

NUMERICAL APPROXIMATION OF POINCARÉ MAPS

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Abstract A classical technique for analyzing dynamical systems is due to Poincaré [4]. It replaces the flow of a n^{th} -order continuous-time system by a $(n - 1)^{\text{th}}$ -order discrete-time system called the Poincaré map. A Poincaré map essentially describes how points on a plane Σ (the Poincaré section), which is transverse to an orbit Γ and which are sufficiently close to Γ , get mapped back onto Σ by the flow [5]. Unfortunately, except under the most trivial circumstances, the Poincaré map cannot be expressed by explicit equations. Here the numerical analysis interpose. In this paper we present an algorithm for constructing the Poincaré map, we build its Maple code, and implement it on an example.

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1. INTRODUCTION

Consider a system of autonomous differential equations

$$\dot{x} = f(x), x \in \mathbb{R}^n. \tag{1}$$

Throughout the text, we assume that the vector field f is sufficiently smooth (C^1 will do nicely). Let $\varphi(x, t)$ denote the solution of (1) satisfying $\varphi(x, 0) = x$. The curve $O(x) = \{\varphi(x, t) : t \in \mathbb{R}\}$ is called the *orbit* or *trajectory* passing through the point x .

In many physical applications and particularly in the theory of dynamical systems, one is often interested in computing a *Poincaré map* P (also known as the *first return map*) of a system such as (1). This map is produced by considering successive intersections of a trajectory with a codimension-one surface Σ - called the *Poincaré section*- of the phase space \mathbb{R}^n .

Given a system (1), the existence of a Poincaré map is far from being obvious; in many cases it simply does not exist. However, in the case when the system admits periodic solutions Γ , the Poincaré map is well-defined. Indeed, let x^* be a point on such a solution ($x^* \in \Gamma$). There exists a positive number T , called the *period* of the orbit, such that $\varphi(x^*, T+t) = \varphi(x^*, t)$ for all $t \in \mathbb{R}$. In particular, $\varphi(x^*, T) = \varphi(x^*, 0) = x^*$, so the point x^* returns to itself after having flowed T time units. Now consider a surface Σ that is transversal to the flow, i.e. the surface normal at x^* satisfies $\langle n_\Sigma(x^*), f(x^*) \rangle \neq 0$, where $\langle \cdot \rangle$ denotes the inner product in \mathbb{R}^n . By the implicit function theorem we can find an open neighborhood $U \subset \Sigma$ of x^* such that for all $x \in U$, there exists a positive number $\tau(x)$ such that if $z = \varphi(x, \tau(x))$ then: (a) $z \in \Sigma$ (x returns to the plane Σ after time $\tau(x)$); (b) $\text{sign} \langle n_\Sigma(x), f(x) \rangle = \text{sign} \langle n_\Sigma(z), f(z) \rangle$ (Σ is crossed from the same direction). The function $\tau : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is continuous, and represents the time taken by the point x to return to Σ according to the condition (b). The point $z = \varphi(x, \tau(x))$ is called the *first return* of x , and the Poincaré map $P : U \rightarrow \Sigma$ is defined by $P(x) = \varphi(x, \tau(x))$. Note, by definition, we have $\tau(x^*) = T$ and $P(x^*) = x^*$.

The advantages of a Poincaré map consists in the facts that it reduces the study of a flow to the study of maps and it also reduce the dimension of the problem by 1. Except for some cases, the Poincaré map can not be expressed by explicit equations, but it is implicitly defined by the vector field f and the section Σ .

2. LOCATING HYPERPLANE CROSSINGS Σ

In practical implementations the (n-1)-dimensional hyperplane Σ can be chosen in several ways, but if the program already knows the position of a limit cycle, it can choose a point in the hyperplane $x_\Sigma = x^*$ and the normal vector $h = f(x^*)$ where x^* is any point on the limit cycle. So the hyperplane is represented by the equation $H(x) = \langle h, x - x_\Sigma \rangle = 0$. In practice the hyperplane Σ divides \mathbb{R}^n in two regions: $\Sigma_- = \{x \in \mathbb{R}^n \mid \langle h, x - x_\Sigma \rangle < 0\}$ and $\Sigma_+ = \{x \in \mathbb{R}^n \mid \langle h, x - x_\Sigma \rangle > 0\}$ and the trajectory will repeatedly cross Σ from Σ_- to Σ_+ to Σ_- etc.

In order to locate the first hyperplane crossing of a trajectory $\varphi(x, t)$, integrate the trajectory and calculate $H(\varphi(x, t))$ at every time-step. We keep integrating until two consecutive points, $x_1 = \varphi(x, t_1)$ and $x_2 = \varphi(x, t_2)$, lie on different sides of Σ , that is $H(x_1)$ and $H(x_2)$ have opposite signs. Once x_1 and x_2 are found, the exact crossing is some point $\alpha = \varphi(x, \tau)$ with $t_1 < \tau < t_2$.

If we want to calculate α and τ , assuming x_1, x_2, t_1 and t_2 are known, we use a simple linear interpolation scheme. Let $\alpha_1 = H(x_1)$ and $\alpha_2 = H(x_2)$. Then $\alpha \approx \frac{\alpha_2}{\alpha_2 - \alpha_1}x_1 + \frac{\alpha_1}{\alpha_1 - \alpha_2}x_2$, $\tau \approx \frac{\alpha_2}{\alpha_2 - \alpha_1}t_1 + \frac{\alpha_1}{\alpha_1 - \alpha_2}t_2$.

3. A NUMERICAL EXAMPLE FOR CONSTRUCTING THE POINCARÉ MAP

Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2), \\ \dot{x}_2 &= x_1 + x_2(1 - x_1^2 - x_2^2). \end{aligned}$$

If we pass to the polar system of coordinates $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$ this system becomes the product system

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1, \quad (2)$$

the general solution of which is

$$\Phi(r, \theta, t) = ([1 + (\frac{1}{r^2} - 1)e^{-2t}]^{-1/2}, t + \theta). \quad (3)$$

We note that the periodic solution is $r(t) = 1$ (with the period $T = 2\pi$) and the transversal section is $\Sigma_p = \{(r, \theta) \in \mathbb{R}^+ \times S^1 | r > 0, \theta = 0\}$.

Consider now the diffeomorphism who help us to pass from polar coordinates to Cartesian $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2, h(r, \theta) = (r \cos \theta, r \sin \theta) = (x_1, x_2)$. So, the two systems are topological conjugate, and the solution of the first system will be $\varphi(x, t) = h \circ \Phi(r, \theta, t)$. Doing the calculations we obtain

$$\varphi(x, t) = [1 + (\frac{1}{x_1^2 + x_2^2} - 1)e^{-2t}]^{-\frac{1}{2}} \frac{1}{\sqrt{x_1^2 + x_2^2}} (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t). \quad (4)$$

The periodic solution $r(t) = 1$ of the system in polar coordinates will be transform in the unit circle $x_1^2 + x_2^2 = 1$ in Cartesian coordinates. The principal period is also $T = 2\pi$ and a point on this solution is $x^* = (1, 0)$. So we

can choose the point in the hyperplane Σ , $x_\Sigma = x^*$ and by using the equation of H we conclude that the Poincaré section is $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = 0\}$. We denote by $\Sigma_+ = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ and $\Sigma_- = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 < 0\}$ the two regions in which Σ divide \mathbb{R}^2 . We also choose the point $x = (0.5, 0)$ in the neighborhood of x^* , and construct the image $P(x)$ of this point under the Poincaré map. We note that in an Euclidian space the point $P(x)$ is not the first point at which $\varphi(x, t)$ intersects Σ ; $\varphi(x, t)$ will pass at least once after returning in the neighborhood U . If Σ is properly chosen, then the trajectory in study will successively cross Σ coming from Σ_+ to Σ_- to Σ_+ etc. The point $x = (0.5, 0)$ situated in the neighborhood of $x^* = (1, 0)$ first returns in the same neighborhood after $\tau = 6.283185307 \approx 2\pi$.

The Maple code of the programm which constructs the Poincaré map and the numerical results are presented below.

```

f := proc(x :: Vector)
return < -x[2]+x[1]*(1-x[1]^2-x[2]^2), x[1]+x[2]*(1-x[1]^2-x[2]^2) >:
end:
H := proc(x :: Vector)
local y, h :: Vector :
y := x - xs : h := f(xs) :
return DotProduct(h, y) :
end:
phi := proc(x :: Vector, t :: float)
local r :: float :
r := 1/(x[1]^2 + x[2]^2) :
return < 1/sqrt(1 + (r - 1) * exp(-2 * t)) * (x[1] * cos(t) * sqrt(r) - x[2] *
sin(t) * sqrt(r)), 1/sqrt(1 + (r - 1) * exp(-2 * t)) * (x[1] * sin(t) * sqrt(r) + x[2] *
cos(t) * sqrt(r)) >:
end:
crossings := proc(x0 :: Vector, t0, tf :: float)
local t1, t2, tau, alfa1, alfa2 :: float :
x1, x2, alfa :: Vector :

```

```

x2 := x0 :
t2 := t0 :
alfa2 := H(x0) :
while (t2 < tf) do
x1 := x2 :
t1 := t2 :
alfa1 := alfa2 :
t2 := t1 + 0.001 :
if t2 > tf then
t2 := tf : fi;
x2 := phi(x, t2) :
alfa2 := H(x2) :
if (alfa1 * alfa2 < 0.0) then
print('There is a crossing between the points:'):
print(x1) : print(x2) :
alfa := alfa2/(alfa2 - alfa1) * x1 + alfa1/(alfa1 - alfa2) * x2 :
tau := alfa2/(alfa2 - alfa1) * t1 + alfa1/(alfa1 - alfa2) * t2 :
print('The point of crossing is'): print(alfa):
print('The time when the crossing took place is'):print(tau):
fi:
od:
end:

```

It	Crossing between points		Crossing point	Time of crossing
1	-0.9972070812 0.0005909984257	-0.9972127303 -0.0004062110483	-0.997210429151753176 $-2.31352480575819364 * 10^{-14}$	3.141592652
2	0.9999947498 -0.0001853062088	0.9999944460 0.0008146884761	0.999994693503674648 $-2.62652584143846513 * 10^{-14}$	6.283185307
3	-0.9999996874 0.0007779606830	-0.999999652 -0.0002220392266	-0.9999990351749724 $2.76520855182651720 * 10^{-14}$	9.424777961
4	0.999999312 -0.0003706143508	0.999998020 0.0006293855992	0.9999988331662348 $-3.07184291024886758 * 10^{-14}$	12.56637061
5	-0.999995360 0.0009632678000	-0.999999992 -0.00003673205104	-0.9999998219571140 $-1.97446090510408624 * 10^{-15}$	15.70796327
6	0.999998456 -0.0005559215100	0.999999014 0.0004440784466	0.9999987622042166 $-2.6994571107613378 * 10^{-14}$	18.84955592

References

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