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ON THE GENERALIZED HALLEY METHOD FOR SOLVING NONLINEAR EQUATIONS

Grigore Albeanu

"Spiru Haret" University, Bucharest

galbeanu@gmail.com

Abstract Halley's method is a famous iteration method for solving nonlinear equations $F(X) = 0$. Some Kantorovich-like theorems have been given, including extensions for general spaces. Quasi-Halley methods were proposed too. This paper uses the generalized inverse approach in order to obtain a robust generalized Halley method.

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1. INTRODUCTION

A famous iteration was presented by Halley [11] in 1694. A large number of papers have been written in time and some interesting extensions were proposed, not only for scalar case but also for multivariate case, for nonlinear systems of equations, even for operator equations in Banach spaces. If f is a function $f : R \to R$ and x^* is a root of f, that is $f(x^*) = 0$, a common approach for obtaining x^* is to use the Newton iteration (also called *the tangent method*):

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{1}$$

for $k = 0, 1, ...$, generating a sequence of iterations which converges to x^* under some condition. It was shown that the order of convergence for the Newton method is 2.

The Halley's method has a cubic order of convergence. It is known also as *the method of tangent hyperbolas*. The Halley clasic iteration is given by [6, 15, 19, 20]

$$x_{k+1} = x_k - \frac{f(x_k)/f'(x_k)}{1 - \frac{1}{2}\frac{f(x_k)f''(x_k)}{f'(x_k)^2}}, \quad k = 0, 1, ... \tag{2}$$

In this paper we consider the case of nonlinear systems of equations and we extend the Halley method in order to use generalized inverse of the Jacobian matrix. The next section is dedicated to the general basic iterative relation. Algorithmic aspects will be discussed in the third section. The concluding section provides remarks and future possible developments.

2. THE GENERALIZED HALLEY METHOD

Let $F : R^n \to R^n$ generate a system of n equations in n unknowns

$$F(x) = 0, \tag{3}$$

where F is sufficiently smooth and the Jacobian $F'(x^*)$ is nonsingular if x^* is a solution of the system. Consider the Halley class of iterations for solving (1)

$$x_{k+1} = x_k - \left\{ I + \frac{1}{2} L_F(x_k)\,[I - \alpha L_F(x_k)]^{-1} \right\} (F'(x_k))^{-1} F(x_k), k = 0, 1, 2, ..., \tag{4}$$

where $L_F(x) = (F'(x))^{-1}F''(x)(F'(x))^{-1}F(x)$ is the degree of logarithmic convexity [12], and α is a real parameter.

For $\alpha = 0$, the classical Chebyshev' s method is obtained

$$x_{k+1} = x_k - \left\{ I + \frac{1}{2} L(x_k) \right\} F'(x_k)^{-1} F(x). \tag{5}$$

When $\alpha = \frac{1}{2}$, we obtain the classical Halley's method. For real-valued function, the iteration is written as in (2). For $\alpha = 1$ is obtained the super-Halley method. It is easy to note that the Newton method is obtained for $\alpha \to -\infty$.

For real valued functions, it was shown that any iterative process given by the expression

$$x_{k+1} = x_k - H(L_F(x_k))(F'(x_k))^{-1}F(x_k), \tag{6}$$

where H is such that $H(0) = 1$, $H'(0) = 1/2$, and $|H''(0)| < \infty$, generates a third order convergence method [9, 19]. An interesting case appears when H is a sum of operators as in the equation

$$H(L) = I + \frac{1}{2}L + \sum_{i=2}^{\infty} A_i L^i, \tag{7}$$

where I is the identity operator, L is the degree of convexity for some function, and $A_i (i = 2, ...,)$ are real constants.

Algorithmic aspects for increasing the confidence in numerical computing of the solution of a system of equations will be detailed in the next section using the two-step iteration model.

3. ALGORITHMIC ASPECTS

In order to implement the computer iteration (2), let us rewrite the equation in the following way (where x_0 is given as a point belonging to a small neighborhood of x^*):

1. solve for u_k: $F'(x_k)u_k = -F(x_k)$;
2. solve for v_k: $(F'(x_k) + \alpha F''(x_k)u_k)v_k = -\frac{1}{2}F''(x_k)u_k u_k$;
3. compute $x_{k+1} = x_k + u_k + v_k$.

In order to have a robust algorithm, without significantly decreasing of the convergence speed, we propose to use Penrose generalized inverse of matrices.

If A is a general real matrix the pseudoinverse $A^\dagger$ of A is defined as the unique matrix satisfying all of the following conditions [5, 16]:

1 $AA^\dagger A = A$;

2 $A^\dagger AA^\dagger = A^\dagger$;

3 $(AA^\dagger)^T = AA^\dagger$;

4 $(A^\dagger A)^T = A^\dagger A$.

. The new iterative process can be described by:

1 compute $u_k = -(F'(x_k))^\dagger F(x_k)$;

2 compute $v_k = -\frac{1}{2}(F'(x_k) + \alpha F''(x_k)u_k)^\dagger F''(x_k)u_k u_k$;

3 compute $x_{k+1} = x_k + u_k + v_k$, for $k = 0, 1, ...$

Such an approach is motivated by the requirement to increase the method reliability while maintaining a degree of convergence greater than 2, even less than 3, when the matrix under inversion did not have a good behaviour. The convergence of the method is guaranteed by theoretical results based on the fact that (2) is an approximation to the Newton equation

$$F'(x_k + u_k)v_k = -F(x_k + u_k), \tag{8}$$

since

$$F(x_k + u_k) \approx F(x_k) + F'(x_k)u_k + \frac{1}{2}F''(x_k)u_k u_k = \frac{1}{2}F''(x_k)u_k u_k,$$

and

$$F'(x_k + u_k) \approx F'(x_k) + \alpha F''(x_k)u_k.$$

From the computational point of view we need second-derivative-free iterations as in the study [14]. In our case $n \geq 1$, and finite difference approximations for the Hessian are required. There are some choices when computing derivatives. The first one consists of establishing a small step $h > 0$ and making differences related to x_k and $x_k \pm h$. The second one can use x_0 as fixed reference point for doing derivatives just like in the univariate case $(f(x_k) - f(x_0))/(x_k - x_0)$. In this case a simplified version like in [2] is used. However, a multipoint approach can be used based on the scheme $(f'(x_k) - f'(x_{k-1}))/(x_k - x_{k-1})$ written for the univariate function f.

Computing generalized inverse is not a difficult task. Some powerful algorithms were already proposed [4, 7, 10, 13, 18] and implementation techniques based on different numerical schemes are available.

The complexity of the generalized Halley method is larger than for the classical iteration, but it is motivated by reliability increasing and robustness of the procedures to be implemented in programming languages.

4. CONCLUSIONS

The Halley's method continues to be an important subject of investigation. In our study we extend the standard iteration in order to obtain robust algorithms based on pseudoinverse computing. There are some options for obtaining second-derivative-free iterations, and it is possible to think about an updating scheme for the Jacobian matrix as in the case of Broyden type methods [1].

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