ROMAI J., **4**, 1(2008), 193–201

CONDITIONS OF SINGLE NASH EQUILIBRIUM EXISTENCE IN THE INFORMATIONAL EXTENDED TWO-MATRIX GAMES

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Abstract The informational aspect in game theory is manifested by: the devise of possession of information about strategy's choice, the payoff functions, the order of moves, and optimal principles of players; the using methods of possessed information in the strategy's choice by players. The player's possession of supplementary information about unfolding of the game can influence appreciably the player's payoffs.

Keywords: two-matrix games, Nash equilibrium.

2000 MSC: 91A40.

An important element for the players is the possession of information about the behaviour of his opponents. Thus for the same sets of strategies and the same payoff functions it is possible to obtain different results, if the players have supplementary information. So the information for the players about the strategy choice by the others players have a significant role for the unfolding of the game.

Consider the two-matrix game in the normal form $\Gamma = \langle N, X_1, X_2, A, B\rangle$, where , $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ (A and B are the payoff matrices for the first and the second player respectively. Each player can choose one of his strategies and his purpose is to maximize his payoff. The player can choose his strategy independently of his opponent and the player does not know the chosen strategy of his opponent.

According to [1] we define the Nash equilibrium.

Definition 1. *The pair* (i^*, j^*), $i^* \in X_1, j^* \in X_2$ *is called the Nash equilibrium (NE) for the game* Γ, *if the relations*

$$\begin{cases} a_{i^*j^*} \geqslant a_{ij^*}, \forall i \in X_1, \\ b_{i^*j^*} \geqslant a_{i^*j}, \forall j \in X_2, \end{cases} \tag{1}$$

hold. Notation: $(i^*, j^*) \in NE\,(\Gamma)$.

There are two-matrix games for which the set of the Nash equilibria is empty: $NE\,(\Gamma) = \emptyset$ (solutions do not exist in pure strategies).

For every two-matrix game we can construct some informational extended games. If one of the players knows the strategy chosen by the other, we consider that it is one form of the informational extended two-matrix game for the initial game. Even if the initial two-matrix game has no solutions in pure strategies, for the informational extended games at least one solution in pure strategies always exists (Nash equilibria). The proof of this assertion can be found in [2], [3]. In the case of informational extended games the player which knows the chosen strategy of his opponent has one advantage and he will obtain one of his greater payoff.

According to [1], let us define two forms of informational extended games $_1\Gamma$ and $_2\Gamma$. We consider that for the game $_1\Gamma$ the first player knows the chosen strategy of the second player, and for the game $_2\Gamma$ the second player knows the chosen strategy of the first player.

If one of the players knows the chosen strategy of the other, then the set of the strategies for this player can be represented by a set of mappings defined on the set of strategies of his opponent.

Definition 2. *(The game* $_1\Gamma$ according to [1]*) The informational extended two-matrix game* $_1\Gamma$ *can be defined in the normal form by:* $_1\Gamma = \left\langle N, \overline{X_1}, X_2, \overline{A}, \overline{B} \right\rangle$, *where* $N = \{1, 2\}$, $\overline{X_1} = \{\varphi_1 : X_2 \longrightarrow X_1\}$, $\overline{A} = \{\overline{a}_{ij}\}$, $\overline{B} = \left\{\overline{b}_{ij}\right\}$, $i = \overline{1, m^n}$, $j = \overline{1, n}$.

For the game $_1\Gamma$ we have $\overline{X_1} = \{1, 2, \ldots, m^n\}$, $X_2 = \{1, 2, \ldots, n\}$, $\left|\overline{X_1}\right| = m^n$, the matrices $\overline{A}$ and $\overline{B}$ have dimension $[m^n \times n]$ and they are formed from elements of initial matrices A and B respectively.

Choosing one element from each of the rows from the matrix A we will build one column in the matrix $\overline{A}$. The columns from the matrix $\overline{B}$ are built in the same manner, choosing one element from each of the rows from the matrix B. Thus, the matrices $\overline{A}$ and $\overline{B}$ have the dimension $[m^n \times n]$.

Definition 3. *(The game ${}_2\Gamma$ according to [1]) The informational extended two-matrix game ${}_2\Gamma$ can be defined in the normal form by:* ${}_2\Gamma = \left\langle N, X_1, \overline{X_2}, \widetilde{A}, \widetilde{B} \right\rangle$, *where* $\overline{X_2} = \{\varphi_2 : X_1 \longrightarrow X_2\}$, $\left|\overline{X_2}\right| = n^m$, $\widetilde{A} = \{\widetilde{a}_{ij}\}$, $\widetilde{B} = \left\{\widetilde{b}_{ij}\right\}$, $i = \overline{1,m}$, $j = \overline{1,n^m}$.

For the game ${}_2\Gamma$ we have $X_1 = \{1, 2, \ldots, m\}$, $\overline{X_2} = \{1, 2, \ldots, n^m\}$, the matrices $\widetilde{A}$ and $\widetilde{B}$ have dimension $[m \times n^m]$ and they are formed from elements of initial matrices A and B respectively.

The extended matrices $\widetilde{A}$ and $\widetilde{B}$ will be built in a similar way as in the case of the game ${}_1\Gamma$. That is each of rows in the matrices $\widetilde{A}$ (or in the matrix $\widetilde{B}$, respectively) will be built choosing one element from each of the columns from the matrix A (or B, respectively).

The following algorithm can be used for the determination of Nash equilibria in the informational extended two-matrix games ${}_1\Gamma$ and ${}_2\Gamma$.

Algorithm.

For the game ${}_1\Gamma$, we determine the maximum element in each column from the matrix A, i. e. $a_{i_j j} = \max_i \{a_{1j}, a_{2j}, \ldots, a_{mj}\}$, for $\forall j = \overline{1,n}$.

For each element $a_{i_j j}$, $j = \overline{1,n}$ thus obtained, we determine the corresponding elements with the same indices from the matrix $B : b_{i_j j}$, $j = \overline{1,n}$.

For each of these pairs $a_{i_j j}, b_{i_j j}$, $\left(j = \overline{1,n}\right)$ we determine if these values can be the payoffs for players for some Nash equilibria.

Thus if $\forall k \in X_2 \backslash \{j\}$ $\exists b_{ik} : b_{ik} \leqslant b_{i_j j}$, then the pair $a_{i_j j}, b_{i_j j}$ can be the payoffs for players for some Nash equilibria in the game ${}_1\Gamma$; consider this pair $a_{i^* j^*}, b_{i^* j^*}$.

It is possible that for the pair $a_{i^* j^*}, b_{i^* j^*}$ there are several Nash equilibria.

If we wish to determine how many Nash equilibria there are in the game ${}_1\Gamma$ for the pair $a_{i^* j^*}, b_{i^* j^*}$ we determine the number of elements which are in each column $k \in X_2 \backslash \{j\}$ from the matrix B for which $b_{ik} \leqslant b_{i^* j^*}$. Denote by n_j, $j = \overline{1,n}$ the number of elements b_{ij} from the column j for which $b_{ij} \leqslant b_{i^* j^*}$. For j^* we have $n_{(j^*)} = 1$.

Then the number of Nash equilibria for which the players have the payoff $a_{i^*j^*}$ and $b_{i^*j^*}$, respectively, can be determined by

$$N_{j^*} = n_1 \cdot n_2 \cdot \ldots \cdot n_{(j^*-1)} \cdot 1 \cdot n_{(j^*+1)} \cdot \ldots n_n. \tag{2}$$

The number of all Nash equilibria in the game ${}_1\Gamma$ can be determined by: $N = \sum\limits_j N_j$.

If the pair of elements $a_{i_j j}, b_{i_j j}$ can be the payoffs of the players for some Nash equilibrium in the informational extended game ${}_1\Gamma$, then j will be the strategy for the second player. And because $\overline{X_1} \neq X_1$, we have to determine the strategy for the first player, for which the elements $a_{i^*j^*}, b_{i^*j^*}$ will correspond to one Nash equilibrium.

In this way we determine the elements $b_{i_1 1}, b_{i_2 2}, \ldots, b_{i_j j}, \ldots, b_{i_n n}$, for which $b_{i_k k} \leqslant b_{i_j j}$, $\forall k \in X_2 \backslash \{j\}$.

Then using the indices of the rows of these elements, we can determine the strategy for the first player by

$$i' = (i_1 - 1)\, m^{n-1} + (i_2 - 1)\, m^{n-2} + \ldots + (i_j - 1)\, m^{n-j} + \ldots + (i_n - 1)\, m^0 + 1. \tag{3}$$

So, the pair i', j is the Nash equilibrium for the informational extended game ${}_1\Gamma : (i', j) \in NE\left({}_1\Gamma\right)$.

Similarly, for the game ${}_2\Gamma$, we can determine the strategy for the second player by

$$j' = (j_1 - 1)\, n^{m-1} + (j_2 - 1)\, n^{m-2} + \ldots + (j_i - 1)\, n^{m-i} + \ldots + (j_m - 1)\, n^0 + 1, \tag{4}$$

where the indices j_i $\left(i = \overline{1, m}\right)$ are determined by the indices of columns of the elements $b_{ij_i} = \max\limits_j \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $\forall i = \overline{1, m}$.

Theorem 1. *Assume that for the initial game Γ the next conditions hold:*

1) for each column $j \in X_2$ there exists a single maximum element in the matrix $A : a_{i^*j} = \max\limits_{i \in X_1} a_{ij}$ and the corresponding element from the matrix B is the minimum element from the matrix $b_{i^*j} = \min\limits_{i \in X_1, j \in X_2} b_{ij}$;

2) there exists a single column $j^* \in X_2$ in the matrix A so that for the maximum element from this column $\max\limits_{i \in X_1} a_{ij^*} = a_{i'j^*}$ the corresponding element from the matrix B is greater than the minimum element from this matrix: $b_{i'j^*} > \min\limits_{i \in X_1, j \in X_2} b_{ij}$;

3) for each column from the matrix B the other elements are greater than $b_{i'j^*}$;

Then the informational extended game ${}_1\Gamma$ has a single Nash equilibrium, so $|NE\left({}_1\Gamma\right)| = 1$.

Theorem 2. *Assume that for the initial game Γ the next conditions hold:*

1) for each row $i \in X_1$ there exists a single maximum element in the matrix $B : b_{ij^*} = \max\limits_{j \in X_2} b_{ij}$ and the corresponding element from the matrix A is the minimum element from the matrix $a_{ij^*} = \min\limits_{i \in X_1, j \in X_2} a_{ij}$;

2) there exists a single row $i^* \in X_1$ in the matrix B so that for the maximum element from this row $\max\limits_{j \in X_2} b_{i^*j} = b_{i^*j'}$ the corresponding element from the matrix A is greater than the minimum element from the matrix $A : a_{i^*j'} > \min\limits_{i \in X_1, j \in X_2} a_{ij}$;

3) for each row from the matrix A the other elements are greater than $a_{i^*j'}$;

Then the informational extended game ${}_2\Gamma$ has a single Nash equilibrium, so $|NE\left({}_2\Gamma\right)| = 1$.

Proof. We prove the theorem for the informational extended game ${}_1\Gamma$.

Consider that all conditions hold.

For the proof we use the algorithm for the determination of Nash equilibria and the relation (formula) for the number of Nash equilibria in the informational extended game.

According to this algorithm, in the informational extended game ${}_1\Gamma$ Nash equilibria will exist for the elements $a_{i^*j} = \max\limits_{i \in X_1} a_{ij}$, $j \in X_2$, if for the corresponding element b_{i^*j} in each column from the matrix B there are elements less than b_{i^*j}.

There are two cases:

1) the element a_{i^*j} is from column j^* (according to the second condition from the theorem);

2) the element a_{i^*j} is not from the column j^*.

Consider the first case. So for $j' \in X_2$, $j' \neq j^*$ we consider the element $a_{i^*j'} = \max\limits_{i \in X_1} a_{ij'}$. Then from first condition of theorem it follows that $b_{i^*j'} = \min\limits_{i \in X_1, j \in X_2} b_{ij} = \min\limits_{i \in X_1} b_{ij'}$, so $\min\limits_{i \in X_1} b_{ij^*} > \min\limits_{i \in X_1, j \in X_2} b_{ij}$. Then in the relation (formula) for the number of Nash equilibria in the informational extended game ${}_1\Gamma$ we have the component with index j^* equal to zero. Thus for the pair of elements $a_{i^*j'}$ and $b_{i^*j'}$ we have the number of Nash equilibria in the game ${}_1\Gamma$: $N_{j'} = 1 \cdot 1 \cdot \ldots \cdot 0 \cdot \ldots \cdot 1 = 0$. So for all pairs of elements for which the element a_{i^*j} is not from the column j^* Nash equilibria do not exist in the informational extended game ${}_1\Gamma$.

Consider now the second case. For the column $j' \in X_2$, $j' = j^*$ we consider the element $a_{i^*j'} = \max\limits_{i \in X_1} a_{ij^*} = a_{i^*j^*}$ and the corresponding element $b_{i^*j^*} = \min\limits_{i \in X_1} b_{ij^*} > \min\limits_{i \in X_1, j \in X_2} b_{ij}$ (according to the second condition from the theorem). Then for each column $j \in X_2$ in the matrix B a single minimum element exists, so $n_1 = \ldots = n_{j^*} = \ldots = n_n = 1$ and $N_{j^*} = 1 \cdot 1 \cdot \ldots \cdot 1 \cdot \ldots \cdot 1 = 1$.

Thus for all pairs of elements $\left(a_{i^*j'}, b_{i^*j'}\right)$ we have a single Nash equilibrium: $\sum\limits_{j \in X_2} N_j = 1$ therefore $|NE\left({}_1\Gamma\right)| = 1$. The theorem is proved.

Remark 1. *From the conditions of this theorem it follows:*

1) for the game ${}_1\Gamma$ there is a single minimum element in each column in the matrix B;

2) for the game ${}_2\Gamma$ there is a single minimum element in each row in the matrix A.

We can do some specifications for the third condition from this theorem.

Remark 2. *For the informational extended game ${}_1\Gamma$ the column j^* in the matrix B can contain some elements $b_{i'j^*}$ less than $b_{i^*j^*}$, and greater than $\min\limits_{i \in X_1, j \in X_2} b_{ij}$, only if $a_{i'j^*} \neq \max\limits_{i \in X_1} a_{ij^*}$.*

For the informational extended game ${}_2\Gamma$ the row i^* in the matrix A can contain some elements $a_{i^*j'}$ less than $a_{i^*j^*}$, and greater than $\min\limits_{i \in X_1, j \in X_2} b_{ij}$, only if $b_{i^*j'} \neq \max\limits_{j \in X_2} b_{i^*j}$.

Example 1. *Consider the initial game Γ with the matrices*

$$A = \begin{pmatrix} 0 & \underline{4} & 2 & 3 \\ \underline{5} & 0 & 3 & 2 \\ 3 & 3 & \underline{6} & 0 \\ 1 & 2 & 1 & \underline{\underline{7}} \end{pmatrix}; \qquad B = \begin{pmatrix} \underline{4} & 0 & 2 & 2 \\ 0 & \underline{5} & 3 & 3 \\ 3 & 3 & 0 & \underline{6} \\ 2 & 2 & \underline{\underline{5}} & 1 \end{pmatrix}.$$

For the initial game Γ, Nash equilibria do not exist.

For the elements of these matrices all conditions of theorem hold and for each of the informational extended games ${}_1\Gamma$ and ${}_2\Gamma$ there exists a single Nash equilibrium.

Consider firstly the informational extended game ${}_1\Gamma$. For this game the matrices have the dimension $[256 \times 4]$.

We determine the maximum elements from each column in the matrix A : $a_{21} = 5$, $a_{12} = 4$, $a_{33} = 6$, $a_{44} = 7$.

For each of these elements we examine the pairs of elements from the matrices A and B. Thus for the pairs of elements $(a_{21}, b_{21}) = (5, 0)$, $(a_{12}, b_{12}) = (4, 0)$, $(a_{33}, b_{33}) = (6, 0)$ Nash equilibria do not exist in the informational extended game ${}_1\Gamma$.

For the pair of elements $(a_{44}, b_{44}) = (7, 1)$ in the extended matrix $\widetilde{A}$ there exists a row formed by elements $(a_{21}, a_{12}, a_{33}, a_{44}) = (5, 4, 6, 7)$ while the extended matrix $\widetilde{B}$ will contain the corresponding row formed by the corresponding elements $(0, 0, 0, 1)$; in the extended matrices these rows will have the index $i^* = 76$. For the determination of this number we use the relation (3). For the determination of Nash equilibrium we determine the maximum element from the row i^* in the extended matrix $\widetilde{B}$. Thus the single Nash equilibrium in the informational extended game ${}_1\Gamma$ is $(i^*, j^*) = (76, 4)$, and the payoffs for the players are the corresponding elements $(a_{44}, b_{44}) = (7, 1)$.

Consider now the informational extended game ${}_2\Gamma$. For this game the matrices have the dimension $[4 \times 256]$.

The maximum elements in each row in the matrix B are: $b_{11} = 4$, $b_{22} = 5$, $b_{34} = 6$, $b_{43} = 5$.

Thus for the pairs of elements $(a_{11}, b_{11}) = (0, 4)$, $(a_{22}, b_{22}) = (0, 5)$, $(a_{34}, b_{34}) = (0, 6)$ Nash equilibria do not exist in the informational extended game ${}_2\Gamma$.

For the pair of elements $(a_{43}, b_{43}) = (1, 5)$ in the extended matrix $\overline{B}$ there exists a column formed by the elements $(b_{11}, b_{22}, b_{34}, b_{43}) = (4, 5, 6, 5)$ and the corresponding column in the extended matrix $\overline{A}$ will be formed by the corresponding elements $(0, 0, 0, 1)$; in the extended matrices these columns will have the index $j^* = 31$ (using the relation (4)). Further we determine the maximum elements in the column j^* from the extended matrix $\overline{A}$. So the single Nash equilibrium in the informational extended game ${}_2\Gamma$ is $(i^*, j^*) = (4, 31)$ and the payoffs for the players are the corresponding elements $(a_{43}, b_{43}) = (1, 5)$. $\square$

Example 2. *We examine a game for which Nash equilibria do not exist and the conditions of theorem hold only for the informational extend game* ${}_2\Gamma$,

$$A = \begin{pmatrix} 0 & \underline{4} & \underline{6} \\ \underline{5} & 0 & \underline{6} \\ \underline{5} & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} \underline{7} & 2 & 1 \\ 1 & \underline{9} & 1 \\ 1 & 4 & \underline{\underline{6}} \end{pmatrix}.$$

For the informational extended game ${}_2\Gamma$ there is a single Nash equilibrium which will be determined in the columns from the extended matrices formed by the elements $(b_{11}, b_{22}, b_{33}) = (7, 9, 6)$ and $(a_{11}, a_{22}, a_{33}) = (0, 0, 3)$; and the payoffs for the players are the elements $(a_{33}, b_{33}) = (3, 6)$ respectively.

For the game ${}_1\Gamma$ there will exist four Nash equilibria, because in the matrix A in each of the first and the third columns there are two maximum elements. So in the extended matrix $\widetilde{A}$ will be four rows formed by the maximum elements from the columns of the matrix $A : (5, 4, 6)$ and in the matrix $\widetilde{B}$ there will exist four rows formed by the corresponding elements $(1, 2, 1)$. Thus the Nash equilibria will be determined by the elements $(a_{12}, b_{12}) = (4, 2)$, because $b_{12} = 2$ is the maximum element in the row $(1, 2, 1)$.

Example 3. *Consider a game in which Nash equilibria do not exist and for which all conditions of theorem hold for both informational extended games* ${}_1\Gamma$ *and* ${}_2\Gamma$, *so for each of these informational extended games exists a single Nash equilibrium*

$$A = \begin{pmatrix} 1 & \underline{\underline{4}} & 2 \\ \underline{5} & 0 & 3 \\ 0 & 3 & \underline{6} \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 & \underline{\underline{5}} \\ 0 & \underline{5} & 3 \\ \underline{3} & 1 & 0 \end{pmatrix}.$$

For informational extended game ${}_1\Gamma$ the Nash equilibrium will be determined by the pair of elements $(a_{12}, b_{12}) = (4, 2)$; and for the game ${}_2\Gamma$ the Nash equilibrium will be determined by the pair of elements $(a_{13}, b_{13}) = (2, 5)$.

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