

## THEORY OF OLIGOPOLIES: DYNAMICS AND STABILITY OF EQUILIBRIA

Konstantinos Andriopoulos, Tassos Bountis, Nikos Papadopoulos

*Centre for Research and Applications of Nonlinear Systems and Department of Mathematics,  
University of Patras, Greece*

kand@aegean.gr; bountis@math.upatras.gr

**Abstract** The theory of oligopolies is a particularly active area of research using applied mathematics to answer questions that arise in microeconomics. It basically studies the occurrence of equilibria and their stability in market models involving few firms and has a history that goes back to the work of Cournot in the 19th century. More recently, interest in this approach has been revived, owing to important advances in analogous studies of Nash equilibria in game theory. In this paper, we first attempt to highlight the basic ingredients of this theory for a concrete model involving two firms. Then, after reviewing earlier work on this model, we describe our modifications and improvements, presenting results that demonstrate the robustness of the approach of nonlinear dynamics in studying equilibria and their stability properties. On the other hand, plotting the profit functions resulting from our modified model we show that its behavior is more realistic than that of other models reported in the literature.

**Keywords:** economic dynamics, nonlinear dynamics, stability of equilibrium points.

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### 1. INTRODUCTION

People constantly want to buy products, choose amongst those which they find more appropriate to their needs and finally pay the price requested. How many of us, however, really understand how the market works? Do persons play a vital role in the determination of the prices for each product, or are they simply recipients of the flow? How can one attack these problems mathematically and how close are the mathematical findings to reality?

Depending on the product a market is born. The number of firms activated in that market depends on the easiness to achieve profitability, the measures followed by Government, the strength of the market itself and many other possibly minor factors. An immediate question is: how do these firms decide how much to produce and at what price? When this question is answered, production takes place and depending upon the evolution of the market new firms enter, while others collapse and exit the market.

In economic theory two extremes are usually studied first: Monopoly, where there is just one firm (most of the times due to government intervention) and the so-called perfect (complete) competition. Somewhere in between lies the case of *oligopoly*, that is few firms (oligo) selling (polo) products, which may be either identical or differentiated.

The theory of oligopoly is an active field of research and has attracted much attention through the last decades. This paper first presents an introduction to oligopoly theory with emphasis on two firms (duopoly) for reasons of simplicity and then focuses on the resulting dynamics, which is of particular interest. Currently, one can find many research papers, in both mathematical and economical journals, approaching the problem from the perspective of nonlinear dynamics based on some ‘reasonable’ assumptions.

One such approach was followed recently by Matsumoto and Szidarovszky (MS) in their paper [6]. They postulated a particular dependence of prices on production levels and proposed a dynamical system based solely on deviations from equilibrium configurations. In the present paper, we begin from a more fundamental set of differential equations describing the dynamics and also introduce more realistic price functions. What we find is that the resulting equilibria are distributed in much the same way as in the MS approach and they are all stable. However, the corresponding profit values at these equilibria are distributed very differently on the parameter plane and suggest that our approach is more ‘natural’ than the one followed by MS.

## **2. FORMULATION OF A REASONABLE MODEL**

We start with a number of assumptions: Consider the case of two firms with the restriction that no other firm can enter the market in the future. Both

firms produce the same product, which cannot be stored and sold later. The process takes place in finite time and the firms have no information on the other's actions/choices and do not cooperate.

Cournot [4] assumed that firms choose their output level (productivity) first and then the market sets the price straightforwardly, based on a demand curve and the total quantity offered. The quantity of production chosen by one firm affects the profit of all others including itself and assumptions are made by each firm, regarding the output of all the others.

A set of quantities sold for which, holding the quantities of all other firms constant, no firm can obtain a higher profit by choosing a different quantity is called a Cournot (Nash) equilibrium. At such an equilibrium, no firm wants to change its behaviour. Each firm is on its best-response curve and attains its maximal profit, given that it has the correct information about its rivals' output.

Stackelberg [12] and Bertrand [3] formulated other models originating from different beliefs concerning the behaviour of the market. The former studied the case of a firm setting its output level first, followed by the other firms, assuming Cournot's ideas for the determination of the equilibrium point, while the latter postulated that firms firstly choose prices and then let the quantities of the product be designated by the market.

In this paper, we study Cournot behaviour in a market where two firms are active: firm 1 and firm 2, which sell the identical product at quantities  $x$  and  $y$ , respectively, at the same price. The objective functions, which represent the overall profit (or loss) of each firm, are

$$u_1 = \{p(x, y) - c_1(x, y)\}x \quad \text{and} \quad u_2 = \{p(x, y) - c_2(x, y)\}y, \quad (1)$$

where  $p(x, y)$  is the price requested per item  $x$  or  $y$  sold and  $c_i, i = 1, 2$ , are the cost functions for each firm. The first-order conditions for the computation of the Cournot equilibrium point are

$$\left\{ \frac{\partial p}{\partial x} - \frac{\partial c_1}{\partial x} \right\} x + \{p(x, y) - c_1(x, y)\} = 0 \quad \text{and} \\ \left\{ \frac{\partial p}{\partial y} - \frac{\partial c_2}{\partial y} \right\} y + \{p(x, y) - c_2(x, y)\} = 0 \Leftrightarrow \quad x = X(y) \quad \text{and} \quad y = Y(x). \quad (2)$$

The solution of the simultaneous system, (2), yields the Cournot (Nash) equilibrium point.

## 2.1. INVERSE DEMAND FUNCTION

The choice of an inverse demand function is of obvious fundamental importance. Driven by economic reasoning, one expects the price to vary as follows: for small demand, the price remains approximately the same, while, as demand increases, the slope decreases steeply and then levels off and falls slowly to zero, as shown in Figure 1.

Puu [7, 8, 9] considered a price function of the form

$$p(x, y) = \frac{1}{x + y}, \quad (3)$$

according to an assumption first made in [5]. Of course, this assumption has a serious drawback due to its singularity as production levels go to zero [1, 2].

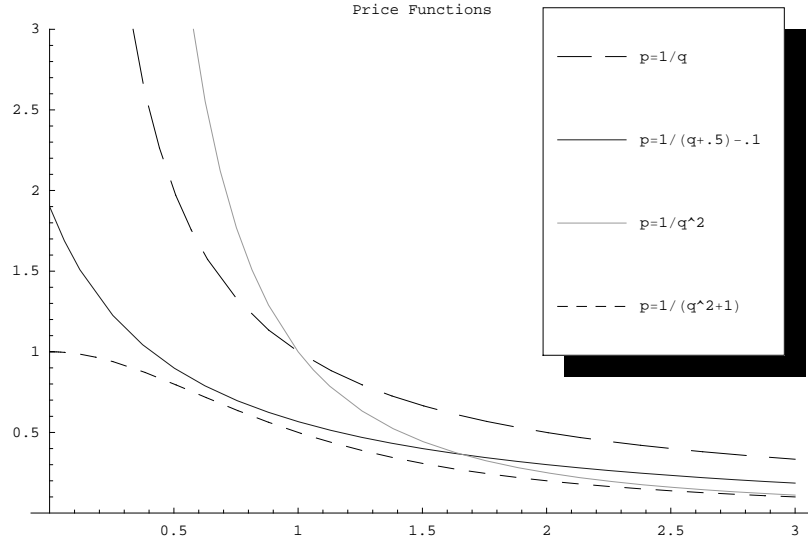


Fig. 1. Four representative choices for the price function vs. output level.

This is ‘amended’ by considering a price function of the form

$$p + p_0 = \frac{1}{q + q_0}, \quad \text{where } q = x + y. \quad (4)$$

Still, this choice suffers from the fact that a steep price decrease occurs already at very small output levels.

We propose a more realistic price function (see Figure 1) given by the expression

$$p = \frac{1}{q^2 + 1}, \quad (5)$$

which overcomes the above difficulties.

Note that in the case of differentiated goods, the way this differentiation enters in the corresponding expressions is very important. A simple and realistic way to achieve this is by introducing two parameters,  $\theta_1$  and  $\theta_2$ , as in [6] and by defining two inverse demand functions as follows

$$p_1 = \frac{1}{x + \theta_1 y} \quad \text{and} \quad p_2 = \frac{1}{\theta_2 x + y}, \quad (6)$$

where  $0 < \theta_i < 1$  keeping  $dp_1/dx < 0$  and  $dp_2/dy < 0$ , an assumption first made in [7] and later extended to differentiated goods [6].

We suggest that (6) may be appropriate for the analysis followed in [6], but it is not optimal. For reasons mentioned above, a better choice might be

$$p_1 = \frac{1}{(x + \theta_1 y)^2 + 1} \quad \text{and} \quad p_2 = \frac{1}{(\theta_2 x + y)^2 + 1}. \quad (7)$$

## 2.2. FORMULATION OF A CONTINUOUS DYNAMICAL SYSTEM

We consider the case of constant marginal costs to simplify the calculations and best illustrate the idea. The objective functions for the two firms are

$$u_1(x, y) = p_1(x, y)x - c_1x \quad \text{and} \quad u_2(x, y) = p_2(x, y)y - c_2y. \quad (8)$$

Let us formulate a continuous dynamical system modeling the situation of a Cournot duopoly. Matsumoto and Szidarovszky [6] introduced the so-called ‘reaction functions’,  $R_1(y)$  and  $R_2(x)$ , by solving the first-order conditions for the system (8) with inverse demand functions as given in (6), i.e.  $\theta_1 y = c_1(x + \theta_1 y)^2$  and  $\theta_2 x = c_2(y + \theta_2 x)^2$  and, solving for  $x$  and  $y$ , respectively, obtained

$$R_1(y) = \sqrt{\frac{\theta_1 y}{c_1}} - \theta_1 y \quad \text{and} \quad R_2(x) = \sqrt{\frac{\theta_2 x}{c_2}} - \theta_2 x. \quad (9)$$

Hence, they postulated that the differential equations generate the continuous dynamics of the system are

$$\dot{x}(t) = k_1(R_1(y(t)) - x(t)) \quad \text{and} \quad \dot{y}(t) = k_2(R_2(x(t)) - y(t)), \quad (10)$$

where the dot denotes differentiation with respect to time  $t$  and  $k_i$ ,  $i = 1, 2$ , are some positive constants (adjustment coefficients).

This choice seems rather *ad hoc*, in the sense that there are several ways to construct a dynamical system. In [6] the authors try to express the change of output level depending on the deviation from the Cournot equilibrium point. An alternative approach, which we investigate here, is to assume that the change of output is proportional to the *rate of change of profits* with respect to the production level, according to the equations

$$\dot{x}(t) = k_1 \frac{\partial u_1}{\partial x} \quad \text{and} \quad \dot{y}(t) = k_2 \frac{\partial u_2}{\partial y} \quad (11)$$

or, explicitly for the case of (6) and (8),

$$\dot{x}(t) = k_1 \left( \frac{\theta_1 y}{(x + \theta_1 y)^2} - c_1 \right) \quad \text{and} \quad \dot{y}(t) = k_2 \left( \frac{\theta_2 x}{(y + \theta_2 x)^2} - c_2 \right). \quad (12)$$

### 2.3. CHOICE OF A COST FUNCTION

Marginal costs determine the dependence of cost functions on amounts of productivity and, in most research papers, are assumed to be constant. This makes the analysis much easier, but lacks sufficient economical justification. Other choices include the introduction of capacity constraints for each firm as proposed, for example, by Puu and Norin [9] and Puu [10, 11]. More specifically, Puu and Norin [9] introduce as *total production cost functions* the expressions  $-\ln(1 - x/v_1)$  for the first firm and  $-\ln(1 - y/v_2)$  for the second, where  $v_1$  and  $v_2$  are the capacity limits for firm 1 and 2, respectively. This is a reasonable assumption since zero production levels result in zero cost, increasing output levels increase the total cost and finally, as production reaches capacity limits, costs go to infinity. These total costs lead to the profit functions

$$\begin{aligned} u_1(x, y) &= xp_1(x, y) + \ln(1 - x/v_1), \\ u_2(x, y) &= yp_2(x, y) + \ln(1 - y/v_2), \end{aligned}$$

see (1). However, this choice turns out to give results similar to those we obtain assuming constant marginal costs (see Sections 3 and 4) and will not be further pursued here.

### 3. PUTTING THESE IDEAS TO WORK

#### 3.1. THE FORMULATION DUE TO MATSUMOTO AND SZIDAROVSKY [6]

Before we use the modified price functions (7) let us consider the problem as expressed by our equations (12), involving the price functions introduced in [6]. Thus, the objective functions for the two firms are given by

$$u_1(x, y) = \frac{x}{x + \theta_1 y} - c_1 x, \quad u_2(x, y) = \frac{y}{\theta_2 x + y} - c_2 y. \quad (13)$$

The vanishing of the first-order partial derivatives

$$\frac{\partial u_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u_2}{\partial y} = 0 \quad (14)$$

determines the Cournot equilibrium point by the relations

$$\begin{aligned} \frac{\theta_1 y}{(x + \theta_1 y)^2} - c_1 &= 0, \\ \frac{\theta_2 x}{(\theta_2 x + y)^2} - c_2 &= 0. \end{aligned} \quad (15)$$

In order to study the stability of these equilibrium points, we need to consider the Jacobian matrix associated with the dynamical system (12)

$$J = \begin{bmatrix} k_1 \frac{\partial^2 u_1}{\partial x^2} & k_1 \frac{\partial^2 u_1}{\partial x \partial y} \\ k_2 \frac{\partial^2 u_2}{\partial x \partial y} & k_2 \frac{\partial^2 u_2}{\partial y^2} \end{bmatrix}$$

and obtain its eigenvalues from the characteristic equation

$$\lambda^2 + 2A\lambda + B = 0, \quad (16)$$

where

$$\begin{aligned} A &= -\frac{1}{2}k_1 \frac{\partial^2 u_1}{\partial x^2} - \frac{1}{2}k_2 \frac{\partial^2 u_2}{\partial y^2}, \\ B &= \left( \frac{\partial^2 u_1}{\partial x^2} \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_1}{\partial x \partial y} \frac{\partial^2 u_2}{\partial x \partial y} \right) k_1 k_2. \end{aligned} \quad (17)$$

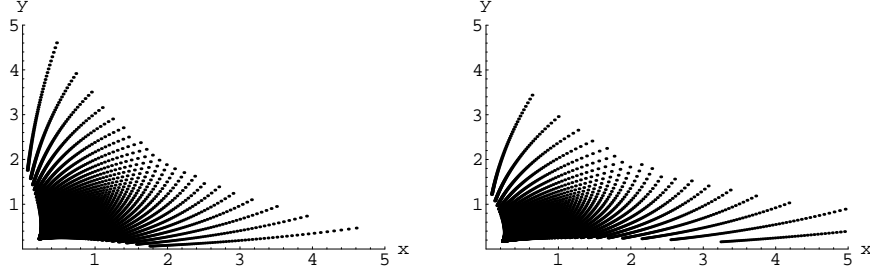


Fig. 2. Stable equilibrium points of the MS model for  $\theta_1 = \theta_2 = 0.5$ ,  $k_1 = 1$ ,  $k_2 = 1.1$  (left) and  $\theta_1 = 0.7$ ,  $\theta_2 = 0.2$ ,  $k_1 = 0.4$ ,  $k_2 = 1.3$  (right).

The eigenvalues are

$$\lambda_{1,2} = -A \pm \sqrt{A^2 - B} \quad (18)$$

and stability is achieved if and only if both  $A > 0$  and  $B > 0$  for every equilibrium point.

Following this approach – which we call the MS formulation – Matsumoto and Szidarovszky found that the equilibrium point of (10) is unique and always locally asymptotically stable [6]. Solving (12) numerically, we also find a unique equilibrium point, which is always asymptotically stable, as follows: The eigenvalues (18) evaluated at the fixed points for every choice of parameters in the set  $(\theta_1, \theta_2, c_1, c_2)$  have negative real part. We then allow the marginal costs  $c_1$  and  $c_2$  to vary in the interval  $(0, 1)$ , compute the corresponding equilibrium point and plot it in the  $(x, y)$  plane of Figure 2 for two choices of proportionality constants  $k_1$  and  $k_2$ .

### 3.2. USING THE PRICE FUNCTIONS (7)

Consider the following objective functions for the two firms

$$\begin{aligned} u_1(x, y) &= \frac{x}{(x + \theta_1 y)^2 + 1} - c_1 x, \\ u_2(x, y) &= \frac{y}{(\theta_2 x + y)^2 + 1} - c_2 y. \end{aligned} \quad (19)$$

The first-order conditions (14) give

$$\frac{1}{(x + \theta_1 y)^2 + 1} - \frac{2x(x + \theta_1 y)}{[(x + \theta_1 y)^2 + 1]^2} = c_1, \quad (20)$$

$$\frac{1}{(\theta_2 x + y)^2 + 1} - \frac{2y(\theta_2 x + y)}{[(\theta_2 x + y)^2 + 1]^2} = c_2. \quad (21)$$

In order to specify the reaction functions explicitly, one must solve (19) and (21) for  $x = R_1(y)$  and  $y = R_2(x)$  analytically, which appears impossible. Assuming that  $x = R_1(y)$  and  $y = R_2(x)$  we determine the first-order derivatives,

$$\frac{\partial x}{\partial y} = \frac{\partial R_1}{\partial y} \quad \text{and} \quad \frac{\partial y}{\partial x} = \frac{\partial R_2}{\partial x}, \quad (22)$$

by differentiating (19) with respect to  $y$  keeping  $x = x(y)$  and (21) with respect to  $x$ . Thus, we obtain

$$\frac{\partial x}{\partial y} = \frac{\theta_1^2 y(3x^2 - \theta_1^2 y^2 - 1) + 2\theta_1 x(x^2 - 1)}{\theta_1 y(3\theta_1 xy + 2\theta_1^2 y^2 + 2) - x(x^2 - 3)} = F(x, y) = \frac{\partial R_1}{\partial y}, \quad (23)$$

$$\frac{\partial y}{\partial x} = \frac{\theta_2^2 x(3y^2 - \theta_2^2 x^2 - 1) + 2\theta_2 y(y^2 - 1)}{\theta_2 x(3\theta_2 xy + 2\theta_2^2 x^2 + 2) - y(y^2 - 3)} = G(x, y) = \frac{\partial R_2}{\partial x}. \quad (24)$$

Consider now the continuous dynamical system of the form (10), with  $k_1 = k_2 = 1$  for simplicity. Its Jacobian matrix is

$$J = \begin{bmatrix} -1 & \frac{\partial R_1(y)}{\partial y} \\ \frac{\partial R_2(x)}{\partial x} & -1 \end{bmatrix},$$

whose eigenvalues satisfy the characteristic equation

$$\lambda^2 + 2\lambda + 1 - \frac{\partial R_2}{\partial x} \frac{\partial R_1}{\partial y} = 0. \quad (25)$$

Again, as in the previous subsection, we locate all equilibrium points,  $(x_e, y_e)$ , of the system, for various choices of the parameter values, solving (20) and (21) numerically. Note that, combining (22), (23) and (24), the characteristic equation (25) takes the form

$$\lambda^2 + 2\lambda + 1 - G(x_e, y_e)F(x_e, y_e) = 0 \quad (26)$$

whose roots are given by

$$\lambda_{\pm} = -1 \pm \sqrt{G(x_e, y_e)F(x_e, y_e)}. \quad (27)$$

Figure 3 shows all the equilibrium points for  $c_1, c_2$  choices ranging from 0 to 1. Clearly all  $(c_1, c_2)$  pairs leading to  $x_e < 0$  or  $y_e < 0$  are rejected as being ‘unphysical’.

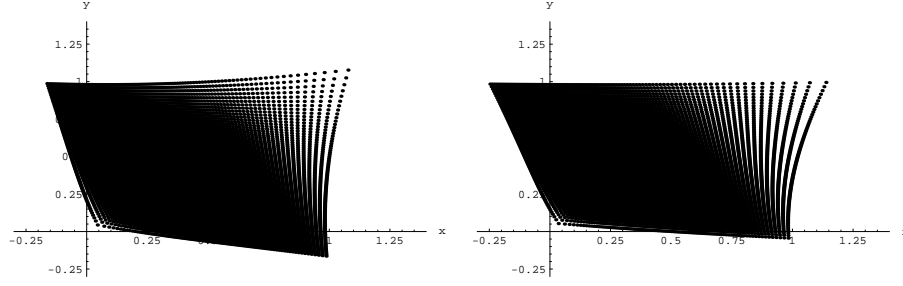


Fig. 3. Stable equilibrium points of our system for  $\theta_1 = \theta_2 = 0.5$ ,  $k_1 = 1$ ,  $k_2 = 1.1$  (left) and  $\theta_1 = 0.7$ ,  $\theta_2 = 0.2$ ,  $k_1 = 1$ ,  $k_2 = 1.1$  (right). Compare with Figure 2.

The important result is that our approach, utilising (7) and (11), also yields that the equilibrium points,  $(x_e, y_e)$ , are *all stable* and have an analogous distribution in the  $x, y$  plane as in the case of the formulation described in subsection 3.1.

#### 4. A STUDY OF THE PROFIT FUNCTIONS

Having determined the Cournot equilibrium points and their stability, firms naturally worry about their profits. Provided that in the short term a given firm is in a position to change its parameters and/or initial conditions, the two firms try to maximise their profits,  $u_1$  and  $u_2$ , as in (13) or (19). Let us now examine how these profits evaluated at the various equilibrium points vary as functions of the parameters  $c_1$  and  $c_2$ :

Following the MS formulation, we plot in Figure 4 for each equilibrium point shown in Figure 2 the corresponding  $u_1$  and  $u_2$  values for the profits of firms 1 and 2, respectively. Observe that these profit pairs lie on two nonintersecting curves, indicating that the MS approach exhibits a rather restrictive and unrealistic distribution of profit values. It seems unnatural to expect that such an extensive variation of the  $c_1$  and  $c_2$  parameters would yield profits belonging to so limited a (1-dimensional) subset of the  $(u_1, u_2)$  plane.

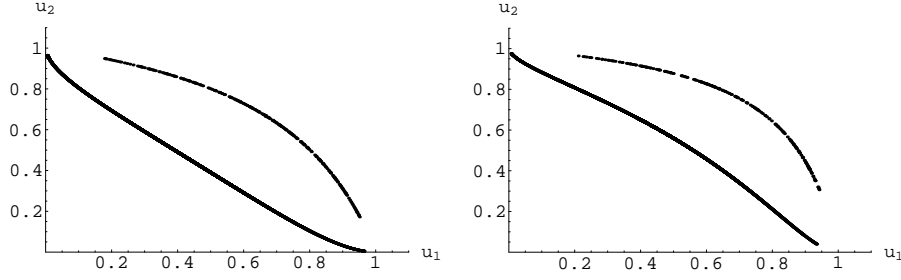


Fig. 4. Profit distributions for the MS model, (13), at  $\theta_1 = \theta_2 = 0.5$  (left) and  $\theta_1 = 0.7$ ,  $\theta_2 = 0.2$  (right).

By contrast, our approach for the corresponding  $c_1, c_2$  values leads to a distribution of profits that occupies a significant (2-dimensional) subset of the  $(u_1, u_2)$  plane, as shown in Figure 5.

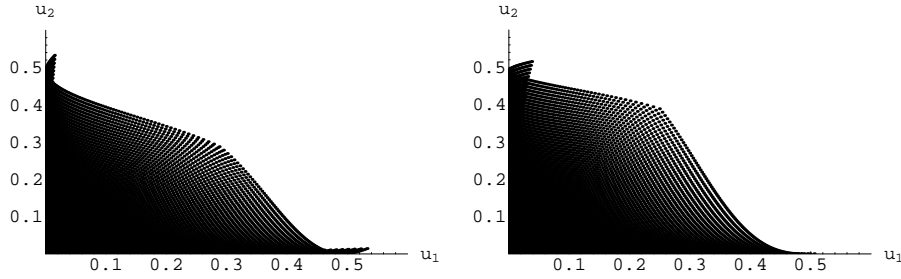


Fig. 5. Profit distributions for our formulation (19) at  $\theta_1 = \theta_2 = 0.5$  (left) and  $\theta_1 = 0.7$ ,  $\theta_2 = 0.2$  (right).

We also plot profits vs. prices for each firm separately in Figure 6. We find that, at  $\theta_1 > \theta_2$ , the profits of firm 2 are considerably *higher* than those of firm 1 for a large set of  $c_1$  and  $c_2$  parameters corresponding to prices of firm 1 that are *lower* than those of firm 2, i.e.  $p_1 < p_2$ .

We believe that this is an interesting observation. It implies, at least for the parameter values studied, that a firm (here firm 2) which pays less attention to the level of production of its rival (here firm 1) and consequently sets its prices with  $\theta_2 < \theta_1$ , achieves *higher profits* when its products are more expensive than those of its rival.

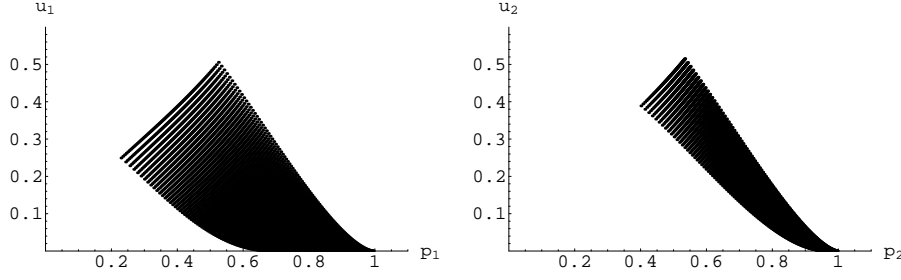


Fig. 6. Profits vs. prices resulting from our approach (19) with  $\theta_1 = 0.7$ ,  $\theta_2 = 0.2$  for firm 1 (left) and firm 2 (right).

## 5. CONCLUDING REMARKS

The operation of a free market is a complex problem whose time evolution depends on many factors and generally relies on choices made by humans. Still, in a market that is well-developed, it appears that competing firms operate on a basis of well-defined principles whose effectiveness has been validated by experience. It follows, therefore, that it would be useful to analyse free market operation using deterministic models of nonlinear dynamics (like systems of nonlinear ordinary differential equations) to describe firm competition.

In this paper, we have adopted such a dynamical approach for a market consisting of two firms producing a single, but differentiated, product. Following a formulation suggested by Matsumoto and Szidarovszky (MS) we have performed a detailed study of a model that uses more realistic price functions and a dynamical system derived from more fundamental principles. Our first conclusion is that in terms of existence and stability of equilibria our approach yields results very similar to MS, demonstrating the robustness of these duopoly models.

However, extending our study to the evaluation of the associated price functions we have found that the results of our model are richer and, therefore, more realistic than those of MS. Furthermore, an important observation can be made concerning the choice of the two parameters,  $\theta_1$  and  $\theta_2$ , that reflect the degree to which each firm takes into account the output level of the other firm, when setting the price of its own product. We have found that if firm

2 sets its prices with less regard for the output level of firm 1, i.e.  $\theta_2 < \theta_1$ , its profits are higher over the full regime where the prices of firm 1 are lower than those of firm 2.

These results may lead to the conclusion that a two-firm market has simple dynamics which in all cases we examined is attracted by a stable equilibrium point. Clearly, to observe more complicated behavior, one has to go beyond duopoly models and investigate systems of  $n \geq 3$  competing firms.

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