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AVOIDING BANKRUPTCY AT ALL COSTS

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Abstract Assuming that the value of the stock of a company at time t can be represented as a controlled one-dimensional Bessel process, we consider the problem of finding the control that minimizes the mathematical expectation of a cost function with quadratic control costs on the way. Two terminal cost functions, which are infinite if the process hits the origin before a given positive boundary, are used and the optimal control is obtained explicitly in each case.

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1. INTRODUCTION

Let $\{X(t), t \geq 0\}$ be a one-dimensional controlled Bessel process defined by the stochastic differential equation

$$dX(t) = \frac{\beta - 1}{2X(t)}\,dt + b u(t)\,dt + dB(t), \tag{1}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, $u(t)$ is the control variable and $\beta > 0$ and $b \neq 0$ are constants. We want to find the value of the control $u^*(t)$ that minimizes the expected value of the cost criterion

$$J(x) = \int_0^{T(x)} \tfrac{1}{2} q u^2(t)\,dt + K[X(T), T],$$

where $x = X(0)$, $T(x)$ is a random variable, q is a positive constant and K is the termination cost function.

Using a result due to Whittle ([4], p. 289), we can state that

$$u^* \; (= u^*(0)) = \frac{1}{b}\frac{G'(x)}{G(x)} \tag{2}$$

with

$$G(x) := E\left[\exp\{-\alpha K[x(\tau), \tau]\} \mid x(0) = x\right], \tag{3}$$

in which

$$\alpha := \frac{b^2}{q},$$

$\{x(t), t \geq 0\}$ is the *uncontrolled* process that corresponds to $\{X(t), t \geq 0\}$ [that is, we set $u(t) \equiv 0$ in (1)] and τ is the same as T, but for $\{x(t), t \geq 0\}$.

Remark. In fact, the condition $P[\tau(x) < \infty] = 1$ must hold for (2) to be valid.

In [3], the author defined the random variable

$$T(x) = \inf\{t > 0 : X(t) = d \mid X(0) = x < d\}$$

and he chose (in particular)

$$K[X(T), T] = K(T) = \begin{cases} 0 & \text{if } T \geq t_0, \\ \infty & \text{if } T < t_0. \end{cases}$$

Hence, the aim was to force the controlled process $\{X(t), t \geq 0\}$ to stay in the continuation region $C := (-\infty, d)$ until a fixed time t_0. We find that

$$u^* = \frac{1}{b}\frac{H'(x)}{H(x)},$$

where

$$H(x) := P[\tau(x) \geq t_0] = 1 - F_{\tau(x)}(t_0).$$

Thus, the optimal control u^* was obtained from the distribution function $F_{\tau(x)}$ of the random variable $\tau(x)$.

In the present paper, we consider the controlled process $\{X(t), t \geq 0\}$ in the interval $[c, d]$, where $c \geq 0$. We want the process to leave the continuation region through its right-hand side d. If $X(t)$ represents the value of the stock of a certain company at time t, and if we choose $c = 0$, then the aim is to find the value of $u(t)$ that enables the company to avoid bankruptcy (corresponding to $X(t) = 0$), while taking the quadratic control costs $\frac{1}{2}qu^2(t)$ into account. In this application, $u(t)$ would be the amount of capital that must be invested in the company to prevent it from going bankrupt.

In Section 2, we will define

$$T(x) = \inf\{t > 0 : X(t) = c \text{ or } d \mid X(0) = x\} \tag{4}$$

and we will take

$$K[X(T), T] = K[X(T)] = -\lambda \ln[X(T)], \tag{5}$$

where $\lambda > 0$. If $c = 0$, we see that we give an infinite penalty if $X(t)$ reaches the origin (before d). The constant d can be chosen as large as we want. The larger it is, the longer it will take $X(t)$ to attain this value, which means that the company will be in business for a long period of time.

By giving an infinite penalty if the final value of $X(t)$ is equal to c, we force the process to avoid this boundary. We assume that there are no constraints on the control variable $u(t)$. Thus, we can state that we are ready to avoid bankruptcy at all costs.

Next, in Section 3, we will change the definition of the final time $T(x)$ to

$$T(x) = \inf\{t > 0 : \{X(t) = d\} \cap \{X(s) \neq c \; \forall s \in (0,t)\} \mid X(0) = x \in (c,d)\}. \tag{6}$$

That is, $T(x)$ is now the time it takes $\{X(t), t \geq 0\}$ to reach the boundary at d, without having ever touched the boundary at c. Moreover, we will choose

$$K[X(T), T] = K(T) = \lambda T, \tag{7}$$

where, as above, $\lambda > 0$.

Remark. If the controlled process $\{X(t), t \geq 0\}$ actually hits c before d, we set $T = \infty$ (because the joint event $\{X(t) = d\} \cap \{X(s) \neq c \; \forall s \in (0,t)\}$ will never occur). Therefore, once again, we give an infinite penalty when $X(T) = c$.

We will obtain an explicit formula for the optimal control u^* in the two cases mentioned above, and we will conclude this work with a few remarks in Section 4.

2. OPTIMAL CONTROL WHEN $K[X(T), T] = -\lambda \ln[X(T)]$

First, we mention that if we want the constant c to take the value 0 in the definition (4) of $T(x)$, then the constant β in (1) must be in the interval $[0, 2)$. Indeed, when $0 < \beta < 2$, the origin is a regular boundary for the Bessel process $\{x(t), t \geq 0\}$ defined by

$$dx(t) = \frac{\beta - 1}{2x(t)} dt + dB(t), \tag{8}$$

while it is an exit boundary if $\beta = 0$ ([2], p. 239). In both cases, $\{x(t), t \geq 0\}$ can reach the origin. However, if $\beta \geq 2$, then 0 is an entrance boundary, which is not attainable in finite expected time.

Remarks. i) In the case $\beta = 1$, $\{x(t), t \geq 0\}$ is a standard Brownian motion defined on $[c, d]$, with $c \geq 0$. Actually, $\{x(t), t \geq 0\}$ is the absolute value of a Brownian motion process if $\beta = 1$.

ii) We could generalize Eq. (8) by multiplying $dB(t)$ by a positive constant σ. However, a Bessel process is traditionally defined with $\sigma = 1$.

Because the interval $[c, d]$ is bounded, it is not difficult to justify that the condition $P[\tau(x) < \infty] = 1$ is satisfied. Therefore, we can get the optimal control u^* from (2).

With the termination cost function defined in (5), the function $G(x)$ in (3) becomes

$$G(x) = E\left[x^{\alpha\lambda}(\tau) \mid x(0) = x\right].$$

We can write that

$$G(x) = c^{\alpha\lambda} P[x(\tau) = c \mid x(0) = x] + d^{\alpha\lambda} P[x(\tau) = d \mid x(0) = x].$$

Let $\pi_d(x) := P[x(\tau) = d \mid x(0) = x]$. The function π_d satisfies the Kolmogorov backward equation

$$\pi_d''(x) + \frac{\beta - 1}{x}\pi_d'(x) = 0,$$

and is such that

$$\pi_d(c) = 0 \quad \text{and} \quad \pi_d(d) = 1.$$

We easily find that, if $\beta \neq 2$,

$$\pi_d(x) = \frac{c^{2-\beta} - x^{2-\beta}}{c^{2-\beta} - d^{2-\beta}} \quad \text{for } c \leq x \leq d.$$

When $\beta = 2$, we obtain that

$$\pi_d(x) = \frac{\ln x - \ln c}{\ln d - \ln c} \quad \text{for } c \leq x \leq d.$$

Since $P[x(\tau) = c \mid x(0) = x] = 1 - \pi_d(x)$, we have

$$G(x) = c^{\alpha\lambda}\left(\frac{x^{2-\beta} - d^{2-\beta}}{c^{2-\beta} - d^{2-\beta}}\right) + d^{\alpha\lambda}\left(\frac{c^{2-\beta} - x^{2-\beta}}{c^{2-\beta} - d^{2-\beta}}\right) \quad \text{if } \beta \neq 2$$

and

$$G(x) = \frac{c^{\alpha\lambda}(\ln d - \ln x) + d^{\alpha\lambda}(\ln x - \ln c)}{\ln d - \ln c} \quad \text{if } \beta = 2.$$

Making use of (2), we can now state the following proposition.

Proposition 2.1. *The optimal control, when the termination cost function is the one defined in (5), is given, for $c < x < d$, by*

$$u^* = \frac{1}{b} \frac{\left(c^{\alpha\lambda} - d^{\alpha\lambda}\right)(2-\beta)x^{1-\beta}}{c^{\alpha\lambda}\left(x^{2-\beta} - d^{2-\beta}\right) + d^{\alpha\lambda}\left(c^{2-\beta} - x^{2-\beta}\right)} \quad \textit{if } \beta \neq 2 \tag{9}$$

and

$$u^* = \frac{1}{bx} \frac{d^{\alpha\lambda} - c^{\alpha\lambda}}{c^{\alpha\lambda}\ln(d/x) + d^{\alpha\lambda}\ln(x/c)} \quad \textit{if } \beta = 2.$$

Remarks. i) If $\beta \in [0, 2)$, we can set $c = 0$. Then (9) reduces to

$$u^* = \frac{1}{b} \frac{2-\beta}{x} \quad \text{for } 0 < x < d.$$

Moreover, in the case when $\{x(t), t \geq 0\}$ is a standard Brownian motion (so that $\beta = 1$), the optimal control is simply $u^* = 1/(bx)$.

ii) If $\beta \in [0, 2)$ and $c = 0$, the optimally controlled process $\{X^*(t), t \geq 0\}$ satisfies the stochastic differential equation

$$dX^*(t) = \frac{3-\beta}{2X^*(t)}dt + dB(t).$$

Hence, we can state that $\{X^*(t), t \geq 0\}$ is also a Bessel process, with $\beta^* = 4-\beta$. Notice that $\beta^* \in (2, 4]$. Therefore, $\{X^*(t), t \geq 0\}$ cannot reach the origin.

In the next section, the case when $K[X(T), T] = K(T) = \lambda T$ will be treated.

3. OPTIMAL CONTROL WHEN $K[X(T), T] = \lambda T$

When the termination cost function $K[X(T), T]$ is given by λT, where λ is a positive constant and T is now the random variable defined in (6), we must calculate $G(x) = E\left[e^{-\gamma\tau} \mid x(0) = x\right]$, where $\gamma := \alpha\lambda$ (> 0). That is, we need the moment generating function of the random variable

$$\tau = \inf\{t > 0 : \{x(t) = d\} \cap \{x(s) \neq c \; \forall s \in (0, t)\} \mid x(0) = x \in (c, d)\}.$$

The function G satisfies the Kolmogorov backward equation

$$\frac{1}{2}G''(x) + \frac{\beta - 1}{2x}G'(x) = \gamma G(x) \quad \text{for } c < x < d, \tag{10}$$

subject to $G(c) = 0 \quad \text{and} \quad G(d) = 1$. We find that

$$G(x) = \frac{d^\theta K_\theta(\sqrt{2\gamma}c)\, I_\theta(\sqrt{2\gamma}x)}{x^\theta \left[K_\theta(\sqrt{2\gamma}c)\, I_\theta(\sqrt{2\gamma}d) - K_\theta(\sqrt{2\gamma}d)\, I_\theta(\sqrt{2\gamma}c)\right]} \quad \text{for } c \le x \le d,$$

where

$$\theta := \frac{\beta}{2} - 1$$

and K_θ and I_θ are modified Bessel functions ([1], p. 374).

Proposition 3.1. *When the termination cost function is* $K[X(T), T] = \lambda T$, *with* T *defined in (6), the optimal control* u^* *is given by (2), where the function* $G(x)$ *has been calculated above.*

Because the general explicit expression for u^* is rather involved, for the sake of simplicity we will present only some particular cases.

I) First, if $\beta = 1$ (so that $\{x(t), t \ge 0\}$ is a standard Brownian motion), we obtain that

$$G(x) = \frac{\sinh[\sqrt{2\gamma}\,(x - c)]}{\sinh[\sqrt{2\gamma}\,(d - c)]} \quad \text{for } c \le x \le d.$$

Hence,

$$u^* = \frac{\sqrt{2\gamma}}{b} \coth[\sqrt{2\gamma}\,(x - c)] \quad \text{for } c < x < d,$$

in which $c \ge 0$.

II) Next, assume that $\beta = 1/2$. Then, with $c = 0$, we calculate

$$G(x) = \left(\frac{x}{d}\right)^{3/4} \frac{I_{3/4}(\sqrt{2\gamma}x)}{I_{3/4}(\sqrt{2\gamma}d)} \quad \text{for } 0 \le x \le d$$

and

$$u^* = \frac{\sqrt{2\gamma}}{b} \frac{I_{-1/4}(\sqrt{2\gamma}x)}{I_{3/4}(\sqrt{2\gamma}x)} \quad \text{for } 0 < x < d.$$

Making use of the following formula, which is valid as z tends to zero ([1], p. 375):

$$I_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu + 1)} \quad (\text{for } \nu \neq -1, -2, \ldots),$$

we can write that, for x very small,

$$u^* \simeq \frac{3}{2bx}.$$

III) Now, with $\beta = 3/2$ and $c = 0$, we get that

$$G(x) = \left(\frac{x}{d}\right)^{1/4} \frac{I_{1/4}(\sqrt{2\gamma}x)}{I_{1/4}(\sqrt{2\gamma}d)} \quad \text{for } 0 \le x \le d,$$

from which we deduce that

$$u^* = \frac{\sqrt{2\gamma}}{b} \frac{I_{-3/4}(\sqrt{2\gamma}x)}{I_{1/4}(\sqrt{2\gamma}x)} \quad \text{for } 0 < x < d.$$

This time, we obtain (if x is very small) that

$$u^* \simeq \frac{1}{2bx}.$$

IV) Finally, if we take $\beta = 2$ (and $c > 0$), the function $G(x)$ is given by

$$G(x) = \frac{I_0(\sqrt{2\gamma}c)\,K_0(\sqrt{2\gamma}x) - K_0(\sqrt{2\gamma}c)\,I_0(\sqrt{2\gamma}x)}{I_0(\sqrt{2\gamma}c)\,K_0(\sqrt{2\gamma}d) - K_0(\sqrt{2\gamma}c)\,I_0(\sqrt{2\gamma}d)} \quad \text{for } c \le x \le d.$$

It follows that

$$u^* = \frac{\sqrt{2\gamma}}{b} \frac{K_0(\sqrt{2\gamma}c)\,I_1(\sqrt{2\gamma}x) + I_0(\sqrt{2\gamma}c)\,K_1(\sqrt{2\gamma}x)}{K_0(\sqrt{2\gamma}c)\,I_0(\sqrt{2\gamma}x) - I_0(\sqrt{2\gamma}c)\,K_0(\sqrt{2\gamma}x)} \quad \text{for } c < x < d.$$

As $z \to 0$ ([1], p. 375),

$$I_0(z) \sim 1 \quad \text{and} \quad K_0(z) \sim -\ln z,$$

while

$$I_1(z) \sim \frac{z}{2} \quad \text{and} \quad K_1(z) \sim \frac{1}{z}.$$

Hence, if x and c are both small enough, then we can write that

$$u^* \simeq \frac{1}{2bx}\left[\frac{2 - x^2\ln(\sqrt{2\gamma}c)}{\ln(x/c)}\right].$$

4. CONCLUSION

We have obtained (exact and) explicit expressions for the optimal control of a Bessel diffusion process when the objective is to prevent the process from ever hitting a boundary at $x = c$ (≥ 0). This work is related to the one presented in [3].

Hitting the boundary at $x = 0$ (if $\beta \in [0, 2)$) meant, in the application mentioned in Section 1, that the company went bankrupt.

The stochastic differential equation (1) could serve as a model in other situations, and the objective could then be to avoid hitting the boundary at $x = d$ instead. We could also consider the same problem, but in the infinite interval $[c, \infty)$. In that case, we would give a reward for survival in the continuation region $[c, \infty)$. To apply the result due to Whittle, one should make sure that $P[\tau(x) < \infty] = 1$. That is, the uncontrolled process must be certain to eventually hit the boundary at $x = c$. Because the interval $[c, \infty)$ is infinite, it is not obvious that $\{x(t), t \geq 0\}$ will necessarily end up at $x = c$.

Finally, we could try to solve the same type of problem as the one presented in the current paper, but for a two-dimensional diffusion process. For example, the continuation region could be a circle centered at the origin and $T(x, y)$ be the first time that the controlled process hits the boundary of this circle. The random variable $X(T)$ would then be a continuous rather than discrete random variable, and computing its mathematical expectation could prove quite difficult.

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