ROMAI J., **4**, 1(2008), 261–270

PSEUDO QUATERNION REPRESENTATIONS IN THE THEORY OF MAPPINGS AND THEIR APPLICATIONS

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Abstract The work deals with setting a correspondence between the main directions, main curvatures and theorema egregium of a surface with imaginary units, coefficients and discriminants of quadratic form, Beltrami equations system. Their applications in describing birth and uptake operators of bosons and fermions are investigated. In transonic gas dynamics it is established the connection between the Mach number and the imaginary units. Pseudoquaternions' matrix representation is investigated. New representation of Beltrami equations system solutions enable us to show that the Riemann theorem on mappings is valid on hyperbolical planes. The issue on the Poincaré's problem solution variant is studied too.

Keywords: systems of equations, fermion, commutative algebra, imaginary units, quaternion.

2000 MSC: 53Z05.

1. INTRODUCTION

The traditional representation of complex numbers is not adequate for the following reasons:

1) according to the Lagrange theorem, the direct sum of a quadratic functional in one-dimensional spaces reads $|z|^2 = x^2 - y^2$ instead of $|z|^2 = x^2 + y^2$. Consequently, there is at least one representation of complex numbers satisfying the Lagrange theorem [1];

2) the ellipticity of a point is connected with the theorema egregium, main surface curvatures. In the representation $z = x + iy$, x, y are the projections of z on the corresponding coordinate axis. The imaginary unit i is

connected with the theorema egregium, demonstrated further. An essential question arises: if the complex numbers represent the points of the surface of elliptic type, then what kind of numbers express the points of the surfaces of hyperbolic and parabolic types? Thus, an essential problem arises of deducing these numbers from equations system of elliptic type, i.e. from the Cauchy-Riemann and Beltrami equations system.

In order to do this, one must represent these equations in the operator form [2]. Factorization of the quadratic form is an example of the necessity of these numbers. These circumstances make us research the new representation of these numbers. Besides, in the following it is demonstrated that by means of these numbers the commutation relations for the birth and uptake operators of bosons can be deduced. In this respect, a fragment from the article of Dirac "Quantum electrodynamics" [3] is illuminating. "We discovered the mathematical description of electromagnetic field in the terms of decreasing and increasing of field components' excitation level by one quant. These operators might also be described as emission and uptake operators of a boson. All η are the birth operators which increase the excitation level by a unit, and all $\overline{\eta}$ represent the uptake operators which decrease the excitation level by a unit: η, $\overline{\eta}$ are the birth and uptake operators correspondingly. There are a pair of variables η^a and $\overline{\eta}^a$ for all boson states. Commutation relations for them include the following: 1) variables, corresponding to different boson states commute each other, i.e. commute all birth operators

$$\eta^a\eta^b - \eta^b\eta^a = 0, \tag{1}$$

and all uptake operators

$$\overline{\eta}^a\overline{\eta}^b - \overline{\eta}^b\overline{\eta}^a = 0; \tag{2}$$

2) the expression $\overline{\eta}^a\eta^b - \eta^b\overline{\eta}^a$ converts to zero when a and b are different and equal to zero when a is equal to b

$$\overline{\eta}^a\eta^b - \eta^b\overline{\eta}^a = \delta^{ab}, \tag{3}$$

where δ^{ab} – Kronecker's symbol.

If we deal with the electromagnetic field or any boson ensemble, we need the variables η and $\overline{\eta}$ for the description of quantum mechanical system which satisfy the abovestated commutation relations.

It is also possible to introduce the operators η and $\overline{\eta}$ for fermions which describe birth and uptake of the fermions as it was with bosons. Now any separate operator η and $\overline{\eta}$ correspond to the fermion state. Consequently, the operators η and $\overline{\eta}$ satisfy commutation relations which are different from the bosons' commutation relations

$$\eta^a\eta^b + \eta^b\eta^a = 0, \tag{4}$$

$$\overline{\eta}^a\overline{\eta}^b + \overline{\eta}^b\overline{\eta}^a = 0, \tag{5}$$

$$\overline{\eta}^a\eta^b + \eta^b\overline{\eta}^a = \delta^{ab}. \tag{6}$$

The equations (4)–(6) are the same to (1)–(3) correspondingly, only the sign minus is replaced with the sign plus. I consider it to be a very amazing mathematical fact. It is not certain what is hidden behind in fact, because we deal with two completely different physical situations. The equations (1)–(3) belong to particles, any number of which may be of any state. The equations (4)–(6) belong to the particles two of which can never be of the same state. Physically these two situations strongly differ, however, despite it, there is a close parallel between the equations corresponding to them".

2. BELTRAMI EQUATIONS

The following lemma is famous from algebra [1]: any commutative algebra with dividing possesses dimensionality of no more than 2. Thus, this lemma is one of the main reasons for the inapplicability of commutation relations in quantum mechanics while algebra with dividing is used. Consequently, the birth and uptake of bosons take place in a two-dimensional space, while fermions in a space of higher dimension.

If on the two-dimensional Riemann manifold V_2 the element of arc length is defined as

$$ds^2 = \gamma dx^2 - 2\beta dxdy + ady^2, \quad \alpha\gamma - \beta^2 = 1, \tag{7}$$

where x, y are curvilinear coordinates and $\gamma,\ \beta,\ \alpha$ are functions of variables, then there are the functions $u = u(x, y)$, $v = v(x, y)$ which map any neighborhood of the point onto the domain of the Euclidean plane in such a way that

$$\gamma dx^2 - 2\beta dxdy + \alpha dy^2 = \sigma(u, v)(du^2 + dv^2),$$

where $\sigma(u, v) = \frac{\partial(x,y)}{\partial(u,v)} > 0$ is the Jacobian mapping. In this case, the functions satisfy Beltrami equations system taking into consideration the surface curvature

$$u_x = \beta v_x + \gamma v_y,\ -u_y = \alpha v_x + \gamma v_y,\ \alpha\gamma - \beta^2 = 1.$$

Further we consider smooth surfaces of more general form with the quadratic form (7) but without the ellipticity condition $\alpha\gamma - \beta^2 = 1$. Consequently, the quadratic form of the surface (7) is indefinite.

Remind the Euler theorem. *In each point of smooth surface there exist two perpendicular tangents l_1, l_2 in the direction of which a normal surface curvature assumes its maximal and minimal values $k_1,\ k_2$. If l is an arbitrary tangent forming the angle θ with the line l_1, then the normal curvature in the direction of l is $k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$.*

Let $K = k_1 k_2$ be the theorem egregium of the surface. Let's consider the Beltrami first order partial differential equations system of mixed type on the surface [4]

$$\beta u_x + \gamma u_y = -(k_1 k_2) v_x = -K v_x,\ \beta v_x + \gamma v_y = u_y. \tag{8}$$

This system is equivalent to the following system of operator equations

$$\begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2 - K}{\gamma} & -\beta \end{pmatrix} \begin{pmatrix} u_x + \sqrt{-K} v_x \\ u_y + \sqrt{-K} v_y \end{pmatrix} = \sqrt{-K} \begin{pmatrix} u_x + \sqrt{-K} v_x \\ u_y + \sqrt{-K} v_y \end{pmatrix}. \tag{9}$$

Let us introduce the following imaginary units $\overrightarrow{i_e}, \overrightarrow{i_h}$ and $\overrightarrow{i_p}$.

i) Obviously, $\alpha\gamma - \beta^2 = K = -|K| i_e^2$, where $\overrightarrow{i_e}$ $(i_e^2 = -1)$ is the elliptic imaginary unit, where $i_e \neq \sqrt{-1}$ if $K > 0$, i.e. if the surface is of elliptic type;

ii) $\alpha\gamma - \beta^2 = K = -Ki_h^2$, where $\overrightarrow{i_h}$, $(i_h^2 = 1)$ is the hyperbolic imaginary unit, where $i_h = \pm 1$ if $K < 0$, i.e. if the surface is of hyperbolic type;

iii) $\alpha\gamma - \beta^2 = K = -Ki_p^2$ where $\overrightarrow{i_p}$, $(i_p^2 = 0)$ is the hyperbolic imaginary unit, where $i_p \neq 0$ if $K = 0$, i.e. if the surface is of parabolic type.

Generalizing the Lavrentev characteristic method for the mixed Beltrami equations system (equations system on smooth surfaces) we obtain

$$\gamma = k_1 \cos^2\theta + k_2 \sin^2\theta,\ \beta = (k_1 - k_2)\cos\theta\sin\theta,\ \alpha = k_1 \sin^2\theta\theta + k_2\cos^2\theta,$$

whence $\alpha\gamma - \beta^2 = k_1 k_2 = K$.

This allows one to make a geometrical interpretation of the coefficients of the system (7) and directions of coordinate axes (x, y). Thus, it is easy to establish the dependence of k_1, k_2 and θ on the coefficients of the system (8)

$$\tan\theta = \frac{2\beta}{\gamma - \alpha},\ k_{1,2} = \frac{\gamma + \alpha \pm \sqrt{(\gamma - \alpha)^2 + 4\beta^2}}{2}. \tag{10}$$

As the main curvatures as well as the theorem egregium of the surface are geometric objects, they do not depend on the introduced by us coordinates. Let us introduce the hyperbolic system of coordinates. Supposing

$$\gamma = k_1 \cosh^2\vartheta + k_2 \sinh^2\vartheta,\ \beta = (k_1 + k_2)\cosh\vartheta\sinh\vartheta,\ \alpha = k_1 \sinh^2\vartheta + k_2\cosh^2\vartheta, \tag{11}$$

we shall receive $\alpha\gamma - \beta^2 = k_1 k_2$. The expressions of k_1, k_2 and ϑ in terms of α, β, γ read

$$\tanh 2\vartheta = \frac{2\beta}{\alpha + \gamma},\ k_{1,2} = \frac{\gamma - \alpha \pm \sqrt{(\gamma + \alpha)^2 - 4\beta^2}}{2}. \tag{12}$$

Consequently the Beltrami equations system of elliptical type arises when we reduce first quadratic form of the surface to the canonical form while the mixed Beltrami equations system arises when the equation of Dupin's indicatrixes are reduced to the canonical form.

The analogue of the Euler's formula, valid for hyperbolic trigonometric functions reads $e^{i_h\vartheta} = \cosh\vartheta + i_h \sinh\vartheta$. All famous interrelations of hyperbolical trigonometry can be studied by using this formula. Thus, we obtain the result

Theorem. *Riemannian metrics of a surface which satisfies the mixed Beltrami equations system defines main directions and main curvatures of the surface uniquely.*

Thus, the homeomorphism of Beltrami equations system forms a network of curves on the surface, i.e. the system of two families of curves, called surface curvature lines, each of which covers the surface pieces completely.

3. APPLICATION TO GAS DYNAMICS

Let us apply of the representations (2) and (10).

Let $u - i_e v = qe^{i\vartheta}$ be the complex potential of a flow. The equation of transonic gas dynamics reads

$$\begin{cases} \theta_x = -\frac{\mu^2}{q}\sin\theta\cos\theta q_x + \frac{1}{q}(1-\mu^2\sin^2\theta)q_y \equiv \beta q_x + \gamma q_y, \\ \theta_y = \frac{1}{q}(1-\mu^2\cos^2\theta)q_x - \frac{\mu^2}{q}\sin\theta\cos\theta q_y \equiv \alpha q_x + \beta q_y, \end{cases} \tag{13}$$

where μ is the Mach number. Obviously, $\alpha\gamma - \beta^2 = \frac{1-\mu^2}{q^2}$. Putting in (10) the coefficients from (13) we obtain: i) θ, which act in (13), are similar; ii) $\alpha + \gamma = \frac{1}{q}(2-\mu^2)$, $\alpha - \gamma = \frac{\mu^2}{q}\cos 2\theta$; iii) $k_1 = \frac{1}{q}$, $k_2 = \frac{1-\mu^2}{q}$; iiii) $k_1 k_2 = \frac{1-\mu^2}{q^2} = K$, whence it is clear that:

i) if the flow is subsonic, i.e. $\mu < 1$, then $K > 0$ therefore the stream surface is of elliptical type;

ii) if the flow is transonic, i.e. $\mu = 1$, then $K = 0$ therefore the stream surface is of parabolic type;

iii) if the flow is supersonic, i.e. $\mu > 1$, then $K < 0$, therefore the stream surface is of hyperbolical type.

It is essential that mixed Beltrami equations system allows us to compute uniquely: i) values of main curvatures; ii) theorema egregium; iii) to define uniquely main directions of the surface flow in terms of the characteristics of the complex potential flow.

4. PSEUDOQUATERNIONS MATRIX REPRESENTATION

The following matrices will be put in correspondence to the quaternions

$$\begin{cases} \vec{i_e} \to \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2+1}{\gamma} & -\beta \end{pmatrix} = A_e, \\ \vec{j_e} \to i_e \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2-1}{\gamma} & -\beta \end{pmatrix} = B_e, \\ \vec{k_e} \to i_e \begin{pmatrix} 1 & 0 \\ -\frac{2\beta}{\gamma} & -1 \end{pmatrix} = C_e, \end{cases} \tag{14}$$

The following matrices will be put in correspondence to the pseudoquaternions

$$\begin{cases} \vec{i_k} \to i_e \begin{pmatrix} \beta & \gamma \\ \frac{\beta^2+1}{\gamma} & -\beta \end{pmatrix} = A_h, \\ \vec{j_k} \to i_h \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2-1}{\gamma} & -\beta \end{pmatrix} = B_h, \\ \vec{k_h} \to i_e i_h \begin{pmatrix} 1 & 0 \\ -\frac{2\beta}{\gamma} & -1 \end{pmatrix} = C_h, \end{cases} \tag{15}$$

Direct computations lead to the following properties of the matrices (14)

$$A_e^2 = -E,\ B_e^2 = -E,\ C_e^2 = -E,\ A_eB_e + B_eA_e = 0,$$

$$B_eC_e + C_eB_e = 0, \quad C_eA_e + A_eC_e = 0.$$

Correspondingly, we have

$$\vec{i_e}\vec{i_e} = \vec{i_e}^2 = -1, \quad \vec{j_e}\vec{j_e} = \vec{j_e}^2 = 1, \quad \vec{k_e}\vec{k_e} = \vec{k_e}^2 = -1,$$

$$\vec{i_e}\vec{j_e} + \vec{j_e}\vec{i_e} = 0, \quad \vec{j_e}\vec{k_e} + \vec{j_e}\vec{k_e} = 0, \quad \vec{k_e}\vec{i_e} + \vec{i_e}\vec{k_e} = 0.$$

The commutation properties of the matrices are defined exactly (15)

$$A_h^2 = E,\ B_h^2 = E,\ C_h^2 = E,\ A_hB_h - B_hA_h = 0,$$

$$B_hC_h - C_hB_h = 0, \quad C_hA_h - A_hC_h = 0.$$

Correspondingly, we have

$$\vec{i_h}\vec{i_h} = \vec{i_h}^2 = 1, \quad \vec{j_h}\vec{j_h} = \vec{j_h}^2 = 1, \quad \vec{k_h}\vec{k_h} = \vec{k_h}^2 = 1,$$

$$\vec{i_h}\vec{j_h} - \vec{j_h}\vec{i_h} = 0, \quad \vec{j_h}\vec{k_h} - \vec{j_h}\vec{k_h} = 0, \quad \vec{k_h}\vec{i_h} - \vec{i_h}\vec{k_h} = 0. \tag{16}$$

Now we introduce the operators conjugate to the mentioned operators as the operators corresponding to the conjugate Beltrami equations system.

$$\bar{A}_e = -A_e, \quad \bar{B}_e = -B_e, \quad \bar{C}_e = -C_e, \bar{A}_h = -A_h, \quad \bar{B}_h = -B_h, \quad \bar{C}_h = -C_h.$$

Obviously, all operators with the index e satisfy (4), (5) and are considered as the birth and uptake operators of fermions, while all operators with the index h satisfy the relations (1), (2) and are considered as the birth and uptake operators of bosons. At the same time, these operators are the permutation operators of fermions and bosons respectively. Therefore, the relations (3) and (6) are replaced by squaring the operators with the indexes e, h.

5. THE POINCARÉ PROBLEM AND THE RIEMANN THEOREM

These results allow us to solve the Poincaré problem (conjecture) by using the Seifert fibration [6]. The domain D serves as the fibration base on the manifold, while all bounded domains which are the images of D through quasi-conformal mappings corresponding to the elliptical type of Beltrami equations system serve as the fibration space. Jacobians (consequently, theorema egregium of M) of these mappings define the fibration layers. The group of the conformal mappings serve as the structural group of layer transformations. Torus-torus in the fibration of Seifert arise as a sequence of the action of the operator B, while the Klein full bottle is the sequence of the action of the operator A. Torus-torus and the Klein full bottle arise because the elements of the considered by us operators-matrices are functions, namely the coefficients of the Beltrami equations system defined in D. The direct product $S^1 \times D$ where S^1 is the one-dimensional sphere, serves as the trivial puff torus while the puff Klein bottle is the direct sum in the representation of the Beltrami equations system [4]

$$w = u + i_e v + i_h(p - i_e q) = W_+ \oplus W_-, \tag{17}$$

which is covered twice by the trivial torus-torus. With the renormalization of (17) one can obtain the direct sum for the representation of the puff torus-torus $w = u + i_h p + i_e(v + i_h q)$, which is finite-to-one covered by the trivial torus-torus.

By the Dupin theorem [7], the surfaces of triple orthogonal system are always pairwise intersected by the curvature lines. The curvature lines are considered as integral curves of the equations

$$dx = 0, \quad dy = 0, \quad \gamma dx - \beta dy = 0.$$

Then, $u = M(\gamma dx - \beta dy)$, $v = N dy$, $d\varphi = N dx$, where $M = M(y)$ is the integration multiplier [8] of the forms $\psi_1 = \gamma dx - \beta dy$, $\psi_2 = dy$, $\psi_3 = dx$. Then, $M^2 \left[(\psi_1)^2 + (\psi_2)^2 + (\psi_3)^2\right] = du^2 + dv^2 + d\psi^2$, that is, the conformal mapping by the Gauss in a three-dimensional Euclidean space. Consequently, two arbitrary surfaces which possess the representation (17) must have common curvature lines. The following surface family serves as the mentioned triple of surface families

$$\omega_1 = X i_e + Y j_h + Z i_e j_h, \quad \omega_2 = X i_h + Y i_e i_h + Z i_e, \quad \omega_3 = X i_e i_h + Y i_e + Z i_h.$$

Obviously, $\omega_1^2 + \omega_2^2 + \omega_3^2 = X^2 + Y^2 + Z^2$ and $X = X(x, y)$, $Y = Y(x, y)$, $Z = Z(x, y)$, where $(x, y) \in D$.

From (2) - (11) and (17) on may conclude that the substitution of the imaginary unit i_e by the unit i_h in the Beltrami equations system gives us the system of a hyperbolic type. Here, the bounded domains D and Δ, between which the mappings corresponding to the Beltrami equations system of elliptic type are considered as the geometrical objects in affine spaces, remain unchanged. In the same way, it can be proved the Riemann theorem on mappings for the hyperbolic Beltrami equations system.

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