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# LYAPUNOV STABILITY ANALYSIS IN TAYLOR-DEAN SYSTEMS

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**Abstract** The Taylor-Dean viscous fluid flow between two rotating cylinders is a combination of a circular Couette flow and azimuthal Poiseuille flow. Its linear stability was investigated analytically and numerically, among others, in the case when the size of the gap between the two cylinders was taken into account [4], [21], [19]. When this parameter becomes important, the existence of a large variety of patterns bifurcated from the Taylor-Dean flow depends on the strata determined in the parameter space. The main interest in most of these studies is for the critical instability conditions. In this paper, the eigenvalue problem governing the Lyapunov stability of the basic Taylor-Dean flow against rotationally symmetric perturbations, previously investigated in [14] by using isoperimetric inequalities, is studied along with some other examples from [24], by means of spectral methods based on Legendre polynomials [12]. In each case the critical Taylor number at which the instability sets in is obtained. All our numerical results agree with those existing in the literature.
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**Keywords:** Taylor-Dean system, spectral methods, Taylor number, eigenvalue problem.

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## 1. STATEMENT OF THE PROBLEM

The simplest example of a steady-circular flow of a viscous fluid between two-rotating cylinders is the Taylor-Couette flow for which the laminar basic state is the circular Couette flow. In the absence of viscosity, the first criterion of stability was given by Rayleigh [23], i.e., a stratification of angular momentum about an axis is stable if and only if increases monotonically outward. For the case of a viscous Couette flow, theoretical and experimental investigations were performed for the first time by G.I. Taylor (1923) [26] for the case of a small gap $d$ ($d = R_2 - R_1$) between the rotating cylinders compared to the mean radius ($(R_1 + R_2)/2$). The analytical expression of its criterion is given in terms of the Taylor number, a dimensionless parameter that characterizes the importance of the centrifugal forces due to rotation of fluid about vertical axis, relative to viscous forces. In the case when the inner cylinder rotates and the outer cylinders is stationary, he found [26] that the flow becomes unstable when the parameter $Re(d/R_1)^{1/2}$ exceeds the value of about 41, where the spacing between the cylinders is small compared to the inner radius $R_1$ and the disturbances were assumed axisymmetric. Here $Re$ represents the Reynolds number. Chandrasekhar [4] also investigated the eigenvalue problem arising in the Lyapunov stability study and obtained expressions for the secular equations leading to the neutral manifolds that separate the stability and instability domains.

Another motion of great interest from the engineering point of view is the Taylor-Couette flow of a viscous fluid in a fully filled annulus of concentric rotating cylinders. Applications from bearing lubrification and viscometry need analytical and numerical results on this motion [1]. Practical applications of the system from [1] exist in the textile industry or paper fabrication.

When a viscous fluid flows in a curved channel under a pressure gradient acting round the cylinders a pumping velocity distribution occur. This type of instability was first considered by Dean (1928)[7], also in the case of a thin annulus compared to the mean radius. Further analytical and numerical investigations of the resulting eigenvalue problems were given by DiPrima [10], Reid [24], Hammerlin [16], to quote but a few of the first works.

In the following we present some results from previous works concerning the Taylor-Dean system, pointing out in each case that the main interest was on the influence of the various fields or of the physical parameters (the gap space for instance) on the stability of the system or on the numerical evaluations of the critical values of the Taylor number.

For a viscous flow between two rotating cylinders in which the basic motion is a combination of a circular Couette flow and an azimuthal Poisseuille flow the system becomes a Taylor-Dean one. Experimental investigations of the Taylor-Dean problem in the case of a small gap between the cylinders compared to the mean radius were first conducted by Brewster and Nissan [2] and then also theoretically and experimentally by Brewster, Grosberg and Nissan in [3]. For the case when the outer cylinder is at rest the analytical investigation of the problem was performed by DiPrima [10] using Fourier series expansions in order to obtain critical values of the Taylor number. In this paper the theoretical analysis of this problem [10] is performed using spectral methods based on Legendre and Chebyshev polynomials expansions and the numerical results are compared to the ones from [10], proving to be similar.

Linear stability analysis were conducted for the arbitrary gap spacing case also. At first, Chen and Chang [5] conducted a complete analysis for the onset of secondary motion of Taylor-Dean flow in between two infinitely long rotating cylinders in the small gap approximation theory. In [6] Chen found, for the arbitrary gap spacing case, that the most stable state correspond either to the case when the instability is about to change from a nonaxisymmetric mode into a symmetric mode, as the pumping velocity is increasing, or when a nonaxisymmetric mode changes both its azimuthal wavelength and its direction of travelling. The numerical evaluations of the two-point eigenvalue problem was based on a shooting technique together with a unit-disturbance method [6].

In [21] the linear and weakly nonlinear analysis of the Taylor-Dean system which consists of the flow between two rotating cylinders in the horizontal position is presented. The numerical computations of the critical Taylor number, wavenumbers and wave speeds for the primary transitions are given at first for a finite gap system and then compared with results obtained in the small gap approximation. The weakly nonlinear analysis of the transition towards the spirals or the stationary rolls shows that this transition is always supercritical if the Poisseuille-Couette flow is induced by a partial filling. When an external pumping is present, however, a supercritical bifurcation for specific values of the rotation ratio is induced leading to the conclusion that these two configurations are not always equivalent either for linear or nonlinear analysis.

In [22] the flow of a Bingham fluid between two rotating cylinders with a pressure gradient in the tangential direction is considered. A nonlinear stability analysis of the Taylor-Couette flow, along with a theoretical study of the Taylor-Dean configuration, are performed. The critical parameters characterizing the occurrence of the instability in the Taylor-Dean configuration for a Bingham fluid were computed.

Hills and Bassom [18], [19] were concerned with a large wavenumber perturbations in small-gap Taylor-Dean flow. They proved that a consistent solution of the governing equations in the main body of the fluid can be constructed by a WKB analysis which links the Taylor and Dean instabilities at exponentially small orders. Although their studies concerned high wavenumbers, the observed asymptotic behaviours could be use to interpret the neutral curves behaviours obtained by Kachoyan [20].

In [8] a linear stability analysis of a viscous flow driven by a constant azimuthal pressure gradient between two horizontal radial temperature concentric porous cylinders when a radial temperature gradient is given. The influence of the temperature field is pointed out for both outflow and inflow cases.

A modified Taylor-Dean system is considered in [13]. The flow of a viscous fluid take place between a inner cylinder, which is rotating about its fixed axis and, in general, the outer cylinder is fixed and noncircular. The gap space is assumed to be small compared to the radius of the inner cylinder. The streamwise growth of a steady disturbance is calculated with a given method that involves considering a steady-state small disturbance of the base flow. Stability characteristic and critical values of the Taylor number are obtained. It is proven that the more the gap width increases in the direction of the flux of the basic flow the more unstable the flow becomes, and conversely when the gap width correspondingly decreases.

The Taylor-Dean flow configurations when the annulus is only partially filled have multiple engineering applications such as rotating drum filter, in electrogalvanizing line in the steel making industry which uses a roller-type cell to plate zinc onto the surface of a steel strip. It has been shown that wall curvature can have a significant effect on the performance of film cooling over turbine blades [1].

## 2. SPECTRAL METHODS

The general Taylor-Dean stability problem leads to the following two-point boundary value problem [10]

$$\begin{cases} (D^2 - a^2)^2 U = F(x)V, \\ (D^2 - a^2)V = -a^2 T G(x) U \end{cases} \tag{1}$$

and the boundary conditions

$$U = DU = V = 0 \text{ at } x = 0, 1, \tag{2}$$

where $D$ denote the differentiation with respect to the variable $x$, i.e. $D = \dfrac{d}{dx}$, $F(x)$ and $G(x)$ are known continuous, indefinitely derivable functions, $a$ is the wavenumber and $T$ is the Taylor number. The unknown functions $U$ and $V$ stand for the amplitude of the perturbation fields of the radial and azimuthal velocity, respectively. In (1)-(2) the vector $(U, V)$ represents the eigenvector and $T$ is the corresponding eigenvalue. The classical Taylor stability problem is of the form (1) - (2) with $g(x) = 1$, $f(x) = 1 - (1 - \mu)x$, $\mu < 1$, $\mu = \dfrac{\Omega_2}{\Omega_1}$ with $\Omega_1$ and $\Omega_2$ the angular velocities of rotation about the axis of the inner and the outer cylinders. In the general case, the basic mathematical problem is: *given $F(x)$ and $G(x)$, determine the smallest value of $T$ for $a > 0$ such that a solution of (1) - (2) exists.* We are interested in the critical value of the Taylor number $T$ for positive values of the wavenumber at which instability sets in.

The main reason for the use of spectral methods is their exponential accuracy. Large classes of eigenvalue problems can be solved numerically using spectral methods, where, typically, the various unknown fields are expanded upon sets of orthogonal polynomials or functions. The convergence of such methods is in most cases easy to assure and they are efficient, accurate and fast. Our numerical study is performed using a weighted residual (Galerkin type) spectral method.

One of the most important characteristic of the Galerkin projection is that the boundary conditions of the problem are treated implicitly by building them into the base functions. In these conditions, the basis functions must be chosen such that each of the functions satisfies the boundary conditions which imply that they are satisfied by a linear combination of such functions also in the case of the linear stability problems from hydrodynamic stability characterized by linear and homogeneous boundary conditions.

Following [17], let us consider the complete orthogonal sets of functions

$$\{\phi_i\}_{i=1,2,...,n} \in L^2(0,1) : \phi_i(x) = \int_0^x Q_i(t)dt,$$

with $\phi_i$, $i = 1, 2, ..., n$ satisfying boundary conditions of the type $\phi_i(0) = \phi_i(1) = 0$ and

$$\{\beta_i\}_{i=1,2,...,n} \in L^2(0,1) : \beta_i(x) = \int_0^x \int_0^s Q_i(t)dtds,$$

with $\beta_i$, $i = 1, 2, ..., n$ satisfying boundary conditions of the type $\beta_i(0) = \beta_i(1) = \beta_i'(0) = \beta_i'(1) = 0$.

we denoted by $Q$ the classical Legendre polynomials defined on $(-1, 1)$.

Let us write the unknown functions $U$ and $V$ as series of the form

$$U = \sum_{i=1}^{n} U_i \beta_i(x), \quad V = \sum_{i=1}^{n} V_i \phi_i(x). \tag{3}$$

In this way, the functions $U$ and $V$ satisfy all boundary conditions imposed on them since each of the expansion function satisfies automatically the boundary conditions.

Replacing the expressions (3) in the system of ordinary differential equations that define the eigenvalue problem (1), we obtain the following algebraic system in the expansion functions

$$\begin{cases} \sum\limits_{i=1}^{n} U_i(D^2 - a^2)^2 \beta_i - \sum\limits_{i=1}^{n} V_i F(x) \phi_i = 0, \\ \sum\limits_{i=1}^{n} V_i(D^2 - a^2)\phi_i + a^2 T \sum\limits_{i=1}^{n} U_i G(x) \beta_i = 0. \end{cases} \tag{4}$$

Following [17], we impose the condition that the equations in (4) be orthogonal on the vector $(\beta_k, \phi_k)$, $k = 1, 2, ..., n$. We get

$$\begin{cases} \sum_{i=1}^{n} [U_i\Big((D^2 - a^2)^2\beta_i, \beta_k\Big) - V_i(F((x)\phi_i, \beta_k)] = 0, \\ \sum_{i=1}^{n} [V_i\Big((D^2 - a^2)\phi_i, \phi_k\Big) + a^2 T U_i(G(x)\beta_i, \phi_k)] = 0. \end{cases} \tag{5}$$

The secular equation, which yields the critical value of the Taylor number $T$ is obtained by imposing the condition that the determinant of the system (5) vanish, i.e.

$$\begin{vmatrix} ((D^2 - a^2)^2\beta_i, \beta_k) & (F(x)\phi_i, \beta_k) \\ & \\ a^2 T(G(x)\beta_i, \phi_k) & ((D^2 - a^2)\phi_i, \phi_k) \end{vmatrix} = 0. \tag{6}$$

The unknown vector field $(U, V)$ from (1) can also be expanded upon complete sequences of functions in $L^2(0, 1)$ defined by using Chebyshev polynomials that satisfy the boundary conditions of the problem. Keeping the above notations, the functions $\phi_i$, $i = 1, 2, ..., N$ are defined, for instance, by $\phi_i(x) = T_i^*(x) - T_{i+2}^*(x)$ and $\beta_i$, $i = 1, 2, ..., n$ by $\beta_i(n) = T_i^*(x) - \dfrac{2(i+2)}{i+3}T_{i+2}^*(x) + \dfrac{i+1}{i+3}T_{i+4}^*(x)$ [25] with $T_i^*$, $i = 1, 2, ..., n$, the shifted Chebyshev polynomials on $(0, 1)$ defined in a similar manner as the shifted Legendre polynomials. All the evaluations of the scalar products were based on the orthogonality relation

$$\int_0^1 T_n^*(z)T_m^*(x)w^*(z)dz = \begin{cases} \dfrac{\pi}{2}c_n\delta_{nm}, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases} \tag{7}$$

with respect to the weight function $w^*(x) = \dfrac{1}{\sqrt{x(1-x))}}$.

**Remark.** We could also work with original Legendre or Chebyshev polynomials by transforming the defined interval on $(a, b) = (-1, 1)$. However, the choice $(a, b) = (0, 1)$ led us to simplified numerical evaluations.

## 3. A VARIATIONAL APPROACH

Since $F(x)$, $G(x) \in \mathcal{C}^\infty$ and $V \in \mathcal{C}^2$, from $(1)_1$ we get that $U \in \mathcal{C}^6$, then $(1)_2$ implies that $V \in \mathcal{C}^4$ and, again from $(1)_1$ it follows that $U \in \mathcal{C}^8$ and so on such that $U, V \in \mathcal{C}^\infty$. As they are defined on a compact set it follows that they are bounded hence $U^2$, $V^2$ are Lebesgue integrable. The problem (1) - (2) can also be written in the form

$$A\mathbf{X} = TB\mathbf{X}, \quad \mathbf{X} \in \mathcal{D}(A - TB),$$

where $\mathbf{X} = (U, V)^t$, the expressions of the matricial differential operators $A$ and $B$ are $A = \begin{pmatrix} (D^2 - a^2)^2 & 0 \\ 0 & D^2 - a^2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & F(x) \\ -a^2 G(x) & 0 \end{pmatrix}$ and $\mathcal{D}(A - TB)$, the domain of definition of the operator $A - TB$, is defined by $\mathcal{D}(A - TB) = \mathcal{D}(A) \cap \mathcal{D}(B)$,

$\mathcal{D}(A) = \{\mathbf{X} \in \overline{\mathcal{C}}^\infty \times \overline{\mathcal{C}}^\infty | U, V$ satisfy the boundary conditions (2)$\}$, $\mathcal{D}(B) = \overline{\mathcal{C}}^\infty$,

therefore $\mathcal{D}(A - TB) = \mathcal{D}(A)$, with $\overline{\mathcal{C}}^\infty = \mathcal{C}^\infty(\overline{(0,1)}; \mathbb{R})$ the space of functions $\widetilde{f} : (0,1) \to \mathbb{R}$ which, together with all their derivatives can be prolonged by continuity on $[0,1]$.

Since $\mathcal{C}_0^\infty$ is dense in $L^2$, the operators $A$ and $B$ may be extended to operators in $L^2$ densely defined on some Hilbert subspace $\mathcal{H}$ of $L^2$ obtained by the closure of $\mathcal{D}(A)$ in $L^2$. It can be proved immediately that $A = A^*$ and $B^* = B^t$, $A^*$ and $B^*$ being the adjoint operators for $A$ and $B$ and that for

$$A\mathbf{X} = TB\mathbf{X}, \ \mathbf{X} \in \mathcal{H} \tag{8}$$

its adjoint reads

$$A\mathbf{X}^* = TB^t\mathbf{X}^*, \ \mathbf{X}^* \in \mathcal{H}. \tag{9}$$

A necessary condition for the existence of the eigenvector $\mathbf{X}$ of (8) is

$$Q_1(\mathbf{X}, \mathbf{X}^*) = ((A - TB)\mathbf{X}, \mathbf{X}^*) = 0, \tag{10}$$

where $\mathbf{X}^*$ is the eigenvector of (9).

The following variational principle for the linear nonselfadjoint operator $A - TB$ holds[9], [27]

$(T, \mathbf{X})$ *and* $(T^*, \mathbf{X}^*)$ *are the eigenvalues and eigenvectors of problem (8) and (9) respectively iff* $T = extQ$ *where* $Q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ *is the functional* $Q(\mathbf{X}, \mathbf{X}^*) = \dfrac{(A\mathbf{X}, \mathbf{X}^*)}{(B\mathbf{X}, \mathbf{X}^*)}$ *and ext stands for the extremum value of* $Q$. (11)

This reduces the calculation of the eigenvalues and eigenvectors of (8) and (9) to the solution of the variational problem $extQ$ in $\mathcal{H} \times \mathcal{H}$.

Due to the complicate form of $Q$ it is preferable to transform the variational problem without constraints (11) into the isoperimetric problem:

*extremal (stationary) points of $Q$ in $\mathcal{H} \times \mathcal{H}$ are equal to the extremal (stationary)*

*points of the functional $Q_1$ in $\mathcal{H}_1 \times \mathcal{H}_1 = \{(\mathbf{X}, \mathbf{X}^*) | (B\mathbf{X}, \mathbf{X}^*) = 1\}$.*

If $\mathbf{X} = (U, V)$ and $\mathbf{X}^* = (U^*, V^*)$ are represented by series, e.g. $U = \sum\limits_{i=1}^{n} U_i \beta_i(x)$, $V = \sum\limits_{i=1}^{n} V_i \phi_i(x)$, $U^* = \sum\limits_{i=1}^{n} U_i^* \beta_i(x)$, $V^* = \sum\limits_{i=1}^{n} V_i^* \phi_i(x)$, then $Q_1$ becomes a function of $4n$ variable $U_1, U_2, ..., U_n$, $V_1, ..., V_n$, $U_1^*, ..., U_n^*$, $V_1^*, ..., V_n^*$ such that in order for $Q_1$ to achieve an extremum value it is necessary that

$$\frac{\partial Q_1}{\partial U_i} = \frac{\partial Q_1}{\partial V_i} = \frac{\partial Q_1}{\partial U_i^*} = \frac{\partial Q_1}{\partial V_i^*} = 0, \quad i = 1, 2, ..., n. \tag{12}$$

The evaluations can be simplified by halving the order of differentiation if the expression of $Q_1$ from (10) is integrated by parts and then taking into account the boundary conditions on $U$, $V$, $U^*$, $V^*$, i.e. $U = DU = V = 0$, $U^* = DU^* = V^* = 0$ at $x = 0$ and $x = 1$. We get

$$Q_1(\mathbf{X}, \mathbf{X}^*) = \int_0^1 \{D^2UD^2U^* + 2a^2DUDU^* + a^4UU^* - F(x)VU^* - DVDV^* -$$

$$-a^2VV^* + a^2TG(x)UV^*\}dx = 0. \tag{13}$$

Since $Q_1(\mathbf{X}, \mathbf{X}^*)$ is then equal to

$$Q_1(U_1, .., U_n, V_1, .., V_n, U_1^*, ..., U_n^*, V_1^*, ..., V_n^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} U_i U_j^* ((D^2\beta_i, D^2\beta_j) +$$

$$+2a^2(D\beta_i, D\beta_j) + a^4(\beta_i, \beta_j)) - V_i U_j^* (F(x)\phi_i, \beta_j) - V_i V_j^* ((D\phi_i, D\phi_j)) + a^2(\phi_i, \phi_j) +$$

$$+a^2TU_iV_j^*(G(x)\beta_i, \phi_j) = 0 \tag{14}$$

(12) will lead to the following relation $detM \cdot detM^t = 0$, where $M$ is a simplified form of the determinant in (6).

## 4. NUMERICAL EVALUATIONS

Some particular examples of (1)-(2) are numerically treated using the spectral method based on polynomials presented in Section 2.

Let be a viscous flow between two concentric cylinders due to a pressure gradient acting round the cylinders. The $z$ axis is the cylinders axis. In the narrow gap approximation and if the principle of exchange of stabilities holds, the stability of the basic flow to rotationally symmetric perturbations, specially periodic in the $z$-direction is governed by the system [14], [24]

$$\begin{cases} (D^2-a^2)^2u = a^2T(4x-4x^2)v, \\ (D^2-a^2)v = (1-2x)u \end{cases} \tag{15}$$

and the boundary conditions

$$v = u = Du = 0 \text{ at } x = 0, 1. \tag{16}$$

In [14] Joseph's method is applied in order to obtain bounds for the stability problem (15)-(16).

Multiplying $(15)_2$ by $-a^2T$ the system (15) is reduced to the general form (1), i.e.

$$\begin{cases} (D^2-a^2)^2\Psi = (4x^2-4x)v, \\ (D^2-a^2)v = -a^2T(1-2x)\Psi. \end{cases} \tag{17}$$

Here, and in all the examples below, $\Psi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ is the vector perturbation function and it represents the eigenfunction in the above eigenvalue problem.

Representing the unknown functions as in the above spectral method, the secular equation is obtained allowing us to obtain numerical evaluations on the critical Taylor number. These numerical results are presented in comparison with the ones from [24] in Table 1.

In [10] DiPrima studied the stability of a viscous flow between rotating cylinders when the inner cylinder is rotating (the Taylor problem) and at the same time the fluid is being pumped round the annulus (the Dean problem), with the possibility of pumping in the direction of rotation or opposed to it. Introducing a pumping parameter $Q = 3(1+a)\dfrac{V_p}{V_r}$, with the quantities $V_p$ and $V_r$ being the average velocities due to rotation and pumping, respectively, he obtained numerical values of the critical Taylor number for different wavenumbers for axisymmetric modes in a small-gap approximation, for various values of the parameter $Q$.

The stability problem for this type of velocity distribution has the form [10]

$$\begin{cases} (D^2-a^2)^2u = \{(1-x)-Q(x^2-x)\}v, \\ (D^2-a^2)v = -Ta^2\{1+Q(2x-1)\}u \end{cases} \tag{18}$$

with the boundary conditions (16). The eigenvalue problem (18) -(16) is solved in [10] using a method from Chandrasekhar [4]. Our numerical evaluations, based on shifted Legendre and Chebyshev polynomials, presented in Table 2, are similar to those from [10].

In [15] the linear stability of the Taylor-Dean flow was investigated in the case of a partially filled rotating confinement and subject to circumferential drop pressure. The eigenvalue problem is defined by a nonselfadjoint matrix differential operators with variable coefficients which lead to the application of numerical methods (e.g. shooting method was used by Mutabazi, Normand, Peerhossainti, Wesfried) rather than towards analytical approaches.

The flow of an incompressible viscous fluid between two coaxial horizontal cylinders rotating in the same direction in the presence of a circumferential pressure gradient is investigated [15]. More over, two cases are analyzed: one case when both cylinders are rotating and another when only the inner cylinder is rotating and the outer one is at rest. In the first one, the problem governing the linear stability of the basic flow $(U, V, W, P) = (0, V(x), 0, P(x))$, where $V(x) = 3(1+\mu)x^2 - 2(2+\mu)x + 1$, with respect to infinitesimal perturbations of the form $f(x, y) = f(y)e^{i\sigma x}$, can be written in the classical form [15]

$$\begin{cases} (D^2-a^2)^2u = Vv, \\ (D^2-a^2)v = -2a^2TDVu, \qquad x \in [0,1] \end{cases} \tag{19}$$

where $D = d/dx$, $u \in \mathcal{C}^4([0,1],\mathbb{R})$, $v \in \mathcal{C}^2([0,1],\mathbb{R})$, $\mu \in [0,1]$, $T > 0$, $a \in \mathbb{R}_+$. Here $\mu$ is the rotation velocities ratio.

For the case when the inner cylinder is rotating with the angular velocity $\Omega_1$ and the other cylinder is at rest, the basic flow velocity distribution is a quadratic polynomial

$$V_0(x) = 2\alpha(1-x) + 6(1-\alpha)(x-x^2) \tag{20}$$

where $\alpha$ is the dimensionless measure of the relative strength of the velocities due to the rotation and the pressure gradient.

The eigenvalue problem has then the form [15]

$$\begin{cases} (D^2 - a^2)^2 u = a^2 T V_0 v, \\ (D^2 - a^2) v = 0.5 D V_0 u \end{cases} \tag{21}$$

with the boundary conditions (16).

The problem was investigated analytically and numerically in [15] using the Chandrasekhar method. The azimuthal eigenfunction $v(x)$ is expanded in a Fourier sine series $v(x) = \sum\limits_{n=1}^{\infty} v_n \sin(n\pi x)$ that satisfies all boundary conditions on $v$. The radial eigenfunction $u(x)$ is obtained by replacing $v(x)$ in $(21)_1$ and solve the obtained ordinary differential equation. Then the functions $u(x)$ and $v(x)$ are replaced in $(21)_2$ and imposing the condition that the left-hand side of the obtained equation to be orthogonal on $\sin(m\pi x)$, $m = 1, 2, ...$, an algebraic infinite system of equations is obtained. Since the solution of this system is nontrivial, the conditions that the determinant of the system to vanish is imposed. The secular equation of the form $F(T, a, \alpha) = 0$ is obtained. The same method was also used in [20]. In [20] it is shown also that the values of the wavenumber on some intervals lead to some discontinuities; in this case the lowest stationary mode does not exists.It is clear that the method we used lead to a simplification not only of the numerical computations but also on the symbolic written.

The numerical results for all the presented cases are given in Tables 1-4. In the following we give the notations used for the tables headings: $T_L$-corresponding to the numerical results obtained here using shifted Legendre polynomials, $T_C$-corresponding to the numerical results obtained here using shifted Legendre polynomials, $T_{[cited\ reference]}$-corresponding to the numerical results obtained in the cited reference. The accuracy of the spectral methods used here is proven also by the small number of terms in the expansion series, $n \leq 6$. Obviously, we performed numerical evaluations for a larger number of terms, but the obtained improvements of the numerical evaluations did not justified the much larger computational time.

| $a$ | $T_L$ | $T_C$ | $T_{[24]}$ |
|---|---|---|---|
| 1 | 126320.1329 | 122194.722 | 117835.7058 |
| 2 | 41110.9207 | 40176.220 | 38786.6952 |
| 2.5 | 31570.5941 | 31045.954 | 30007.6002 |
| 3 | 27007.4955 | 26728.576 | 25869.0258 |
| 3.8 | 24449.0427 | 24480.515 | 23764.18005 |
| 3.9 | 24443.38079 | 24452.04647 | 23748.48592 |
| 4 | 25051.82227 | 24467.073 | 23774.64568 |
| 4.5 | 25550.4704 | 25120.706 | 23840.10808 |
| 5 | 26893.4293 | 26612.164 | 26005.6818 |
| 5.5 | 28976.0245 | 28849.428 | 28298.2050 |
| 6 | 31758.7613 | 31802.477 | 31375.1250 |
| 8 | 50185.97265 | 51218.709 | 51520.5000 |

*Table 1* **Numerical values of the critical Taylor number for various values of the wavenumber for the model (15)-(16).**

| $Q$ | $a$ | $T_L$ | $T_C$ | $T_{[10]}$ |
|---|---|---|---|---|
| $-2$ | 3.80 | 13576.5545 | 12098.7883 | 12594.1346 |
| $-0.5$ | 3.17 | 4279.4433 | 4291.879 | 4182.021 |
| 0 | 3.12 | 3474.8687 | 3532.390395 | 3393.6866 |
| 0.5 | 3.13 | 2915.5634 | 2975.603180 | 2845.6975 |
| 3 | 3.14 | 1525.6674 | 1582.064085 | 1480.5318 |
| 10 | 3.45 | 472.0859 | 484.4711193 | 455.6961 |
| 21 | 3.70 | 154.8957 | 157.1818753 | 149.0145 |

*Table 2* **Numerical values of the critical Taylor number for various values of the parameters $Q$ and $a$ for the model (18)-(16).**

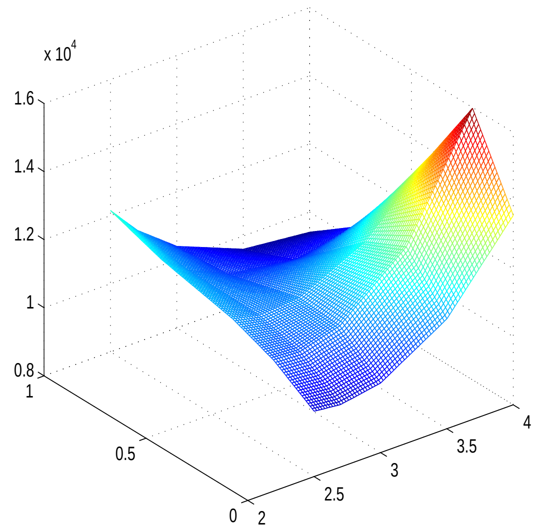


Fig.1.: Neutral surface for the model (19)-(16).

The neutral surface separating the stability and instability domain is presented in Figure 1. It was obtained in MATLAB using a two dimensional

| $\mu$ | $a$ | $T_L$ | $T_C$ |
|---|---|---|---|
| 0 | 2.5 | 9919.7149 | 11460.31073 |
| 0 | 2.7 | 9835.469 | 11532.63127 |
| 0 | 3 | 10045.964 | 12016.05962 |
| 0.2 | 3 | 10995.292 | 12409.86652 |
| 0.2 | 4 | 15979.639 | 16915.09562 |
| 0.4 | 2.7 | 10998.024 | 11539.55249 |
| 0.4 | 2.8 | 10984.887 | 11554.30105 |
| 0.4 | 3 | 11088.561 | 11707.50378 |
| 0.75 | 3 | 10504.836 | 9860.645306 |
| 0.75 | 3.5 | 10224.794 | 9790.854202 |
| 0.75 | 3.75 | 10301.214 | 9968.810226 |
| 1 | 3 | 10395.189 | 8801.452055 |
| 1 | 4 | 9403.829 | 8387.847645 |
| 1 | 4.15 | 9426.951 | 8480.124589 |
| 1 | 4.5 | 9606.823 | 8819.413100 |

*Table 3* **Numerical values of the critical Taylor number for various values of the parameters $\mu$ and $a$ for the model (19)-(16).**

| $\alpha$ | $a$ | $T \times 10^3_{[20]}$ | $T \times 10^3_L$ | $T \times 10^3_C$ |
|---|---|---|---|---|
| 0.125 | 3.68 | 4.75 | 4.9569 | 5.05001 |
| 0.5 | 3.14 | 2.96 | 3.0513 | 3.17117 |
| 0.75 | 3.13 | 2.162 | 2.222 | 2.29017 |
| 1.5 | 3.24 | 1.205 | 1.233 | 1.20967 |
| 3 | 3.80 | 0.731 | 0.7542 | 0.62713 |
| $-3$ | 5.7 | 2.553 | 6.358 | 2.41603 |

*Table 4* **Numerical values of the critical Taylor number for various values of the parameters $\alpha$ and $a$ for the model (21)-(16).**

interpolation procedure. The graphical representation gives us the critical Taylor number as a function of $a$ and $\mu$, $T = T(a, \mu)$. It can be noticed that for fixed values of $\mu$ (a projection on this space) the neutral curve given by $T = T(a)$ has the classical well known form from [4].

## 5. CONCLUSIONS

In this paper we present a few models of Taylor-Dean system, the analytical and numerical investigation performed on them. The practical importance of these models was pointed out and the physical characteristics on each case are emphasized. We were interested in the critical values of the Taylor number, $T_c$ (proportional to the inner cylinder rate) at which the system changes from a spatially uniform state to a travelling wave state. We also completed some of them with our analytical and numerical results. The evaluations were each time compared to the existing ones and they proved to be similar.

We can conclude that an experimental, analytical and numerical study is of great theoretical and practical interest due to the large variety of patterns that this system exhibits.

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