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ON A DIFFUSION PROCESS INTERMEDIATE BETWEEN STANDARD BROWNIAN MOTION AND THE ORNSTEIN-UHLENBECK PROCESS

Mario Lefebvre

École Polytechnique de Montréal, Canada

mlefebvre@polymtl.ca

Abstract We first consider a time-inhomogeneous diffusion process that is a generalization of the standard Brownian motion. We find that it has a Gaussian probability density function with the same mean as an Ornstein-Uhlenbeck process, and variance that generalizes that of the standard Brownian motion. Next, the problem of finding diffusion processes having a Gaussian $\mathrm{N}(0, t)$ probability density function is treated.

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1. INTRODUCTION AND THEORETICAL RESULTS

Arguably the two most important diffusion processes are the Wiener process $\{W(t), t \geq 0\}$ and the Ornstein-Uhlenbeck process $\{U(t), t \geq 0\}$ defined respectively by the stochastic differential equations

$$dW(t) = \mu\, dt + \sigma\, dB_1(t)$$

and

$$dU(t) = -\alpha U(t)\, dt + \sigma\, dB_2(t),$$

where $\{B_i(t), t \geq 0\}$, $i = 1, 2$, is a standard Brownian motion, and $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants. As is well known, conditional on $W(t_0) = w_0$ and $U(t_0) = u_0$, we may write that $W(t) \sim \mathrm{N}\left(w_0 + \mu(t - t_0), \sigma^2(t - t_0)\right)$ and $U(t) \sim \mathrm{N}\left(u_0 e^{-\alpha(t-t_0)}, \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha(t-t_0)}\right]\right)$ see Lefebvre (2007), pp. 184 and 203, for instance).

In this note, we first consider the time-inhomogeneous diffusion process $\{X(t), t \geq 0\}$ that satisfies the stochastic differential equation

$$dX(t) = -\frac{k}{2} X(t)\, dt + (1 + kt)^{1/2}\, dB(t), \tag{1}$$

where k is a non-negative constant and $\{B(t), t \geq 0\}$ is a standard Brownian motion. Notice that it generalizes the standard Brownian motion, which corresponds to the case when $k = 0$. That is, if $k = 0$, then $\{X(t), t \geq 0\}$ is a Wiener process with infinitesimal parameters $\mu = 0$ and $\sigma = 1$.

In Section 2, we find the probability density function of the random variable $X(t)$. We see that if $t_0 = 0$ and $x_0 = 0$, then $X(t)$ has the same probability density function as a standard Brownian motion, namely a Gaussian $\mathrm{N}(0, t)$ distribution. Then, in Section 3, we consider the problem of finding other diffusion processes having a Gaussian $\mathrm{N}(0, t)$ distribution. Finally, a few remarks conclude this work in Section 4.

2. PROBABILITY DENSITY FUNCTION OF *X*(*T*)

Let $\Phi(t)$ be the function that satisfies the ordinary differential equation

$$\frac{d}{dt}\Phi(t) = -\frac{k}{2}\Phi(t), \quad \text{for } t > t_0,$$

subject to the initial condition $\Phi(t_0) = 1$. Its solution is $\Phi(t) = \exp\left\{-\frac{k}{2}(t - t_0)\right\}$, for $t \geq t_0$. We can state the following proposition.

Proposition 2.1. *Conditional on $X(t_0) = x_0$, the distribution of the random variable $X(t)$ is given by*

$$X(t) \sim N\left(x_0\, e^{-(k/2)(t-t_0)}, t - t_0\, e^{-k(t-t_0)}\right) \quad \textit{for } t \geq t_0.$$

Proof. This result is an application of Proposition 4.3.1, p. 211, in Lefebvre (2007) [see Remark iii), p. 212]. Indeed, we deduce from this proposition that $X(t) \mid \{X(t_0) = x_0\}$ has a Gaussian distribution with mean

$$m(t) = \Phi(t)\left(x_0 + \int_{t_0}^{t} \Phi^{-1}(u) \cdot 0\, du\right) = x_0\, e^{-(k/2)(t-t_0)}$$

and variance $\sigma^2(t) = \Phi^2(t)\int_{t_0}^t \Phi^{-2}(u)\,(1+k\,u)\,du$. That is,

$$\sigma^2(t) = e^{-k(t-t_0)}\int_{t_0}^t e^{k(u-t_0)}\,(1+k\,u)\,du = t - t_0\,e^{-k(t-t_0)} \quad \text{for } t \geq t_0. \quad \blacksquare$$

Remarks. i) We can also obtain the probability density function of $X(t)$ by proceeding as follows: the function

$$f(x,t)\ (= f(x,t;x_0,t_0))\ := \frac{P\,[X(t) \in (x, x+dx) \mid X(t_0) = x_0]}{dx} \tag{2}$$

satisfies the Kolmogorov forward equation (also called Fokker-Planck equation; see Cox and Miller (1965), for instance)

$$\begin{aligned} & \frac{1}{2}\frac{\partial^2}{\partial x^2}\{(1+k\,t)\,f(x,t)\} - \frac{\partial}{\partial x}\left\{-\frac{k}{2}\,x\,f(x,t)\right\} = \frac{\partial}{\partial t}f(x,t) \\ \iff \quad & \frac{1+k\,t}{2}\,f_{xx} + \frac{k}{2}\,(f + x\,f_x) = f_t. \end{aligned}$$

Taking the Fourier transform on both sides of this partial differential equation, we obtain that $F(\omega,t) := \int_{-\infty}^{\infty} e^{i\omega x}\,f(x,t)\,dt$, where $\omega \in \mathbb{R}$, is a solution of

$$F_t + \frac{k}{2}\,\omega\,F_\omega + \frac{\omega^2}{2}\,(1+k\,t)F = 0. \tag{3}$$

Moreover, the function F is such that

$$F(0,t) = 1 \tag{4}$$

and

$$\lim_{t \downarrow t_0} F(\omega,t) = e^{i\omega x_0}. \tag{5}$$

This last condition follows from the fact that

$$\lim_{t \downarrow t_0} f(x,t) = \delta(x - x_0).$$

Next, the general solution of equation (3) can be written as $F(\omega,t) = e^{-\omega^2 t/2}\,G\left(\omega\,e^{-kt/2}\right)$, where G is an arbitrary function. We then infer from (4) and (5) that $F(\omega,t)$ is of the form

$$F(\omega,t) = \exp\left\{i\,\omega\,x_0\,e^{-k(t-t_0)/2} - \frac{\omega^2}{2}\,\left[t - t_0\,e^{-k(t-t_0)}\right]\right\}.$$

Finally, the result in the proposition is obtained by remembering that if $X \sim \mathrm{N}(\mu, \sigma^2)$, then the characteristic function of X is given by $C_X(\omega) := E\left[e^{i\omega X}\right] = \exp\left\{i\omega\mu - \frac{\sigma^2}{2}\omega^2\right\}$.
ii) When the initial time is $t_0 = 0$, the distribution of $X(t)$ reduces to

$$X(t) \mid \{X(0) = x_0\} \sim \mathrm{N}\left(x_0 \, e^{-kt/2}, t\right).$$

Notice that the mean of $X(t)$ is the same as that of an Ornstein-Uhlenbeck process with $\alpha = k/2$, while its variance corresponds to that of a standard Brownian motion. Furthermore, if the process starts at $x_0 = 0$, then $X(t) \sim \mathrm{N}(0, t)$. In the next section, the problem of finding diffusion processes having the same probability density function as a standard Brownian motion starting at the origin will be treated.

iii) From the previous remark, we can state that the diffusion process $\{X(t), t \geq 0\}$ defined by (1) is intermediate between the Wiener process with $\mu = 0$ and $\sigma = 1$, and the Ornstein-Uhlenbeck process with $\alpha = k/2$. In applications where the mean of $X(t)$ tends to 0 with increasing t, rather than remaining constant, and the variance of $X(t)$ is a linear function of t, this process would be a model better than either the Wiener or the Ornstein-Uhlenbeck process. In the case of the Ornstein-Uhlenbeck process, its variance is bounded (from above) by $\sigma^2/(2\alpha)$ $(= \sigma^2/k)$.

3. DIFFUSION PROCESSES HAVING A GAUSSIAN PROBABILITY DENSITY FUNCTION

In the preceding section, we found that the diffusion process $\{X(t), t \geq 0\}$ defined by (1) has the same probability density function as a standard Brownian motion starting from the origin, if $x_0 = 0$ and $t_0 = 0$. Now, we try to find other diffusion processes having a Gaussian $\mathrm{N}(0, t)$ probability density function.

Let $m(x, t)$ and $v(x, t) \geq 0$ be the infinitesimal parameters of $\{X(t), t \geq 0\}$. These functions must be such that (see Lamberton and Lapeyre (1997, p. 58), in particular), for all $s \geq 0$,

$$\int_0^s |m(x,t)|\,dt < \infty \quad \text{and} \quad \int_0^s v(x,t)\,dt < \infty. \tag{6}$$

Then, the function $f(x,t)$ defined in (2) satisfies the Kolmogorov forward equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x,t)f(x,t)\} - \frac{\partial}{\partial x}\{m(x,t)f(x,t)\} = \frac{\partial}{\partial t}f(x,t). \tag{7}$$

When $v(x,t) \equiv 1$ and $m(x,t) \equiv 0$, we know that (if $x_0 = 0$ and $t_0 = 0$)

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}$$

for $x \in \mathbb{R}$ and $t > 0$. Substituting the function $f(x,t)$ into (7), we obtain that

$$\frac{1}{2}\left\{v\left(\frac{x^2}{t^2} - \frac{1}{t}\right) + 2v_x\left(-\frac{x}{t}\right) + v_{xx}\right\} - \left\{m_x + m\left(-\frac{x}{t}\right)\right\} = -\frac{1}{2t} + \frac{x^2}{2t^2}.$$

Since we have only one differential equation and two unknown functions, there are many possible solutions for which $X(t)$ is a diffusion process.

First, notice that we cannot have $m(x,t) = m(t)$ and $v(x,t) = v(t)$ at the same time, except when $m(x,t) \equiv 0$ and $v(x,t) \equiv 1$. That is, when $X(t)$ is a standard Brownian motion. Assume that $v(x,t) = v(t)$, but that $m(x,t)$ depends on x. We find that m satisfies the ordinary differential equation

$$m_x - \left(\frac{x}{t}\right)m + \frac{v(t)-1}{2t}\left(1 - \frac{x^2}{t}\right) = 0,$$

whose general solution is

$$m(x,t) = c_1 \exp\left\{\frac{x^2}{2t}\right\} - x\left(\frac{v(t)-1}{2t}\right).$$

Let us choose the constant $c_1 = 0$. We see that the process considered in the previous section corresponds to the infinitesimal variance $v(t) = 1 + kt$, with k a non-negative constant. Indeed, we then have $m(x,t) = -\frac{k}{2}x$. Note that the conditions in (6) are satisfied with this choice of infinitesimal parameters. There are however other interesting possibilities. For example, we could take $v(x,t) = v(t) = 1 + kt^2$ and $m(x,t) = -\frac{k}{2}tx$. Furthermore, we can of course consider the case when $m(x,t) = m(t)$, but $v(x,t)$ depends on x, as well as the general case when both $m(x,t)$ and $v(x,t)$ depend on x (and t).

4. CONCLUSION

In this note, we first considered the diffusion process $\{X(t), t \geq 0\}$ whose infinitesimal parameters are $m(x,t) = -kx/2$ and $v(x,t) = 1 + kt$, where the constant k is non-negative. Although this process is time-inhomogeneous, we were able to calculate explicitly the probability density function of the random variable $X(t)$. We saw that $X(t)$ is normally distributed and that its parameters are related to those that correspond to the standard Brownian motion and the Ornstein-Uhlenbeck process.

The diffusion process $\{X(t), t \geq 0\}$ is a good compromise between the Wiener and Ornstein-Uhlenbeck processes, in that it behaves partly like these two very important diffusion processes. Moreover, if $\{X(t), t \geq 0\}$ starts from the origin at time $t_0 = 0$, then $X(t) \sim \mathrm{N}(0, t)$, exactly like a standard Brownian motion.

In Section 3, we saw that there are other time-inhomogeneous diffusion processes $\{X(t), t \geq 0\}$ for which $X(t)$ has a Gaussian $\mathrm{N}(0, t)$ distribution. This is true when $t_0 = 0$ and $X(0) = 0$. Making use of the proposition in Lefebvre (2007) mentioned above, we could calculate their probability density function in the general case when the initial time is $t_0 \geq 0$ and $X(t_0) = x_0 \in \mathbb{R}$.

As a sequel to this work, we could, in particular, try to find diffusion processes having a lognormal probability density function, like the geometric Brownian motion. This diffusion process is used extensively in financial mathematics. Moreover, it would be nice to have some real-life data for which the diffusion process $\{X(t), t \geq 0\}$ introduced in Section 1 would be a good model. Finally, we could also study first passage time problems involving $\{X(t), t \geq 0\}$.

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