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THE RECOGNITION OF COMPLETENESS RELATIVE TO IMPLICIT REDUCIBILITY IN THE CHAIN EXTENSIONS OF INTUITIONISTIC LOGIC

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Abstract A.V. Kuznetsov [1] introduced the notion of implicit expressibility and implicit reducibility, which are natural generalizations of usual expressibility.

The research of the mentioned generalizations of expressibility in nonclassical logics is an actual problem. The criterion of completeness with respect to implicit reducibility in these logics is given. This criterion is based on 13-classes of formulas.

Formulas (of propositional logic) are constructed from variables p, q, r (possibly with indices) by means of logical operations: & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation). The formulas are designated with capital letters of the Latin alphabet. Using the mark $\rightleftharpoons$, and reading it as "means" we introduce designations for seven formulas: $1 \rightleftharpoons (p \supset p)$, $0 \rightleftharpoons (p \,\&\, \neg p)$, $\perp F \rightleftharpoons (F \vee \neg F)$ (ternondation), $(F \sim G) \rightleftharpoons ((F \supset G) \,\&\, (G \supset F))$ (equivalence), $(F \cdot G) \rightleftharpoons ((F \sim G) \,\&\, \neg\neg G)$, $(F \,\&'\, G) \rightleftharpoons ((F \,\&\, G) \sim \perp (F \sim G)$ and $(F, G, H) \rightleftharpoons ((F \,\&\, G) \vee (F \,\&\, H) \vee (G \,\&\, H))$ (median). The symbol $F[\alpha_1, \ldots, \alpha_n]$ designates the result of substitution in the formula F of the values $\alpha_1, \ldots, \alpha_n$ for the variables $p_1, \ldots, p_n$, respectively.

Intuitionistical and classical propositional calculus are based on the mentioned concept of formula. By these calculuses the intuitionistical and classical logics are defined. Thus we determine the logic of that calculus as the set of all formulas deducible in the given calculus. In this sense, the classical logic coincides, as it is known, with the set of formulas valid on the classical matrix.

We examine logics that are intermediary between classical logic and intuitionistic one [2]. They are constructed on finite or infinite chains (i.e. linear ordered sets) of true values. It is known that the logic is called a chain if the formula $((p \supset q) \vee (q \supset p))$ is true in it. In the considered m-valued logic ($m = 2, 3, \ldots$) the variables will take values from the set E_m, where $E_m = \{0, 1, \tau_1, \tau_2, \ldots, \tau_{m-2}\}$ if m is finite and $E_m = \{0, 1, \tau_1, \tau_2, \ldots\}$ if m is infinite. Instead of τ_1 and τ_2 we will write τ and ω, respectively. We remind that the set of all functions as mappings from E_m into E_m is usually called the general m-valued logic P_m. Further we consider the linear ordering on the set E_m by the relation $0 < \tau_1 < \tau_2 < \ldots \ldots < \tau_{m-2} < 1$. We define the operations $\&, \vee, \supset$, and $\neg$ on E_m as follows:

$$\begin{array}{l} p \,\&\, q = \min(p, q), \\ p \vee q = \max(p, q), \end{array} \qquad p \supset q = \begin{cases} 1 & \text{if } p \leq q, \\ q & \text{if } p > q, \end{cases} \qquad \neg p = p \supset 0.$$

In the considered interpretation of symbols $\&, \vee, \supset$ and $\neg$ each formula expresses some function of general m-valued logic. Let us note that the function $\lrcorner p$ of P_3 defined by the equalities $\lrcorner 0 = \lrcorner \tau_1 = 1$ and $\lrcorner 1 = 0$ is not expressed by any formula. We remind [4] that the pseudo-Boolean algebra is the system $\mathfrak{A} =< M; \&, \vee, \supset, \neg >$ that is a lattice by $\&$ and $\vee$, where $\supset$ is the relative pseudo-complement and $\neg$ is the pseudo-complement. The logic of this algebra is defined as the set of all formulas that are true on $\mathfrak{A}$, i.e. formulas identically equal to the greatest element 1 of this algebra. We will denote the algebra $< E_m; \&, \vee, \supset, \neg >$ $(m = 2, 3, \ldots)$ by Z_m. The logic of this algebra LZ_m is denoted by C_m. It is also possible to define the logic C_1 of one-element algebra which includes the set of all formulas and is contradictory. The smallest chain logic, called Dummett logic [2], coincides with the intersection of all m-valued chain logics with m positive integer number.

Two formulas F and G are called equivalent in logic L (and write $L \vdash (F \sim G)$) if the equivalence $F \sim G$ in L is true. Two formulas are equivalent in the logic C_m $(m = 1, 2, \ldots)$ if and only if the operators of algebra Z_m, expressed by them, are equal. Therefore instead of the relation $C_m \vdash (F \sim G)$ we sometimes will use the equality $F = G$ on Z_m. If the formula $F \sim G$ contains only the variables $p_1, p_2, \ldots, p_n$ and the inequality $(F \sim G)\,[p_1/\alpha_1, \ldots, p_n/\alpha_n] \neq 1$ is true on Z_m, then we will use the notation $(F \neq G)\,[p_1/\alpha_1, \ldots, p_n/\alpha_n]$. The formula F is called explicitly expressible in the logic L by the system of formulas of Σ [1] if it is possible to obtain the formula F from variables and formulas of Σ using a finite number of times the weak substitution rule, and the rule of replacement by equivalents in L. The relation of explicit expressibility is transitive. The formula F is called directly expressible via the system of formulas of Σ if it is possible to obtain F from variables and formulas of Σ by using a finite number of times the weak substitution rule. The relation of direct expressibility is transitive.

The formula F is called implicitly expressible in the logic L [1] via the system of formulas Σ if there exist the formulas G_i and H_i $(i = 1, \ldots, k)$ explicitly expressible in L by Σ such that the predicate $L \vdash (F \sim q)$, where q is a variable not contained in F, is equivalent to the predicate $L \vdash ((G_1 \sim H_1) \,\&\, \ldots \,\&\, (G_k \sim H_k))$.

Because the relation of implicit expressibility, generally speaking, is not transitive, we are going to introduce a new concept. The formula F is called implicitly reducible in the logic L via formulas of Σ if there exists a finite sequence of formulas $G_1, G_2, \ldots, G_l$, where G_l coincides with F and each term of this sequence can be implicitly expressible in L by Σ and terms of the sequence placed before it. We say that the system Σ' of formulas is implicitly reducible in L to the system Σ if each formula of Σ' is implicitly reducible in L to Σ. It is clear that the relation of implicit reducibility is transitive. The system Σ of formulas is called complete with respect to implicit reducibility in the logic L if each formula (in the language of this logic) is implicitly reducible in L to Σ. The system Σ of formulas is said to be pre-complete with respect to the implicit reducibility in L if Σ is not complete by this reducibility in L, but the system $\Sigma \cup \{F\}$ is complete relative to the implicit reducibility in L, for any formula F.

Two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_k)$ of P_m are called permutable [3] if the identity $f(g(x_{11}, \ldots, x_{1k}), \ldots, g(x_{n1}, \ldots, x_{nk})) =$ $g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1k}, \ldots, x_{nk}))$ is true. The set of all functions of P_m, permutable with the given function f, is called the centralizer of the function f (that is denoted $\prec f \succ$)[3]. The set of all formulas which in the interpretation on Z_m are permutable with the function f (from P_m) is called the formula centralizer on the algebra Z_m of the function f. We say the function $f(x_1, \ldots, x_n)$ of P_m preserves the predicate (relation) $R(x_1, \ldots, x_w)$ if for any possible values of the variables $x_{ij} \in E_m$ $(i = 1, \ldots, w;\ j = 1, ..n)$, from the truth of $R(x_{11}, x_{21}, \ldots, x_{w1}), \ldots, R(x_{12}, x_{22}, \ldots, x_{w2}), \ldots, R(x_{1n}, x_{2n}, \ldots, x_{wn})$ follows the truth of $R(f(x_{11}, x_{12}, \ldots, x_{1n}), \ldots, f(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, f(x_{w1}, x_{w2}, \ldots, x_{wn}))$. The centralizer $\prec f(x_1, \ldots, x_n) \succ$ coincides with the set of all functions of P_m which preserve the predicate $f(x_1, \ldots, x_n) = x_{n+1}$, where the variable x_{n+1} differs from $x_1, \ldots, x_n$ [1]. We say that the formula F preserves, on the algebra Z_m, the predicate R if the function of the logic C_m, expressed by formula F, preserves R. The predicate could be replaced by the corresponding to it matrix (α_{ij}) $(i = 1, \ldots, w;\ j = 1, \ldots, t)$ of elements of the algebra Z_m such that the predicate R is true on all those and only those sets of elements that are columns in this matrix. Let us remark that each formula of the system $\{p \,\&\, q, p \vee q, p \supset q, \neg p\}$ preserves on the algebra Z_m $(m = 3, 4, \ldots)$ the below predicates and matrices, therefore any formula preserves them too:

$$\neg x = \neg y, \quad x \neq \tau_j \quad (j = 1, 2, \ldots, m-2),$$

$$\begin{pmatrix} 0 & \tau & 1 \\ 0 & \tau_j & 1 \end{pmatrix} \quad (j = 1, 2, \ldots, m-2),$$

$$\begin{pmatrix} 0 & \tau & \omega & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (v,\, w = 1, 2, \ldots, m-2;\ v < w),$$

$$\begin{pmatrix} 0 & \tau_j & 1 & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (j,\, v,\, w = 1, 2, \ldots, m-2;\ v < w).$$

Let us note that the class of all formulas that preserve on Z_m some predicate is closed relative to the explicit expressibility in logic C_m, but it is not necessarily closed relative to the implicit expressibility in this logic [1]. It is easy to see that any class of formulas is closed relative to the implicit reducibility in the logic C_m if and only if it is closed relative to the implicit expressibility. We remind that the centralizer of one or another function is closed relative to the implicit expressibility. It is obvious that for each $m = 1, 2, \ldots$, if the class of functions K is closed relative to the implicit expressibility in the logic C_m, then K is closed relative to the implicit expressibility in any logic C_n where $n \geq m$.

Let us define the functions f_1 and f_2 from P_4 as follows:

$$f_1(0) = 0, \quad f_1(\tau) = 1, \quad f_1(w) = \omega, \quad f_1(1) = 1,$$
$$f_2(0) = 0, \quad f_2(\tau) = \omega, \quad f_2(w) = \omega, \quad f_2(1) = 1.$$

We denote the classes of formulas preserving the predicates $x = 0,\ x = 1,\ \neg x = y,\ x \,\&\, y = z,\ x \vee y = z,\ (x \sim (y \sim z)) = u$ on Z_2, $\lrcorner\lrcorner x = y,\ \perp x = \perp y,\ (x \,\&\, y = z) \,\&\, (\neg x = \neg y), ((x \sim y) \,\&\, \neg\neg y = z) \,\&\, (\neg x = \neg y),\ ((x \,\&\, y) \sim ((x \sim y) \vee \neg(x \sim y)) = z) \,\&\, (\neg x = \neg y)$ on Z_3, $f_1(x) = y,\ f_2(x) = y$ on Z_4 by symbols $\Omega_0, \Omega_1, \ldots, \Omega_{12}$,respectively. Let us note that the class Ω_5 on the algebra Z_2 coincides with the known class of linear Boolean functions. Remind that the closure relative to the implicit expressibility in C_2 of classes $\Omega_0, \ldots, \Omega_5$ is based on the fact that they are centralizers of some functions. A similar closure in C_3 of classes $\Omega_6, \ldots, \Omega_{10}$ is shown in [5]. It follows that these classes are closed relative to the implicit expressibility in any other logic C_m, where $m \geq 3$.

Assertion 1 (A.V. Kuznetsov [1]). *In order that the system Σ of formulas could be complete by the implicit reducibility in the logic C_2 it is necessary and sufficient that Σ be not included in any of clases $\Omega_0, \ldots, \Omega_5$.*

According to [5] the next criterion of completeness relative to the implicit reducibility in logic of First Iaśkowski's Matrix is true:

Assertion 2. *In order that the system* Σ *of formulas could be complete with respect to the implicit reducibility in logic* C_3 *it is necessary and sufficient that for each* $i = 0, \ldots, 10$ *should exist a formula of* Σ *which does not belong to the class* Ω_i.

The following criteria of completeness with respect to the reducibility in any chain logic included in C_4 are true.

Theorem 1. *For any* $m = 4, 5, \ldots,$ *in order that the system* Σ *of formulas could be complete by the implicit reducibility in logic* C_m *it is necessary and sufficient that* Σ *be complete by implicit reducibility in the logic* C_3 *and be not included in the following two formula centralizers on the algebra* Z_4:

$$\prec f_1(p) \succ, \quad \prec f_2(p) \succ.$$

Theorem 2. *In order that the system of formulas* Σ *could be complete relative to the implicit reducibility in any chain logic* L*, including the Dummett logic, it is necessary and sufficient that the next conditions be satisfied simultaneously:*

1) if $L \subseteq C_2$*, then the system* Σ *is included neither in* Ω_0*, nor in* Ω_1*, nor in* Ω_2*, nor in* Ω_3*, nor in* Ω_4*, nor in* Ω_5*;*

2) if $L \subseteq C_3$*, then* Σ *is also included neither in* Ω_6*, nor in* Ω_7*, nor in* Ω_8*, nor in* Ω_9*, nor in* Ω_{10}*;*

3) if $L \subseteq C_4$*, then* Σ *is also included neither in* Ω_{11}*, nor in* Ω_{12}.

From this criterion the following corollaries follow.

Theorem 3. *For any chain logic* L *(including the Dummett logic) there exists an algorithm that allows us to recognize for any finite system* Σ *of formulas if* Σ *is complete relative to the implicit reducibility in the logic* L *or not.*

From Assertions 1 and 2 it follows that the classes $\Omega_0, \Omega_1, \ldots, \Omega_5$ and only they are pre-complete relative to the implicit reducibility in C_2, and the classes

$\Omega_0, \Omega_1, \ldots, \Omega_{10}$ and only they are pre-complete by the implicit reducibility in C_3.

Theorem 4. *The following 13 classes: $\Omega_0, \Omega_1, \ldots, \Omega_{12}$ of formulas and only they are pre-complete relative to the implicit reducibility in the logic C_m, for any $m = 4, 5, \ldots$.*

The logics L_1 and L_2 are called equal relative to the completeness by the implicit reducibility if any system Σ of formulas is complete by implicit reducibility in L_1 if and only if this system is complete by the implicit reducibility in L_2.

Theorem 5. *Any chain logic is equal relative to the completeness with respect to the implicit reducibility to one and only one of the following 4 logics: the absolute contradictory logic, the classical logic, the logic C_3 and the C_4 logic.*

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