z_{rj}^{(1)} + \langle\langle\cdot, \bar{\psi}_{ij}^{(1)}\rangle\rangle\bar{z}_{rj}^{(1)}\right],$$

$$\mathbf{Q}_{x_0} : Z \to\ span\{z_{rj}^{(1)}\} = Z_{2n},$$

generating the expansions of Banach spaces Y and Z into the direct sums $Y = Y^{2n}\dot{+}Y^{\infty-2n}$, $Z = Z_{2n}\dot{+}Z_{\infty-2n}$.

According to A. Lyapounov approach, finding the real solutions of the equation (3) in the form

$$y = (\mathbf{P}_{x_0} + \bar{\mathbf{P}}_{x_0})y \equiv v(x_0, \xi, \bar{\xi}) + u_{x_0} = \xi \cdot \varphi + \bar{\xi} \cdot \bar{\varphi} + u_{x_0} = v_{x_0}(\xi, \bar{\xi}) + u_{x_0},$$
$$\xi \cdot \varphi = \xi_{11}\varphi_{11}^{(1)} + \cdots + \xi_{n_1,1}\varphi_{n_1,1}^{(1)} + \cdots + \xi_{\nu,1}\varphi_{\nu,1}^{(1)} + \cdots + \xi_{\nu,n_\nu}\varphi_{\nu,n_\nu}^{(1)}, \xi_{rj}^{(1)} = \langle\langle y, \gamma_{rj}\rangle\rangle$$

come to the following equivalent system

$$\widehat{\mathfrak{B}}u_{x_0} = (I - \mathbf{Q}_{x_0} - \bar{\mathbf{Q}}_{x_0})\mathcal{R}(x_0, \frac{d}{d\tau}(u_{x_0} + v_{x_0}), u_{x_0} + v_{x_0}, \mu, \varepsilon), \tag{4}$$

$$f(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) \equiv (\mathbf{Q}_{x_0} + \bar{\mathbf{Q}}_{x_0})\mathcal{R}(x_0, \frac{d}{d\tau}(u_{x_0} + v_{x_0}), u_{x_0} + v_{x_0}, \mu, \varepsilon) = 0,$$

where $\widehat{\mathfrak{B}} = (I - \mathbf{P}_{x_0} - \bar{\mathbf{P}}_{x_0}) : Y^{\infty-2n} \to Z_{\infty-2n}$ is the restriction of the operator $\mathfrak{B}$ to $Y^{\infty-2n}$. The first equation (4) is the problem about 2π-periodic solutions in pair of Banach spaces $Y^{\infty-2n}$, $Z_{\infty-2n}$ with invertible operator $\widehat{\mathfrak{B}}$. It has the unique small solution $u_{x_0} = u(v(x_0, \xi, \bar{\xi}), \mu, \varepsilon)$ depending on parameters ξ, μ, ε. Its substitution into the second equation (4) gives the A. Lyapounov BEq

$$\begin{aligned} f(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) &= \mathbf{Q}_{x_0}\mathcal{R}(x_0, \frac{d}{d\tau}[u(v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) + v(x_0, \xi, \bar{\xi})], \\ &\quad u(v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) + v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = 0, \\ \bar{f}(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) &= \bar{\mathbf{Q}}_{x_0}\mathcal{R}(x_0, \frac{d}{d\tau}[u(v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) + v(x_0, \xi, \bar{\xi})], \\ &\quad u(v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) + v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = 0. \end{aligned} \tag{5}$$

E. Schmidt approach uses the generalized E. Schmidt lemma [7], according to which there exists the bounded operator Γ, inverse to the operator $\widetilde{\mathfrak{B}}_{x_0} = \mathfrak{B}_{x_0} + \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}[\langle\langle \cdot, \gamma_{rj}^{(1)}\rangle\rangle z_{rj}^{(1)} + \langle\langle \cdot, \bar{\gamma}_{rj}^{(1)}\rangle\rangle \bar{z}_{rj}^{(1)}]$. Then the equivalent to (3) system reads

$$\widetilde{\mathfrak{B}}y = \mathcal{R}(x_0, \frac{dy}{d\tau}, y, \mu, \varepsilon) + \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}[\xi_{rj}z_{rj} + \bar{\xi}_{rj}\bar{z}_{rj}], \tag{6}$$

$$\xi_{rj} = \langle\langle y, \gamma_{rj}\rangle\rangle, \qquad \bar{\xi}_{rj} = \langle\langle y, \bar{\gamma}_{rj}\rangle\rangle$$

By the implicit operator theorem the first equation (6) has the unique solution of the form $y = w + v_{x_0}, w = w(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon)$. Then the second equations (6) give the E. Schmidt BEq

$$t(x_0, v_{x_0}, \mu, \varepsilon) = \mathbf{P}_{x_0} w(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = 0, \tag{7}$$

$$\bar{t}(x_0, v_{x_0}, \mu, \varepsilon) = \bar{\mathbf{P}}_{x_0} w(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = 0.$$

Remark 1.1. *Acting similarly to article [11] the branching equation in the root subspaces can be constructed, which is essentially used in [11] at the investigation of the stability of bifurcating solutions*

In the articles [12], [13] for the stationary equation $F(x, \varepsilon) = 0$, $F(x_0, 0) = 0$ with potential operator $F(x, \varepsilon)$ and also in [14] for branching equations of potential type under group symmetry conditions the theorem about the BEq dimension lowering (reduction) is proved. Here, as in [14], on the base of the general theorem [15] about the equation (1) group symmetry inheritance by the relevant branching equation, when it is of potential type, necessary and sufficient conditons for the invariance of the BEq potential are established.

The cosymmetric identity of the BEq left-hand side with Lie algebra operators of the group representation is obtained, that allows us to prove the theorem of the BEq reduction together with corollaries about BEq general form construction for the case of the isolated branching point x_0 (the case of invariant kernels). The terminology and notations of [1–9] are used. The obtained results are announced in [16] and supported by Russian Fond of Basic Research and Romanian Academy, grant No. 07-01-91680.

2. GROUP SYMMETRY INHERITANCE THEOREM, BRANCHING EQUATION OF POTENTIAL TYPE, COSYMMETRIC IDENTITY

Further it is supposed that the operator F in the equation (1) allows the group G, i.e. there exist the representations L_g in E_1 and K_g in E_2 intertwining the operator F

$$K_g F(p, x, \varepsilon) = F(L_g p, L_g x, \varepsilon). \tag{8}$$

Differentiation of the identity (8) with respect to x at the branching point x_0 yields the relations

$$K_g F'_p(0, x_0, \varepsilon) = F'_p(0, L_g x_0, \varepsilon) L_g \Leftrightarrow K_g A_{x_0} = A_{L_g x_0} L_g, \tag{9}$$

$$K_g F'_x(0, x_0, \varepsilon) = F'_x(0, L_g x_0, \varepsilon) L_g \Leftrightarrow K_g B_{x_0} = B_{L_g x_0} L_g,$$

whence the Fredholm operators A_{x_0} and B_{x_0} possess the symmetry only with respect to stationary subgroup of the point x_0.

The action of the representations L_g (K_g) in E_1 (E_2) is naturally extended to the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$. Everywhere below it is supposed that the Lie group $G = G_l = G_l(a)$ $a = (a_1, \dots, a_l)$ is l–dimensional differentiable manifold satisfying the following conditions [12, 13, 17]:

c_1) the mapping $a \to L_{g(a)} x_0$ acting from the neighbourhood of the unit element of $G_l(a)$ in the space E_1 belongs to the class C^1. Therefore $X x_0 \in E_1$ for all infinitesimal operators $Xx = \lim_{t \to 0} t^{-1}(L_{g(a(t))} x - x)$ in the tangent to $L_g(a)$ manifold $T^l_g(a)$;

c_2) the stationary subgroup of the element $x_0 \in E_1$ defines the representation $L(G_s)$ of the local Lie group $G_s \subset G_l$, $s < l$ with s-dimensional subalgebra $T^s_{g(a)}$ of infinitesimal operators.

Being $\pm i\alpha_r$ as n_r–dimensional eigenvalue at the point spectrum $P\sigma_A(B)$ means that for the operator $\mathbf{B}(\alpha_r) = \begin{pmatrix} B & \alpha_r A \\ -\alpha_r A & B \end{pmatrix} : E_1\dot{+}E_1 \to E_2\dot{+}E_2$ the zero-subspaces are $2n_r$–dimensional, i.e.

$$N(\mathbf{B}(\alpha_r)) = span\left\{\Phi^{(1)}_{1rj} = \begin{pmatrix} u_{1rj} \\ u_{2rj} \end{pmatrix},\ \Phi^{(1)}_{2rj} = \begin{pmatrix} -u_{2rj} \\ u_{1rj} \end{pmatrix},\ j = 1,\dots,n_r\right\},$$

$$N^*(\mathbf{B}(\alpha_r)) = span\left\{\Psi^{(1)}_{1rj} = \begin{pmatrix} v_{2rj} \\ -v_{1rj} \end{pmatrix},\ \Psi^{(1)}_{2rj} = \begin{pmatrix} v_{1rj} \\ v_{2rj} \end{pmatrix},\ j = 1,\dots,n_r\right\}.$$

Consequently the condition c_2) in the space $\mathcal{E}_1$ means that the elements $(\xi_{rk}\varphi_{rk}+\bar{\xi}_{rk}\bar{\varphi}_{rk}) = X_{rk}x_0$, $X_{rk} \in T^l_{g(a)}$ form in every $N(\mathcal{B}^{(r)}_{x_0}) = span\{\varphi_{r1}, \bar{\varphi}_{r1}, \dots, \varphi_{r,n_r}, \bar{\varphi}_{r,n_r}\}$ some $2æ = 2(l-s)$–dimensional subspace and also the bases in $N(\mathfrak{B}^{(r)}_{x_0})$ with fixed r (not for all $r = 1,\dots,\nu$ simultaneously) and in the algebra $T^l_{g(a)}$ can be ordered so that $\xi_{rk}\varphi_{rk} + \bar{\xi}_{rk}\bar{\varphi}_{rk} = X_{rk}x_0$, $1 \le k \le h$, $X_{rj}x_0 = 0$ for $j \ge æ + 1$.

As in [12, 13] the dense embeddings $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{H}$ ($E_1 \subset E_2 \subset H$) in the Hilbert spaces $\mathcal{H}$ (H) are supposed with estimates $\|y\|_{\mathcal{H}} \le \alpha_2\|y\|_{\mathcal{E}_2} \le \alpha_1\|y\|_{\mathcal{E}_1}$;

c_3) for every $X \in T^l_{g(a)}$ the map $X : \mathcal{E}_1 \to \mathcal{H}$ is bounded in $\mathcal{L}(\mathcal{E}, \mathcal{H})$-topology.

By virtue of the shift symmetry on the time (the original nonlinear equation (1) is autonomous) the BEq admits the rotation group $SO(2)$ in every pair of variables ξ_{n_rj}, $\bar{\xi}_{n_rj}$; f_{n_rj}, $\bar{f}_{n_rj}$ with matrix representation $\mathcal{A}_0(a_0) = \mathcal{B}_0(a_0) = diag\begin{pmatrix} e^{ia_0} & 0 \\ 0 & e^{-ia_0} \end{pmatrix}$ and the infinitesimal operator $\widehat{X}_0 = \{\widehat{X}_0(\xi), \widehat{X}_0(f)\}$, $\widehat{X}_0(\xi) = \sum\limits_{r=1}^{\nu}\sum\limits_{j=1}^{n_r}(\xi_{rj}\partial_{\xi_{rj}} - \bar{\xi}_{rj}\partial_{\bar{\xi}_{rj}})$, $\partial_{\xi_{rj}} = \frac{\partial}{\partial\xi_{rj}}$.

The problem of small solutions construction for the equation (3) for small ε and $\mu = \mu(\varepsilon)$ is equivalent to the finding of small solutions for finite dimensional nonlinear A. Lyapounov branching equation (1) or E. Schmidt branching equation (7).

First consider A. M. Lyapounov BEq construction for nonlinear problem (3) under group symmetry conditions (8). Set $(\mathfrak{B}_{L_g x_0} y)(\tau) = (B_{L_g x_0} y)(\tau) - \alpha A_{L_g x_0}\frac{dy}{d\tau}$ and $\mathcal{R}(x_0, \frac{dy}{d\tau}, y, \mu, \varepsilon) = F((\alpha+\mu)\frac{dy}{d\tau}, y, \varepsilon) - \alpha A_{x_0}\frac{dy}{d\tau} + B_{x_0} y$, where $F(0, x_0, \varepsilon) = 0$, $(x_0; 0)$ is the bifurcation point. Then the relations (9) take the form

$$K_g \mathfrak{B}_{x_0} y = K_g(B_{x_0} y - \alpha A_{x_0}\frac{dy}{d\tau}) = B_{L_g x_0} L_g y - \alpha A_{L_g x_0}\frac{dL_g y}{d\tau} = \mathfrak{B}_{L_g x_0} L_g y \Rightarrow$$

$$\varphi_{rj}(L_g x_0, \tau) = L_g \varphi_{rj}(x_0, \tau), \quad \varphi_{rj}(x_0, \tau) = \varphi_{rj},$$

$$\gamma_{rj}(L_g x_0, \tau) = L_g^{*^{-1}} \gamma_{rj}(x_0, \tau), \quad \gamma_{rj}(x_0, \tau) = \gamma_{rj}, \ r = 1, \dots, \nu, \ j = 1, \dots, n_r,$$

$$K_g \mathcal{R}(x_0, \frac{dy}{d\tau}, y, \mu, \varepsilon) = K_g F((\alpha+\mu)\frac{dy}{d\tau}, y, \varepsilon) - \alpha K_g A_{x_0}\frac{dy}{d\tau} + K_g B_{x_0} y =$$
$$= F((\alpha+\mu)\frac{dL_g y}{d\tau}, L_g y, \varepsilon) - \alpha A_{L_g x_0}\frac{dL_g y}{d\tau} + B_{L_g x_0} L_g y = \mathcal{R}(L_g x_0, \frac{dL_g y}{d\tau}, L_g y, \mu, \varepsilon).$$

For the range $R(\mathfrak{B}_{x_0})$ of the operator $\mathfrak{B}_{x_0}$ the relation

$$R(\mathfrak{B}_{x_0}) = R(K_g \mathfrak{B}_{x_0} L_g^{-1}) = K_g R(\mathfrak{B}_{x_0}).$$

is valid. Then for the defect-subspace of the operator $\mathcal{B}_{x_0}$ it follows

$$N^*(\mathfrak{B}_{x_0}) = span\{\psi_{rj}, \bar{\psi}_{rj}\} \ \Rightarrow \ N^*(\mathfrak{B}_{L_g x_0}) = span\{K_g^{*^{-1}}\psi_{rj}, K_g^{*^{-1}}\bar{\psi}_{rj}\},$$
$$z_{rj}(L_g x_0) = K_g z_{rj}(x_0), \ r = 1, \dots, \nu, \ j = 1, \dots, n_r$$

For the projectors $\mathbf{P}_{x_0}$, $\bar{\mathbf{P}}_{x_0}$ and $\mathbf{Q}_{x_0}$, $\bar{\mathbf{Q}}_{x_0}$ the following relations are true

$$\begin{aligned} \mathbf{P}_{L_g x_0} &= L_g \mathbf{P}_{x_0} L_g^{-1} \text{or} \mathbf{P}_{L_g x_0} L_g = L_g \mathbf{P}_{x_0}, \\ \bar{\mathbf{P}}_{L_g x_0} &= L_g \bar{\mathbf{P}}_{x_0} L_g^{-1} \text{or} \bar{\mathbf{P}}_{L_g x_0} L_g = L_g \bar{\mathbf{P}}_{x_0}, \\ \mathbf{Q}_{L_g x_0} &= K_g \mathbf{Q}_{x_0} K_g^{-1} \text{or} \mathbf{Q}_{L_g x_0} K_g = K_g \mathbf{Q}_{x_0}, \\ \bar{\mathbf{Q}}_{L_g x_0} &= K_g \bar{\mathbf{Q}}_{x_0} K_g^{-1} \text{or} \bar{\mathbf{Q}}_{L_g x_0} K_g = K_g \bar{\mathbf{Q}}_{x_0}. \end{aligned} \tag{10}$$

Lyapounov–Schmidt method uses the expansion of the Banach spaces $Y = Y^{2n}(x_0)\dot{+}Y^{\infty-2n}(x_0)$ and $Z = Z_{2n}(x_0)\dot{+}Z_{\infty-2n}(x_0)$ satisfying the properties

$Y^{2n}(L_g x_0) = L_g Y^{2n}(x_0)$, $Y^{\infty-2n}(L_g x_0) = (I - \mathbf{P}_{L_g x_0} - \bar{\mathbf{P}}_{L_g x_0})Y = L_g(I - \mathbf{P}_{x_0} - \bar{\mathbf{P}}_{x_0})L_g^{-1}Y = L_g(I - \mathbf{P}_{x_0} - \bar{\mathbf{P}}_{x_0})Y = L_g Y^{\infty-2n}(x_0)$, $Z_{2n}(L_g x_0) = K_g Z_{2n}(x_0)$, $Z_{\infty-2n}(L_g x_0) = K_g Z_{\infty-2n}(x_0)$.

Lemma 1. *If the subspace $Y^{\infty-2n}(x_0) = I - P_{x_0}$ is invariant to operators L_g (condition I of the articles [9]- [10]), then the subspace $Y^{\infty-2n}(L_g x_0)$ is also invariant to operators L_g.*

Proof. In fact, the inheritance of $Y^{\infty-2n}(x_0)$ to operators L_g is equivalent to the invariance of the subspace $span\{\gamma_{11}, \overline{\gamma}_{11}, \dots, \gamma_{1n_1}, \overline{\gamma}_{1n_1}, \dots, \gamma_{\nu 1}, \overline{\gamma}_{\nu 1}, \dots, \gamma_{\nu n_\nu}, \overline{\gamma}_{\nu n_\nu}\}$ to operators L_g^*. Then the assertion follows from the definition of the functionals $\gamma_{rj}(L_g x_0, \tau)$ and (2).

Similarly to [14]- [15] the following assertion about A. M. Lyapounov and E. Schmidt BEqs group symmetry inheritance can be proved

Theorem 2.1. *The following relations are true*

$$\boldsymbol{f}(L_g x_0, L_g v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = \boldsymbol{f}(L_g x_0, v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon) = K_g \boldsymbol{f}(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) \tag{11}$$

$$\boldsymbol{t}(L_g x_0, L_g v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) = \boldsymbol{t}(L_g x_0, v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon) = L_g \boldsymbol{t}(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon). \tag{12}$$

Proof. According to [10] rewrite the equation (1) near the branching point $L_g x_0$ in projections

$$\begin{gathered} \widehat{\mathfrak{B}}_{L_g x_0} \widetilde{u} = (I - \mathbf{Q}_{L_g x_0} - \bar{\mathbf{Q}}_{L_g x_0}) \mathcal{R}\left(L_g x_0, \tfrac{d}{d\tau}\left[\widetilde{u} + v(L_g x_0, \xi, \bar{\xi})\right], \widetilde{u} + v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon\right) \\ 0 = (\mathbf{Q}_{L_g x_0} + \bar{\mathbf{Q}}_{L_g x_0}) \mathcal{R}\left(L_g x_0, \tfrac{d}{d\tau}\left[\widetilde{u} + v(L_g x_0, \xi, \bar{\xi})\right], \widetilde{u} + v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon\right) \end{gathered} \tag{13}$$

For the restriction of the operator $\mathfrak{B}_{x_0}$ to the subspace $\mathcal{Y}^{\infty-2n}$ the symmetry relation

$$\begin{aligned} K_g \widehat{\mathfrak{B}}_{x_0} = K_g \mathfrak{B}_{x_0}(I - \mathbf{P}_{x_0} - \bar{\mathbf{P}}_{x_0}) \overset{(9)}{=} \mathfrak{B}_{L_g x_0} L_g (I - \mathbf{P}_{x_0} - \bar{\mathbf{P}}_{x_0}) = \\ = \mathfrak{B}_{L_g x_0}(I - \mathbf{P}_{L_g x_0} - \bar{\mathbf{P}}_{L_g x_0}) L_g = \widehat{\mathfrak{B}}_{L_g x_0} L_g \end{aligned} \tag{14}$$

is fulfilled. Application of $K_g^{-1} = K_{g^{-1}}$ to the first equation (13) gives

$$K_g^{-1}\widehat{\mathfrak{B}}_{L_g x_0}\widetilde{u} \overset{(14)}{=} \widehat{\mathfrak{B}}_{x_0} L_g^{-1}\widetilde{u} \overset{(10)}{=}$$
$$= (I - \mathbf{Q}_{x_0} - \bar{\mathbf{Q}}_{x_0})K_g^{-1}\mathcal{R}\left(L_g x_0, \frac{d}{d\tau}\left[\widetilde{u} + v(L_g x_0, \xi, \bar{\xi})\right], \widetilde{u} + v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon\right) =$$
$$= (I - \mathbf{Q}_{x_0} - \bar{\mathbf{Q}}_{x_0})\mathcal{R}\left(x_0, \frac{d}{d\tau}\left[L_g^{-1}\widetilde{u} + v(x_0, \xi, \bar{\xi})\right], L_g^{-1}\widetilde{u} + v(x_0, \xi, \bar{\xi}), \mu, \varepsilon\right).$$

According to implicit operators theorem the first equation (13) has the unique solution

$$L_g^{-1}\widetilde{u} = u(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon) \in \mathcal{Y}^{\infty - 2n} \Longrightarrow \widetilde{u} = L_g u(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon).$$

The substitution of the found solution in the second equation (13) gives the A. Lyapounov BEq at the bifurcation point $(L_g x_0, 0)$ and its group symmetry (11)

$$\boldsymbol{f}(L_g x_0, v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon) =$$
$$(\mathbf{Q}_{L_g x_0}\mathcal{R}(L_g x_0, \frac{d}{d\tau}(L_g u_{x_0} + v(L_g u_{x_0}, \xi, \bar{\xi})), L_g u_{x_0} + v(L_g u_{x_0}, \xi, \bar{\xi}), \mu, \varepsilon) =$$
$$= (\mathbf{Q}_{L_g x_0} K_g \mathcal{R}(x_0, \frac{d}{d\tau}(u_{x_0} + v(u_{x_0}, \xi, \bar{\xi})), u_{x_0} + v(u_{x_0}, \xi, \bar{\xi}), \mu, \varepsilon) \overset{(10)}{=}$$
$$= K_g \boldsymbol{f}(x_0, v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon);$$
$$\bar{\boldsymbol{f}}(L_g x_0, v(L_g x_0, \xi, \bar{\xi}), \mu, \varepsilon) = K_g \bar{\boldsymbol{f}}(x_0, v(x_0, \xi, \bar{\xi}), \mu, \varepsilon).$$

For the proof of the relation (12) rewrite the equation (1) near the bifurcation point $(L_g x_0; 0)$ in the form of the system

$$\widehat{\mathfrak{B}}_{L_g x_0} L_g y = \mathcal{R}(L_g x_0, \frac{d}{d\tau}(L_g y), L_g y, \mu, \varepsilon) + \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}\left[\xi_{rj} z_{rj}(L_g x_0) + \bar{\xi}_{rj}\bar{z}_{rj}(L_g x_0)\right],$$
$$\xi_{rj} = \left\langle\left\langle L_g y, \gamma_{rj}(L_g x_0)\right\rangle\right\rangle, \qquad z_{rj} = \left\langle\left\langle L_g y, \bar{\gamma}_{rj}(L_g x_0)\right\rangle\right\rangle. \qquad (15)$$

Setting $L_g y = L_g v(x_0,\xi,\overline{\xi}) + \widetilde{w} = v(L_g x_0,\xi,\overline{\xi}) + \widetilde{w}$, by virtue of the group symmetry of the operator $\widetilde{\mathfrak{B}}_{x_0}$

$$K_g\widetilde{\mathfrak{B}}_{x_0}h = K_g\mathfrak{B}_{x_0}h + \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}\left[\left\langle\left\langle h,\gamma_{rj}(x_0)\right\rangle\right\rangle K_g z_{rj}(x_0) + \left\langle\left\langle h,\overline{\gamma}_{rj}(x_0)\right\rangle\right\rangle K_g\overline{z}_{rj}(x_0)\right] =$$
$$\overset{(9),(10)}{=} \mathfrak{B}_{L_gx_0}L_gh + \sum_{r=1}^{\nu}\sum_{j=1}^{n_r}\left[\left\langle\left\langle L_gh,\gamma_{rj}(L_gx_0)\right\rangle\right\rangle z_{rj}(L_gx_0) + \right.$$
$$\left. + \left\langle\left\langle L_gh,\overline{\gamma}_{rj}(L_gx_0)\right\rangle\right\rangle \overline{z}_{rj}(L_gx_0)\right] = \widetilde{\mathfrak{B}}_{L_gx_0}L_gh$$

it follows

$$0 = \widetilde{\mathfrak{B}}_{L_gx_0}\widetilde{w} - \mathcal{R}(L_gx_0, \frac{d}{d\tau}\left[L_gv(x_0,\xi,\overline{\xi}) + \widetilde{w}\right], L_gv(x_0,\xi,\overline{\xi}) + \widetilde{w},\mu,\varepsilon) =$$
$$= K_g\left(\widetilde{\mathfrak{B}}_{x_0}L_g^{-1}\widetilde{w} - \mathcal{R}(x_0, \frac{d}{d\tau}\left[v(x_0,\xi,\overline{\xi}) + L_g^{-1}\widetilde{w}\right], v(x_0,\xi,\overline{\xi}) + L_g^{-1}\widetilde{w},\mu,\varepsilon)\right).$$

Hence $L_g^{-1}\widetilde{w} = w(x_0, v(x_0,\xi,\overline{\xi}),\mu,\varepsilon)$ or $\widetilde{w} = L_g w(x_0, v(x_0,\xi,\overline{\xi}),\mu,\varepsilon)$ is valid. Then the second equation (15) gives E. Schmidt BEq at the point L_gx_0 and its group symmetry (12)

$$\boldsymbol{t}(L_gx_0, v(L_gx_0,\xi,\overline{\xi}),\mu,\varepsilon) = P_{L_gx_0}\widetilde{w} =$$
$$= P_{L_gx_0}L_g w(x_0, v(x_0,\xi,\overline{\xi}),\mu,\varepsilon) \overset{(10)}{=} L_gP_{x_0}w(x_0,\xi,\overline{\xi}),\mu,\varepsilon) = L_g\boldsymbol{t}(x_0, v(x_0,\xi,\overline{\xi}),\mu,\varepsilon);$$
$$\overline{\boldsymbol{t}}(L_gx_0, v(L_gx_0,\xi,\overline{\xi}),\mu,\varepsilon) = L_g\overline{\boldsymbol{t}}(x_0, v(x_0,\xi,\overline{\xi}),\mu,\varepsilon).$$

Remark 2.1. *Note that Theorem 1 takes place for discrete or continuous group G without the requirements c_1)–c_3).*

Remark 2.2. *Let x_0 be an isolated bifurcation point of the equation* (1)*. Then the zero-subspace $N(\mathfrak{B}_{x_0})$ is invariant with respect to the matrix representation $\mathcal{A}_g$: $\mathcal{A}_g\varphi_{ri} = \sum_{j=1}^{n_r} a_{r,ji}\varphi_{rj}$, $\mathcal{A}_g\overline{\varphi}_{ri} = \sum_{j=1}^{n_r} a_{r,ji}\overline{\varphi}_{rj} \Rightarrow \tilde{\xi}_{ri} = \mathcal{A}_g\xi_{ri} = \sum_{j=1}^{n_r} a_{r,ij}\xi_{rj}$ and $v(L_gx_0,\xi,\overline{\xi}) = L_gv(x_0,\xi,\overline{\xi}) = v(x_0,\tilde{\xi},\tilde{\overline{\xi}}) = v(x_0,\mathcal{A}_g\xi,\mathcal{A}_g\overline{\xi})$. Analogously the subspace $N^*(\mathfrak{B}_{x_0})$ is invariant to the representation $\mathcal{B}_g$: $\mathcal{B}_g\psi_{rk} = \sum_{j=1}^{n_r}\beta_{r,kj}\psi_j$.*

Corollary 2. *For the invariant kernel the symmetry inheritance theorem takes the form*

$$\boldsymbol{f}(x_0, \mathcal{A}_g\xi, \mathcal{A}_g\overline{\xi}, \mu, \varepsilon) = \mathcal{B}_g\boldsymbol{f}(x_0, \xi, \overline{\xi}, \mu, \varepsilon), \quad \textit{for A. Lyapounov BEq}$$

$$\boldsymbol{t}(x_0, \mathcal{A}_g\xi, \mathcal{A}_g\overline{\xi}, \mu, \varepsilon) = \mathcal{B}_g\boldsymbol{t}(x_0, \xi, \overline{\xi}, \mu, \varepsilon), \quad \textit{for E. Schmidt BEq.}$$

As in [14], [18]–[20] introduce the following

Definition 2.1. *BEq* (1) *(resp.* (7)*) is the potential type equation if in a neighbourhood of the point* $(x_0, 0)$ *for the vector* $\mathbf{f}(y, v(y, \xi, \overline{\xi}), \mu, \varepsilon) = (f_{11}, \overline{f}_{11}, \ldots, f_{1n_1}, \overline{f}_{1n_1}, \ldots, f_{\nu 1}, \overline{f}_{\nu 1}, \ldots, f_{\nu n_\nu}, \overline{f}_{\nu n_\nu})$ *the equality*

$$\mathbf{f}(y, v(y, \xi, \overline{\xi}), \mu, \varepsilon) = d\ grad_y U(y, \xi, \overline{\xi}, \mu, \varepsilon) \tag{16}$$

with invertible operator d *is fulfilled.*

Then the functional $U(y, \xi, \overline{\xi}, \mu, \varepsilon)$ is called the potential of BEq (1) (resp (7)) and the operator $\mathbf{f}$ (resp $\mathbf{t}$) is the pseudogradient of the functional U.

Theorem 2.2. *Let the conditions* $c_1)$*–*$c_3)$ *be satisfied and BEq* (1) *is potential type one. Its potential* U *is invariant with respect to representation* L_g *iff*

$$L_g^* d^{-1} K_g = d^{-1} \tag{17}$$

Proof. Due to the BEq (1) potentiality and by Lagrange mean value theorem

$$U(y, \xi, \overline{\xi}, \mu, \varepsilon) = \int_0^1 \langle\langle d^{-1}\mathbf{f}(\vartheta y, v(\vartheta y, \xi, \overline{\xi}), \mu, \varepsilon), y)\rangle\rangle_{\mathcal{H}} d\vartheta,$$

$$U(L_g y, \xi, \overline{\xi}, \mu, \varepsilon) = \int_0^1 \langle\langle d^{-1}\mathbf{f}(L_g\vartheta y, v(L_g\vartheta y, \xi, \overline{\xi}), \mu, \varepsilon), L_g y)\rangle\rangle_{\mathcal{H}} d\vartheta$$

take place. According to (11), (16) $grad_y U(y, \xi, \overline{\xi}, \mu, \varepsilon) = K_g^{-1}\mathbf{f}(L_g y, v(L_g y, \xi, \overline{\xi}), \mu, \varepsilon)$, whence again by Lagrange theorem

$U(y, \xi, \overline{\xi}, \mu, \varepsilon) = \int_0^1 \langle\langle d^{-1}K_g^{-1}\mathbf{f}(L_g\vartheta y, v(L_g\vartheta y, \xi, \overline{\xi}), \mu, \varepsilon), y)\rangle\rangle_{\mathcal{H}} d\vartheta$. The potential $U(y, \xi, \overline{\xi}, \mu, \varepsilon)$ is L_g-invariant if $U(y, \xi, \overline{\xi}, \mu, \varepsilon) = U(L_g y, \xi, \overline{\xi}, \mu, \varepsilon)$, that is possible iff

$$\int_0^1 \langle\langle (L_g^* d^{-1} - d^{-1} K_g^{-1}) \mathbf{f}(L_g \vartheta y, v(L_g \vartheta y, \xi, \overline{\xi}), \mu, \varepsilon), y)\rangle\rangle_{\mathcal{H}} d\vartheta = 0$$

The last relation is equivalent to equality (17).

Corollary 3. *Necessary and sufficient condition of invariance for the potential type BEq* (7) *is the equality* $L_g^* d^{-1} = d^{-1} L_g^{-1}$

Corollary 4. *[22] Let* x_0 *be an isolated bifurcation point of* G*–invariant equation* (1) *and the relevant A. Lyapounov (E. Schmidt) BEq be of potential type one, i.e.*

$$\mathbf{f}(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\ grad_{\xi,\overline{\xi}} U(\xi, \overline{\xi}, \mu, \varepsilon)),$$
$$\mathbf{t}(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\ grad_{\xi,\overline{\xi}} U(\xi, \overline{\xi}, \mu, \varepsilon)),$$

with invertible matrix d*. Then the potential* $U(\xi, \overline{\xi}, \mu, \varepsilon)$ *is invariant to the representation* $\mathcal{A}_g$ *iff*

$$\mathcal{A}'_g d^{-1} \mathcal{B}_g = d^{-1}, \text{ for A. Lyapounov and}$$
$$\mathcal{A}'_g d^{-1} \mathcal{A}_g = d^{-1}, \text{ for E. Schmidt BEq.}$$

Corollary 5. *If at the isolated bifurcation point* x_0 *the equalities* $f(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\, grad_{\overline{\xi}} U(\xi, \overline{\xi}, \mu, \varepsilon)$ *and* $\overline{f}(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\, grad_{\xi} U(\xi, \overline{\xi}, \mu, \varepsilon)$ *(the same relations for E. Schmidt BEq* $t(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\, grad_{\overline{\xi}} U(\xi, \overline{\xi}, \mu, \varepsilon)$ *and* $\overline{t}(x_0, \xi, \overline{\xi}, \mu, \varepsilon) = d\, grad_{\xi} U(\xi, \overline{\xi}, \mu, \varepsilon)$*) hold, then the relevant BEqs have partial potentiality in every pair of variables* $(\xi_j, \overline{\xi}_j, f_j, \overline{f_j})$.

Theorem 2.3. *The preudogradient* $\mathbf{f}$ *(resp* $\mathbf{t}$*) of the invariant functional is* (L_g, K_g) *(resp* (L_g, L_g)*) equivariant in the sense of* (8)*. For every* $X_{rj} \in T^l_{g(a)}$ *in some neighbourhood of the point* $(x_0, 0)$ *the cosymmetric identity*

$$\langle\langle d^{-1}\mathbf{f}(y,v(y,\xi,\bar{\xi}),\mu,\varepsilon),\ \sum_{r=1}^{\nu} X_{rk}\left[y+v(y,\xi,\bar{\xi})+u(y,v(y,\xi,\bar{\xi},\mu,\varepsilon),\mu,\varepsilon)\right]\rangle\rangle_{\mathcal{H}}=0,$$

$$(\langle\langle\langle d^{-1}\mathbf{t}(y,v(y,\xi,\bar{\xi}),\mu,\varepsilon),\ \sum_{r=1}^{\nu} X_{rk}\left[y+v(y,\xi,\bar{\xi})+u(y,v(y,\xi,\bar{\xi},\mu,\varepsilon),\mu,\varepsilon)\right]\rangle\rangle_{\mathcal{H}}=0), \tag{18}$$

$k=1,\dots,l$ *is true*

Proof. Let the norm $\|h\|_{\mathcal{H}}$ be sufficiently small. Since the operator $\mathbf{f}$ of BEq left-hand side is of potential type in the sense (16) of Definition 1, then

$$U(y+\lambda L_g^{-1}h,\xi,\overline{\xi},\mu,\varepsilon)-U(y,\xi,\overline{\xi},\mu,\varepsilon)=$$

$$=\lambda\langle\langle d^{-1}\mathbf{f}(y,v(y,\xi,\overline{\xi}),\mu,\varepsilon,y),L_g^{-1}h\rangle\rangle_{\mathcal{H}}+o(\|\lambda L_g^{-1}h\|),$$

$$U(L_gy+\lambda h,\xi,\overline{\xi},\mu,\varepsilon)-U(L_gy,\xi,\overline{\xi},\mu,\varepsilon)=$$

$$=\lambda\langle\langle d^{-1}\mathbf{f}(L_gy,v(L_gy,\xi,\overline{\xi}),\mu,\varepsilon),h\rangle\rangle_{\mathcal{H}}+o(\|\lambda h\|).$$

The potential U invariance gives the coincidence of the left-hand side of these equalities. Then

$$\langle\langle L_g^{*-1}d^{-1}\mathbf{f}(y,v(y,\xi,\overline{\xi}),\mu,\varepsilon)-d^{-1}\mathbf{f}(L_gy,v(L_gy,\xi,\overline{\xi}),\mu,\varepsilon),h\rangle\rangle_{\mathcal{H}}\overset{(17)}{=}$$
$$=\langle\langle d^{-1}(K_g\mathbf{f}(y,v(y,\xi,\overline{\xi}),\mu,\varepsilon)-\mathbf{f}(L_gy,v(L_gy,\xi,\overline{\xi}),\mu,\varepsilon)),h\rangle\rangle_{\mathcal{H}}=0$$

whence the first part of the assertion.

Let $X_{rj}\in T^l_{g(a)}$ and $L_{g(a)}$ be one-parametric subgroup of $L_{g(a)}$. According to the invariance of the functional U it follows

$$0=U(L_{g(a(t))}y,\xi,\overline{\xi},\mu,\varepsilon)-U(y,\xi,\overline{\xi},\mu,\varepsilon)=$$
$$=\langle\langle d^{-1}\mathbf{f}(y,v(y,\xi,\overline{\xi},\mu,\varepsilon),(L_{g(a(t))})-I)(y+v(y,\xi,\overline{\xi})+u(y,v(y,\xi,\overline{\xi}),\mu,\varepsilon)\rangle\rangle+$$
$$+o(\|(L_{g(a(t))})-I)(y+v(y,\xi,\overline{\xi})+u(y,v(y,\xi,\overline{\xi}),\mu,\varepsilon)\|).$$

Passage to the limit at $\|h\|_{\mathcal{H}} \to 0$ gives (18). The relevant assertion for E.Schmidt BEq can be proved analogously.

Corollary 6. *For the case of invariant kernel the BEqs of potential type* (1) *and* (7) *satisfy the cosymmetric identities*

$$\langle d^{-1}\mathbf{f}(\xi,\bar{\xi},\mu,\varepsilon), \sum_{r=1}^{\nu} X_{rj}\xi\rangle_{\Xi^{2n}} = 0 \tag{19}$$

$$\langle d^{-1}\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon), \sum_{r=1}^{\nu} X_{rj}\xi\rangle_{\Xi^{2n}} = 0, \tag{20}$$

where X_{rj}, $j = 1,\ldots,l$ are the infinitesimal operators of l–dimensional representation $\mathcal{A}_g$ in the $2n$–dimensional space Ξ^{2n} of coefficients ξ, $\bar{\xi}$ in the expansion of an arbitrary element $\varphi \in N(\mathcal{B}_{x_0})$ with respect to the basis $\{\varphi_{rj}, \bar{\varphi}_{rj},\ j = 1,\ldots,n_r,\ r = 1,\ldots,\nu\}$

3. BRANCHING EQUATION REDUCTION THEOREM AND ITS APPLICATIONS

A. Everywhere below $\nu = 1$, $n_1 = n$. Following [12]–[14] sufficient condition can be obtained for the reduction of potential type BEq of Andronov-Hopf bifurcation under group symmetry of the nonlinear equation (1) ((3)).

Theorem 3.1. *Let the conditions c_1)–c_3) be fulfilled, let A. Lyapounov BEq* (1) *(resp E. Schmidt BEq* (7)*) be of potential type one with the potential U belonging to the class C^2 in some neighbourhood of the bifurcation point* $(0, x_0; 0)$ *and being invariant to the representation $L_{g(a)}$ of the group $G_l(a)$, let s be a dimension of stationary subgroup of the element x_0, and $æ = l - s > 0$. Then:*

1. if $æ = n$, then for all $(\xi(\varepsilon), \bar{\xi}(\varepsilon), \mu(\varepsilon), \varepsilon)$ (or $v(x_0, \xi(\varepsilon), \bar{\xi}(\varepsilon), \mu(\varepsilon), \varepsilon)$) from some neighbourhood of zero in $\Xi^{2n} \times R^2$ BEq (1) *(resp* (7)*) is satisfied identically;*

2. if $æ < n$ and $n \geq 2$, then partial reduction of BEq takes place: $æ$ of its equations are linear combinations of other $n - æ$ ones.

Proof. Accept the agreement about the enumeration of the basis elements in $N(\mathfrak{B}_{x_0})$ of Section 2. Then according to cosymmetric identity (18) in some zero neighborhood in $\Xi^{2n} \times R^2$ it follows

$$0 = \langle\langle d\mathbf{f}(x_0, v(x_0,\xi,\overline{\xi}), \mu, \varepsilon), X_k(x_0 + v(x_0,\xi,\overline{\xi}) + u(x_0, v(x_0,\xi,\overline{\xi})), \mu, \varepsilon)\rangle\rangle_{\mathcal{H}} =$$

$$= \sum_{j=1}^{n}[f_j(x_0, v(x_0,\xi,\overline{\xi}), \mu, \varepsilon)\langle\langle dz_j, \varphi_k\rangle\rangle + \overline{f}_j(x_0, v(x_0,\xi,\overline{\xi}), \mu, \varepsilon)\langle\langle d\overline{z}_j, \overline{\varphi}_k\rangle\rangle] +$$

$$+ \langle\langle dz_j + d\overline{z}_j, X_k(v(x_0,\xi,\overline{\xi}) + u(x_0, v(x_0,\xi,\overline{\xi})), \mu, \varepsilon)\rangle\rangle,$$

where $k = 1, \ldots, æ$ and the rank of $2n \times æ$–matrix $[\langle\langle dz_j, \varphi_k\rangle\rangle, \langle\langle d\overline{z}_j, \overline{\varphi}_k\rangle\rangle]$ is equal to æ.

Remark 3.1. *For the case of $r = 1, \ldots, \nu$ Theorem 4 is true, but the BEq reduction can be made only relative to æ equations, i.e. not for all r simultaneously.*

Remark 3.2. *Theorems 2–4 are valid also under usual potentiality of BEqs* (1), (7)*, i.e. when $d = I$.*

Remark 3.3. *For the case of invariant kernel there is the reduction of potential type BEq when the BEq potential $U(\xi,\bar{\xi},\mu,\varepsilon)$ is the invariant of l–dimensional representation $\mathcal{A}_{g(a)}$ having a complete system $\{I_j(\xi,\bar{\xi})\}_1^{l_1}$, $n - l_1 \leq l$ of functionally independent invariants.*

In fact, if $U(\xi,\bar{\xi},\mu,\varepsilon) = \Phi(I_1(\xi,\bar{\xi}), \ldots, I_{l_1}(\xi,\bar{\xi}), \mu, \varepsilon)$, then Lyapounov (Schmidt) BEq takes the form

$$\mathbf{f}(\xi,\bar{\xi},\mu,\varepsilon) = d\, grad_{\xi,\bar{\xi}} U(\xi,\bar{\xi},\mu,\varepsilon) = d\, grad\, \Phi(I_1(\xi,\bar{\xi}), \ldots, I_{l_1}(\xi,\bar{\xi}), \mu, \varepsilon).$$

Since the matrix d is invertible and the system of invariants $\{I_j(\xi,\bar{\xi})\}_1^{l_1}$ is functionally independent, then the BEq is reduced to $2l_1 \times 2l_1$ system.

B. Let x_0 be an isolated bifurcation point, i.e. $L_g x_0 = x_0$ and let the BEq invariant to the matrix representation $\mathcal{A}_g : \mathcal{A}_g \varphi_i = \sum_{j=1}^{n} \alpha_{ji}(a)\varphi_j \Rightarrow \tilde{\xi}_i = \mathcal{A}_g \xi = \sum_{j=1}^{n} \alpha_{ij}(a)\xi_j$.

Here the BEq left-hand side presents the nonlinear operator acting in finite dimensional subspaces remaining invariant under the action of the representations L_g and K_g.

The operator K_g generates the finite-dimensional representation $\mathcal{B}_g = \|\beta_{ij}(a)\|_{i,j=1}^n$ in the defect-subspace of functionals $N^*(\mathfrak{B}_{x_0}) = span\{\psi_1, \dots, \psi_n\}$. Thus the problem of the general form BEq construction on allowed group symmetry arises [9, 10].

By virtue of the shift symmetry on the time the BEqs admit the rotation group $SO(2)$ in any pair of variable $(\xi_k, \bar{\xi}_k; f_k, \bar{f}_k)$, i.e. $\mathcal{A}_0(a_0) = \mathcal{B}_0(a_0) = diag\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$, $X_0 = (\widehat{X}_0(\xi); \widehat{X}_0 f)$, $\widehat{X}_0(\xi) = \sum_{j=1}^n \left(\xi_j \partial_{\xi_j} - \bar{\xi}_j \partial_{\bar{\xi}_j}\right)$.

The cosymmetric identity (18) gives the new approach to its solution. Consider here some examples of potential and partially potential BEqs with rotation groups $SO(2)$ symmetry in spatial variables.

Theorem 3.2. *Four dimensional continuously differentiable BEq* $(f_1, \bar{f}_1, f_2, \bar{f}_2)$ *with* $SO(2)$ *symmetry has the form*

$$f_j(\xi, \bar{\xi}, \mu, \varepsilon) \equiv \xi_j |\xi_j|^{-1} U_j(|\xi_1|, |\xi_2|, \mu, \varepsilon) = 0, \ |\xi_j| = \sqrt{\xi_j \bar{\xi}_j}, \ j = 1, 2 \qquad (21)$$

with some additional conditions on the functions $U_j, j = 1, 2$. *If to* $SO(2)$ *the reflection symmetry* $(\xi) : (\xi_1, \xi_2) \to (\xi_2, \xi_1)$ *is added, i.e. the BEq admits the group* $O(2)$, *then*

$$U_2(s_1, s_2, \mu, \varepsilon) = U_1(s_2, s_1, \mu, \varepsilon). \qquad (22)$$

The functions U_j *are infinitesimal at* $s_k, \mu, \varepsilon \to 0$. *The analytic BEq is represented by the series*

$$f_j(\xi, \bar{\xi}, \mu, \varepsilon) \equiv \xi_j \sum_{l=1}^{\infty} a_l^{(j)}(\mu, \varepsilon)(\xi_1 \bar{\xi}_1)^{l_1} (\xi_2 \bar{\xi}_2)^{l_2} = 0, \ a_{l_1 l_2}^{(2)} = a_{l_2 l_1}^{(1)}, \ j = 1, 2 \quad (23)$$

Proof. The system of infinitesimal operators here has the form $X_0 = \xi_1 \partial_{\xi_1} - \bar{\xi}_1 \partial_{\bar{\xi}_1} + \xi_2 \partial_{\xi_2} - \bar{\xi}_2 \partial_{\bar{\xi}_2}$, $X_1 = \xi_1 \partial_{\xi_1} - \bar{\xi}_1 \partial_{\bar{\xi}_1} - \xi_2 \partial_{\xi_2} + \bar{\xi}_2 \partial_{\bar{\xi}_2}$. The cosymmetric identities $\langle f, X_0\xi\rangle = f_1\bar{\xi}_1 - \bar{f}_1\xi_1 + f_2\bar{\xi}_2 - \bar{f}_2\xi_2 = 0$ and $\langle f, X_1\xi\rangle = f_1\bar{\xi}_1 - \bar{f}_1\xi_1 - f_2\bar{\xi}_2 + \bar{f}_2\xi_2 = 0$ give the relations $\frac{f_k}{\xi_k} = \frac{\bar{f}_k}{\bar{\xi}_k}$, $k = 1, 2$. They allow to express the real magnitudes (quantities) $\frac{f_k}{\xi_k}$ through arbitrary continuously differentiable functions depending on invariants $|\xi_1|$ and $|\xi_2|$, i.e. to obtain the general form of the BEq (21)([9], th.2). It is not difficult to verify the equalities $\frac{\partial f_k}{\partial \bar{\xi}_k} = \frac{\partial \bar{f}_k}{\partial \xi_k}$ guaranteeing the partial potentiality of the BEq (21).

For its complete potentiality, i.e. for the symmetry of the matrix

$$\begin{matrix} \frac{\partial f_1}{\partial \bar{\xi}_1}, & \frac{\partial f_1}{\partial \xi_1}, & \frac{\partial f_1}{\partial \bar{\xi}_2}, & \frac{\partial f_1}{\partial \xi_2} \\ \frac{\partial \bar{f}_1}{\partial \bar{\xi}_1}, & \frac{\partial \bar{f}_1}{\partial \xi_1}, & \frac{\partial \bar{f}_1}{\partial \bar{\xi}_2}, & \frac{\partial \bar{f}_1}{\partial \xi_2} \\ \frac{\partial f_2}{\partial \bar{\xi}_1}, & \frac{\partial f_2}{\partial \xi_1}, & \frac{\partial f_2}{\partial \bar{\xi}_2}, & \frac{\partial f_2}{\partial \xi_2} \\ \frac{\partial \bar{f}_2}{\partial \bar{\xi}_1}, & \frac{\partial \bar{f}_2}{\partial \xi_1}, & \frac{\partial \bar{f}_2}{\partial \bar{\xi}_2}, & \frac{\partial \bar{f}_2}{\partial \xi_2} \end{matrix} \tag{24}$$

the other four equalities

$$\frac{\partial f_1}{\partial \bar{\xi}_2} = \frac{\partial f_2}{\partial \bar{\xi}_1}, \ \frac{\partial f_1}{\partial \xi_2} = \frac{\partial \bar{f}_2}{\partial \bar{\xi}_1}, \ \frac{\partial \bar{f}_1}{\partial \bar{\xi}_2} = \frac{\partial f_2}{\partial \xi_1}, \ \frac{\partial \bar{f}_1}{\partial \xi_2} = \frac{\partial \bar{f}_2}{\partial \xi_1} \tag{25}$$

should be satisfied, which give the additional restrictions on the functions $U_j, j = 1, 2$. The remaining assertions of the theorem are evident.

Remark 3.4. *In this way to obtain the explicit form of the functions U_j (i.e. to determine the potential of the continuously differentiable BEq) is not presented to be possible.*

Further determine the potential of the real analytic four dimensional BEq (23) with $O(2)$ symmetry, taking into account its expansion

$$f_1 = a_{00}\xi_1 + \sum_{n=l}^{\infty} \left[a_{n0}\xi_1(\xi_1\bar{\xi}_1)^n + a_{n-1,1}\xi_1(\xi_1\bar{\xi}_1)^{n-1}(\xi_2\bar{\xi}_2) + \dots \right.$$

$$\left. + a_{n-s,s}\xi_1(\xi_1\bar{\xi}_1)^{n-s}(\xi_2\bar{\xi}_2)^s + \dots + a_{0n}\xi_1(\xi_2\bar{\xi}_2)^n\right], \tag{26}$$

$$f_2 = a_{00}\xi_2 + \sum_{n=l}^{\infty} \left[a_{n0}\xi_2(\xi_2\bar{\xi}_2)^n + a_{n-1,1}\xi_2(\xi_2\bar{\xi}_2)^{n-1}(\xi_1\bar{\xi}_1) + \dots\right.$$

$$+a_{n-s,s}\xi_2(\xi_2\overline{\xi}_2)^{n-s}(\xi_1\overline{\xi}_1)^s+\ldots+a_{0n}\xi_2(\xi_1\overline{\xi}_1)^n\big]$$

The equalities (25) lead to relations which are sufficient for the complete potentiality of the vector field (3)

$$\begin{gathered} a_{n-1,1}=na_{0n},\quad 2a_{n-2,2}=(n-1)a_{1,n-1},\ldots, \\ sa_{n-s,s}=(n-s+1)a_{s-1,n-s+1},\ldots,(n-1)a_{1,n-1}=2a_{n-2,2}. \end{gathered} \tag{27}$$

Now the formula $\sum\limits_{k=1}^{2}\left[\int\limits_0^1 f_k(t\xi_1,t\xi_2,\mu,\varepsilon)\overline{\xi}_k dt+\int\limits_0^1 \overline{f}_k(t\xi_1,t\xi_2,\mu,\varepsilon)\xi_k dt\right]$ from the monograph [18] allows us to determine the potential of BEq (23)

$$\begin{aligned}
&U(\xi_1,\xi_2,\mu,\varepsilon)=\textstyle\sum_{n=0}^{\infty}U_n(\xi_1,\xi_2,\mu,\varepsilon),\quad s_1=\xi_1\overline{\xi}_1,\quad s_2=\xi_2\overline{\xi}_2,\quad a_{jk}=a_{jk}(\mu,\varepsilon),\\
&U_0=a_{00}(s_1+s_2),\quad U_1=\tfrac{1}{2}[a_{10}(s_1^2+s_2^2)+2a_{01}s_1s_2],\\
&U_2=\tfrac{1}{3}[a_{20}(s_1^3+s_2^3)+3a_{02}(s_1^2s_2+s_1s_2^2)]\\
&U_3=\tfrac{1}{4}[a_{30}(s_1^4+s_2^4)+4a_{03}(s_1^3s_2+s_1s_2^3)+2a_{12}s_1^2s_2^2],\\
&U_4=\tfrac{1}{5}[a_{40}(s_1^5+s_2^5)+5a_{04}(s_1^4s_2+s_1s_2^4)+\tfrac{5}{2}a_{13}(s_1^3s_2^2+s_1^2s_2^3)],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&U_{2m}=\tfrac{1}{2m+1}\left[a_{2m,0}(s_1^{2m+1}+s_2^{2m+1})+(2m+1)a_{0,2m}(s_1^{2m}s_2+s_1s_2^{2m})+\right.\\
&+\tfrac{2m+1}{2}a_{1,2m-1}(s_1^{2m-1}s_2^2+s_1^2s_2^{2m-1})+\tfrac{2m+1}{3}a_{2,2m-2}(s_1^{2m-2}s_2^3+s_1^3s_2^{2m-2})+\ldots+\\
&\left.+\tfrac{2m+1}{m}a_{mm}(s_1^{m+1}s_2^m+s_1^ms_2^{m+1})\right].\\
&U_{2m+1}=\tfrac{1}{2m+2}\left[a_{2m+1,0}(s_1^{2m+2}+s_2^{2m+2})+(2m+2)a_{0,2m+1}(s_1^{2m+1}s_2+s_1s_2^{2m+1})+\right.\\
&+(m+1)a_{1,2m}(s_1^{2m}s_2^2+s_1^2s_2^{2m})+\tfrac{2m+2}{3}a_{2,2m-1}(s_1^{2m-1}s_2^3+s_1^3s_2^{2m-1})+\ldots\\
&+\tfrac{m+1}{2}a_{3,2m-2}(s_1^{2m-2}s_2^4+s_1^4s_2^{2m-2})+\ldots+\tfrac{2m+2}{m}a_{m-1,m+2}(s_1^{m+2}s_2^m+s_1^ms_2^{m+2})+\ldots\\
&\left.+\tfrac{2m+2}{m+1}a_{m,m+1}s_1^{m+1}s_2^{m+1}\right].
\end{aligned} \tag{28}$$

Thus the following assertion is proved

Theorem 3.3. *Real analytic four dimensional potential BEq with* $O(2)$ *symmetry has the form* (3) *with coefficients satisfying the relation* (27). *The expansion of its potential is* (28).

More complicated cases of partially potential and potential BEqs occur in symmetry breaking problems for Andronov–Hopf bifurcation. In such problems nonlinear equation admits rotation groups and as the bifurcational parameter crosses its critical value solutions with crystallographic group symmetries arise.

Introduce the assumption about the enumeration of elementary cell of periodicity: if to one of the tops is prescribed an odd number then to the opposite top the subsequent even number corresponds. At such assmption BEq turns out to be partially potential (potential type) relative to every pair of variables ξ_{2k-1}, ξ_{2k}.

Theorem 3.4. *Continuously differentiable* $2l$*-dimensional BEq of Andronov–Hopf bifurcation with symmetry* $O(2)$ *in* j*-th pair of variables* $\xi_{2j-1}, \overline{\xi}_{2j-1}, \xi_{2j}, \overline{\xi}_{2j}$ *at the independent group parameters for different* j *and with* $2l$*-dimensional representation of* l*-dimensional cube group symmetry has the form*

$$\begin{aligned}
&f_{2k-1}(\xi,\overline{\xi},\mu,\varepsilon) = \xi_{2k-1}U(|\xi_k|,|\xi_2|,\ldots,|\xi_{k-1}|,|\xi_1|,|\xi_{k+1}|,\ldots,|\xi_l|,\mu,\varepsilon) = 0,\\
&f_{2k}(\xi,\overline{\xi},\mu,\varepsilon) = \xi_{2k}U(|\xi_k|,|\xi_2|,\ldots,|\xi_{k-1}|,|\xi_1|,|\xi_{k+1}|,\ldots,|\xi_l|,\mu,\varepsilon) = 0,\\
&\overline{f}_{2k-1}(\xi,\overline{\xi},\mu,\varepsilon) = 0, \quad \overline{f}_{2k}(\xi,\overline{\xi},\mu,\varepsilon) = 0, \quad |\xi_k| = \sqrt{\xi_k\overline{\xi}_k}, \quad k = 1,\ldots.
\end{aligned} \tag{29}$$

The function U *is invariant relative to pairwise permutation of arguments with numbers more than one accordingly to* l *-dimensional cube discrete symmetry. The BEq* (29) *has partial potentiality on* k*-th fourth of variables* $\xi_{2k-1}, \overline{\xi}_{2k-1}, \xi_{2k}, \overline{\xi}_{2k}$.

Branching equation (29) possesses only partial potentiality.

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ON GAME PROBLEMS OF CONTROL OF PENCIL TRAJECTORIES

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Abstract The delay linear differential games in the L. S. Pontryagin's formalization, from the point of view of the pursuer are analyzed. Our constructions generalize the L. S. Potryagin's mode [1] in linear differential games case like (1). This note uses results from [2–8].

1. Introduction. Basic definitions and problem. The pursuing delay linear differential game

$$\dot{z}(t) = Az(t) + Bz(t-h) - u + v, \ t > 0, \tag{1}$$

is analyzed, where $z, u, v \in \mathbb{R}^n$, $n \geq 1$, A, B are constant quadratic matrixes of order n; h is a fixed positive number, the vectors u, v are control parameters of pursuer and fleeing. They are chosen from the class of measurable functions and satisfy restrictions like

$$u(t) \in P, \ \ v(t) \in Q, \ \ 0 \leq t < \infty, \tag{2}$$

where P and Q are nonempty compact subsets of $\mathbb{R}^n$.

Further the measurable functions $u = u(t)$ and $v = v(t)$, $0 \leq t < \infty$, satisfying (2), are called possible control of pursuer and fleeing respectively.

In $\mathbb{R}^n$ area market terminal set is M. As a basis of the initial set $N(R(\cdot))$ it is chosen a large number of measurable branches of many-valued representations $R(s)$, $-h \leq s \leq 0$.

Let $u = u(t)$, $0 \leq t < \infty$ and $v = v(t)$, $0 \leq t < \infty$, be arbitrary allowable controls of pursuer and fleeing respectively. The functions $u(\cdot), v(\cdot)$

and the initial set $N\left(R\left(\cdot\right)\right)$ are placed in accordance with pencil of trajectories $z\left(u\left(\cdot\right),v\left(\cdot\right),N\left(R\left(\cdot\right)\right)\right)$ of the equation $\dot{z}\left(t\right)=Az\left(t\right)+Bz\left(t-h\right)-u\left(t\right)+v\left(t\right),$ and coming out from condition $z_0\left(\cdot\right)\in N\left(R\left(\cdot\right)\right)$ [5].

In studying the game (1) we equate ourselves with the pursuer. In this case our aim is to reduce the pencil of trajectories $z\ \left(u\left(\cdot\right),v\left(\cdot\right),N\left(R\left(\cdot\right)\right)\right)$ to the set M.

The aim of control of pencil of trajectories is to find the number $T\geq 0$, and in every $t\in\ [0,\,T]$, to construct the meanings $u\left(t\right)$ of parameter u so that each trajectory $z(u\left(\cdot\right),v\left(\cdot\right),N\left(R\left(\cdot\right)\right))$ hit the set M till it exceeds T. That is for each trajectory $z\left(t\right),\ 0\leq\ t\ \leq\ T$, belongs to $z\ \left(u\left(\cdot\right),v\left(\cdot\right),N\left(R\left(\cdot\right)\right)\right)$ at some $t\ =\ t'\ \in\ [0,\,T]$. If there exists inclusion $z\left(t\right)\ \in\ M$, the number T is called transfer period.

When the problem of the control of pencil of trajectories is solvable, we say that in the game (1) the pencil of trajectory can be transferred from the initial set $N\left(R\left(\cdot\right)\right)$ to set M during T.

In this note we find sufficient conditions of solving problems of control of pencil of trajectories (see Theorems 1–4.).

2. Main results. Further it is supposed: a) terminal set M has the form $M\ =\ M_0+\ M_1$, where M_0 is a linear subspace of $\mathbb{R}^n$, M_1 is a subset of the subspace L, which is orthogonal to M_0 in $\mathbb{R}^n$, i.e. $M_0\oplus\ L=\mathbb{R}^{\,n}$;
b) π is the matrix of operator of orthogonal projection of $\mathbb{R}^n$ to the subspace L;
c) by simple or multi-valued function (multi-valued representation) it is understood its Lebesgue integral [1];
d) by operation $\underline{*}$ it is understood the geometrical difference [1].

Let $\tau\ \geq\ 0$ and let allowable controls $u\ =\ u\ (r)\,,v\ =\ v\ (r)$ be defined in interval $[0\,,\,\tau]$. In that case it is possible to solve the system with delay state (1) in initial condition $z_0\left(\cdot\right)\in N\left(R\left(\cdot\right)\right)$, $\left(z\left(s\right)=z_0\left(s\right),\ -h\leq s\leq 0\ \right)$ by the

Cauchy formula (see 201 [5])

$$z(\tau) = K(\tau) z_0(0) + \int\limits_{-h}^{0} K(\tau - r - h) B z_0(r) dr - \int\limits_{0}^{\tau} K(\tau - r)[u(r) - v(r)] dr, \tag{3}$$

where $K(r)$ is the matrix function, with the following properties:

a) $K(r) = \tilde{0}$, $r < 0$, $\tilde{0}$ – null matrix of order n;

b) $K(0) = E$, E – unitary matrix of order n;

c) elements of the matrix $K(r)$ belong to $C[0, \tau]$;

d) $K(r)$ satisfies the equation $\dot{K}(r) = AK(r) + BK(r-h)$ for $r \in [0, \tau]$.

Existence and uniqueness of matrix function $K(r)$, satisfying conditions a)–d) can be proved by the simple method of successive integration.

Let us consider the geometric difference $\underline{*}$ of the sets $\pi K(r) P$ and $\pi K(r) Q$ [1]

$$\hat{w}(r) = \pi K(r) P \,\underline{*}\, \pi K(r) Q, \ 0 \le r \le \tau \tag{4}$$

and introduce the notation

$$W_1[M_1 \underline{*} H(\tau, N(R(\cdot))), \tau] = [M_1 \underline{*} H(\tau, N(R(\cdot)))] + \int_0^\tau \hat{w}(r) dr, \tag{5}$$

where $H(\tau, N(R(\cdot))) = \pi K(\tau) R(0) + \int_{-h}^{0} \pi K(\tau - r - h) B R(r) dr$.

Theorem 1. *Suppose that for some $\tau = \tau_1$ it is possible to have*

$$0 \in W_1[M_1 \underline{*} H(\tau, N(R(\cdot))), \tau]. \tag{6}$$

Then in the game (1) the pencil of trajectories can be transferred from the set $N(R(\cdot))$ to set M during the period $T = \tau_1$. At the same time for the construction of $u(t)$ pursuer at the moment t the meanings of $v(t)$ parameter v is used.

Introduce the following set

$$\overline{W}_1(\tau) = \left[M_1 + \int\limits_0^\tau \hat{w}(r) dr \right] \underline{*}\, H(\tau, N(R(\cdot))).$$

Theorem 2. *Suppose that for some $\tau = \tau_2$ it is possible to have*

$$0 \in \overline{W}_1(\tau).$$

Then in the game (1) the pencil of trajectories can be transferred from set $N(R(\cdot))$ to set M during the period $T = \tau_2$.

Consider the first approximation of the L. S. Pontryagin's alternative interval [1]. Let

$$W_2(\tau) = \left[\left(M_1 + \int_0^\tau \pi K(r-h)\,Pdr\right)\right] \underline{*}$$
$$\underline{*} \left[H(\tau,\, N(R(\cdot))) + \int_0^\tau \pi K(r-h)\,Qdr\right].$$

Theorem 3. *Suppose that for some $\tau = \tau_3$ it is possible to have $0 \in W_2(\tau)$. Then in the game (1) the pencil of trajectories can be transferred from the set $N(R(\cdot))$ to the set M during the period $T = \tau_3$.*

In the above the control $u(\cdot)$ depended, in general, on $z_0(\cdot)$ and $v(\cdot)$. Now suppose the sufficient condition, where control $u(r) \in P$, $0 \le r \le T$ and period T do not depend on the point $z_0(\cdot) \in N(R(\cdot))$.

Consider the following set

$$\overline{W}_2[M_1 \underline{*} H(\tau, N(R(\cdot))), \tau] = \left[(M_1 \underline{*} H(\tau, N(R(\cdot)))) + \int_0^\tau \pi K(r-h)\,Pdr\right] \underline{*}$$
$$\underline{*} \int_0^\tau \pi K(r-h)\,Qdr, \quad \tau > 0.$$

Theorem 4. *Suppose that for some $\tau = \tau_4$ it is possible to have*

$$0 \in \overline{W}_2[M_1 \underline{*} H(\tau, N(R(\cdot))), \tau].$$

Then in the game (1) the pencil of trajectories can be transferred from the set $N(R(\cdot))$ to the set M during the period $T = \tau_4$.

3. Proof of theorems. Proof of Theorem 1. Let the condition of Theorem 1 be fulfilled, i.e $0 \in [M_1 \underline{*} H(\tau_1, N(R(\cdot)))] + \int_0^{\tau_1} \hat{w}(r)\,dr$. Then there can be found the vectors $d \in (M_1 \underline{*} H(\tau_1, N(R(\cdot))))$ and $w \in \int\limits_0^{\tau_1} \hat{w}(r)\, dr$ such that (see (5), (6)) $d + w = 0$. Further, by the definition of integral $\int\limits_0^{\tau_1} \hat{w}(r)\, dr$ we can sum the vector function $w(r) \in \hat{w}(r)$, $0 \le r \le \tau_1$, so that $w = \int\limits_0^{\tau_1} w(r)\, dr$. We have $w(\tau_1 - r) \in \hat{w}(\tau_1 - r) = \pi K(\tau_1 - r) P \underline{*} \; \pi K(\tau_1 - r) Q$, $0 \le r \le \tau_1$.

Here, by the definition of the geometrical difference $\underline{*}$ we have

$$\pi K(\tau_1 - r) v(r) + w(\tau_1 - r) \subset \pi K(\tau_1 - r) P,$$

for every $v(r) \in Q$, $0 \le r \le \tau_1$. So, knowing the pursuer $v(r)$ we can built the control $u(r) \in P$, that satisfies the equation

$$\pi K(\tau_1 - r) u(r) = \pi K(\tau_1 - r) v(r) + w(\tau_1 - r). \tag{7}$$

Allowable controls $u(r) \in P$, $0 \le r \le \tau_1$, satisfying equation (7), can be not unique. If it is so, let us take among them that one, which has the least first coordinate, and if there are many of them, which has the least second coordinate etc. That is, from every solution of equation (7), we choose the least in the lexicographic sense and mark it by $u(r)$. That is why for an arbitrary measurable the function $v = v(r) \in Q$, $0 \le r \le \tau_1$, function $u(r)$, $0 \le r \le \tau_1$, will be a Lebesgue measurable function [6] .

Put $u = u(r)$, $0 \le r \le \tau_1$, and show, that in this method of control with parameter u all trajectories of pencil $z(u(\cdot), v(\cdot), N(R(\cdot)))$ will hit the set M during the period that not exceeds $T = \tau_1$.

Supposing $\tau = \tau_1$ in formula (3), and applying to both sides of equation (3) the operator π, according to our method of choosing the control of the pursuer we get

$$\pi z(\tau_1) = \pi K(\tau_1) z_0(0) + \int_{-h}^{0} \pi K(\tau_1 - r - h) B z_0(r) dr - \int_{0}^{\tau_1} \pi K(\tau_1 - r)\{u(r) -$$

$$-v(r)\} dr = \pi K(\tau_1) z_0(0) + \int_{-h}^{0} \pi K(\tau_1 - r - h) B z_0(r)\, dr - \int_{0}^{\tau_1} w(\tau_1 - r)\, dr =$$

$$= \pi K(\tau_1) z_0(0) + \int_{-h}^{0} \pi K(\tau_1 - r - h) B z_0(r) dr - w =$$

$$= \pi K(\tau_1) z_0(0) + \int_{-h}^{0} \pi K(\tau_1 - r - h) B z_0(r) dr + d,$$

as $d + w = 0$. Next, we have

$$\pi z(\tau_1) = d + \pi K(\tau_1) z_0(0) + \int_{-h}^{0} \pi K(\tau_1 - r - h) B z_0(r)\, dr \in d+$$

$$+H(\tau_1, N(R(\cdot))) \subset M_1$$

in accordance with the definition of the geometrical difference $\underline{*}$ [1].

Thus, for every $z_0(\cdot) \in N(R(\cdot))$ it is possible to find $\pi z(\tau_1) \in M_1$, which is equivalent to $z(\tau_1) \in M$. It means that the control of the pencil of trajectories is solved and $T = \tau_1$ is the transfer period. Theorem is proved. **Proof of Theorem 2.** By theorem, for $\tau = \tau_2$ it is possible to have

$$0 \in \left[M_1 + \int_{0}^{\tau_2} \hat{w}(r)\, dr \right] \underline{*}\ H(\tau_2, N(R(\cdot))).$$

Then, by the definition of the geometrical difference we have

$$H(\tau_2, N(R(\cdot))) \subset M_1 + \int_{0}^{\tau_2} \hat{w}(r)\, dr. \qquad (8)$$

Let $z_0(\cdot) \in N(R(\cdot))$ be an arbitrary initial state of system (1). Then from the definition of the algebraic sum of two sets (8), we find the vectors $m \in M_1$ and $w \in \int_0^{\tau_2} \hat{w}(r)\,dr$ such that

$$\pi K(\tau_2) z_0(0) + \int_{-h}^{0} \pi K(\tau_2 - r - h) B z_0(r)\,dr = m + w.$$

Further, by the definition of the integral of multi-valued representation it follows that there is sum up the function $w(r) \in \hat{w}(r)$, $0 \le r \le \tau_2$ such that $w = \int_0^{\tau_2} w(r)\,dr$. Then

$$\pi K(\tau_2) z_0(0) + \int_{-h}^{0} \pi K(\tau_2 - r - h) B z_0(r)\,dr = m + \int_0^{\tau_2} w(\tau_2 - r)\,dr. \quad (9)$$

From the definition of multi-valued function $\hat{w}(\tau_2 - r), 0 \le r \le \tau_2$ ((see (4)), and the properties of the geometrical difference it follows

$$w(\tau_2 - r) + \pi K(\tau_2 - r) Q \subset \pi K(\tau_2 - r) P, \ \ 0 \le r \le \tau_2. \quad (10)$$

Let $v(r)$, $0 \le r \le \tau_2$ be an arbitrary allowable control of fleeing player. Taking into consideration (10), let us see the following equation

$$w(\tau_2 - r) + \pi K(\tau_2 - r) v(r) = \pi K(\tau_2 - r) u, \quad (11)$$

in the unknown vector $u \in P$. By Fillipov-Kasten lemma [6] it follows that the equation (11) has a measurable solution $u(r) \in P$, $0 \le r \le \tau_2$. Replacing the chosen control $u(r)$, $0 \le r \le \tau_2$ by u and the function $v(r)$, $0 \le r \le \tau_2$ by v in equation (1), we get the delay differential equation $\dot{z}(r) = Az(r) + Bz(r-h) - u(r) + v(r)$, $0 \le r \le \tau_2$.

Then to solve the delay systems $z = z(r)$, $0 \le r \le \tau_2$ under the initial condition $z_0(\cdot) \in N(R(\cdot))$, after its projection to L, under $r = \tau_2$, we have (see (3), (9),(11))

$$\pi z(\tau_2) = \pi K(\tau_2) z_0(0) + \int\limits_{-h}^{0} \pi K(\tau_2 - r - h) B z_0(r) dr - \int\limits_{0}^{\tau_2} \pi K(\tau_2 - r) \{u(r) -$$

$$- v(r)\} dr = \pi K(\tau_2) z_0(0) + \int\limits_{-h}^{0} \pi K(\tau_2 - r - h) B z_0(r) dr - \int\limits_{0}^{\tau_2} w(\tau_2 - r) dr =$$

$$= m + \int\limits_{0}^{\tau_2} w(\tau_2 - r) dr - \int\limits_{0}^{\tau_2} w(\tau_2 - r) dr = m \in M_1 .$$

Thus, obviously $\pi z(\tau_2) \in M_1$, which is equivalent to $z(\tau_2) \in M$. Theorem is proved.

Proofs of Theorems 3,4 are analogous to proofs of Theorems 1,2.

4. Example. The game "simple pursue-deviation", is described by equation $\dot{z}(t) = z(t-h) - u + v$, where $z, u, v \in \mathbb{R}^n$, $n \geq 1$; h is a positive number, $|u| \leq \rho$, $|v| \leq \sigma$. As the initial set we take $R(\cdot) = a + \varepsilon S$, and as terminal set M we choose $M = \{ z : |z| \leq l \}$; ρ, σ, ε, l are positive constants, a is a point of $\mathbb{R}^n$, S is a closed single sphere in $\mathbb{R}^n$ with the center at the origin of coordinates.

Compute the sets $H(\tau, N(R(\cdot)))$, $W_1 [M_1 \underline{*} H(\tau, N(R(\cdot))), \tau]$ and the matrix function $K(\tau)$:

$H(\tau, N(R(\cdot))) = (a + \varepsilon S) + \int_{-h}^{0} (a + \varepsilon S) dt = a + \varepsilon S + ah + \varepsilon h S = (1+h) a + (1+h) \varepsilon S.$

$$W_1 [M_1 \underline{*} H(\tau, N(R(\cdot))), \tau] = [lS - (1+h) \varepsilon S] - (1+h) a + (\rho - \sigma) \tau S =$$

$$= [l - (1+h) \varepsilon] S - (1+h) a + (\rho - \sigma) \tau S = [l - (1+h) \varepsilon + (\rho - \sigma) \tau] S - (1+h) a.$$

$K(\tau) = 0$ at $\tau < 0$ and $K(\tau) = \sum\limits_{j=0}^{N} \frac{(\tau - j)^j}{j!}$, $N \leq \tau \leq N+1$, $N = 0, 1, 2, \ldots$

.

By applying Theorems 1-4 to this example we can state that: if $\rho > \sigma$, $l \geq \varepsilon(1+h)$, $l \leq (1+h)(|a| + \varepsilon)$, i.e. $\varepsilon(1+h) \leq l \leq (1+h)(|a| + \varepsilon)$. Then we can transfer the pencil of trajectories from the initial set $N(R(\cdot))$ to the terminal set M during the period

$$T = \frac{(1+h)[|a| + \varepsilon] - l}{\rho - \sigma}.$$

5. Application. Now let us quote the theorem that establishes the connection between the sets $\overline{W}_1(\tau)$, $W_2(\tau)$, $W_1\left[M_1 \underline{*} H\left(\tau,\ N\left(R\left(\cdot\right)\right)\right),\tau\right]$, $\overline{W}_2\left[M_1 \underline{*} H\left(\tau, N\left(R\left(\cdot\right)\right)\right),\tau\right]$.

Lemma. *For each $\tau \geq 0$ it is possible to have:*

a) $\overline{W}_1\ (\tau) \subset\ W_2(\tau)$;

b) $\overline{W}_2\left[M_1\ \underline{*}\ \ H\left(\tau, N\left(R\left(\cdot\right)\right)\right),\tau\right] \subset\ W_2(\tau)$;

c) $W_1\left[M_1\ \underline{*}\ H\left(\tau, N\left(R\left(\cdot\right)\right)\right),\tau\right] \subset\ W_1(\tau)$;

d) $W_1\left[M_1\ \underline{*}\ H\left(\tau, N\left(R\left(\cdot\right)\right)\right),\tau\right]\ \subset\ \overline{W}_2\left[M_1\ \underline{*}\ \ H\left(\tau, N\left(R\left(\cdot\right)\right)\right),\tau\right]$.

Proof. Let us prove a). Let

$$z\ \in\ \left[M_1 + \int_0^\tau \hat{w}\ (r)\, dr\right]\ \underline{*}\ H\ \left(\tau,\ N\left(R\left(\cdot\right)\right)\right)\ .$$

According to the definition of the geometrical difference $\underline{*}$ we have

$$z\ +\ H\ \left(\tau,\ N\left(R\left(\cdot\right)\right)\right)\ \subset\ M_1 + \int_0^\tau\ \left[\pi\, K\left(r-h\right)P \underline{*}\ \pi\, K\left(r-h\right)Q\,\right]dr.$$

As it is known [1], for arbitrary multi-valued representatives $A\left(r\right), B\left(r\right)$, $0 \leq r \leq \tau$, it is possible to have

$$\int_0^\tau\ \left[\,A\left(r\right)\ \underline{*}\ \ B\left(r\right)\right]dr\ \ \subset\ \int_0^\tau A\left(r\right)dr \ \underline{*}\ \ \int_0^\tau B\left(r\right)dr\,.$$

Hence, we have

$$M_1 + \int_0^\tau \left[\pi K\left(\tau-r\right)P\ \underline{*}\ \ \pi K\left(\tau-r\right)Q\right]dr \subset$$

$$\subset M_1 + \left(\int_0^\tau \pi K\left(\tau-r\right)Pdr\ \underline{*}\ \ \int_0^\tau \pi K\left(\tau-r\right)Qdr\right).$$

From these relations, by virtue of $A + \left(B\ \underline{*}\ C\right) \subset \left(A+B\right)\ \underline{*}\ C$, we get

$$M_1 + \left(\int_0^\tau \pi K\left(\tau-r\,\right)Pdr\ \underline{*} \int_0^\tau \pi K\left(\tau-r\right)Qdr\right)\ \subset$$

$$\subset \left(M_1 + \int_0^{\tau} \pi K(\tau - r)\, P dr \right) \underline{*} \int_0^{\tau} \pi K(\tau - r)\, Q dr.$$

Hence,

$$z + H(\tau, N(R(\cdot))) \subset \left(M_1 + \int_0^{\tau} \pi K(\tau - r)\, P dr \right) \underline{*} \int_0^{\tau} \pi K(\tau - r)\, Q dr$$

and from the definition of the geometrical difference $\underline{*}$ we have

$$z + \left(H(\tau, N(R(\cdot))) + \int_0^{\tau} \pi K(\tau - r)\, Q dr \right) \subset M_1 + \int_0^{\tau} \pi K(\tau - r)\, P dr,$$

or

$$z \in \left[M_1 + \int_0^{\tau} \pi K(\tau - r)\, P dr \right] \underline{*} \left[H(\tau, N(R(\cdot))) + \int_0^{\tau} \pi K(\tau - r)\, Q dr \right].$$

So a) is proved. Proofs of b), c), d) are similar.

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RECURSIVE UNSOLVABILITY OF A PROBLEM OF EXPRESSIBILITY IN THE LOGIC OF PROVABILITY

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Abstract It is notified the recursive unsolvability of a problem of syntactical expressibility in the Gödel-Löb logic of provability.

In studies lying on the intersection of mathematical logic and its applications an important role is often played by the relation of (functional) expressibility. That is a relation among logical operations (functions) signifying the possibility to obtain some of them from others by means of compositions. In the case of classical logic the expressibility relation was stadied by Post [1-3].

In particular, Post obtained the description of all classes of Boolean functions, closed with respect to expressibility, the so-called *Post classes* [3]. On the basis of the survey of these classes it is not difficult to obtain a not complicate algorithm for determining the expressibility. For any Boolean function f, given by a table or by formula, and for any finite system $\sum$ of such functions, this algorithm enables us to recognize whether the given function f is expressible through $\sum$ by means of superpositions.

At the same time there are some well-known logics represented by means of logical calculus (i.e. systems of axioms and inference rules). For example,

1) the (propositional) intuitionistic logic,

2) the dual intuitionistic logic,

3) the modal logics S4 and S5,

4) the Gödel-Löb logic of provability.

These logics can not be represented, as well as like classical logic by means of some finite system of finite truth tables. But any of them can be formulated by means of a corresponding logical calculus (i.e. syntactically), namely:

I. the intuitionistic calculus of Heyting [4];

II. the dual intuitionistic calculus of Moisil [5,6];

III. the modal systems S4 and S5 of Lewis [7];

IV. the calculus of provability of Gödel-Löb [8-12].

The language of the calculus of provability consists of formulas constructed in the usual way by means of the connectives & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation) and Δ (Gödel provability), and brackets, starting with propositional variables p, q, r (possibly with subscripts).

The calculus of provability is given by the axioms of the classical prepositional calculus and by the two Δ-axioms: $\Delta(\Delta p \supset p) \supset \Delta p$ (axiom of Löb), $\Delta(p \supset q) \supset (\Delta p \supset \Delta q)$, and by the three rules of inference: the substitution rule, the modus ponens rule, and the necessitation rule (permiting the transition from a formula F to ΔF).

The logic of provability of Gödel-Löb (in short, the logic GL) is defined as the set of all formulas derivable in the corresponding calculus.

Two formulas F and G are said to *be equivalent* in a logic L, if their equivalence $(F \sim G) = ((F \supset G)\&(G \supset F))$ is true in L (that is, belongs to the set L).

Now, let us remind the notion of expressibility in syntactical version, proposed by Kuznetsov [13].

Let consider a formula F and a system of formulas Σ. Then formula F is said to be *expressible in the logic L by means of the system* Σ, if F can be obtained from variables and from formulas of Σ by a finite number of applications of the weak rule of substitution $A, B/A[\pi/B]$ and the rule of replacement by equivalents in L A/B if $(A \sim B) \in L$.

By the *problem of expressibility in the logic* L we mean the algorithmic problem requiring the construction of an algorithm which, for every formula F and for every finite system of formulas Σ, enables us to recognize whether F is expressible in L by means of Σ.

Theorem 1. *There is no algorithm solving the problem of expressibility in the Gödel-Löb logic of provability.*

The problem of expressibility in a logic L is said to be recursively solvable if there exists an algorithm solving it in L.

The Theorem 1 is equivalent [14,15] to the following

Theorem 2. *The problem of expressibility in the Gödel-Löb logic of provability is recursive unsolvable.*

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THE χ^2 HOMOGENEITY TEST APPLIED TO DETERMINE THE EFFICIENCY OF MULTIMEDIA TECHNOLOGY IN THE TEACHING-LEARNING PROCESS

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Abstract The experimental implementation and the determination of the efficiency of multimedia teaching-learning technologies was done with the purpose of establishing the necessity of transformations that are paramount for the educational system, in order to synchronize it with the general development tendencies of contemporary society. This article highlights some ways in which various statistical methods are applied in view of comparing modern teaching methods, based on implementing informational technologies, with traditional ones tested, for a period of two years, in parallel groups: experimental and control groups.

The immediacy of the investigation topic consists in the fact that, last few years, society has forwarded ever more persistent demands regarding everything related to computers: training specialists in applying computer systems by developing abilities in finding, gathering and comprehending information, in applying information and communication technology in its processing, in constructing virtual models of objects and actions from the real world. Advanced information technology has contributed to the increase of motivation in studying certain subject matters, that have nothing in common with informatics, because these facilitate learning, due to the fact that individual characteristics, abilities and preferences of the student are taking into consideration, thus ensuring "the existence of reverse connections (feedback) between the student and the program [2], increasing the efficiency of the learning process."

Consequently, information technology is more and more frequently used in various spheres of human activity: medicine, finance, mass media, science, including education.

By bringing forward these arguments we can state that the reinvigoration of the educational process is basically impossible, without implementing advanced information technology, including multimedia technology. Within the scientific experiments, we propose various alternatives for organizing experimental groups. Organizing the experiments will be performed according to the following methods: the single group technique, the parallel or equivalent groups technique, the factor variation technique.

In the research experiment, we have applied the technique of parallel groups which requires the involvement of four groups of second year students, majoring in Finance and Banking.

In the measuring activity, the objects or their characteristics are assigned numbers, i.e. arbitrary amounts, with the condition that the established measurement rules ensure the viability of the admitted measuring function [1]. Measuring is an indispensable condition for later processing and interpretation of research results. In order for the measurement to observe the validation conditions it is compulsory to determine the characteristics of the phenomena, which we intend to measure and to use the most proper measuring tool. In educational research we can distinguish several steps of the evaluation:

1 the first step is the recording, which consists of detecting the presence or lack of an objective behavioral traits. Within this stage, the subjects and their answers will be counted, grades and averages of the same size will be counted etc.;

2 the second step of quantitative evaluation is ranking or classifying. It consists of outlining the research objects in a ascending or descending succession. Ranking the parameters based on certain traits which are common to all of them is called the procedure of the rank. Ranking the elements of the string is done based on a determined criterion. The position of each case in the string represents its rank within the group, which is attributed a number. If the evaluation of the exam performances is reported to the score of a test, the number of points obtained by each student indicates the rank. "The number obtained with the aid of ranks can serve as guidelines for evaluating the degree in which various categories of skills, behaviors and knowledge have been acquired." [4]

The data indicating the score obtained by each student of the two groups compared and based on the test: for experimental groups and the results of the control group are outlined in Table 1.

Experimental group			Control Group		
Student code	Score for the questions of the test	Grade	Student code	Score for the questions of the test	Grade
IE1	24	8.00	IC1	19	6.33
IE2	22	7.33	IC2	18	6.00
IE3	29	9.67	IC3	25	8.33
IE4	30	10.00	IC4	28	9.33
IE5	15	5.00	IC5	15	5.00
IE6	26	8.67	IC6	24	8.00
IE7	18	6.00	IC7	18	6.00
IE8	23	7.67	IC8	22	7.33

IE9	27	9.00	IC9	21	7.00
IE10	19	6.33	IC10	14	4.67
IIE1	25	8.33	IIC1	20	6.67
IIE2	29	9.67	IIC2	17	5.67
IIE3	28	9.33	IIC3	26	8.67
IIE4	30	10.00	IIC4	21	7.00
IIE5	17	5.67	IIC5	16	5.33
IIE6	26	8.67	IIC6	23	7.67
IIE7	18	6.00	IIC7	18	6.00
IIE8	26	8.67	IIC8	20	6.67
IIE9	30	10.00	IIC9	22	7.33
IIE10	21	7.00	IIC10	13	4.33
IIE11	24	8.00	**Sum/Avg**	**400**	**6.67**
IIE12	30	10.00			
Sum/Avg	**537**	**8.14**			

Table 1. Scores and grades obtained by the students of the two groups.

The χ^2 test is applied when:

- we want to test if it exists the differences between the distributions for a control group and an experimental group. It is a classic research situation where two groups are implied. The data from each group are considered separate patterns [5];
- we want to test if the preferences for a particular subject is different from a zone to other.

In the homogeneity χ^2 test we have a pattern for each group which we want to compare (two groups at least). In our case we considered the experimental group with 22 students and the control group with 20 students.

In the following we present the contingency table.

	Experimental group	**Control group**	
Grades Interval	Observably frequencies (Fo)	Observably frequencies (Fo)	**Total**
4...6	4	8	**12**
6.1...8	6	9	**15**
8.1...10	12	3	**15**
Total	**22**	**20**	**42**

Table 2.Frequency grade for experimental group and control group.

In the case of the homogeneity χ^2 test the totals of the groups are fixed (they represent the measure of the patterns which are established before the beginning of the research). Only the marginal totals of the interest qualitative variable depend on the observably frequencies from cells.

The expected frequencies for the homogeneity χ^2 test are obtained from the formula

$$Fa = \frac{Tr * Tc}{n} \tag{1}$$

where:

- Tr represents the total of the rows;
- Tc represents the total on the columns;
- n is the measure of the patterns.

Using this formula (unique for a contingency table) we compute the expected frequencies:

Grades Interval	**Experimental group**	**Control Group**	**Total**
4...6 Fa	4 6.285	8 5.714	**12**
6.1...8 Fa	6 7.857	9 7.142	**15**
8.1... Fa	12 7.857	3 7.142	**15**
Total	**22**	**20**	**42**

Table 3. The observation frequencies and the waiting frequencies for the grades from the two groups

The degrees of freedom for the homogeneity test χ^2 are computed from the formula $g.l. = (R-1)(C-1)$ where: R is number of the rows and C is number of the columns. In order to construct this test we follow the following steps:

1 the specification of the null and alternative hypothesis.

H_0**:** the two groups have the same homogenous distributions. This means that the changes of the increases of the notes have a occasional character,

H_1**:** the two groups have not homogenous distributions. This means that the increases of the notes as the result of the application of the multimedia courses in the teaching-learning process have a possible character;

2 the computing the degrees of freedom. Our contingency table has three rows and two columns, therefore $g.l = 2 * l = 2$;

3 the determination of the significance level and the computing of the critical value of the χ^2. We take a generic significance level $\alpha = 0.05$. Because the homogeneity test χ^2 is made only on the right part, the critical value for which the surface from the right part is equal to 5 percentage at two degrees of freedom, is $\chi^2_{critic} = 5.99$;

4 the drawing of the distribution χ^2 and of the repugnant zone. First of all, from fig. 1, we see that the curve has a less usually form. Generally, the curve χ^2 is a oblong curve at right, but in our figure this curve is not the same. This is due to the small number of the degrees of freedom. For two degrees of freedom only, the top of the curve is on the Oy axes;

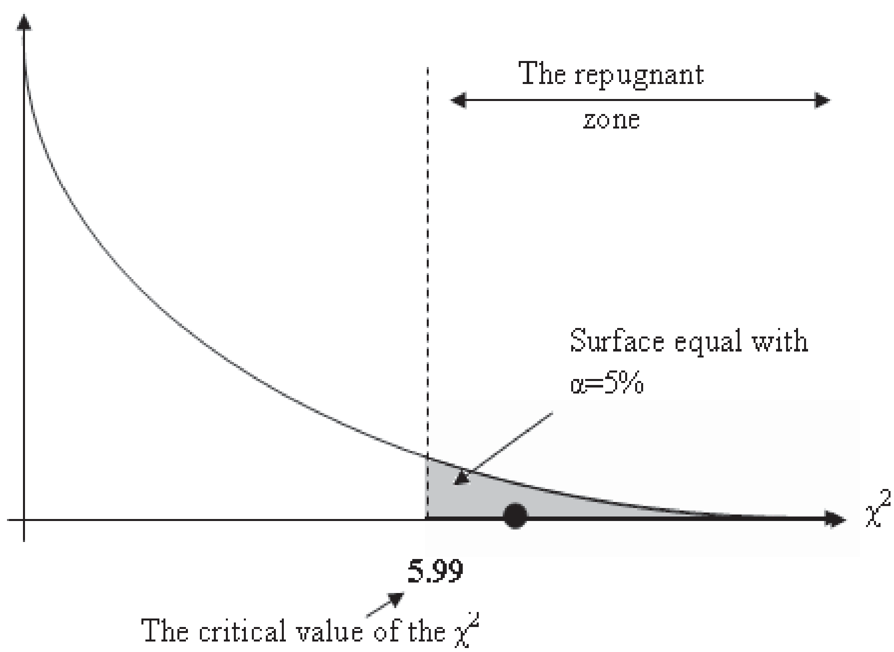

Fig.1. χ^2 curve.

5 the determination of the χ^2 by formula $\chi^2 = \sum_{i=1}^{k} \frac{(Fo-Fa)^2}{Fa}$ where:

- Fo represents observation frequencies,

- Fa represents the waiting frequencies,
- $k = R * C$, in our case $k = 2 * 3 = 6$.

$Fo - Fa$	$(Fo - Fa)^2$	$(Fo - Fa)^2/Fa$
-2.285	5.224489796	0.831169
-1.857	3.448979592	0.438961
4.142	17.16326531	2.184416
2.285	5.224489796	0.914286
1.857	3.448979592	0.482857
-4.142	17.16326531	2.402857

Table 4. The computing of the value χ^2.

After summing we obtain $\chi^2 = 7.2545$ (the point from fig. 1);

6 taking of the decision. Because the calculated χ^2 is larger than critical χ^2, it is in the repulsive zone, therefore we repulse the null hypothesis [3] about the homogeneity of the two groups. In conclusion, the H_1 hypothesis is verified, so, the growing of the notes in the result of the application of the multimedia courses in teaching-learning process has a possible character.

Conclusions

Applying the statistical methods to process experimental data has confirmed hypotheses about the positive impact (influence) of implementing multimedia courses in the teaching-learning in experimental groups, compared to the traditional method, applied in control groups.

This conclusion is made for the significance level $\alpha = 0,05$. Taking into consideration the fact that the elements of statistical populations are non-homogenous, the conclusion is considered statistical only for elements involved in the experiment. For other experimental elements, this statement is accepted by analogy.

The research in question has tried to propose a new perspective for performing the learning-teaching process, corresponding to present requirements, which, by using information technology, offers new possibilities to stimulate interest, new ways for active involvement of the student in the knowledge process.

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PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO FINANCE

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As a component part of the national market or the international market, the financial market comprises, in its turn, two distinct segments: the bank market and the proper financial market, and the capital market or capitals' market respectively [2].

As part of the capital market, an important place is taken by the derivate financial products or derivatives (e.g. options, forward contracts, futures contracts). These are financial instruments whose value depends on the underlying asset value (stocks, foreign currencies, commodities etc.). The derivatives are used by investors, mainly, in the following purposes: speculation (with the hope of earning a great amount of money with an initial small investment), hedging (covering the risk through buying other contracts which have as a support the same underlying asset [8]) and arbitration (fulfilling income bringing transaction so that the investor should not assume risks or invest capital [1].

The simplest derivative is a forward or futures contract. The forward contract is a sale and purchase contract of a financial asset, commodity or foreign currency, at a certain future date (denoted by T), established through contract and to a price (called strike price and further denoted by E), established at a moment of the transaction end.

At the emission, the premium of a forward contract is zero.

A futures contract has the same features as a forward contract, with a few differences:

- the futures contract is standardized. All futures contracts having the same standardized asset form a sort of futures contracts [3];

1 the price vary daily depending on the demand and supply;

2 it is traded on a secondary market. The options are standardized contracts which give their buyer the right to sell or to buy a certain quantity of underlying asset at a fixed price, at the option expiry date (denoted by T) or before this date.

There are two types of options:

1 buying options (CALL), which give the buyer the right (removing the obligation) to buy the underlying asset at a fixed price at the moment of the transaction end and at a certain established date;

2 selling options (PUT), which give the buyer the right (removing the obligation) to sell the underlying asset at a fixed price at the moment of the transaction end and at a certain established date.

As this is a right and not an obligation, the buyer of the option can choose not to exercise this right and let the option expire. To exchange this right, the buyer of the option pays the option seller a certain amount of money, called premium.

For a CALL option, the payoff function is max(S-E, 0), where S is the underlying asset value. Thus, if the price of the underlying asset, at the T time, is higher than the practicing price, then the option will be exercised. If not, the option will be not exercised.

For a PUT option, the payoff function is max(E-S, 0), the option being exercised only if the price of the underlying asset, at the T time, is lower than the exercise price.

Depending on the moment of her exertion, we distinguish two great categories of options:

1 European options, whose exertion is allowed only at the date of payment;

2 placecountry-region American options, whose exertion is allowed any time before the date of payment.

As time passes, the option value modifies not only because the expiry date comes nearer but also because of the variations of the underlying asset's price. Therefore, the value of an option (denoted by V), at the T expiry moment, depends exclusively on S and T. It is also supposed the same thing for a time moment t<T and we write Vt=V(S, t). If the law of movement for S is known, namely we can write the equation which determines dS, applying Ito's lemma [7], we determine dV_t, where the price increases are denoted by dV_t [8].

One of the most important methods used in finances for evaluating the derivative products consists in using partial differential equations.

Neftci, in [6], shortly presents the logic behind the derivatives evaluating method which led to the use of partial differential equations. A wide accepted model in finances, ever since the moment of its publishing (1973), is the Black_Scholes model of option pricing.

Thus, starting with the stochastic differential equation

$$dS = \mu \cdot Sdt + \sigma \cdot SdB_t, \tag{1}$$

where μ = drift, σ = volatility, B = Brownian motion, building a free risk portfolio and applying Ito's 1_dimensional formula, the Black_Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r \cdot \frac{\partial V}{\partial S} \cdot S - r \cdot V = 0), \qquad (2)$$

is obtained, where r is the interest rate.

By adding the final condition (written for a placeEuropean CALL option):

$$Vt = max(S - E, 0), \qquad (3)$$

and boundary conditions

$$\lim_{S \to 0} V(S,t) = 0$$

and

$$\lim_{S \to \infty} V(S,t) = \infty$$

to a partial differential equation, allows us an accurate determination of the solution. In financial problems, limit conditions show the way in which the solution behaves for $S \to 0$ or $S = 0$. Generally, the final conditions consist in payoff function. The second-order partial differential equation written in the general form

$$a\frac{\partial^2 V}{\partial S^2} + b\frac{\partial^2 V}{\partial S \partial t} + c\frac{\partial^2 V}{\partial t^2} + d\frac{\partial V}{\partial S} + e\frac{\partial V}{\partial t} + f = 0$$

is of parabolic type if $b^2 - 4ac = 0$. Thus, the Black_Scholes partial differential equation is parabolic and linear (the sum of two solutions of the equation is another solution of the equation). Black and Scholes solved this equation and obtained the solution

$$V(S,t) = SN(d_1) - E \cdot e^{-r \cdot (T-t)} \cdot N(d_2),$$

where $d_1 = \frac{\ln\frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $N(d_i) = \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} dx$.

The partial differential equation obtained by Black_Scholes is relevant if a series of hypotheses are fulfilled [1], [6], [8]:

1 returns are log normally distributed;

2 the stock pays no dividends during the option life;

3 no commissions are charged;

4 markets are efficient;

5 the interest rate and volatility remain constant;

6 there are no arbitrage opportunities.

For an placeEuropean PUT option with final condition $V_t = max(E - S, 0)$ and boundary conditions $\lim_{S\to 0} V(S,t) = E \cdot e^{-r(T-t)}$ and $\lim_{S\to\infty} V(S,t) = 0$, the Black_Scholes formula reads

$$V(S,t) = -SN(-d_1) + E \cdot e^{-r\cdot(T-t)} \cdot N(-d_2)$$

with $N(-d) = 1 - N(d)$.

However, there are many situations in which one or more of these hypotheses are not fulfilled, case in which a new partial differential equation is obtained.

If the underlying asset generates constant and continuous share, denoted by δ, it is obtained the equation [1]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) \cdot \frac{\partial V}{\partial S} \cdot S - r \cdot V = 0$$

which has the solutions

$$V(S,t) = Se^{-\delta(T-t)}N(d_1) - E \cdot e^{-r\cdot(T-t)} \cdot N(d_2) \text{ (for placeStateCALL options)}$$

$$V(S,t) = -Se^{-\delta(T-t)}N(-d_1) + E \cdot e^{-r\cdot(T-t)} \cdot N(-d_2)\ (forPUToptions)$$

where

$$d_1 = \frac{\ln \frac{S}{E} + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln \frac{S}{E} + (r-\delta-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Numerical example. Consider an European CALL option on a Microsoft stock which has the following specific features[12]: the current price S=28,22, the striking price E=27.5, volatility (%) 19,37, the risk_free interest rate (%) 5.32 and continuous dividend rate 2,1%. If the date of payment is 30 days, it is obtained a value for an option of 1,09.

In Chart 1 there are graphically represented the values of the option calculated by means of the Black_Scholes formula, in the situation in which the current price of the stock has the values given in Table 1, the values of the other specific features remaining constant.

Share price	Option price
25,840	0,1067
25,910	0,1174
25,930	0,1206
25,980	0,1289
26,330	0,2004
26,860	0,3591
27,200	0,4973
27,440	0,6130
28,22	1,0901

Table 1

Chart 1. The graph of the pricing option value.

The options pricing on futures contracts leads to an equation analogous to the equations (2)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - r \cdot V = 0,$$

where F = futures prices. The solutions of these equations are

$$V(F,t) = e^{-r(T-t)}[FN(d_1) - E \cdot N(d_2)], \text{ (for placeStateCALL options)}$$

$$V(F,t) = e^{-r(T-t)}[-FN(-d_1) + E \cdot N(-d_2)], \text{ (for PUT options)}$$

where $d_1 = \dfrac{\ln \frac{F}{E} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$. Therefore, solving the partial differential equations and their analysis plays a very important role in derivatives theory.

Notes

1. In concordance with the information taken by the author from the address: www.nasdaq.com.
2. Share prices for the MSFT assets (in the interval 01.09.2006-29.09.2006). See www.nasdaq.com

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ASYMPTOTIC STABILITY OF TRIVIAL SOLUTION OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED LINEARITY, IN BANACH SPACES

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This year the mathematical scientific community in Russia celebrated the 150-th birthday of the outstanding mathematician - academician of St-Petersburg Academy Alexandr Mikhaylovich Lyapunov. International Congress of Nonlinear Analysis in June 2007 was devoted to this jubilee. On World Congress ICIAM this significant event was marked in reports of the minisymposium organized by our scientific group. Practically during all my life I worked in branching and bifurcation theory of solutions of nonlinear equations, one of initiator of which was Academician Lyapunov (see, for example, [1-2]). Another among his important contribution to the mathematical science was the creation of contemporary stability theory. Before preparing the 150-th Lyapunov jubilee I began a series of articles, trying to over-understand some of this fundamental results in stability theory from the functional analysis point of view. This allows to carry over the Lyapunov approach from ordinary differential equations to functional-differential and integro-partial-differential equations.

At first our attention is payed to the generalization of the Lyapunov theorem about stability on the first approximation. In a Banach space, a differential equation with linear unbounded operator and nonlinear part depending on time is considered. Non-analytic and analytic cases are studied. In this communication the more general case considered by us is suggested. The nonlinear operator contained in the differential equation can be increasing when the time goes to infinity. This fact essentially extends the application possibilities of our method.

Some separate fragments of this communication were reported, discussed and published on various kinds of international conferences. Note, in particular, International Congress in Steklov Mathematical Institute RAN, devoted to the 100th Birthday of Academician Nikol'sky [6], ISAAC Congress in Sicilia [5], [8], seminar of Academician Il'in on Computation and Cybernetics Faculty of Moscow State Lomonosov University [7], World Congress ICIAM-2007.

In a real or complex Banach space X let us consider on the semiaxis $\Re^+ = [0, \infty)$ the Cauchy problem for the differential equation

$$\dot{x} = Ax + R(t, x), \quad x(0) = x_0. \tag{1}$$

We make the following assumptions.

I. A is a closed linear operator, densely defined on $D(A) \subset X$ with values in X. It is the generator of a strongly continuous semigroup $U(t) = exp(At)$ exponentially decreasing, i.e. there exist $M > 0$ and $\alpha > 0$ such that on $\Re^+$ the inequality $||U(t)|| \leq Me^{-\alpha t}$ is valid.

II. The nonlinear operator $R(t, x)$ is continuous by the set of variables $t \in \Re^+, \quad x \in S = \{x \in X| \quad ||x|| < p\}$, and $R(t, 0) = 0$ for all $t \in \Re^+$. Let further exist $\beta > 0$ and the continuous on $\Re^+$ function $C(t) > 0$ such that for all $t \in \Re^+, \quad x_1, x_2 \in S$ the following inequality

$$||R(t, x_1) - R(t, x_2)|| \leq C(t) max^{\beta}(||x_1||, ||x_2||)||x_1 - x_2||$$

is fulfilled.

For the problem (1) define its classical solution as the function $x(.) : \Re^+ \to S$ such that on $\Re^+$, $x(t) \in D$ is continuously differentiable and its substitution in (1) turns it in identity.

The generalized solution of (1) is understood as a continuous on $\Re^+$ solution of the integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)R(s,x(s))ds. \tag{2}$$

In addition, the weakened solution of the problem (1) can be considered, when x(t) is continuous on $\Re^+$ and continuously differentiable on the semi-axis $(0, +\infty)$.

Note that from the condition II the inequality

$$||R(t,x)|| \le C(t)||x||^{1+\beta}, \forall x \in S, t \in \Re^+. \tag{3}$$

follows. The differential equation from (1) has the trivial classical solution $x(t) = 0$.

If it would be established (under some additional conditions), that the problem (1) for all sufficiently small $||x_0||$ has unique small together with small $||x_0||$ solution and $||x(t)|| \to 0,\ t \to \infty$, then by this the asymptotic stability of the trivial solution would be established.

Introduce two complementary restrictions on the function C(t) in condition II.

III. There exist $\gamma \in (0, \alpha)$, $C^* > 0$ such that $C(t)e^{-\gamma\beta t} \le C^*, \forall t \Re^+$.

III'. $\int_0^{+\infty} C(s)e^{-\alpha\beta s}ds = C_1^* < \infty$ (convergent).

Further our aim is the study of solutions to integral equation (2) on $\Re^+$, which are exponentially decreasing as $t \to +\infty$. For the achievement of this aim introduce a suitable family of Banach spaces of abstract functions.

Definition. For $\gamma > 0$, will be called C_γ the set of all defined and continuous

on $\Re^+$ abstract functions $x(t)$ taking the values in X with the natural operations of their addition and multiplication on scalars, for which the following norm is finite $|||x|||_\gamma = sup_{\Re^+}||x(t)||e^{\gamma t}$ will be called C_γ.
Note that C_γ is a Banach space.
From this definition it follows that if $x(t) \in C_\gamma$, then $||x(t)|| \leq |||x|||_\gamma e^{-\gamma t}$.
Note also that if $\gamma_1 < \gamma_2$ then the space C_{γ_2} is embedded in the space C_{γ_1}.

Lemma 1. *For $\gamma \in (0, \alpha)$ define on X the linear operator in the form $(Dx_0)(t) = U(t)x_0$. Then $D \in L(X, C_\gamma)$, $||D|| \leq M$ and $Dx_0(t)$ is the generalized solution of Cauchy problem*

$$\dot{x} = Ax, \quad x(0) = x_0.$$

Proof. $||e^{\gamma t}U(t)x_0|| \leq M||x_0||$, and it means $|||Dx_0|||_\gamma \leq M||x_0||$.□

Consider in C_γ the open ball $S_\gamma = \{x(t) \in C_\gamma;\ |||x|||_\gamma < p\}$.

Lemma 2. *Assume $\gamma \in (0, \alpha]$, one of the conditions III or III' be satisfied and $x(t) \in S_\gamma$. Define the nonlinear operator F by formula*

$$F(x)(t) = \int_0^t U(t-s)R(s, x(s))ds.$$

Then the abstract function $F(x)(t)$ belongs to C_γ and there exists an independent on x constant $K > 0$ such that $|||F(x)|||_\alpha \leq K|||x|||_\gamma^{1+\beta}$.

Proof. From the condition II and inequality (3) it follows the estimate

$$\begin{aligned} ||e^{\gamma t}F(x)(t)|| &\leq Me^{-(\alpha-\gamma)t}\int_0^t e^{\alpha s}C(s)||x(s)e^{\gamma s}||^{1+\beta}e^{-\gamma(1+\beta)s}ds \\ &\leq Me^{-(\alpha-\gamma)t}\int_0^t C(s)e^{(\alpha-\gamma)s}e^{-\gamma\beta s}ds|||x|||_\gamma^{1+\beta}. \end{aligned}$$

For $\gamma = \alpha$ the passage to supremum on $\Re^+$ gives $|||F(x)|||_\alpha \leq MC_1^*|||x|||_\alpha^{1+\beta}$.
Let be $\gamma < \alpha$. Consider the quotient of $\int_0^t C(s)e^{(\alpha-\gamma)s}e^{-\gamma\beta s}ds$ and $e^{(\alpha-\gamma)t}$.
Application of the L'Hopital rule gives the expression

$$(\alpha-\gamma)^{-1}\lim_{t\to\infty} C(t)e^{\gamma\beta t} \leq (\alpha-\gamma)^{-1}C_1^*.$$

From the proved lemma it follows that the operator $\Phi(x, x_0) = Dx_0 + F(x)$ is acting in C_γ, mapping every closed ball from S_γ of sufficiently small radius in a closed ball. The subsequent lemma shows that this operator satisfies the special form of the Lipschitz condition.

Lemma 3. *Let $x_1(t), x_2(t) \in S_\gamma$. Then the following inequality*

$$|||F(x_1) - F(x_2)|||_\gamma \leq Kmax^\beta(|||x_1||_\gamma, |||x_2|||_\gamma)|||x_1 - x_2|||_\gamma$$

is true.

The proof is carried out by the scheme of the proof of Lemma 2.
Further, for simplicity of presentation, restrict oneself to the case $\gamma = \alpha$.

Theorem. *There exist the numbers $r_* > 0, \quad \rho_* > 0$ such that for any $x_0, \quad ||x_0|| \leq \rho_*$ the equation $x = Dx_0 + F(x)$ has in the ball $|||x|||_\alpha \leq r_*$ the continuous in the ball $||x_0|| \leq \rho_*$ unique solution $x = x(x_0), \quad x(0) = 0$.*

The proof of this lemma is contained in the article [7].

Coming back to the integral equation (2) we see that the existence of its continuous on $\Re^+$ solution is proved and it satisfies the inequality $||x(t)|| \leq r_* e^{-\alpha t}$. If applied to Cauchy problem (1) this means the proof of existence of its exponentially decreasing generalized solution for initial values of sufficiently small norm. This fact means the asymptotic stability of the trivial solution of (1) in the generalized sense.

Remarks. Under the assumption of the C(t) boundedness in the work [7] it is proved that the generalized solution of the problem (1) is its classical solution if R(x,t) satisfies some special Hölder condition.

We plan to investigate this question in the hypotheses of this paper. More difficult is the generalization on the case when operator A is dependent on t. Such situation is closely connected with possibilities of the developments of Lyapunov exponents method and Lyapunov-Floquet theory to the case of unbounded operators in Banach spaces. The author hopes, that all these problems will be resolved in the collaboration at the passage of work on the research grant Romania-Russia.

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BOUNDEDNESS PROBLEMS FOR JUMPING PETRI NETS

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Abstract The goal of this paper is to extend a decidability result for the classical boundedness problem from the class of finite jumping Petri nets to the class of reduced-computable jumping Petri nets, and also to establish a new one for the generalized boundedness problem.

Keywords:parallel/distributed systems, Petri nets, decidability.

1. INTRODUCTION

A Petri net ([8]) is a mathematical model used for the specification and the analysis of parallel and distributed systems.

Petri nets proved to be a powerful language for system modeling and validation and they are now in widespread use for many different practical and theoretical purposes in various fields of software and hardware development.

One type of problems related to Petri nets is that of finding algorithms which take a Petri net Σ and a property π as input and answer, after a finite number of steps, whether or not Σ satisfies π. For instance, the Karp-Miller graph for Petri nets allows us to decide the boundedness problem (BP), the finiteness reachability set/tree problem (FRSP/FRTP), and the quasi-liveness problem (QLP), or the equivalent problem called the coverability problem (CP) (see [5, 9] for more details).

It is well-known that the behaviour of some distributed systems cannot be adequately modeled by classical Petri nets. Many extensions which increase the computational and expressive power of Petri nets have been thus introduced. One direction has led to various modifications of the firing rule of nets. One of these extension is that of jumping Petri net, introduced in [10]. A jumping Petri net is a classical net Σ equipped with a (recursive) binary relation R on the markings of Σ. The meaning of a pair $(m, m') \in R$ is that the net Σ may "spontaneously jump" from m to m' (this is similar to λ-moves in automata theory).

Previous results (see [10]) showed that the decision problems related to reachability, coverability and quasi-liveness are undecidable for general jumping nets and are decidable only for finite jumping nets, by using the techniques of Karp-Miller coverability graphs in a similar manner as for classical P/T nets ([5]).

In [12] we introduced a larger class of jumping nets than the finite jumping nets, called reduced-computable jumping nets, for which we could define finite Karp-Miller coverability graphs. Based on them, in this paper we will extend a decidability result about the classical boundedness problem to the class of reduced-computable jumping Petri nets, and we shall establish a new one for the generalized boundedness problem.

The paper is organized as follows. Section 2 presents the basic terminology and notation, and also previous results concerning Petri nets and jumping Petri nets. In Section 3, we use the Karp-Miller coverability structures to establish the decidability of some boundedness problems for reduced-computable jumping Petri nets. Finally, in Section 4 we conclude this paper and formulate some open problems.

2. PRELIMINARIES

In this section we will establish the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites for the understanding of this paper (for details the reader is referred to [1, 8, 9, 4]). Mainly, we will follow [4, 10].

2.1. PETRI NETS

A *Place/Transition net*, shortly *P/T-net* or *net*, (finite, with infinite capacities), abbreviated *PTN*, is a 4-tuple $\Sigma = (S, T; F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbb{N}$ is the *weight function* of Σ satisfying $W(x, y) = 0$ iff $(x, y) \notin F$.

A *marking* of a *PTN* Σ is a function $m : S \to \mathbb{N}$; it will be sometimes identified with a $|S|$-dimensional vector. The operations and relations on vectors are defined component-wise. $\mathbb{N}^S$ denotes the set of all markings of Σ. A *marked PTN*, abbreviated *mPTN*, is a pair $\gamma = (\Sigma, m_0)$, where Σ is a *PTN* and m_0, called the *initial marking* of γ, is a marking of Σ.

In the sequel we often use the term "Petri net"(*PN*) or "net" whenever we refer to a *PTN* (*mPTN*) and it is not necessary to specify its type (i.e. marked or unmarked).

Let Σ be a net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbb{N}$ and $\Delta t, \Delta w : S \to \mathbb{Z}$ are defined by $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$ and

$$\Delta w(s) = \begin{cases} 0, & \text{if } w = \lambda, \\ \sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \ldots t_n \ (n \geq 1), \end{cases} \qquad \text{for all } s \in S.$$

The sequential behaviour of a net Σ is given by the so-called *firing rule*, which consists of

(ER) the *enabling rule*: a transition t is *enabled* at a marking m in Σ (or t is *fireable* from m), abbreviated $m[t\rangle_\Sigma$, iff $t^- \leq m$;

(CR) the *computing rule*: if $m[t\rangle_\Sigma$, then t may *occur* yielding a new marking m', abbreviated $m[t\rangle_\Sigma m'$, defined by $m' = m + \Delta t$.

In fact, for any transition t of Σ we have a binary relation on $\mathbb{N}^S$, denoted by $[t\rangle_\Sigma$ and given by: $m[t\rangle_\Sigma m'$ iff $t^- \leq m$ and $m' = m+\Delta t$. If $t_1, t_2, \ldots, t_n$, $n \geq 1$, are transitions of Σ, the classical product of the relations $[t_1\rangle_\Sigma, \ldots, [t_n\rangle_\Sigma$ will be denoted by $[t_1 t_2 \ldots t_n\rangle_\Sigma$; i.e. $[t_1 t_2 \ldots t_n\rangle_\Sigma = [t_1\rangle_\Sigma \circ \ldots \circ [t_n\rangle_\Sigma$. Moreover, we also consider the relation $[\lambda\rangle_\Sigma$ given by $[\lambda\rangle_\Sigma = \{(m,m) | m \in \mathbb{N}^S\}$.

Let $\gamma = (\Sigma, m_0)$ be a marked Petri net, and $m \in \mathbb{N}^S$. The word $w \in T^*$ is called a *transition sequence* from m in Σ if there exists a marking m' such that $m[w\rangle_\Sigma m'$. Moreover, the marking m' is called *reachable* from m in Σ. We denote by $RS(\Sigma, m) = [m\rangle_\Sigma = \{m' \in \mathbb{N}^S | \exists w \in T^* : m[w\rangle_\Sigma m'\}$ the set of all reachable markings from m in Σ. In the case $m = m_0$, the set $RS(\Sigma, m_0)$ is abbreviated by $RS(\gamma)$ (or $[m_0\rangle_\gamma$) and it is called *the set of all reachable markings* of γ.

We shall assume to be known other notions from P/T-nets, like coverable marking, bounded place, simultaneously unbounded set of places, pseudo-markings etc. For more details about these notions, and about the basic decision problems for P/T-nets, the reachability structures and the Karp-Miller coverability structures for them, and also about the case of P/T-nets with infinite initial markings, the reader is referred to Appendix 4.

2.2. JUMPING PETRI NETS

Jumping Petri nets ([10, 11]) are an extension of classical P/T-nets, which allows them to perform "spontaneous jumps" from one marking to another (this is similar to λ-moves in automata theory).

A *jumping P/T-net*, abbreviated *JPTN*, is a pair $\gamma = (\Sigma, R)$, where Σ is a *PTN* and R, called the *set of (spontaneous) jumps* of γ, is a binary relation on the set of markings of Σ (i.e. $R \subseteq \mathbb{N}^S \times \mathbb{N}^S$). In what follows the set R of jumps of any *JPTN* will be assumed to be *recursive*, that is for any couple of markings (m, m') we can effectively decide whether or not (m, m') is a member of R.

A *marked jumping P/T-net*, abbreviated *mJPTN*, is defined similarly as a *mPTN*, by changing "Σ" into "Σ, R".

Let $\gamma = (\Sigma, R)$ be a *JPTN*. The pairs $(m, m') \in R$ are referred to as *jumps* of γ. If γ has finitely many jumps (i.e. R is finite) then we say that γ is a *finite jumping net*, abbreviated *FJPTN*.

We shall use the term "*jumping net*" (*JN*) ("*finite jumping net*" (*FJN*), resp.) to denote a *JPTN* or a *mJPTN* (a *FJPTN* or a *mFJPTN*, resp.) whenever it is not necessary to specify its type (i.e. marked or unmarked).

Pictorially, a jumping Petri net will be represented as a classical net and, moreover, the relation R will be separately listed.

The behaviour of a jumping net γ is given by the *j-firing rule*, which consists of

(jER) the *j-enabling rule*: a transition t is *j-enabled* at a marking m (in γ), abbreviated $m[t\rangle_{\gamma,j}$, iff there exists a marking m_1 such that $mR^*m_1[t\rangle_\Sigma$ (Σ being the underlying net of γ and R^* the reflexive and transitive closure of R);

(jCR) the *j-computing rule*: if $m[t\rangle_{\gamma,j}$, then the marking m' is *j-produced* by occurring t at the marking m, abbreviated $m[t\rangle_{\gamma,j}m'$, iff there exist markings m_1, m_2 such that $mR^*m_1[t\rangle_\Sigma m_2R^*m'$.

The notions of *transition j-sequence* and *j-reachable marking* are defined similarly as for Petri nets (the relation $[\lambda\rangle_{\gamma,j}$ is defined by $[\lambda\rangle_{\gamma,j} = \{(m, m') | m, m' \in \mathbb{N}^S, mR^*m'\}$). The *set of all j-reachable markings* of a marked jumping net γ is denoted by $RS(\gamma)$ (or by $[m_0\rangle_{\gamma,j}$).

All other notions from P/T-nets (i.e. coverable marking, bounded place, bounded (or safe) net, simultaneously unbounded set of places, pseudo-markings, etc.) are defined for jumping Petri nets similarly as for Petri nets, by considering the notion of *j-reachability* instead of *reachability* from P/T-nets. Also, all the decision problems from P/T-nets, like (RP), (CP), (BP) and (SUbP), are defined for jumping Petri nets similarly as for P/T-nets.

Some jumps of a marked jumping net may be never used. Thus we say that a marked jumping net $\gamma = (\Sigma, R, m_0)$ is *R-reduced* ([10]) if for any jump $(m, m') \in R$ of γ we have $m \neq m'$ and $m \in [m_0\rangle_{\gamma,j}$. The *reduction problem* (RedP) is: Given γ a *JPTN*, is γ R-reduced?

Remark 2.1. *The following decidability results were proved in [10, 4]: i) the problems (RP), (CP), (BP) are undecidable for mJPTN; ii) the problems (RP), (RedP), (CP), (BP) are decidable for mFJPTN.*

Coverability structures for jumping Petri nets The previous positive decidability results from [10] were based on defining Karp-Miller coverability trees only for the subclass of *finite* jumping Petri nets. Therefore, we were interested in extending the class of jumping Petri nets for which we can define finite Karp-Miller coverability structures. Having such a larger class of nets, afterwards we can solve the above decidability problems for it based on these finite coverability structures.

In [12] we succeeded to introduce a class of jumping nets larger than the finite jumping nets, called *reduced-computable jumping nets*, for which we could define finite Karp-Miller coverability structures (trees and graphs), and also minimal coverability structures; moreover, we extended the results about the minimal coverability structures for P/T-nets from [3] to this class of jumping nets.

Let us recall from [12] the definition of reduced-computable jumping nets.

Let $\gamma = (\Sigma, R)$ be an arbitrary jumping net. We associated with γ a finite subset of jumps $R_{\omega-max}$ (which is maximal in a sense specified below, and which can be used instead of the whole set of jumps R to construct the coverability graphs) as follows. We denoted by "ω-jumps" the set

$$R_\omega = \Big\{ r \in \mathbb{N}_\omega^{2|S|} - \mathbb{N}^{2|S|} \,;\, \exists \{r_n\}_{n\geq 0} \subseteq R \text{ strictly increasing with } \lim_{n\to\infty} r_n = r \Big\}.$$

Let $\overline{R} = R \cup R_\omega$. We defined the *set of ω-maximal jumps* of γ as $R_{\omega-max} = maximal(\overline{R}) = \{r' \in \overline{R} \mid \forall r \in \overline{R} - \{r'\} : r' \not\leq r\}$.

The following are obvious properties of the set of ω-maximal jumps of a jumping net (the proofs are easy and can be found in [12]):

Proposition 2.1. *(1) $R_{\omega-max}$ is finite; (2) $\forall\, r \in \overline{R}, \exists\, r' \in R_{\omega-max}$ such that $r \leq r'$; (3) $\forall\, r \in R_{\omega-max}, \exists\, \{r_n\}_{n\geq 0} \subseteq R$ such that $\lim_{n\to\infty} r_n = r$.*

A marked jumping net is called *reduced-computable jumping net* ([12]), abbreviated *mRCJPTN*, if it is R-reduced and the set $R_{\omega-max}$ is computable.

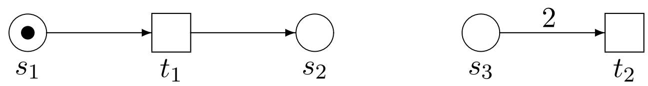
Example 2.1. *Let $\gamma = (\Sigma, R, m_0)$ be the jumping Petri net from Fig. 1; the initial marking is $m_0 = (1,0,0)$ and the set of jumps is $R = \{\ ((0,1,0), (0,0,2\,n)) \mid n \geq 0\}$. Notice that γ is an infinite R-reduced jumping net, and that the only transition sequences in γ are the following ones:*

$$(1,0,0)\ [t_1\rangle_\Sigma\ (0,1,0)\ R\ (0,0,2\,n)\ [t_2\rangle_\Sigma\ (0,0,2\,n-2)\ [t_2\rangle_\Sigma \ldots [t_2\rangle_\Sigma\ (0,0,0),$$

for all $n \geq 0$. Thus, the transition sequence set is $TS(\gamma) = \{t_1 t_2^n \mid n \geq 0\}$, and the reachability set is $RS(\gamma) = \{(1,0,0),(0,1,0)\} \cup \{(0,0,2\,n) \mid n \geq 0\}$. Notice that the reachability set is infinite.

In [12] we also introduced reachability trees and graphs for jumping nets, by a straightforward extension of these structures from classical P/T-nets (i.e. by adding arcs, labelled by "j", for all the jumps of the net).

The reachability graph $\mathcal{RG}(\gamma)$ of the jumping net from example 2.1 is shown in Fig. 2; notice also that it is an infinite graph. This net γ has only one



$$R = \{\ ((0,1,0),(0,0,2\,n)) \mid n \geq 0\}$$

Fig. 1. A jumping Petri net.

"ω-jump", namely $((0,1,0),(0,0,\omega))$, which is also the only ω-maximal jump of the net γ, i.e. $R_{\omega-max} = \{\ ((0,1,0),(0,0,\omega))\ \}$. Thus, γ is a reduced-computable jumping net.

Now let us recall from [12] the definition of coverability trees and graphs generalized for reduced-computable jumping Petri nets.

Let $\gamma = (\Sigma, R, m_0)$ be a *mRCJPTN* with $R \neq \emptyset$. Then the set of ω-maximal jumps is non-empty and finite, i.e. $R_{\omega-max} = \{\ (m'_i, m''_i)\ |1 \leq i \leq n\}$, with $n \geq 1$. Following the same line as in [10], we associated with γ the following P/T-nets: $\gamma_0 = (\Sigma, m_0)$ and $\gamma_i = (\Sigma, m''_i)$, for each $1 \leq i \leq n$, and we defined the notions of *Karp-Miller coverability trees / graphs* of the jumping net γ as being the tuples of the coverability trees / graphs of the P/T-nets $\gamma_0, \gamma_1, \ldots, \gamma_n$:
$\mathcal{KMT}(\gamma) = <\mathcal{KMT}(\gamma_0), \mathcal{KMT}(\gamma_1), \ldots, \mathcal{KMT}(\gamma_n)>$ and
$\mathcal{KMG}(\gamma) = <\mathcal{KMG}(\gamma_0),\ \mathcal{KMG}(\gamma_1), \ldots,\ \mathcal{KMG}(\gamma_n)\ >$ respectively. Notice

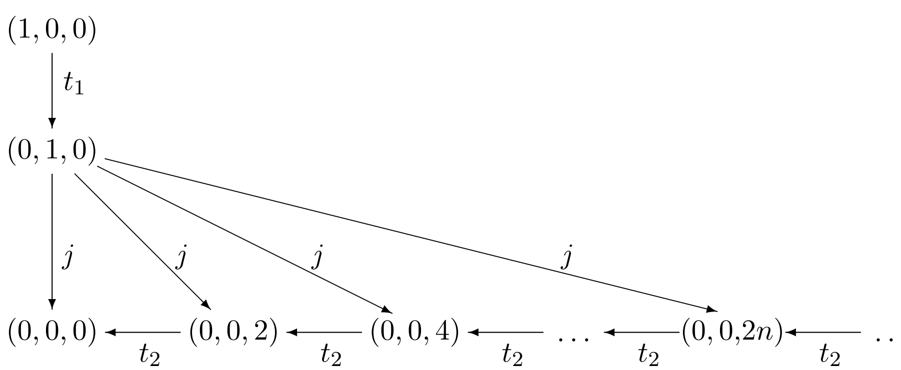


Fig. 2. The reachability graph of γ.

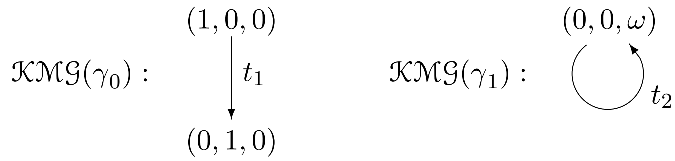


Fig. 3. The Karp-Miller coverability graph of γ.

that it is possible that some of the P/T-nets γ_0,γ_1, $\ldots$,γ_n to have initial markings with ω-components.

Example 2.2. *The P/T-nets associated with the jumping net γ from Example 2.1 are $\gamma_0 = (\Sigma, (1, 0, 0))$ and $\gamma_1 = (\Sigma, (0, 0, \omega))$. Their sets of transition sequences are $TS(\gamma_0) = \{t_1, \lambda\}$ and $TS(\gamma_1) = \{t_2^n \mid n \geq 0\}$, respectively, and their reachability sets are $RS(\gamma_0) = \{(1, 0, 0), (0, 1, 0)\}$ and $RS(\gamma_1) = \{(0, 0, \omega)\}$ respectively. The Karp-Miller coverability graph of the jumping net γ is $\mathcal{KMG}(\gamma) = < \mathcal{KMG}(\gamma_0), \mathcal{KMG}(\gamma_1) >$, and it is shown in fig. 3.*

3. BOUNDEDNESS PROBLEMS FOR JUMPING PETRI NETS

In this section we shall show how we can use the Karp-Miller coverability graph $\mathcal{KMG}(\gamma)$ to solve some boundedness problems for reduced-computable jumping Petri nets.

First of all, let us give a technical result for P/T-nets, more precisely a property of infinite converging sequences of markings, which we will need later for the proofs of some results.

Proposition 3.1. *Let Σ be a P/T-net and $\{m_n\}_{n\geq 0} \subseteq \mathbb{N}^S$ an infinite sequence of markings. If there exists $\lim_{n\to\infty} m_n = m$, with $m \in \mathbb{N}_\omega^S$, and if $m[w\rangle_\Sigma m'$, with $w \in T^*$, then there exist an integer $n_0 \geq 0$ and an infinite sequence of* *markings* $\{m'_n\}_{n\geq n_0} \subseteq \mathbb{N}^S$ *such that*

$$m_n[w\rangle_\Sigma m'_n, \forall n \geq n_0, \quad \textit{and} \quad \lim_{n\to\infty} m'_n = m'.$$

Proof. This result is easy to be proved (see the proof in Appendix 4). ∎

We have the following result:

Lemma 3.1. *Let γ be a mRCJPTN, let $s \in S$ be a place and $S' \subseteq S$ a set of places of γ.*
(i) S' is simultaneously unbounded in γ iff $\exists\, 0 \leq i \leq n$ such that S' is simultaneously unbounded in γ_i;
(ii) s is unbounded in γ iff $\exists\, 0 \leq i \leq n$ such that s is unbounded in γ_i,
where $\gamma_0, \gamma_1, \ldots, \gamma_n$ are the P/T-nets associated with the net γ.

Proof. (i) First, we shall prove the direct implication. Let $S' \subseteq S$ be a simultaneously unbounded set of places in γ. This means that

$$\forall k \geq 0, \exists m_k \in RS(\gamma) \text{ such that } \forall s \in S', m_k(s) \geq k. \tag{1}$$

Let us assume by contradiction that S' is not simultaneously unbounded in γ_i, for all $0 \leq i \leq n$. This means that

$$\forall 0 \leq i \leq n, \exists k_i \geq 0 \text{ such that } \forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k_i. \tag{2}$$

Let $k' \in \mathbb{N}$ be an arbitrary integer, satisfying $k' \geq 1 + \max\{k_i | 0 \leq i \leq n\}$; thus, we have $k' > k_i$, for each $0 \leq i \leq n$.
From the relation (1), for $k = k'$, we conclude that

$$\exists m_{k'} \in RS(\gamma) \text{ such that } \forall s \in S', m_{k'}(s) \geq k' \tag{3}$$

Since $m_{k'} \in RS(\gamma)$, we distinguish two cases:
a) $\exists w \in T^*$ such that $m_0[w\rangle_\Sigma m_{k'}$ (i.e. the marking $m_{k'}$ is reachable from m_0 without jumps).

In this case $m_{k'} \in RS(\gamma_0)$ and from relation (3) we have $m_{k'}(s) \geq k' > k_0, \forall s \in S'$. Thus, we can conclude that $\exists m_{k'} \in RS(\gamma_0)$ such that $\forall s \in S', m_{k'}(s) > k_0$, which contradicts (2), for $i = 0$;

b) $\exists w_1, w_2 \in T^*$ and $r = (m', m'') \in R : m_0\ [w_1\rangle_{\gamma,j}\ m'\ r\ m''\ [w_2\rangle_\Sigma\ m_{k'}$ (i.e. $m_{k'}$ is reachable from m_0 through jumps, r being the last jump).

By Proposition 2.1(2), we have that $\exists\, 1 \leq i' \leq n$ such that $r \leq r_{i'}$. Thus, $m'' \leq m''_{i'}$ and $m''_{i'}[w_2\rangle_\Sigma m'_{k'}$, where $m'_{k'} = m''_{i'} + \Delta w_2 = m''_{i'} - m'' + m_{k'} \geq m_{k'}$. It follows that $m'_{k'} \in RS(\gamma_{i'})$ and $m'_{k'}(s) \geq m_{k'}(s) \geq k' > k_{i'}, \forall s \in S'$. Therefore, we can conclude that $\exists m'_{k'} \in RS(\gamma_{i'})$ such that $\forall s \in S', m'_{k'}(s) > k_{i'}$, which contradicts (2), for $i = i'$.

Now, we prove the inverse implication. So, assume that there exists $0 \leq i \leq n$ such that S' is simultaneously unbounded in γ_i. Let us assume by contradiction that S' is not simultaneously unbounded in γ. This means that

$$\exists k' \geq 0 \text{ such that } \forall m \in RS(\gamma), \exists s_m \in S' : m(s_m) \leq k' \tag{4}$$

Let $i \in \mathbb{N}$ be arbitrary, satisfying $0 \leq i \leq n$. We distinguish two cases:

a) $r_i \in R_{\omega-max} \cap R$, i.e. the jump $r_i = (m'_i, m''_i)$ does not contain ω-components. Since the net γ is R-reduced, we have that $m''_i \in RS(\gamma)$, and then for each $m \in RS(\gamma_i)$ it follows that $m \in RS(\gamma)$, i.e. $RS(\gamma_i) \subseteq RS(\gamma)$. Therefore, from (4) it follows that

$$\forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k'; \tag{5}$$

b) $r_i \in R_{\omega-max} \cap R_\omega$, i.e. the jump $r_i = (m'_i, m''_i)$ contains ω-components. Thus, proceeding from the definition of the set R_ω, we have

$$\exists \{r_n\}_{n \geq 0} \subseteq R \text{ strictly increasing sequence with } \lim_{n\to\infty} r_n = r_i.$$

Let $r_n = (m'_n, m''_n), \forall n \geq 0$.

Let us consider an arbitrary (pseudo-)marking $m \in RS(\gamma_i)$. So, we have

$$\lim_{n\to\infty} m''_n = m''_i \text{ and } m''_i[w\rangle_\Sigma m, \text{ with } w \in T^*.$$

Proceeding from Proposition 3.1 it follows that there exists an integer $n_0 \geq 0$ and an infinite sequence of markings $\{m_n\}_{n\geq n_0}$ such that

$$m''_n[w\rangle_\Sigma m_n, \forall n \geq n_0, \quad \text{and} \quad \lim_{n\to\infty} m_n = m.$$

Since the net γ is R-reduced, we have that $m''_n \in RS(\gamma), \forall n \geq 0$, and, therefore, $m_n \in RS(\gamma), \forall n \geq n_0$. Then, from (4) it follows that: $\forall n \geq n_0, \exists s_n \in S'$ such that $m_n(s_n) \leq k'$, and, since from the fact that $\lim_{n\to\infty} m_n = m$ we have

$$\forall n \geq n_0 : \begin{cases} m_n(s) = m(s), & \forall s \in S - \Omega(m) \\ m_n(s) \geq n, & \forall s \in \Omega(m), \end{cases}$$

we can infer, by denoting $n_1 = max\{n_0, k'+1\}$, that $s_n \in S - \Omega(m)$ and $m(s_n) = m_n(s_n) \leq k'$, for all $n \geq n_1$.
Thus, for $n = n_1$ and by denoting $s_m = s_{n_1}$, we have

$$\forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k' \tag{6}$$

Therefore, from relations (5) and (6) for these two possible cases, it follows that we proved the following fact

$$\exists k' \geq 0 : \forall 0 \leq i \leq n, \forall m \in RS(\gamma_i), \exists s_m \in S' : m(s_m) \leq k'.$$

From this fact follows that for each $0 \leq i \leq n$, S' is not simultaneously unbounded in γ_i, which contradicts the hypothesis. Thus, S' is simultaneously unbounded in γ.

The statement (ii) follows easily from (i), by considering the set $S' = \{s\}$, proceeding from the fact that the place s is unbounded in γ' iff the set of places $\{s\}$ is simultaneously unbounded in γ', which holds for P/T-nets (with finite or infinite initial markings), as well as for jumping Petri nets. ∎

Theorem 3.1. *Let γ be a mRCJPTN, and $\mathcal{KMG}(\gamma)$ its Karp-Miller coverability graph.*

(1) A set of places S' is simultaneously unbounded iff there is at least one node (pseudo-marking) m in at least one graph from $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$, for all $s \in S'$.
(2) A place s is unbounded iff there is at least one node (pseudo-marking) m in at least one graph from $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$.

Proof. These statements follow easily from the definition of the Karp-Miller coverability graph $\mathcal{KMG}(\gamma)$, from the previous lemma, and from the similar results for P/T-nets (with finite or infinite initial markings). ∎

Example 3.1. *For the jumping Petri net from Example 2.1 we have that the places s_1 and s_2 are bounded and the place s_3 is unbounded; the trivial set $\{s_3\}$ is the only simultaneously unbounded set of places.*

Theorem 3.1 holds for every finite coverability graph of γ, not only for the Karp-Miller graph, and so to decide the properties listed in the theorem it is sufficient to compute any finite coverability graph, particularly the minimal one.

From Theorem 3.1 and the similar one from P/T-nets [5] we conclude that the following decision problems are solvable by using the Karp-Miller coverability graph for marked reduced-computable jumping Petri nets (or any other finite coverability graph):

Corollary 3.1. *The boundedness problems (SUbP) and (BP) are decidable for mRCJPTN.*

The use of the minimal coverability graph for solving these decision problems is important from the computational point of view because it is, generally speaking, smaller than the Karp-Miller graph.

4. CONCLUSIONS AND FUTURE WORK

In this paper we extended some decidability results from the class of finite jumping Petri nets (*mFJPTN*) to the class of reduced-computable jumping Petri nets (*mRCJPTN*).

More precisely, we have shown some boundedness problems which are decidable by using the Karp-Miller coverability graph for *mRCJPTN*.

An open problem which remains, is to study if there are more eficient algorithms for these decision problems.

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A P/T-nets – some more basic notions and notations

In this appendix we recall the definitions of some more basic notions from P/T-nets, the reachability structures and the Karp-Miller coverability structures, the basic decision problems regarding P/T-nets, and also the case of P/T-nets with infinite initial markings.

Let γ be a Petri net. The marking m is *coverable* in γ if there exists a marking $m' \in [m_0\rangle_\gamma$ such that $m \leq m'$.

A place $s \in S$ is *bounded* (or *safe*) if there exists an integer $k \in \mathbb{N}$ such that we have $m(s) \leq k$, for all $m \in [m_0\rangle_\gamma$. A subset of places $S' \subseteq S$ is *bounded* (or *safe*) if all its places $s \in S'$ are bounded. The net γ is *bounded* (or *safe*) if the set S is bounded.

A subset of places $S' \subseteq S$ is *simultaneously unbounded* if for every integer $k \in \mathbb{N}$ there is a reachable marking $m_k \in [m_0\rangle_\gamma$ such that we have $m_k(s) \geq k$, for all $s \in S'$. The net γ is *simultaneously unbounded* if the set S is simultaneously unbounded. Obviously, if a subset of places S' is simultaneously unbounded then it is also unbounded, but the converse is not always true.

In order to define coverability structures for Petri nets we add to the set of nonnegative integers $\mathbb{N}$ a new symbol, denoted by ω, giving $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$, and extend the operations $+$ and $-$ and the relation $<$ to the set $\mathbb{N}_\omega$ by: a) $n+\omega=\omega+n=\omega$; b) $\omega-n=\omega$; c) $n<\omega$, for all $n\in\mathbb{N}$.

Functions $m:S\to\mathbb{N}_\omega$ will be called *pseudo-markings*; they will be sometimes identified with $|S|$-dimensional vectors. $\mathbb{N}_\omega^S$ denotes the set of all pseudo-markings. If $m(s)=\omega$, then the component s of m will be called an ω-component; the set of all ω-components of m will be denoted by $\Omega(m)$, i.e. $\Omega(m)=\{s\in S\mid m(s)=\omega\}$. Obviously, any marking is a pseudo-marking. The firing rule is extended to pseudo-markings in the straightforward way: (ER) $m[t\rangle_\Sigma$ iff $t^-\leq m$; (CR) $m[t\rangle_\Sigma m'$ iff $m[t\rangle_\Sigma$ and $m'=m+\Delta t$. The other notions from markings (i.e. transition sequence, reachable marking etc.) are extended similarly to pseudo-markings.

We say that an infinite sequence of pseudo-markings $\{m_n\}_{n\geq 0}$ *converges* to the pseudo-marking m, and we write $\lim_{n\to\infty} m_n=m$, if we have

$$\forall n\geq 0:\begin{cases} m_n(s)=m(s), & \text{for all } s\in S-\Omega(m),\\ m_n(s)\geq n, & \text{for all } s\in\Omega(m).\end{cases}$$

The *reachability tree* of a marked Petri net $\gamma=(\Sigma,m_0)$ is denoted by $\mathcal{RT}(\gamma)$. It is a $(\mathbb{N}^S,T)$-labelled tree (V,E,l_V,l_E) (i.e., a tree with the set of nodes V and the set of arcs E, and the labelling functions $l_V:V\to\mathbb{N}^S$ and $l_E:E\to T$), which satisfies the following properties:

(i) the root node is labelled by the initial marking, i.e. $l_V(v_0)=m_0$;

(ii) for each node $v\in V$, the number of direct successors of v in the tree is equal to the number of transitions of the net γ which are fireable from the marking $l_V(v)$;

(iii) for each node $v\in V$ which has successors in the tree and for each transition $t\in T$ which is fireable from the marking $l_V(v)$, there exists an arc $(v,v')\in E$, labelled by $l_E(v,v')=t$, and, moreover, the label of v' is given by $l_V(v')=l_V(v)+\Delta t$.

The *reachability graph* of the net γ is denoted by $\mathcal{RG}(\gamma)$. It is a labelled directed graph (V, T, E), which is obtained from the reachability tree $\mathcal{RT}(\gamma)$ by identifying nodes with the same label; so, the set of nodes V of $\mathcal{RG}(\gamma)$ is the set of the markings which appear in $\mathcal{RT}(\gamma)$ as labels of nodes, i.e. $V = [m_0\rangle_\gamma$, and the set of arcs is given by $\forall m_1, m_2 \in [m_0\rangle_\gamma, \forall t \in T: \quad (m_1, t, m_2) \in E \ \Leftrightarrow m_1[t\rangle_\gamma m_2$.

The reachability tree/graph of a Petri net could be infinite. That is why people were interested in some refinements of these structures that can produce finite (sub)structures by preserving as much as possible properties.

The first, and well-known, reduced reachability tree/graph was that introduced by Karp and Miller [5]. The Karp-Miller coverability tree of a net γ will be denoted by $\mathcal{KMT}(\gamma)$ and it is a finite $(\mathbb{N}^S, T)$-labelled tree defined by the algorithm given by Karp and Miller in [5]. The Karp-Miller coverability graph of γ will be denoted by $\mathcal{KMG}(\gamma)$ and it is a finite labelled directed graph, obtained from the tree $\mathcal{KMT}(\gamma)$ by identifying nodes with the same label. Later, A. Finkel introduced in [3] the minimal coverability tree and graph. For the general definitions of coverability sets, trees, forests, and graphs for P/T-nets one can see [5, 3].

Some basic decision problems related to P/T-nets are the following:

• (RP) The *Reachability Problem* : Given γ a *mPTN* and m a marking of γ, is m reachable in γ?

• (CP) The *Coverability Problem* : Given γ a *mPTN* and m a marking of γ, is m coverable in γ?

• (BP) The *Boundedness Problem* : Given γ a *mPTN* and s a place of γ, is the place s bounded in γ? We may also ask if: Given γ a *mPTN* and S' a set of places of γ, is the set S' bounded in γ? Or if: Given γ a *mPTN*, is γ a bounded net?

• (SUbP) The *Simultaneously Unboundedness Problem*: Given γ a *mPTN* and S' a set of places, is the set S' simultaneously unbounded in γ? We may also ask if: Given γ a *mPTN*, is the net γ simultaneously unbounded?

Remark 4.1. *It is well-known that: i) the problem (RP) is decidable [7, 6]; ii) the problem (SUbP) is decidable [2]; iii) the problems (CP) and (BP) are decidable by using the Karp-Miller graph [9, 4]; the minimal coverability graph can also be used to solve these problems (see [3]).*

Petri nets with infinite initial markings In [12] we presented the case of P/T-nets with infinite initial markings, and we generalized the notions of coverability structures to them, because we needed them to define coverability structures for jumping Petri nets.

A *marked P/T-net with an infinite initial marking* is a *mPTN* $\gamma = (\Sigma, m_0)$ such that the initial marking has ω-components, i.e. $m_0 \in \mathbb{N}_\omega^S - \mathbb{N}^S$ (or, equivalent, $\Omega(m_0) \neq \emptyset$).

All the notions from P/T-nets with finite initial markings (i.e. firing rule, transition sequence, the set $TS(\gamma, m)$ of transition sequences from a marking, reachable marking, the reachability set $RS(\gamma)$, the reachability tree $\mathcal{RT}(\gamma)$, the reachability graph $\mathcal{RG}(\gamma)$, coverable marking, bounded place, simultaneously unbounded set of places etc.), and all the decision problems (i.e. (RP), (BP), (SUbP) etc.) are defined similarly for P/T-nets with infinite initial markings, with the remark that the initial marking m_0 is actually a pseudo-marking (because m_0 has ω-components), and, consequently, all reachable markings of γ are actually pseudo-markings. Indeed, it is easy to notice that $\Omega(m) = \Omega(m_0), \forall m \in RS(\gamma)$, which means that the ω-components of m_0 are preserved by the firing rule.

Moreover, the coverability structures for P/T-nets with an infinite initial marking are extended from classical nets with finite initial markings, by having the pseudo-markings which appear in these structures extended with ω-components on the set of ω-components of the initial marking. We can use the Karp-Miller graph of a Petri net with an infinite initial marking, to solve the same decision problems as those solved for P/T-nets with finite initial markings. Indeed, we showed the following result [12]:

Theorem 4.1. *Let $\gamma = (\Sigma, m_0)$ be a marked P/T-net (with a finite or an infinite initial marking), and $\mathcal{KMG}(\gamma)$ its Karp-Miller coverability graph.*
(1) A place s is unbounded iff there is at least one node m in $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$;
(2) A set of places S' is simultaneously unbounded iff there is at least one node m in $\mathcal{KMG}(\gamma)$ such that $m(s) = \omega$, for all $s \in S'$.

B Some proofs

In this appendix we prove Proposition 3.1 from Section 3.

Proof of proposition 3.1. We proceed by induction on $k = |w|$. If $k = 0$, then the proposition is trivially satisfied ($m' = m$ and we can take $n_0 = 0$ and $m'_n = m_n$).

If $k = 1$, then $w = t$, with $t \in T$, and from $m[t\rangle_\Sigma m'$, we conclude that $m \geq t^-$ and $m' = m + \Delta t$. Since $\lim_{n\to\infty} m_n = m$, it follows that

$$\forall n \geq 0 : \quad \begin{cases} m_n(s) \geq n, & \forall s \in \Omega(m), \\ m_n(s) = m(s), & \forall s \in S - \Omega(m). \end{cases}$$

Let $n_0 = \max\{t^-(s) | s \in \Omega(m)\}$. Then, it follows that

$$\forall n \geq n_0 : \quad \begin{cases} m_n(s) \geq n \geq n_0 \geq t^-(s), & \forall s \in \Omega(m), \\ m_n(s) = m(s) \geq t^-(s), & \forall s \in S - \Omega(m). \end{cases}$$

and, therefore, we have that $m_n \geq t^-$, for all $n \geq n_0$. Thus, we obtain that $m_n[t\rangle_\Sigma m'_n, \forall n \geq n_0$, by taking $m'_n = m_n + \Delta t$. It is easy to verify that $\lim_{n\to\infty} m'_n = m'$.

The induction step. Let $w = w't$ with $|w'| = k$ and $t \in T$. Then, there exists the marking m'' (namely, $m'' = m + \Delta w'$) such that: $m[w'\rangle_\Sigma m''[t\rangle_\Sigma m'$. By applying the induction hypothesis to the sequence of markings $\{m_n\}_{n\geq 0}$ for which $\lim_{n\to\infty} m_n = m$ and $m[w'\rangle_\Sigma m''$, it follows that: there exists an integer $n_1 \geq 0$ and an infinite sequence of markings $\{m''_n\}_{n\geq n_1}$ (namely, $m''_n = m_n + \Delta w', \forall n \geq n_1$) such that

$$m_n[w'\rangle_\Sigma m''_n, \forall n \geq n_1, \quad \text{and} \quad \lim_{n\to\infty} m''_n = m'' \tag{7}$$

Taking the sequence: $\{\tilde{m}''_n\}_{n\geq 0}$, with $\tilde{m}''_n = m''_{n+n_1}, \forall n \geq 0$, we have

$$\lim_{n\to\infty} \tilde{m}''_n = \lim_{n\to\infty} m''_n = m'' \quad \text{and} \quad m''[t\rangle_\Sigma m'.$$

Therefore, by applying to the sequence $\{\tilde{m}''_n\}_{n\geq 0}$ the statement of this proposition (i.e., the step $k = 1$), we conclude that: there exists an integer $n_2 \geq 0$ and an infinite sequence of markings $\{\tilde{m}'_n\}_{n\geq n_2}$ (namely, $\tilde{m}'_n = \tilde{m}''_n + \Delta t, \forall\, n \geq n_2$) such that $\tilde{m}''_n[t\rangle_\Sigma \tilde{m}'_n, \forall\, n \geq n_2$, and $\lim_{n\to\infty} \tilde{m}'_n = m'$.
But $\tilde{m}''_n = m''_{n+n_1}$, and so, by considering $n_0 = n_1 + n_2$ and the sequence of markings $\{m'_n\}_{n\geq n_0}$, with $m'_n = \tilde{m}'_{n-n_1}, \forall\, n \geq n_0$, we obtain

$$m''_n[t\rangle_\Sigma m'_n, \forall\, n \geq n_0, \quad \text{and} \quad \lim_{n\to\infty} m'_n = m' \tag{8}$$

From (7) and (8) we conclude that $m_n[w\rangle_\Sigma m'_n, \forall\, n \geq n_0$, and $\lim_{n\to\infty} m'_n = m'$, which completes the induction. $\square$

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The inequalities (7), (8) in the cited article should read:

$$|\mathbf{B}(\mathbf{u},\mathbf{v})| \leq \quad c_2 \|\mathbf{u}\| \|\mathbf{v}\| \left[1 + \ln\left(\frac{|\mathbf{\Delta u}|^2}{\lambda_1 \|\mathbf{u}\|^2}\right)\right]^{\frac{1}{2}}, (\forall)\ \ \mathbf{u} \in D(\mathbf{A})\backslash\{\mathbf{0}\}, \mathbf{v} \in \mathcal{V}_1, \tag{1}$$

$$|\mathbf{B}(\mathbf{u},\mathbf{v})| \leq \quad c_3 \|\mathbf{u}\| \|\mathbf{v}\| \left[1 + \ln\left(\frac{|\mathbf{\Delta v}|^2}{\lambda_1 \|\mathbf{v}\|^2}\right)\right]^{\frac{1}{2}}, (\forall)\ \ \mathbf{u} \in \mathcal{V}_1, \mathbf{v} \in D(\mathbf{A})\backslash\{\mathbf{0}\}. \tag{2}$$

It is also important to give some additional explanations concerning their proof, since from the text of the article it follows that they are presented in [1], [2] and [3]. The first one is indeed re-called in the three cited works, is often used in the literature dedicated to the Navier-Stokes equations and is a consequence of the Brézis-Gallouët inequality

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\| \left(1 + \ln \frac{\|u\|_2}{\lambda_1^{1/2}\|u\|}\right)^{1/2}.$$

The second one is not met in the three works cited. It is a direct consequence of the inequality

$$|(\mathbf{B}(\mathbf{u},\mathbf{v}), \mathbf{A}\mathbf{w})| \leq c_3 \|\mathbf{u}\| \|\mathbf{v}\| |\mathbf{A}\mathbf{w}| \left[1 + \ln\left(\frac{|\mathbf{\Delta v}|^2}{\lambda_1 \|\mathbf{v}\|^2}\right)\right]^{\frac{1}{2}},\ (\forall)\ \ \mathbf{u} \in \mathcal{V}_1,\ \mathbf{v},\ \mathbf{w} \in D(\mathbf{A}) \tag{3}$$

enounced in [4], and proved in [5].

The inequalities (12) and (13) from our article should read

$$|B(\mathbf{u},c)| \leq \quad c_2 \|\mathbf{u}\| \|c\| \left[1+\ln\left(\frac{|\mathbf{\Delta u}|^2}{\lambda_1 \|\mathbf{u}\|^2}\right)\right]^{\frac{1}{2}}, (\forall)\ \mathbf{u} \in D(\mathbf{A})\backslash\{\mathbf{0}\}, c \in \mathcal{V}_2, \tag{4}$$

$$|B(\mathbf{u},c)| \leq \quad c_3 \|\mathbf{u}\| \|c\| \left[1+\ln\left(\frac{|\Delta c|^2}{\lambda_1 \|c\|^2}\right)\right]^{\frac{1}{2}}, (\forall)\ \mathbf{u} \in \mathcal{V}_1, c \in D(\mathbf{A})\backslash\{\mathbf{0}\}. \tag{5}$$

and are obtained as consequences of (1) and (2) above.

We apologize to the reader for these imprecisions in our paper.

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