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GENERALIZED QUATERNIONS, CORRESPONDING TO THEM SYSTEM OF EQUATIONS AND APPLICATIONS

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Abstract In topology it is proved that every two-dimensional compact, connected and oriented manifold is diffeomorphic to the sphere with $n \geq 0$. Three-dimensional manifolds are investigated very little. For example, so far it has not been proved if every compact one-connected three-dimensional manifold is diffeomorphic to the sphere S^3, (Poincaré's conjecture) or at least homeomorphic to it [1]. Poincaré's conjecture [2] about homeomorphism is positively solved [3]. The present paper deals with the analogue of Riemann's theorem on mappings in the three-dimensional space. Moreover, famous properties of Beltrami equations system (BES) solutions as well as new results concerning these systems and generalizations are revealed and essentially used.

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BES is system of first-order linear elliptic equations of first order

$$\begin{cases} u_x = -\frac{g_{12}}{g} v_x + \frac{g_{11}}{g} v_y, \\ u_y = -\frac{g_{22}}{g} v_x + \frac{g_{12}}{g} v_y, \end{cases} \tag{1}$$

$$g^2 = g_{11} g_{22} - g_{12}^2.$$

It may be viewed as a condition that the mapping $(x, y) \to (u, v)$ be conformal with respect to Riemann's metric

$$ds^2 = g_{11}(x, y)dx^2 + 2g_{12}(x, y) + g_{22}(x, y)dy^2, \ g_{11} \neq 0, \ \forall (x, y) \in D \subset R^2$$

that is, in this mapping any angle θ on the plane (x, y), determined in Riemann's metrics is transformed to the angle θ on the plane (u, v), determined in the essential way. Two Riemann's metrics are named conformal-equivalent if their coefficients are proportional, where the proportionality multiplier can be a function from (x, y). The two metrics set the same BES [4]. BES can be thought of as a system connecting the old and new coordinate systems at rewriting the positively defined quadratic form in the canonical form.

Here, we follow some results of the author [5]. Consider another interpretation of the generalized mixed BES and demonstrate that these equations systems are related to the main curvatures of the surfaces and the theorema egregium (Gaussian curvature) [5]. Let k_1, k_2 be the main curvatures and Ox, Oy be directed along the corresponding main directions of the smooth surface M at the point O.

From the Euler's formula it follows

$$\begin{aligned} g_{11} &= k_1 \cos^2\theta + k_2 \sin^2\theta, \\ g_{12} &= (k_1 - k_2)\cos\theta\sin\theta, \\ g_{22} &= k_1 \sin^2\theta + k_2 \cos^2\theta, \end{aligned} \tag{2}$$

where $g_{11}g_{22} - g_{12}^2 = k_1k_2 = K$, where K is the Gaussian curvature. From (2) we obtain

$$\tan 2\theta = \frac{2g_{12}}{g_{11} - g_{22}}, \quad k_{1,2} = \frac{g_{11} + g_{22} \pm \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}}{2}. \tag{3}$$

Consider the first-order partial differential equations : $u_x = g_{12}v_x + g_{11}v_y$, $-\, u_y = g_{22}v_x + g_{12}v_y$, where $g_{11}g_{22} - g_{12}^2 = k_1k_2 = K$, which, in the matrix form reads

$$\begin{pmatrix} g_{12} & g_{22} \\ -\frac{g_{12}^2 - K}{g_{22}} & -\, g_{12} \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix} = i_e \begin{pmatrix} w_x \\ w_y \end{pmatrix}. \tag{4}$$

where

$$w = u(x,y) + i_e v(x,y) + i_h r(x,y) + i_e i_h q(x,y) \Rightarrow w_x = u_x + i_e v_x + i_h r_x + i_e i_h q_x,$$

$$w_y = u_y + i_e v_y + i_h v_y + i_h r_y + i_e i_h q_y$$

(related to the solutions of system (4); see also [5]).

The system (4) is of: elliptic type if $g_{11}g_{22} - g_{12}^2 = k_1k_2 = K$ is positive; hyperbolic type if K is negative; parabolic type if K is equal to zero.

From the representation (2) it follows that if $g_{12}(x,y)=0$, then the point (x,y) is the point of rounding where it is not possible to define main directions. Therefore hereinafter it is assumed that $g_{12}\neq 0$. Thus, the following is proved

Lemma. *Let the coefficients of system (4) be defined in a domain D of a smooth surface $M\subset R^3$. Then at any point $O(x,y)\in D$, at which the main curvatures of the surface k_1,k_2 and θ are determined, the coefficients of system (4) (and, consequently, the tensor of Riemann's metric g_{ij}) are defined by system (2) at the point $O(x,y)\in D$. Conversely, if g_{ij} are determined , then the main curvatures of the surface k_1,k_2 and θ are calculated with formula (3) at the point $O(x,y)\in D$.*

Remark. If we divide by $\sqrt{|K|}$ equality (4) (for $K\neq 0$) and introduce the notations

$$\gamma=\frac{g_{11}}{\sqrt{|K|}},\ \beta=\frac{g_{12}}{\sqrt{|K|}},\ \alpha=\frac{g_{22}}{\sqrt{|K|}}, \tag{5}$$

then system (5) reads

$$u_x=\beta v_x+\gamma v_y,\ -u_y=\alpha v_x+\beta v_y,\ \alpha\gamma-\beta^2=signK,$$

or, in matrix form,

$$\begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2-signK}{\gamma} & -\beta \end{pmatrix}\begin{pmatrix} w_x \\ w_y \end{pmatrix}=i_e\begin{pmatrix} w_x \\ w_y \end{pmatrix}. \tag{6}$$

The equations systems (6) are called *mixed Beltrami equations* system. It belongs to: 1) elliptic type for $K>0$, 2) hyperbolic type vor $K<0$, 3) parabolic type for $K=0$.

For the surfaces of hyperbolical type, instead of (2) it is possible to use the following equalities system:

$$\begin{aligned} g_{11}&=k_1\cosh^2\theta+k_2\sinh^2\theta,\\ g_{12}&=(k_1-k_2)\cosh\theta\sinh\theta,\\ g_{22}&=k_1\sinh^2\theta+k_2\cosh^2\theta, \end{aligned} \tag{7}$$

where $g_{11}g_{22}-g_{12}^2=k_1k_2=K$, (theorema egregium of the surface), yielding

$$\tanh 2\theta=\frac{2g_{12}}{g_{11}+g_{22}},\ k_{1,2}=\frac{g_{11}-g_{22}\pm\sqrt{(g_{11}+g_{22})^2-4g_{12}^2}}{2}. \tag{8}$$

Mixed Beltrami equations system is denoted by *MES* (Mixed Equation System). Further we demonstrate that it is possible to obtain *MES* from the Beltrami equations system by using the statement of the famous main lemma.

From (2),(3), (6), (7) it follows an obvious, but very important

Theorem 1. *MES coefficients depend on surface points, at which they are defined, and do not depend on parameters of the introduced coordinate system.*

The next theorem can be considered as an analogue of Riemann's theorem on mappings, as a homeomorphic solution of *MES*.

Theorem 2. *Any three-dimensional closed smooth simply connected manifold with a smooth surface layer is diffeomorphic to the three-dimensional sphere and this diffeomorphism satisfies MES.*

Proof. Let M be the base of fiber bundle. It is a three-dimensional sphere S^3 in the space of quaternions H. In this case the fibration layers are the space of quaternions, i.e. we deal with quaternion fibration. The quaternions imaginary units are isomorphic to the matrices [7]

$$\mathbf{i} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv A^0 \equiv A_e^0, \ \ \mathbf{j} \to \begin{pmatrix} 0 & i_e \\ i_e & 0 \end{pmatrix} \equiv i_e B^0 \equiv B_e^0,$$

$$\mathbf{k} \to \begin{pmatrix} i_e & 0 \\ 0 & i_e \end{pmatrix} \equiv i_e A^0 B^0 \equiv A_e^0 B_e^0 \equiv C_e^0, \tag{9}$$

i.e. there exists the isomorphism $\mathbf{i} \to A_e^0, \mathbf{j} \to B_e^0, \mathbf{k} \to C_e^0$ between the imaginary units of quaternions and the operators.

Let $G^0 = \pm\{E, A_e^0, B_e^0, C_e^0\}$, where E is the unit matrix. It is known that G^0 is a group, where elements form the basis of the complex-valued 2×2 space of matrices.

Let M' be a three-dimensional smooth manifold, which can be considered the base of other fibration $p' : E' \to M'$ [8], where E' is a space, and let F' be the layer of this fibration. Assume that at on each point of M' the Riemann's metric (g_{ij}) is defined $\alpha = \alpha(x, y, z),\ \beta = \beta(x, y, z),\ \gamma = \gamma(x, y, z)$, where (x, y, z) are the coordinates of the points related to the atlas map of the manifold M'. Denote by $\widetilde{i},\ \widetilde{j},\ \widetilde{k}$ the generalized surface quaternions which are isomorphic to the matrices

$$\widetilde{i} \to \begin{pmatrix} \beta & \gamma \\ -\frac{\beta^2+1}{\gamma} & -\beta \end{pmatrix} \equiv A \equiv A_e, \ \widetilde{j} \to \begin{pmatrix} i_e\beta & i_e\gamma \\ -i_e\frac{\beta^2-1}{\gamma} & -i_e\beta \end{pmatrix} \equiv i_e B \equiv B_e,$$

$$\widetilde{k} \to \begin{pmatrix} i_e & 0 \\ -i_e\frac{2\beta}{\gamma} & -i_e \end{pmatrix} \equiv A_e B_e \equiv C_e. \tag{10}$$

Consequently, it is a fixed isomorphism between the imaginary units of the generalized quaternions and the matrices A_e, B_e, C_e. It is easy to prove that $A_e^2 = B_e^2 = C_e^2 = -E$. In terms of g_{ij}, equality (10) reads

$$\widetilde{i} \to \frac{1}{\sqrt{|K|}} \begin{pmatrix} g_{12} & g_{11} \\ -\frac{g_{12}^2+K}{g_{11}} & -g_{12} \end{pmatrix} \equiv I_e, \ \widetilde{j} \to \frac{i_e}{\sqrt{|K|}} \begin{pmatrix} g_{12} & g_{11} \\ -\frac{g_{12}^2-K}{g_{11}} & -g_{12} \end{pmatrix} \equiv J_e,$$

$$\widetilde{k} \to i_e \begin{pmatrix} 1 & 0 \\ -\frac{2g_{12}}{g_{11}} & -1 \end{pmatrix} \equiv K_e,$$

where K is the theorema egregium at the corresponding points of the manifold M'. The matrices $\pm\{E, I_e, J_e, K_e\}$, where E is the unit matrix, form a group. It is easy to prove that: 1) $I_e^2 = J_e^2 = K_e^2 = -E$; 2) $I_e J_e = K_e, \ J_e K_e = -I_e, \ K_e I_e = J_e$; 3) $det I_e = det J_e = det K_e = 1$.

However, for the sake of simplicity of calculations we use the identities (10), assuming $K = 1$. Consequently, it is established the isomorphism between the imaginary units of the generalized quaternions and the matrices A_e, B_e, C_e. Let $G \equiv \pm\{E, A_e, B_e, C_e\}$. Obviously, G forms a Lie group. Now, define the manifold corresponding to this group. As in the quaternion space the bases A_e^0, B_e^0, C_e^0 induce the families of orthonormalized repers at each point of S^3, the basis matrices A_e, B_e, C_e induce the triorthogonal system at the corresponding point of the manifold. The considered matrices act on the tangent planes M' and as the matrices A_e^0, B_e^0, C_e^0 act on the tangent planes of the sphere S^3. This statement follows from the representations (2), (6) of the *MES* coefficients. If $\beta = 0 \Rightarrow k_1 = k_2$, i.e the surface consists of rounding points, but it realizes in a two-dimensional sphere or in a two-dimensional plane on the Euclidean plane $k_1 = k_2 = 0$. However, the Cauchy-Riemann's system arises when of introducing the curvilinear system of coordinates into the plane (x, y). Moreover, from the representation (2), (6) we can conclude that the operators

A_e, B_e, A_eB_e act on tangent planes of the three-dimensional manifold, each point of which is characterized by the layer of manifold possessing main directions and curvatures and theorem egregium. These matrices possess the same properties of the multiplication table as the matrices A_e^0, B_e^0, C_e^0. From these matrices and unit matrix we have formed the group G. Obviously, assuming in $G : \beta = 0, \gamma = 1$ we obtain all elements of the group G^0 [5]. Conversely, it is possible to obtain G from G^0. We shall assume that the matrix $A_e^0 \in G^0$ corresponds to the matrix $T = \begin{pmatrix} t' & x' \\ y' & z' \end{pmatrix} \in G$.

If $(A_e^0)^2 = -E$, where E is the unit matrix, then the square of the corresponding to it matrix from G must be equal to the negative unit matrix too. We can obtain $z' = -t', y' = -\frac{t'^2+1}{x'}$ from it. If in matrix B_e^0 correspondingly we put the matrix $S = \begin{pmatrix} g_{12} & g_{11} \\ -\frac{g_{12}^2+K}{g_{11}} & -g_{12} \end{pmatrix}$ and equate its square to unit matrix, then we obtain $z'' = -t'', y'' = -\frac{t''^2-1}{x''}$.

Anticommutative relations between these matrices define $t' = t'', z' = z''$. Assuming $t' = \beta,\ x' = \gamma$ we obtain $T = A_e \in G,\ S = B_e$. In the same way the correspondence $A_e^0B_e^0 \to A_eB_e$ can be proved. Thus, G^0 and G are homeomorphic and the elements of these sets uniquely satisfy the group operations. Consequently, G^0 and G are isomorphic and form the Lie group. The elements of the group G act on the tangent planes of the manifold M'. These operators transform the field of tangent planes to itself, i.e. they are the symmetry operators. On one hand, these operators are the operators of *MES*. Consider these operators from another point of view. Any three-dimensional closed manifold M' is considered the Seifert's fibration only if it is possible to represent it in the form of lamination on a circumference. It means that M' is divided into non-intersecting in pairs circumferences, union of some parts of which define the neighborhoods which are isomorphic to the entire torus or nonorientable torus [2]. Relating to the map atlas, some manifolds act as a smooth surface. We will consider that the layer circumferences of Seifert's fibration are circumferences which define the main curvatures of the surface in two inter-orthogonal main directions at any point of M'. As a consequence of formulae (2), (3), (4) the operators A_e, B_e, C_e at these points of the manifold are essentially defined. From the fact that M' is representable in view of a fibration on circumferences it follows that the manifold M' is closed.

From the definition of the fibration $E' \to M'$ of the function gluings, define smooth mappings of the domain U_{ij} onto the group G, where $U_{ij} = U_i \bigcap U_j$ and U_i is the domain of base cover M'. Maps U_j are domains of definition of the functions $\alpha = \alpha(p), \beta = \beta(p), \gamma = \gamma(p), p \subset M'$.

The elements of the group G satisfy the conditions

$$1)\ A_eB_e = C_e = (B_eA_e)^{-1}, B_e(A_eB_e) = B_eC_e = ((A_eB_e)B_e)^{-1} = (C_eB_e)^{-1},$$

$$(A_eB_e)A_e = C_eA_e = (A_e(A_eB_e))^{-1} = (A_eC_e)^{-1}$$

$$2)\ A_eB_eB_e(A_eB_e)(A_eB_e)A_e = A_eB_eB_eC_eC_eA_e = -E, \tag{11}$$

$$3)\ detA_e = detB_e = detC_e = 1.$$

As the Lie groups are simultaneously manifold, the condition 2) is satisfied at the points of the threefold overlapping $U_{ijk} = U_i \bigcap U_j \bigcap U_k$. It is easily verified that the following relations for the Lie brackets of G group

$$\frac{1}{2}[A_e, B_e] = \frac{1}{2}(A_eB_e - B_eA_e) = A_eB_e \equiv C_e,$$

$$\frac{1}{2}[B_e, C_e] = \frac{1}{2}(B_eC_e - C_eB_e) = A_e, \tag{12}$$

$$\frac{1}{2}[C_e, A_e] - \frac{1}{2}(C_eA_e - A_eC_e) = B_e,$$

hold, i.e. the Lie brackets are expressed linearly by the elements of the same group G. The index e indicates that the corresponding points of the surface are of elliptic type. The corresponding operators are defined in the following way at the points of hyperbolic type: $A_h = i_eA,\ B_h = i_hB,\ C_h = i_ei_hC$.

Whence, the properties

$$1)\ A_h^2 = B_h^2 = C_h^2 = E;\ A_hB_h = C_h,\ B_hC_h = A_h,\ C_hA_h = B_h;$$

$$2)\ (A_hB_h)^{-1} = C_h^{-1} = A_hB_h,\ (B_hC_h)^{-1} = A_h^{-1} = B_hC_h,\ (C_hA_h)^{-1} = B_h^{-1} = C_hA_h; \tag{13}$$

$$3)\ A_hB_hB_hC_hC_hA_h = E; \qquad 4)\ detA_h = detB_h = detC_h = -1.$$

In the same way as in (11), condition 3) is satisfied at the points of the threefold overlapping $U_{ijk} = U_i \bigcap U_j \bigcap U_k$. The sign minus in the latest equality (11) and the signs plus in 1), 2) in (13) certify the existence of the spin structure [9], as they exist in any three-dimensional oriented manifolds.

From the equations systems (4), (6) and (10) we can conclude that the operators A, B are symmetry operators together with every following from them traces. From (13) we partially obtain the following properties of the operators A, B. Any operator A, B rotates ζ by 180° with respect to a certain center defined by the symmetry of the same operators. The operators A, B act on the fundamental domain - plane square. As after two times application of operator B to ζ, (see (4)), ζ is transformed into itself, it follows that the vector ζ is directed along the cylinder's parallel. The operator A if applied two times transforms ζ to $-\zeta$, and after its four times applications transforms ζ to itself, i.e. ζ is a spinor. We conclude that the flag ζ is directed to the parallel of the Möbius leaf. Consequently, a torus corresponds to the product $B \times B$, whereas a nonorientable torus corresponds to the product $A \times B$.

Formally, the operators B_e, C_e turned into the hyperbolic operators B, C by multiplying them by the imaginary unit i_e. This is why further we call B_e, C_e the operators of hyperbolic type.

Each point $p \subset M'$ is covered at least by two maps. From (7), (8) and the fact that U_i is the domain of definition of the functions α, β, γ, we deduce that (U_i, φ_i) defines one of the three matrices A_e, B_e, C_e rather than A_e, and from 1) (11) it follows that on U_i, the two matrices are defined. As the functions $w = u(p) + i_e v(p)$ in the representation (6) depend on the points of the base M', and do not depend on the introduced coordinate systems, it follows that the derivatives W_x, w_y are the Lie derivatives along two main directions of the base.

On the other hand, the operator B_e essentially arises from the following lemma.

Main lemma. [11] *If the linear operator $L : W \to W$ acting in complex or real linear space W is involute, i.e. $L^2 = E$, then its eigenvalues are equal to ± 1 and diagonalizable, i.e. the space W is considered the direct sum*

$W = W_+ \oplus W_-$, *of the eigenspace* W_+, *corresponding to the eigenvalue +1, and the eigenspace* W_- *corresponding to the eigenvalue -1.*

The operator A_e satisfies the condition of the main lemma, from the statement of which the operator B_e arises. In order to obtain the Lie group, considering its product we obtain the operator C_e. Thus, w possesses the representation

$$w = u(p) + i_e v(p) + i_h r(p) + i_e i_h q(p) = u(p) + i_e v(p) + i_e(r(p) - i_e q(p)).$$

This representation can be considered as a doubling of complex numbers with the imaginary unit i_h. The matrix A_e plays an important role in forming the group G. In the meanwhile the group G induces the triorthogonal system of coordinates. Therefore in (14) essentially occur the partial derivatives with respect to the two variables (x, y). From the invariance of the obtained results with respect to the transformations of coordinate systems, instead of the pair of variables (x, y) it is possible to consider any other pairs among (y, z), (z, x). Finally, we have three equations systems with six first order partial differential equations. All of them form the pair equations system in the three-dimensional manifold domain. In the three-dimensional space *MES* is represented as

$$D_m \begin{pmatrix} u_x + i_e v_x + i_h r_x + i_e i_h q_x \\ u_y + i_e v_y + i_h r_y + i_e i_h q_y \end{pmatrix} = i_m \begin{pmatrix} u_x + i_e v_x + i_h r_x + i_e i_h q_x \\ u_y + i_e v_y + i_h r_y + i_e i_h q_y \end{pmatrix}, \tag{14}$$

where $u, v, p, q : M' \to S^3$, i.e. they are functions of the points of the manifold M' and $u_x, v_x, \dots q_x$ are their Lie derivatives along the main directions M' in the definition point. D_m are real-valued functional matrices, and i_m are their eigenvalues. Simple calculations demonstrate that if $m = 1$ and $i_m = i_1 = i_e$, then D_1 turns into A, (see (10)), if $m = 2$ and $i_m = i_2 = i_h$, then D_2 turns into B, and if $m = 3$ and $i_m = i_3 = i_e i_h$, then we obtain the matrix D_3 in the form of $AB = C$ and conversely. System (14) consists of six equations whose pairs form one elliptic and two hyperbolic systems. Here the pairs $(u, v), (r, -q)$ satisfy the equations system of elliptic and conjugate to it elliptic type, the pairs $(u, r), (v, q)$ satisfy the equations system of hyperbolic and conjugate to it hyperbolic types, the pairs $(u, q), (v, -r)$ satisfy the second equations system hyperbolic and conjugate to hyperbolical types. We are interested in three-dimensional manifolds, therefore, one variable, for example r, can be considered dependent on total four variables and is subject to elimination. Therefore three types of brackets remain: $(u,v),(v,q),(q,u)$. Representing them as cartesian products of the corresponding maps and using (14) we obtain the mapping $F' \times U_{ij} \to G$ [12]. At every point $\mathbf{x} \in M'$ the functions $\beta = \beta(\mathbf{x}), \gamma = \gamma(\mathbf{x})$ are defined, and by means of (14), the layer F' is defined. For each fixed m, $m = 1,2,3$, the layer F' coincides with one of A_e, B_e, C_e at each point $\mathbf{x} \in M'$.

The fibration $p' : E' \to M'$ is called by us the generalized quaternion fibration. The system of equations (14) plays the same role for mappings in the three-dimensional space as in Riemann's theorem on quasiconformal mappings, Beltrami equations system and at $\beta = 0, \gamma = 1$ Cauchy-Riemann's equations system at conformal mappings in two-dimensional space. Let us come back to the Lie brackets. Assuming that at each point of the manifold M' the two-dimensional distribution A_e is defined, automatically using the abovestated procedure we obtain the distribution A_e, B_e, C_e, which satisfies relations (12). It means that the condition of Frobenius theorem is satisfied. From the theorem of Frobenius [12], [13] it follows that in manifold M' a layer is defined. It is known that a class of orthonormalized respects τ^3 in the fibration are extracted. Generally, from the anticommutative relations of the G group elements it follows that the defined layer is triorthogonal space families. Then, by Dupin's theorem, the surfaces of the triorthogonal systems intersect in pairs along the curvature lines of these surfaces. But it allows to build *MES* from which we obtain the matrices A_e, B_e, C_e. The elements of the group $G : A_e, B_e, C_e \in G$ correspond to three orthogonal (one elliptic, two hyperbolic) surface families which form a triorthogonal system. By means of the corresponding (4) *MES* these surfaces are mapped on three orthogonal planes round S^3. Essentially the 3-reper fibrations $p : E \to M$ [12] are defined, where the manifold E points are the pairs $(\mathbf{x}, \tau^3)$, consisting of the points $\mathbf{x} \in M$ and the tangent 3-reper τ^3 at the point $\mathbf{x}$. Here the layer F and a structural group G^0 coincide (main fibration). Taking into account that in the fibration $p' : E' \to M'$ the quaternion space is defined and consequently, any vector $\mathbf{\Omega} \in M'$ is represented in the form

$$\mathbf{\Omega} = G\mathbf{x} = Eu + A_e v + B_e r + C_e q = \begin{pmatrix} t^c & x^c \\ y^c & z^c \end{pmatrix} =$$

$$= \begin{pmatrix} u + \beta v + i_e \beta r + i_e q & \gamma(v + i_e r) \\ -\frac{\beta^2+1}{\gamma} v - \frac{\beta^2-1}{\gamma} r - 2i_e \frac{\beta}{\gamma} q & u - \beta v - i_e \beta r - i_e q \end{pmatrix}, \tag{15}$$

where $(u, v, r, q) \in S^3 \equiv M$ and $det\mathbf{\Omega} = 1$ (the sign shows that all these numbers are complex). By direct calculations we can find that $det\mathbf{\Omega} = u^2 + v^2 + r^2 + q^2 = 1$. This is the three-dimensional sphere S^3 in the four-dimensional space of quaternions H. Therefore, the real functions u, v, r, q are dependent. Solving the last matrix identity with respect to $(u, v, r, q) \in S^3 \equiv M$ we obtain

$$u = \frac{t^c + z^c}{2}, \ v = \frac{1}{\gamma} Rex^c, \ r = \frac{1}{\gamma} Imx^c, \ q = \frac{\gamma(t^c - z^c) - 2\beta x^c}{2\gamma i_e}. \tag{16}$$

We demonstrate that t^c, x^c, y^c, z^c form a three dimensional manifold. In general, the variables t^c, x^c, y^c, z^c are complex and possess eight real variables. From the last identity it follows that $t^c + z^c$ is constant, i.e. $Im(t^c + z^c) = 0$, which requires one condition from t^c, x^c, y^c, z^c. It is easy to calculate the equality

$$y = \frac{\beta^2 x^c - \overline{x}^c + \beta\gamma(z^c - t^c)}{\gamma^2}$$

from which it follows that the variable y is not considered to be dependent. It also requires two more conditions from t^c, x^c, y^c, z^c. Moreover, $det\mathbf{\Omega} = 1$ defines two conditions: $Re(t^c z^c - x^c y^c) = 1, \ Im(t^c z^c - x^c y^c) = 0$. We conclude that only three functions from the eight real and imaginary parts of the complex functions t^c, x^c, y^c, z^c are independent, namely: $t^c + z^c, Rex^c, Imx^c$. As for the triorthogonal system, at each point on M' there exist three orthogonal main directions along which the curvature lines are directed, i.e. coordinate lines. *BES* (4) satisfy these conditions which in matrix writing sets the matrix A_e. The operator A_e satisfies the conditions of the main lemma from statement of which the operator B_e arises. In the meanwhile, in order to obtain the Lie groups and considering their product we obtain the operator C_e. Consequently, the group G essentially arises only from the matrix A_e. In the meanwhile, the group G induces the triorthogonal system of coordinates.

Therefore, the partial derivatives with respect to x, y essentially participate in (14). The invariance of the obtained results concerning the coordinate system enables us to consider any other pairs among $(y, z), (z, x)$ instead of x, y.

As $v + i_e r$ realizes quasiconformal mappings of the maps $M' \supset U_i' \to U_k \subset S^3$ we can conclude that the second and the third equations of system (16) is solvable in U_k and looks like $x = x(v, r),\ y = y(v, r)$. Thus, at any point $(u, v, r, q) \in S^3 \equiv M$ we correspondingly put the points $(u, v, r, q, \alpha, \beta, \gamma) \subset R^6$ of the six-dimensional space. It means that the manifold M' is realized in the form of the three dimensional surface of six-dimensional space at least in a local manner [14]. Different points S^3 are correspondingly substituted by different points of the surface. Following this one-to-one correspondence we consider all possible points (t, x, y, z) corresponding to the points $(u, v, r, q) \in S^3 \equiv M$. The space of these points is denoted by M'. Simultaneously, as stated above, M' is the considered Seifert's fibration. Thus, M' is a three-dimensional closed manifold and any of the operators E, A_e, B_e, C_e acting on the tangent space M' form a cyclic group of four order cycle. Thus, the morphism between the fibrations

$$E' \to M',\ E \to M \equiv S^3,\ E \to E'$$

and an inverse mapping

$$M' \to E',\ M \to E,\ E' \to E,$$

i.e. the correspondence between the points M' and S^3 are established. Thus, the obtained two fibrations on the same base S^3 are quaternionic and generalized quaternionic.

In conclusion we note that Dirac's matrices and Yang-Mill's field can be generalized if we use in an appropriate way the matrices $B - i_e A,\ C = AB$ instead of Pauli's matrices.

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