ROMAI J., **4**, 1(2008), 61–67

ALGEBRAS OF CONTINUOUS FUNCTIONS AND COMPACTIFICATIONS OF SPACES

Mitrofan M.Choban, Laurenţiu I.Calmuţchi

Tiraspol State University, Chişinău, Republic of Moldova

mmchoban@email.ro

Keywords: g-compactification, compactification, algebra of functions.

2000 MSC: 54D35, 54D40.

1. INTRODUCTION

The notion of compactness is one of the most important notions.

A generalized compactification or a g-compacification of a space X is a pair (Y, e_Y), where Y is a compact Hausdorff space and $e_Y : X \to Y$ is a continuous mapping such that the set $e_Y(X)$ is dense in Y. If e_Y is an embedding, then Y is called a compactification of X, we identify $x = e_Y(x)$ for any $x \in X$ and consider that $X \subseteq Y$.

Let (Y, e_Y) and (Z, e_Z) be two g-compactifications of a space X. We consider that $(Y, e_Y) \leq (Z, e_Z)$ if there exists a continuous mapping $f : Z \to Y$ such that $e_Y = f \circ e_z$, i.e. $e_Y(x) = f(e_Z(x))$ for any $x \in X$. If $(Y, e_Y) \leq (Z, e_Z)$ and $(Z, e_Z) \leq (Y, e_Y)$, then f is a homeomorphism and we say that the g-compactifications (Y, e_Y), (Z, e_Z) are equivalent. We identify the equivalent g-compactifications. In this case the set $GC(X)$ of all g-compactifications of the space X is a complete lattice with the maximal element $(\beta X, \beta_X)$. The minimal element of the lattice $GC(X)$ is the one-point space. The compactification βX is the Stone-Čech compactification of the space X.

Let L be a non-empty subset of the lattice $GC(X)$. Then the maximal element $\vee L$ and the minimal element $\wedge L$ are determined in $GC(X)$. Problems connected with the g-compactifications $\vee L, \wedge L$ are among the most interesting problems of the topology.

We use the terminology from [3]. Denote by $|A|$ the cardinality of the set A, by $cl_X A$ or clA the closure of the set A in the space X.

2. PARTIAL ALGEBRAS

Denote by K the ring of complex numbers.

A set A is called a partial algebra if for some pairs $x, y \in X$ there are determined the sum $x + y$ and the multiplication $x \cdot y$ such that:

1. the multiplication and sum are associative, commutative and distributive.

2. for every $x \in A$ and $\alpha \in K$ it is defined $\alpha X \in A$ such that:

2.1. $\alpha(x + y) = \alpha x + \alpha y$ for all $x, y \in A$ and $\alpha \in K$;

2.2. $(\alpha + \beta)x = \alpha x + \beta x$ for all $x \in A$ and $\alpha, \beta \in K$;

2.3. $1 \cdot x = x$ for any $x \in A$;

3. $\alpha(\beta x) = (\alpha\beta)x$ for all $x \in A$ and $\alpha, \beta \in K$;

4. there exist two distinct elements $0, 1 \in A$ such that $0 \cdot x = 0$ and $0 + x = 1 \cdot x = x$ for any $x \in X$.

Let A be a partial algebra. A subset $I \subseteq A$ is called an ideal of A if: $I \neq \emptyset$; $x + y \in I$ provided $x, y \in I$ and $x + y$ is defined in A; $x \cdot y \in I$ provided $x \in I$ and $x \cdot y$ is defined in A; if $\alpha \in K$ and $x \in I$, then $\alpha x \in I$. A maximal ideal of A is called a proper ideal of A if it is not contained in other proper ideal of A. Let $M(A)$ be the set of maximal ideals of A. For every $x \in A$ we put $M(x, A) = \{I \in M(A) : x \in I\}$. Then $M(x_1, ..., x_n, A) = \cap\{M(x_i, A) : i \leq n, n \in \mathbb{N} = \{1, 2, ...\}\}$. If $L \subseteq A$, then $M(L, A) = \cap\{M(x, A) : x \in L\}$.

The family $\{M(L, A) : L \subseteq A\}$ is a closed basis of the topology of the space $M(A)$.

Theorem 2.1. *The ideal space $M(A)$ is a compact T_1-space.*

Proof. Let $\{M(L_\lambda, A) : \lambda \in \Gamma\}$ be a given family of closed sets and $\cap\{M(L_\lambda, A) : \lambda \in P\} \neq \emptyset$ for any non-empty finite subset $P \subseteq \Gamma$. Assume that: for every two elements $\alpha, \beta \in \Gamma$ there exists $\gamma \in \Gamma$ such that $M(L_\gamma, A) \subseteq M(L_\alpha, A) \cap M(L_\beta, A)$; if the set $P \subseteq A$ is finite and $M(P, A) \cap M(L_\alpha, A) \neq \emptyset$ for every $\alpha \in \Gamma$, then $M(P, A) = M(L_\beta, A)$ for some $\beta \in \Gamma$.

We put $I = \cup\{L_\lambda : \lambda \in \Gamma\}$. Then $I \in M(A)$ and $I \in \cap\{M(L_\alpha, A) : \alpha \in \Gamma\}$. It is obvious that $M(A)$ is a T_1-space. The proof is complete.

3. SPECIAL COMPACTIFICATION OF THE FIELD *K*

Let $C = \{z \in K :\mid z \mid= 1\}$. For every $z \in C$ we fix an improper number ∞_z such that:

1. $\infty_z \neq \infty_x$ provided $x \neq z$;

2. if $x, y \in C,\ x \neq -y$ and $z = \alpha(x + y)$ for some positive number α, then we consider that $\infty_z = \infty_x + \infty_y$;

3. if $x, y \in C$ and $z = xy$, then $\infty_z = \infty_x \cdot \infty_y$;

4. $x + \infty_y = \infty_y + x$ for any $x \in K$ and $y \in C$;

5. if $x \in K, y \in C, z \in C$ and $z = \lambda xy$ for some positive λ, then $\infty_z = x \cdot \infty_y = \infty_y \cdot x$;

6. $0 \cdot \infty_z = \infty_z \cdot 0 = 0$ for any $z \in C$.

We put $\bar{K} = K \cup \{\infty_z : z \in C\}$ and $\Omega = \{\infty_z : z \in C\}$. There exists an one-to-one mapping $\varphi : \bar{K} \to B = \{x \in K :\mid x \mid\leq 1\}$ such that $\varphi(0) = 0, \varphi(\infty_z) = z$ for any $z \in C, \varphi(x) = x \cdot (1+ \mid x \mid)$ for any $x \in K$. On $\bar{K}$ we consider the topology with respect to which φ is a homeomorphism. Then $\bar{K}$ is a compactification of the space K and $\bar{K}$ is a partial algebra. If $x, y \in C$ and $x = -y$, then $\infty_x + \infty_y$ is not determined. We consider that $-\infty_x = \infty_{(-x)}$. If R is the field of reals, then $+\infty = \infty_1$, $-\infty = \infty_{(-1)}$ and $R \cup \{-\infty, +\infty\} \subseteq K$. We put $\bar{R} = R \cup \{-\infty, +\infty\} = [-\infty, +\infty]$. Thus $\bar{R}$ is a compactification of the reals and $\bar{R}$ is homeomorphic to the closed interval.

4. PARTIAL ALGEBRAS OF FUNCTIONS

Fix a topological space X. Denote by $C(X, \bar{K})$ the set of all continuous functions of X into $\bar{K}$ and $C^0(X, K) = \{f \in C(X, K) : f(X)$ is a bounded subset of $K\}$. If $f \in C^0(X, K)$, then $cl_K f(X)$ is a compact subset.

A function $f \in C^0(X, K)$ possesses a compact support if there exists a compact subset $F \subseteq X$ such that $f^{-1}(K \backslash \{0\}) \subseteq F$. In this case $supp(f) = cl_X f^{-1}(K \backslash \{0\})$ is a compact subset. Let $C_0(X, K) = \{f \in C(X, K) : supp(f)$ is a compact set$\}$.

Definition 4.1. *A subset $A \subseteq C(X, \bar{K})$ is an algebra of functions on X if*:

- $0 \in A$, *where* $0(x) = 0$ *for any* $x \in X$;

- *if* f *is a constant function, then* $f \in A$;
- *if* $f, g \in A$ *and* $f + g \in C(X, \bar{K})$, *then* $f + g \in A$;
- *if* $f, g \in A$ *and* $f \cdot g \in C(X, \bar{K})$, *then* $f \cdot g \in A$;
- *if* $f \in A$, *then* $-f \in A$ *and* $\bar{f} \in A$;
- *if* $f \in A$, *then* $\lambda f \in A$ *for any* $\lambda \in K$.

For every subset $L \subset C(X, \bar{K})$ there exists a minimal algebra $a(L)$ generated by the set L.

Definition 4.2. *Let* (Y, e_Y) *be a g-compactification of a space* X. *Then* $C(X, K, Y, e_Y) = \{f \circ e_y : f \in C(X, \bar{K})\}$ *is called the algebra of functions on* X *continuously extendable on* Y.

Theorem 4.3. *Let* X *be a topological space* $X_0 = \cup\{U : U$ *is open in* X *and* $cl_X U$ *is a compact Hausdorff subspace*$\}$ and $L \subseteq C(X, \bar{K})$. *Then there exists a unique g-compactification* (Y, e_Y) *of the space* X *with the following properties:*

1. *every function* $f \in L$ *is continuously extendable on* Y, *i.e. there exists a unique continuous function* $ef \in C(Y, K)$ *such that* $f = ef \circ e_Y$.

2. *if* $y_1, y_2 \in Y \backslash e_y(X_0)$ *and* $y_1 \neq y_2$, *then there exists* $f \in L$ *such that* $ef(y_1) \neq ef(y_2)$.

3. *if the set* U *is open in* X *and* $cl_X U$ *is a Hausdorff compact subset of* X, *then* $e_y(U)$ *is an open subset of* Y *and* $e_Y \mid U : U \rightarrow e_Y(U)$ *is a homeomorphism.*

Proof. If X is a compact Hausdorff space, then $Y = X$. Suppose that the space X is not a compact Hausdorff space. Let $\{(x_\alpha, F_\alpha) : \alpha \in A\}$ be the set of all pairs (x, F), where $x \in X_0, F$ is a closed subset of X and $x \notin F$. For every $\alpha \in A$ fix a continuous function $\varphi_\alpha : X \rightarrow [0, 1] \subseteq K$ such that $\varphi_\alpha(x_\alpha) = 0$ and $F_\alpha \cup (X \backslash X_0) \subseteq \varphi_\alpha^{-1}(1)$. Let X_α be the closure of the set $\varphi_\alpha(X)$ in K. Then $(X_\alpha, \varphi_\alpha) \in GC(X)$.

For every $f \in L$ denote by Y_f the closure of the set $f(X)$ in $\bar{K}$. Then $(Y_f, f) \in GC(X)$.

If $X_0 \cup L = \emptyset$, then (Y, e_Y) is the one-point g-compactification.

Suppose that $X_0 \cup L \neq \emptyset$.

Denote by (Y, e_Y) the minimal g-compactification with the following properties:

- $(Y, e_Y) \geq (X_\alpha, \varphi_\alpha)$ for any $\alpha \in A$;

- $(Y, e_Y) \geq (Y_f, f)$ for any $f \in L$.

By construction, the functions from $L \cup \{\varphi_\alpha : \alpha \in A\}$ are continuously extendable on (Y, e_Y). Hence, $e_Y \mid Z : Z \to e_Y(Z)$ is a homeomorphism. If $\alpha \in A$ and $e\varphi_\alpha$ is the extension of φ_α on Y, then $x_\alpha\alpha \in e\varphi_\alpha^{-1}[0,1)$ and $e\varphi_\alpha^{-1}[0,1)$ is an open subset of Y. Thus $e_Y(X_0)$ is an open subset of Y.

Thus (Y, e_Y) satisfies all conditions of Theorem 4.3.

Fix $(S, e_S) \in GC(X)$ with the properties of the Theorem 4.3. Then the functions from $L \cup \{\varphi_\alpha : \alpha \in A\}$ are continuously extendable on S and $(S, e_S) \geq sup(\{(Y_f, f) : f \in L\} \cup \{(X_\alpha, \varphi_\alpha) : \alpha \in A\}) = (Y, e_Y)$ and there exists a continuous mapping $g : S \to Y$ such that $e_Y = g \circ e_S$.

We affirm that g is a homeomorphism and the g-compactifications (Y, e_Y), (S, e_S) are equivalent. Suppose that g is not a homeomorphism. Then there exists two distinct points $s, s_2 \in S$ such that $g(s_1) = g(s_2)$.

If $x_1 \in X \setminus X_0$ and $x_2 \in X\ X_0$, then $\varphi_\alpha(x_1) \neq \varphi_\alpha(x_2)$ provided $x_1 = x_2$. In this case $e_Y(x_1) = e_Y(x_2)$. Therefore $s_1, s_2 \in S \backslash e_S(X_0)$. Let ef be the continuous extension of the function $f \in L$ on (S, e_S). Thus $ef(s_1) = ef(s_2)$ for every $f \in L$, a contradiction with the condition 2. The proof is complete.

Corollary 4.4 *Let X be a locally compact Hausdorff space and $L \subseteq C(X, K)$. Then there exists a unique compactification Y of the space X such that:*

1. *every function $f \in L$ is continuously extendable on Y;*

2. *if $y_1, y_2 \in Y \setminus X$ and $y_1 \neq y_2$, then there exists $f \in L$ such that $ef(y_1) \neq ef(y_2)$;*

3. *X is an open and dense subspace of the space Y.*

Theorem 4.5. *Let X be a topological space, $X_0 = \cup\{U : U$ is open in X and $cl_X U$ is a compact Hausdorff subspace$\}$, $L \subseteq C(X, K)$ and $L \cup \{f \in C(X, K) : supp(f) \subseteq X_0\}$. Then the maximal ideal space $M(\bar{L})$ of the algebra L is the g-compactification with the properties from Theorem 4.3.*

Proof. Let (Y, e_f) be the g-compactification from Theorem 4.3. For every

$f \in I$ denote by ef the continuous extension of f on (Y, e_Y). We consider that $e_Y(Z) = Z \subseteq Y$. From the Stone-Weierstrass theorem ([3], Theorem 3.2.21) the algebra $\{ef : f \in \bar{L}\}$ is dense in the Banach algebra $C(Y, K)$ of all continuous functions of Y into K. The maximal ideal spaces $C(\bar{L})$ and $M(C(Y, K))$ are homeomorphic to the space Y [4]. The proof is complete.

Remark 4.6 For the Riemanian surfaces X and functions $L \subseteq C(X, \bar{R})$ the Corollary 4.4 was proved by C. Constantinescu and A. Cornea in [2], while for any locally compact X Hausdorff space, by M. Brelot [1].

Remark 4.7. The set $L \subseteq C^0(X, K)$ and the subalgebra $I \subseteq C^0(X, K)$ generate the same g-compactification of the Brelot-Constantinescu-Cornea type.

Let $A(X, K)$ be the set of all closed subalgebras of the Banach algebra $C^0(X, K)$ with the sup-norm $||f|| = sup\{|f(x)| : x \in X\}$. Then there exist an one-to-one correspondence $k : A(X, K) \to GC(X)$ and a mapping $c : A(X, K) \to GC(X)$ such that:

1. if $A \in A(X, K)$ and $k(A) = (Y, e_Y)$, then $A = \{f \circ e_Y : f \in C(Y, K)\}$ and Y is the maximal ideal space of the algebra A;
2. $A, B \in A(X, K)$, then $A \subseteq B$ iff $k(A) \leq k(B)$;
3. if $A \in A(X, K)$ and $c(A) = (Y, e_Y)$, then (Y, e_Y) is the g-compactification of the Brelot-Constantinescu-Cornea type generated by the algebra A;
4. by construction, $k(A) \leq c(A)$ for any algebra $A \in A(X, K)$;
5. $c(A(X, K))$ is the set of all g-compactifications of the Brelot-Constantinescu-Cornea type;
6. if X is a locally compact Hausdorff space, then $c(A(X, K))$ is the set of all compactifications of the space X;
7. if $|X| \leq 1$, then $c = k$.

Example 4.8. Let X be a locally compact non-compact Hausdorff space and A be the algebra of constant functions. Then $c(A)$ is the one-point Alexandroff compactification of X and $k(A)$ is the one-point minimal g-compactification of X.

Example 4.9. Let X be the space of reals and $A = \{f \in C^0(X, K) : [-3, 3] \subseteq f^{-1}(0)$. Then $c(A)$ is the Stone-Čech compactification βX of X and

$k(A) = (Y, e_Y)$ is a g-compactification. In this case $e_Y([-3, 3])$ is an one-point subset of the space Y.

Example 4.10. Let $X = [0, 1)$ be endowed with the topology generated by the open base $\{X \cap [x, x+\epsilon) : x \in X, \epsilon > 0\}$, Y be the space $[0, 1]$ in the natural topology, $e_Y(x) = x$ for any $x \in X$ and $A = C^0(Y, K) \subseteq C^0(X, K)$. Then (Y, e_Y) is a g-compactification of the space X and $c(A) = k(A) = (Y, e_Y)$. In this case $c = k$.

References

[1] M. Berlot, *On topologies and boundaries in potential theory*, Springer, Berlin, 1971.

[2] C. Constantinescu, A. Cornea, *Ideale Raunder Riemannscher Flachen*, Ergeb. Math., **32**, Springer, Berlin, 1963.

[3] R. Engelking, *General topology*, PWN, Warszava, 1977.

[4] I. M. Gelfand, A. N. Kolmogoroff, *On rings of continuous functions on topological spaces*, Doklady Akad. Nauk SSSR, **22**(1939), 11-15.