ROMAI J., **4**, 1(2008), 85–92

SYSTEMS OF EQUATIONS INVOLVING ALMOST PERIODIC FUNCTIONS

Silvia - Otilia Corduneanu

Dept. of Math., "Gheorghe Asachi" Technical University of Iaşi

scorduneanu@yahoo.com

Abstract The theory of Fourier series for almost periodic functions for solving systems of equations which are linear with respect to convolution by functions belonging to $L^1(G)$ is used.

Keywords: almost periodic function, functional equation, Fourier series.

2000 MSC: 43A60.

1. INTRODUCTION

The theory of almost periodic functions on groups was elaborated by John Von Neumann. The main results which were developed by Harald Bohr for almost periodic functions on the real line, as the existence of mean value, the theory of the Fourier series, were extended to this class of functions. The set $AP(G)$ of all almost periodic functions on a Hausdorff locally compact abelian group G is a Banach algebra with respect to the supremum norm. If $f \in AP(G)$ then the Fourier series of f is

$$\sum_{n=1}^{\infty} c_{\gamma_n}(f)\gamma_n,$$

where $\hat{G}$ is the dual of the group G and $\{\gamma_n \in \hat{G} |\ n \in \mathbb{N}\}$ is the set of those characters with the property that for every $n \in \mathbb{N}$ the Fourier coefficient $c_{\gamma_n}(f)$ is different by the value 0. Based on the property that two almost periodic functions coincide if they have the same Fourier coefficients, we solve the following system of equations

$$f_i = \sum_{j=1}^{p} g_{ij} * f_j + h_i, \quad i = 1, 2, ..., p. \tag{1}$$

In this context the functions $g_{ij},\ (i,j) \in \{1,2,...,p\}^2$ belong to $L^1(G)$ and the functions $h_i,\ i \in \{1,2,...,p\}$ are in $AP(G)$. A solution of the system (1) is an element $F = (f_1, f_2, ..., f_p) \in (AP(G))^p$ such that the almost periodic functions $f_1, f_2, ..., f_p$ satisfy the relations (1). In the same manner we can discuss the system of functional equations

$$f_i = \sum_{j=1}^{p} \nu_{ij} * f_j + h_i, \quad i = 1,2,...,p, \tag{2}$$

where $\nu_{ij},\ (i,j) \in \{1,2,...,p\}^2$ are bounded measures and $h_i,\ i \in \{1,2,...,p\}$, are almost periodic functions.

2. PRELIMINARIES

Consider a Hausdorff locally compact Abelian group G and let λ be the Haar measure on G. Let us denote by $\mathcal{C}(G)$ the set of all bounded continuous complex-valued functions on G. The space of Haar measurable functions f on G, with $\int_G |f(x)|d\lambda(x) < \infty$, will be denoted by $L^1(G)$. As usually the norm $\|\cdot\|_1$ is defined by

$$\|f\|_1 = \int_G |f(x)|d\lambda(x), \quad f \in L^1(G).$$

We use $m_F(G)$ to denote the space of all bounded measures on G. We denote by $|\nu|$ the variation measure which corresponds to a measure ν on G. For $f \in \mathcal{C}(G)$ and $a \in G$, the translate of f by a is the function $f_a(x) = f(xa)$ for all $x \in G$. In [2], [5], [6], [7], there are defined the almost periodic functions.

Definition 2.1. *A function $f \in \mathcal{C}(G)$ is called an almost periodic function on G, if the family of translates of f, $\{f_a : a \in G\}$ is relatively compact in the sense of uniform convergence on G.*

The set $AP(G)$ of all almost periodic functions on G is a Banach algebra with respect to the supremum norm, closed to conjugation. There exists a unique positive linear functional $M : AP(G) \to \mathbb{C}$ such that $M(f_a) = M(f)$, for all $a \in G$, $f \in AP(G)$ and $M(\mathbf{1}) = 1$. We denote by $\mathbf{1}$ the constant function which is 1 for all $x \in G$. If $f \in AP(G)$ we define the mean of f as being the above complex number $M(f)$. Denote by $\hat{G}$ the dual of G. It is easy to see that $\hat{G} \subset AP(G)$. We put $c_\gamma(f) = M(\overline{\gamma} f)$ for all $\gamma \in \hat{G},\ f \in AP(G)$ and we call $c_\gamma(f)$, the Fourier coefficient of f corresponding to $\gamma \in \hat{G}$. Next, we recall the definition of the Fourier series of an almost periodic function. First, let us say that if $f \in AP(G)$, then there exists only a countably subset of $\hat{G}$, denoted by $\{\gamma_n \in \hat{G} |\ n \in \mathbb{N}\}$, such that $M(\overline{\gamma}_n f) \neq 0,\ n \in \mathbb{N}$.

Definition 2.2. *Let $f \in AP(G)$ and $\{\gamma_n \in \hat{G} |\ n \in \mathbb{N}\}$ be the subset of $\hat{G}$ such that $M(\overline{\gamma}_n f) \neq 0,\ n \in \mathbb{N}$. We define the Fourier series of f by*

$$\sum_{n=1}^{\infty} c_{\gamma_n}(f)\gamma_n.$$

If $g \in L^1(G)$ and $f \in AP(G)$, their convolution, $g * f$, belongs to $AP(G)$. We recall that

$$g * f(x) = \int_G f(xy^{-1})g(y)d\lambda(y), \quad x \in G,$$

and that the Fourier transform of $g \in L^1(G)$, denoted by $\hat{g}$ is given by

$$\hat{g}(\gamma) = \int_G g(x)\overline{\gamma}(x)d\lambda(x), \quad \gamma \in \hat{G}.$$

We also recall that for $\nu \in m_F(G)$ and $f \in AP(G)$ we have that their convolution belongs to $AP(G)$; we use the notation $\nu * f$ for the convolution between f and ν and the meaning of that is

$$\nu * f(x) = \int_G f(xy^{-1})d\nu(y), \quad x \in G.$$

The Fourier - Stieltjes transform of $\nu \in m_F(G)$, denoted by $\hat{\nu}$, is given by

$$\hat{\nu}(\gamma) = \int_G \overline{\gamma}(x)d\nu(x), \quad \gamma \in \hat{G}.$$

3. SYSTEMS OF EQUATIONS

Consider $p \in \mathbb{N},\ p \geq 2$, the functions $g_{ij} \in L^1(G)$, $(i,j) \in \{1,2,...,p\}^2$ and the almost periodic functions $l_i,\ i = 1,2,...,p$. Denote by $[g]$ the matrix having the elements $g_{ij},\ (i,j) \in \{1,2,...,p\}^2$. Consider $i \in \{1,2,...,p\}$. The subset of $\hat{G}$ which contains the characters with the property that the corresponding Fourier coefficient of l_i is different by the value 0, is

$$\mathcal{S}_{l_i} = \{\gamma_n^i |\ n \in \mathbb{N}\}.$$

This means $M\left(\overline{\gamma_n^i} l_i\right) \neq 0,\ n \in \mathbb{N}$. The Fourier series of l_i is

$$\sum_{n=1}^{\infty} c_{\gamma_n^i}(l_i)\gamma_n^i.$$

Consider

$$\mathcal{S} = \bigcup_{i=1}^{p} \mathcal{S}_{l_i},$$

and denote $\mathcal{S} = \{\gamma_n \in \hat{G} |\ n \in \mathbb{N}\}$. Let $(a_n)_n$ be a sequence of complex numbers such that the series $\sum_{n=1}^{\infty} |a_n|^2$ is convergent.

Lemma 3.1. *For every $i = 1, 2, ..., p$ the Fourier series*

$$\sum_{n=1}^{\infty} c_{\gamma_n^i}(l_i) a_n \gamma_n^i$$

is uniformly convergent in the space $AP(G)$.

Proof. For every $i \in \{1, 2, ..., p\}$ we have that the function l_i satisfies the Parseval equality

$$\sum_{n=1}^{\infty} |c_{\gamma_n^i}(l_i)|^2 = M(|l_i|^2). \tag{3}$$

The conclusion follows from (3) and from the inequalities

$$\left(\sum_{k=n}^{n+p} |c_{\gamma_k^i}(l_i) a_k \gamma_k^i(x)|\right)^2 = \left(\sum_{k=n}^{n+p} |c_{\gamma_k^i}(l_i) a_k|\right)^2 \leq$$

$$\leq \sum_{k=n}^{n+p} |c_{\gamma_k^i}(l_i)|^2 \sum_{k=n}^{n+p} |a_k|^2, \quad x \in G,\ n \in \mathbb{N},\ p \in \mathbb{N}. \quad \square$$

Using Lemma 3.1 we can easily obtain the following result.

Corollary 3.1. *There exist the functions $h_1, h_2, ..., h_p$ such that for every $i = 1, 2, ..., p$ we have*

$$h_i = \sum_{n=1}^{\infty} c_{\gamma_n^i}(l_i) a_n \gamma_n^i. \tag{4}$$

in the space $AP(G)$.

Notation 3.1. *For every $n \in \mathbb{N}$ we make the following notation*

$$C_{(h_1,h_2,...,h_p,\gamma_n)} = \begin{bmatrix} c_{\gamma_n}(h_1) \\ \\ c_{\gamma_n}(h_2) \\ \\ \vdots \\ \\ c_{\gamma_n}(h_p) \end{bmatrix}.$$

For $n \in \mathbb{N}$ we consider the matrix

$$M_{([g],\gamma_n)} = \begin{bmatrix} \hat{g}_{11}(\gamma_n) - 1 & \hat{g}_{12}(\gamma_n) & \dots & \hat{g}_{1p}(\gamma_n) \\ \\ \hat{g}_{21}(\gamma_n) & \hat{g}_{22}(\gamma_n) - 1 & \dots & \hat{g}_{2p}(\gamma_n) \\ \\ \dots & \dots & \dots & \dots \\ \\ \hat{g}_{p1}(\gamma_n) & \hat{g}_{p2}(\gamma_n) & \dots & \hat{g}_{pp}(\gamma_n) - 1 \end{bmatrix} \tag{5}$$

and we denote the determinant of the matrix $M_{([g],\gamma_n)}$ by Δ_n. We also consider

$$\max_{(i,j)\in\{1,2,...,p\}^2} \|g_{ij}\|_1 = V_{\max}.$$

Theorem 3.1. *Consider the functions $g_{ij} \in L^1(G),\ (i,j) \in \{1,2,...,p\}^2$ and the functions $h_i,\ i \in \{1,2,...,p\}$ defined in* (4).
If there exists $\delta > 0$ such that $\inf_{n\in\mathbb{N}} |\Delta_n| > \delta$, then the system (1) *has a solution $(f_1, f_2, ..., f_p) \in (AP(G))^p$.*

Proof. Consider $n \in \mathbb{N}$. The Fourier coefficients of the convolutions $g_{ij} * f_j,\ (i,j) \in \{1,2,...,p\}^2$ are given by

$$c_{\gamma_n}(g_{ij} * f_j) = \hat{g}_{ij}(\gamma_n) c_{\gamma_n}(f_j).$$

Taking into account the relations (1), we calculate the Fourier coefficients of the functions $f_1, f_2, ..., f_p$, so, we obtain the linear algebric system

$$c_{\gamma_n}(f_i) = \sum_{j=1}^{p} \hat{g}_{ij}(\gamma_n) c_{\gamma_n}(f_j) + c_{\gamma_n}(h_i), \quad i = 1,2,...,p, \tag{6}$$

which has the matrix (5). From (6) it follows that for every $n \in \mathbb{N}$ we have

$$C_{(f_1,f_2,...,f_p,\gamma_n)} = -{M_{([g],\gamma_n)}}^{-1} C_{(h_1,h_2,...,h_p,\gamma_n)}, \tag{7}$$

where

$$C_{(f_1,f_2,...,f_p,\gamma_n)} = \begin{bmatrix} c_{\gamma_n}(f_1) \\ \\ c_{\gamma_n}(f_2) \\ \\ \vdots \\ \\ c_{\gamma_n}(f_p) \end{bmatrix}.$$

We prove that for every $i \in \{1, 2, ..., p\}$, the Fourier series

$$\sum_{n=1}^{\infty} c_{\gamma_n}(f_i)\gamma_n \tag{8}$$

is uniformly convergent in the space $AP(G)$. For every $(i, j) \in \{1, 2, ..., p\}^2$ we have

$$|\hat{g}_{ij}(\gamma_n)| = |\int_G g_{ij}(x)\overline{\gamma}_n(x)d\lambda(x)| \leq \|g_{ij}\|_1 \leq V_{\max}$$

and

$$|\hat{g}_{ij}(\gamma_n) - 1| \ \leq 1 + V_{\max}.$$

If β_{ij}^n is the element of the matrix ${M_{([g],\gamma_n)}}^{-1}$ situated on the line i and the column j then

$$|\beta_{ij}^n| \leq \frac{(p-1)!}{\delta} (1 + V_{\max})^{p-1}.$$

Taking into account (7) it follows that for every $i \in \{1, 2, ..., p\}$

$$\begin{aligned} &|c_{\gamma_n}(f_i)\gamma_n(x)| \leq \\ &\leq \frac{(p-1)!}{\delta} (1 + V_{\max})^{p-1} \sum_{k=1}^{p} |c_{\gamma_n}(h_k)|, \quad n \in \mathbb{N}, \ x \in G. \end{aligned} \tag{9}$$

On the other hand, for every $k \in \{1, 2, ..., p\}$ the series

$$\sum_{n=1}^{\infty} |c_{\gamma_n}(h_k)| \tag{10}$$

is convergent. Therefore, from (9) we see that the Fourier series (8) are uniformly convergent. Based on the property that two almost periodic functions coincide if they have the same Fourier coefficients, we conclude that the sums of these series satisfy the equations of the system (1). $\square$

In the same manner we can treat the system (2). Consider the measures $\nu_{ij} \in m_F(G),\ (i,j) \in \{1,2,...,p\}^2$. We denote by $[\nu]$ the matrix having the elements $\nu_{ij},\ (i,j) \in \{1,2,...,p\}^2$. For $n \in \mathbb{N}$ we consider the matrix

$$M_{([\nu],\gamma_n)} = \begin{bmatrix} \hat{\nu}_{11}(\gamma_n) - 1 & \hat{\nu}_{12}(\gamma_n) & \dots & \hat{\nu}_{1p}(\gamma_n) \\ \\ \hat{\nu}_{21}(\gamma_n) & \hat{\nu}_{22}(\gamma_n) - 1 & \dots & \hat{\nu}_{2p}(\gamma_n) \\ \\ \dots & \dots & \dots & \dots \\ \\ \hat{\nu}_{p1}(\gamma_n) & \hat{\nu}_{p2}(\gamma_n) & \dots & \hat{\nu}_{pp}(\gamma_n) - 1 \end{bmatrix} \tag{11}$$

and we denote the determinant of the matrix $M_{([\nu],\gamma_n)}$ by Ω_n. We have the inequalities

$$|\hat{\nu}_{ij}(\gamma)| \leq |\nu|(G), \quad \gamma \in \hat{G}, \quad (i,j) \in \{1,2,...,p\}^2.$$

Using formulas for calculating the Fourier coefficients, as

$$c_\gamma(\nu_{ij} * f_j) = \hat{\nu}_{ij}(\gamma) c_\gamma(f_j), \quad \gamma \in \hat{G}, \quad (i,j) \in \{1,2,...,p\}^2,$$

and performing similar steps as in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Consider the measures $\nu_{ij} \in m_F(G),\ (i,j) \in \{1,2,...,p\}^2$ and the functions $h_i,\ i \in \{1,2,...,p\}$ defined in* (4).
If there exists $\omega > 0$ such that $\inf_{n\in\mathbb{N}} |\Omega_n| > \omega$, then the system (2) *has a solution $(f_1, f_2, ..., f_p) \in (AP(G))^p$.*

References

[1] L. N. Argabright, J. Gil de Lamadrid, *Fourier analysis of unbounded measures on locally compact Abelian groups*, Mem. Amer. Math. Soc. **145** (1974).

[2] L. N. Argabright, J. Gil de Lamadrid, *Almost Periodic Measures*, Mem. Amer. Math. Soc. **428** (1990).

[3] C. Corduneanu, *Almost periodic functions*, New York, 1968.

[4] N. Dinculeanu, *Integration on locally compact spaces*, Ed. Academiei R.P.R., Bucureşti, 1965 (Romanian).

[5] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic Functions*, Trans Amer. Math. Soc. **67** (1949), 217-240.

[6] E. Hewitt, K. A. Ross, *Abstract harmonic analysis* **I**, Springer, Berlin, 1963.

[7] J. Gil de Lamadrid, *Sur les mesures presque périodiques*, Astérisque **4** (1973), 61-89.

[8] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, **12**, Interscience Publishers − John Wiley and Sons, New York, 1962.