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SEMIREFLEXIVE SUBCATEGORIES

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Abstract In the topological locally convex Hausdorff vector spaces category, the semireflexive subcategories - a categorial notion which generates some well-known cases of semireflexivity, are examined.

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1. INTRODUCTION

The results of the article are formulated and proved for the category $\mathcal{C}_2\mathcal{V}$ of topological locally convex Hausdorff vector spaces. We denote by $\mathbb{R}$ the lattice of the non-null reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$. Supposing that $\mathbb{R}_m$ is the sublattice of the lattice $\mathbb{R}$ of those $\mathcal{R}$ elements that possess the property: $\mathcal{R}$-replique of the category $\mathcal{C}_2\mathcal{V}$ objects can be realized in two steps - first the topology is weakened, second it is completed somehow. The definded semireflexive spaces have such a property defined in different ways.

In the lattice $\mathbb{R}$ two more complete sublattices are indicated.

$\mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}$, $\mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}$ where $\mathcal{S}$ (respectively Γ_0) is the subcategory of the weak topology spaces (respectively-complete).

In Section 2 some properties of the three lattices $\mathbb{R}_b$, $\mathbb{R}_p$ and $\mathbb{R}_m$ are examined. In Section 3 the next issues are discussed (3.5 - 3.8).

1. Which elements of the lattice $\mathbb{R}_m$ can be realized as a semireflexive product of one element of the lattice $\mathbb{R}_b$ and of one element of the lattice $\mathbb{R}_p$?

2. Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$, where $\mathcal{R} \in \mathbb{R}_b$, and $\mathcal{A} \in \mathbb{R}_p$. The subcategory $\mathcal{R}$ is compulsory c - reflective, does $\mathcal{S} \subset \mathcal{R}$ and that mean the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact?

3. Let $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$. Are the factors $\mathcal{R}$ and $\mathcal{A}$ determined in a unique way?

4. Let $\mathcal{R}_1$, $\mathcal{R}_2 \in \mathbb{R}_b$, $\Gamma_1, \Gamma_2 \in \mathbb{R}_p$ and $\mathcal{R}_1 \times_{sr} \Gamma_1 = \mathcal{R}_2 \times_{sr} \Gamma_2$. What are the relations of inclusion between subcategories $\mathcal{R}_1$ and $\mathcal{R}_2$ or Γ_1 and Γ_2?

1.1. TERMINOLOGY AND NOTATIONS IN LOCALLY CONVEX SPACES THEORY

The c-reflective subcategories were studied in [9] and [7]. Left and right products were defined and studied in [6]. Other authors results concerning semireflexive subcategories can be found in [2].

In the category $\mathcal{C}_2\mathcal{V}$ we consider the following bicategory structures:

$(\mathcal{E}pi, \mathcal{M}_f)$=(the class of epimorphisms, the class of strict monomorphisms);

$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of precise monomorphisms)=(the class of surjective mappings, the class of topological embedings);

$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of precise epimorphisms, the class of universal monomorphisms) [3], [7];

$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of strict epimorphisms, the class of monomorphisms).

We will consider the following subcategories:

Π, the subcategory of complete spaces with weak topology [8];

$\mathcal{S}$, the subcategory of spaces with weak topology [8];

$s\mathcal{N}$, the subcategory of strict nuclear spaces [5];

$\mathcal{N}$, the subcategory of nuclear spaces [10];

$\mathcal{S}c$, the subcategory of Schwartz spaces [8];

Γ_0, the subcategory of complete spaces [11];

$q\Gamma_0$, the subcategory of quasicomplete spaces [12];

$s\mathcal{R}$, the subcategory of semireflexive spaces [8];

$i\mathcal{R}$, the subcategory of inductive semireflexive spaces [4];

$\mathcal{M}$, the subcategory of spaces with Mackey topology [11];

The last subcategory is coreflective and the others are reflective.

Definition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is $\mathcal{B}$-hereditary, if $fg \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$.*

Dual notion: the class $\mathcal{B}$-cohereditary.

2. THE FACTORIZATION OF THE REFLECTOR FUNCTORS

The results of this section can be found in [1] in Russian (see also [14]).

2.1. The lattice $\mathbb{R}$ of the non-null subcategories of the category $\mathcal{C}_2\mathcal{V}$ is divided into three complete sublattices:

a) The sublattice $\mathbb{R}_b$ of $\mathcal{E}_u$-reflective subcategories. A subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective if the $\mathcal{R}$-replique of any object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Moreover,

$$\mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}.$$

b) The sublattice $\mathbb{R}_p$ of $\mathcal{M}_p$-reflective subcategories, the class of those reflective subcategories $\mathcal{R}$ for $\mathcal{R}$-replique for any object of the category $\mathcal{C}_2\mathcal{V}$ is a topological embeding:

$$\mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}.$$

c) $\mathbb{R}_m = (\mathbb{R} \setminus (\mathbb{R}_b \cup \mathbb{R}_p)) \cup \{\mathcal{C}_2\mathcal{V}\}$.

We mention that $\mathbb{R}_m$ is a complete sublattice with the first element Π and the last element $\mathcal{C}_2\mathcal{V}$.

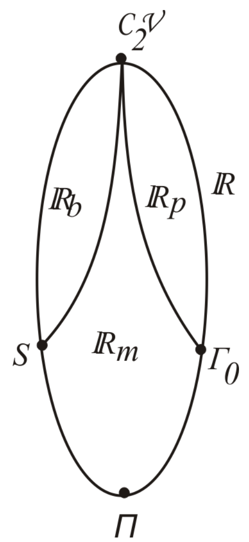

Figure 2.1

2.2. Let $\mathcal{L}$ be an element of lattice $\mathbb{R}_m$. For any object X of category $\mathcal{C}_2\mathcal{V}$ let

$$\begin{tikzcd} X \arrow[r, "b^X"] \arrow[rr, bend left, "l^X"] & bX \arrow[r, "p^X"] & lX \end{tikzcd}$$

Figure 2.2

$l^X : X \longrightarrow lX$ be its $\mathcal{L}$-replique, and $l^X = p^X b^X$ its $(\mathcal{E}_u, \mathcal{M}_p)$-factorization. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{L})$ the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ consisting of all bX form objects and those isomorphic to these. We also can say that $\mathcal{B}$ is the subcategory of all $\mathcal{M}_p$-subobjects of the objects $\mathcal{L}$. It is clear that $\mathcal{B}$ is a $\mathcal{E}_u$-reflective subcategory, and b^X is $\mathcal{B}$-replique of the objects X. Therefore $\mathcal{B} \in \mathbb{R}_b$.

2.3. Let $\Gamma'' = \Gamma''(\mathcal{L})$ be the full subcategory of all objects Y of the category $\mathcal{C}_2\mathcal{V}$, having the property:

Any morphism $f : bX \longrightarrow Y$ is extended through p^X:

$$f = gp^X$$

for some morphism g. The subcategory Γ'' is closed under $\mathcal{M}_f$-subobjects and products. Further, $\Gamma_0 \subset \Gamma''$, therefore $\Gamma'' \in \mathbb{R}_p$. It is obvious that p^X is Γ''-replique of the object bX.

We denote by $G(\mathcal{L})$ the class of all the $\mathcal{M}_p$-reflective subcategories for which p^X is the replique of the object bX. The class $G(\mathcal{L})$ has a minimal element

$$\Gamma' = \Gamma'(\mathcal{L}) = \cap\{\Gamma \mid \Gamma \in G(\mathcal{L})\}.$$

Thus $G(\mathcal{L})$ is a complete lattice with first element $\Gamma'(\mathcal{L})$ and the last element $\Gamma''(\mathcal{L})$.

We can write

$$G(\mathcal{L}) = \{\Gamma \in \mathbb{R}_p \mid \Gamma'(\mathcal{L}) \subset \Gamma \subset \Gamma''(\mathcal{L})\}.$$

2.4. For any element $\Gamma \in G(\mathcal{L})$ the morphism p^X is Γ-replique of the object bX. Therefore if $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{B}$ and $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ are the reflective functors, then

$$l = gb.$$

Theorem. *Let* $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ *and* $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ *be two reflective functors with* $\mathcal{R} \in \mathbb{R}_b$ *and* $\Gamma \in \mathbb{R}_p$. *The following affirmations are equivalent:*

1. $l = gr$.
2. $\mathcal{R} = \mathcal{B}$ *and* $\Gamma \in G(\mathcal{L})$.

2.5. Example. Let us examine the case $\mathcal{L} = \Pi$. Then

$$\mathcal{B}(\Pi) = S$$

and

$$\Gamma'(\Pi) = \Gamma_0.$$

Theorem. *The subcategory* $\Gamma''(\Pi)$ *contains all the normal spaces.*

Proof. Let X be a weak topology space: $X \in\mid \mathcal{S} \mid$, and $g_0^X : X \longrightarrow g_0X$ its Γ_0-replique. Then g_0^X is also the Π-replique of object X. In this case $g_0X \sim K^\tau$, where K is the field of numbers over which the vector spaces from the category $\mathcal{C}_2\mathcal{V} : K = R$ or $K = \mathbb{C}$ are examined. Let $f : X \longrightarrow Y \hookrightarrow \widehat{Y}$, where Y is a normal space, and $\widehat{Y}$ is his completion.

$$\begin{tikzcd}
X \arrow[r, "g_0^X"] \arrow[d, "f"'] & g_0X=K^\tau \arrow[d, dashed, "g"] \arrow[dl, dashed, "g_1"'] \\
Y \arrow[r, "i"] & \hat{Y}
\end{tikzcd}$$

Figure 1.3

Then

$$if = gg_0^X$$

for some morphism g, where i is a canonical embedding. Since $g_0X \sim K^\tau$, we conclude that $g(g_0X)$ is a finite dimensional subspace in $\widehat{Y}$. Then subspace $f(X)$ of the Y space as a finite dimensional space is complete and

$$f = g_1g_0^X$$

for some morphism g_1. The theorem is proved.

2.6. Let X and Y be two normal incomplete subspaces, the algebraical dimension of which is:

$$\aleph_0 \leq dimX < dimY$$

Let Γ_1 (respectively Γ_2) be the smallest reflective subcategory which contains the subcategory Γ_0 and X space (respectively Y space). Then the subcategory Γ_1 is not contained in the subcategory Γ_2.

Theorem. *Lattice $G(\Pi)$ contains a proper class of elements.*

2.7. Remark. *On another side we have*

$$\mathcal{B}(\mathcal{C}_2\mathcal{V}) = \mathcal{C}_2\mathcal{V}, \quad G(\mathcal{C}_2\mathcal{V}) = \{\mathcal{C}_2\mathcal{V}\}.$$

3. SEMIREFLEXIVE SUBCATEGORIES

3.1. Definition. *Let $\mathcal{R}$ and $\mathcal{A}$ be two subcategories of the category $\mathcal{C}_2\mathcal{V}$, where $\mathcal{R}$ is a reflective subcategory. Object X of the category $\mathcal{C}_2\mathcal{V}$ is called $(\mathcal{R}, \mathcal{A})$-semireflexive, if his $\mathcal{R}$-replique belongs to the subcategory $\mathcal{A}$.*

We denote by

$$\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$$

the subcategory of all $(\mathcal{R}, \mathcal{A})$-semireflexive objects. The subcategory $\mathcal{L}$ is called the semireflexive product of the subcategories $\mathcal{R}$ and $\mathcal{A}$.

3.2 In the lattice $\mathbb{R}$ there are elements $\mathcal{R}$ such that their reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact. These kind of elements that belong to sublattice $\mathbb{R}_b$ are called the c-reflective subcategories.

The subcategories $\mathcal{S}$, $s\mathcal{N}$ and $\mathcal{S}c$ are c-reflective. The subcategory $\mathcal{N}$ belongs to class $\mathbb{R}_b$ but is not c-reflective.

We mention that also in lattices $\mathbb{R}_p$ and $\mathbb{R}_m$ there are elements of which reflector functor is left exact. For example, the functors

$$g_0 : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_0,$$

$$\pi : \mathcal{C}_2\mathcal{V} \longrightarrow \Pi$$

have this property.

3.3. Theorem. *Let $\mathcal{R}$ and $\mathcal{A}$ be two reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$ and the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact. Then the subcategory*

$$\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A}$$

is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.

Proof. It is easily to verify that $\mathcal{L}$ is closed under $\mathcal{M}_f$-subobjects and products (see [2]). So it is reflective.

3.4. From the definition we can deduce:

1. Let $\mathcal{R}$, $\mathcal{A} \in \mathbb{R}_b$. Then $\mathcal{S} \subset \mathcal{R} \times_{sr} \mathcal{A}$.
2. Let $\mathcal{R}$, $\mathcal{A} \in \mathbb{R}_p$. Then $\Gamma_0 \subset \mathcal{R} \times_{sr} \mathcal{A}$.
3. Let $\mathcal{R} \in \mathbb{R}_b$, $\mathcal{A} \in \mathbb{R}_p$ and $\mathcal{R} \times_{sr} \mathcal{A}$ be a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. As a rule $\mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$.

3.5. Well known examples of semireflexive subcategories are represented by a semireflexive product of an element of the lattice $\mathbb{R}_b$ and of one element of the lattice $\mathbb{R}_p$. Thus we formulate the following problem.

Problem. *Which elements of the lattice* $\mathbb{R}_m$ *can be realized as a semireflexive product of one element of the lattice* $\mathbb{R}_b$ *and of one element of the lattice* $\mathbb{R}_p$*?*

3.6. Another problem concerning this topic is the following one.

Problem. *Let* $\mathcal{L} = \mathcal{R} \times_{sr} \mathcal{A} \in \mathbb{R}_m$*, where* $\mathcal{R} \in \mathbb{R}_b$*, and* $\mathcal{A} \in \mathbb{R}_p$*. Is the subcategory* $\mathcal{R}$ *necessarily c-reflective?*

3.7. Problem. *Let* $\mathcal{L} = \mathcal{R}\times_{sr}\mathcal{A} \in \mathbb{R}_m$*. Are the factors* $\mathcal{R}$ *and* $\mathcal{A}$ *determined in a unique way?*

3.8. Problem. *Let* $\mathcal{R}_1$*,* $\mathcal{R}_2 \in \mathbb{R}_b$*,* Γ_1*,* $\Gamma_2 \in \mathbb{R}_p$ *and* $\mathcal{R}_1 \times_{sr} \Gamma_1 = \mathcal{R}_2 \times_{sr} \Gamma_2$*. What relations of inclusion are between the subcategories* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *or* Γ_1 *and* Γ_2*?*

3.9. Let (E, t) be a locally convex Hausdorff space, $m(t)$- Mackey topology [11] compatible with t topology. Thus $(E, m(t))$ is $\mathcal{M}$-coreplique of the object (E, t). For the elements of the lattice $\mathbb{R}$ we will analyze the following condition

$(\mathcal{SR})$. Let $(E, t) \in\mid \mathcal{L} \mid$, $\mathcal{L} \in \mathbb{R}$. Then for any locally convex topology u on the vector spaces E

$$t \leq u \leq m(t),$$

the space (E, u) belongs to the subcategory $\mathcal{L}$.

3.10 Categorial, the condition $(\mathcal{SR})$ can be written this way

$(\mathcal{SR})$. Let $X \in\mid \mathcal{L} \mid$, and $b : Y \longrightarrow X \in \mathcal{E}_u \cap \mathcal{M}_u$. Then $Y \in\mid \mathcal{L} \mid$.

3.11. a) In the lattice $\mathbb{R}_b$ the elements $\mathcal{S}$, $s\mathcal{N}$, $\mathcal{N}$, $\mathcal{S}c$ do not satisfy the condition $(\mathcal{SR})$. There are elements that satisfy this condition.

b) In the lattice $\mathbb{R}_m$ there are both elements that have the $(\mathcal{SR})$ property and elements that do not have this property.

The subcategory Π has the $(\mathcal{SR})$ property.

Indeed, let $(E, t) \in \mid \Pi \mid$. Then the topology t is a Mackey topology: $t = m(t)$ [12].

3.12. Theorem. *Any element of the lattice* $\mathbb{R}_p$ *has the property* $(\mathcal{SR})$.

Proof. Let $\Gamma \in \mathbb{R}_p$, $X \in \mid \Gamma \mid$ and $b : Y \longrightarrow X \in \mathcal{E}_u \cap \mathcal{M}_u$. Further, let $g^Y : Y \longrightarrow gY$ be the Γ-replique of the object Y. Then

$$b = fg^Y$$

for some morphism f.

$$\begin{tikzcd}
Y \arrow[rr, "g^Y"] \arrow[drr, "b"'] & & gY \arrow[d, dashed, "f"] \\
& & X
\end{tikzcd}$$

Figure 2.1

Since $b \in \mathcal{M}_u$, $g^Y \in \mathcal{E}pi$ and the class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary it follows that $f \in \mathcal{M}_u$. Also, from the above equality it follows that $f \in \mathcal{E}_u$. Thus in this equality the mappings b and f are bijections. So g^Y also is a bijection, in particulary $g^Y \in \mathcal{E}_u$. Therefore $g^Y \in \mathcal{M}_p \cap \mathcal{E}_u = \mathcal{I}so$. The theorem is proved.

3.13. Theorem. *Given an element* $\mathcal{L} \in \mathbb{R}_m$, *the following affirmations are equivalent*

1. *the subcategory* $\mathcal{L}$ *satisfies condition* $(\mathcal{SR})$*;*
2. $\mathcal{L} = \mathcal{B} \times_{sr} \Gamma$, *where* $\mathcal{B} = \mathcal{B}(\mathcal{L})$ *and* $\Gamma \in G(\mathcal{L})$*;*
3. *there is an element* $\mathcal{R} \in \mathbb{R}_b$ *and an element* $\Gamma \in \mathbb{R}_p$ *such that* $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$.

Proof. We prove the following implications $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

$1 \Longrightarrow 2$. We verify the embedding $\mathcal{L} \subset \mathcal{B} \times_{sr} \Gamma$. Let $X \in \mid \mathcal{L} \mid$. Then in $(\mathcal{E}_u, \mathcal{M}_p)$-factorization of the morphism $l^X = p^X b^X$ both b^X and p^X morphisms are isomorphisms.

$$\begin{tikzcd} X \arrow[r, "b^X=1"] \arrow[rr, bend left, "l^X=1"] & bX \arrow[r, "p^X=1"] & lX \end{tikzcd}$$

Figure 2.2

Thus $bX \in\mid \Gamma \mid$. So $X \in\mid \mathcal{B} \times_{sr} \Gamma \mid$.

Converse. We verify the embedding $\mathcal{B} \times_{sr} \Gamma \subset \mathcal{L}$. Let $(E, t) \in\mid \mathcal{B} \times_{sr} \Gamma \mid$. Then $b(E, t) = (E, b(t)) \in\mid \Gamma \mid$, where the topologies t and $b(t)$ are compatible with the same duality.

$$\begin{tikzcd} (E,t) \arrow[r, "b^E"] \arrow[rr, bend left, "l^E"] & (E,b(t)) \arrow[r, "p^E"] & l(E,t) \end{tikzcd}$$

Figure 2.3

Thus $p^E \in \mathcal{I}so$, and $(E, b(t)) \in\mid \mathcal{L} \mid$. By condition $(\mathcal{SR})$, we have $(E, t) \in\mid \mathcal{L} \mid$.

$2 \Longrightarrow 3$. Obviously.

$3 \Longrightarrow 1$. Let $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$, $(E, t) \in\mid \mathcal{L} \mid$, and (E, u) be a locally convex space where $t \leq u \leq m(t)$

$$\begin{tikzcd} & & (E,r(u))=r(E,u) \arrow[dr] & \\ (E,m(t)) \arrow[r] & (E,u) \arrow[ur] \arrow[r] & (E,t) \arrow[r] & (E,r(t)) \end{tikzcd}$$

Figure 2.4

Let $(E, r(u))$ be the $\mathcal{R}$-replique of the object (E, u). Then

$$r(t) \leq r(u) \leq m(t)$$

and Theorem 3.12 implies that the space $(E, r(u))$ belongs to the subcategory Γ. So $(E, u) \in\mid \mathcal{L} \mid$. Theorem is proved.

3.14. Theorem. *Assume $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3$, where $\mathcal{R}_i \in \mathcal{R}_b$, $i = 1, 2, 3$, and $\Gamma \in \mathbb{R}_p$. Then*

1. $\mathcal{R}_1 \times_{sr} \Gamma \subset \mathcal{R}_2 \times_{sr} \Gamma$;
2. *if* $\mathcal{R}_1 \times_{sr} \Gamma = \mathcal{R}_3 \times_{sr} \Gamma$, *then* $\mathcal{R}_1 \times_{sr} \Gamma = \mathcal{R}_2 \times_{sr} \Gamma$ *also.*

Proof. Let X be a object of the category $\mathcal{C}_2\mathcal{V}$. Since $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3$, we deduce that between the respective repliques of the object X the following relations

$$\begin{tikzcd}
X \arrow[r, "r_3^X"] \arrow[rrr, bend left, "r_1^X"] \arrow[rr, bend right, "r_2^X"'] & r_3X \arrow[r, dashed, "f"] & r_2X \arrow[r, dashed, "g"] & r_1X
\end{tikzcd}$$

Figure 2.5

$$r_2^X = fr_3^X, \tag{1}$$

$$r_1^X = gr_2^X = gfr_3^X, \tag{2}$$

for some morphisms f and g, hold.

1. Let $X \in\mid \mathcal{R}_1 \times_{sr} \Gamma \mid$. Then $r_1X \in\mid \Gamma \mid$, and from equality (2), and Theorem 3.12 we deduce that $r_2X \in\mid \Gamma \mid$. Thus $X \in\mid \mathcal{R}_2 \times_{sr} \Gamma \mid$.

2. Assume $X \in\mid \mathcal{R}_2 \times_{sr} \Gamma \mid$. Then $r_2X \in\mid \Gamma \mid$, and from Theorem 3.12 and equality (1) it follows that $r_3X \in\mid \Gamma \mid$, so $X \in\mid \mathcal{R}_3 \times_{sr} \Gamma \mid$.

3.15. Theorem. *For any reflective subcategory $\mathcal{R}$ with the property $\mathcal{S} \subset \mathcal{R} \subset \mathcal{N}$, we have*

$$\mathcal{R} \times_{sr} q\Gamma_0 = s\mathcal{R},$$

in particular,

$$\mathcal{S} \times_{sr} q\Gamma_0 = s\mathcal{N} \times_{sr} q\Gamma_0 = \mathcal{N} \times_{sr} q\Gamma_0 = s\mathcal{R}.$$

Proof. Following the previous theorem, it is enough to prove that $\mathcal{N} \times_{sr} q\Gamma_0 = s\mathcal{R}$, since, from the definition of the subcategory $s\mathcal{R}$ we have

$$\mathcal{S} \times_{sr} q\Gamma_0 = s\mathcal{R}.$$

Let $X \in\mid \mathcal{N} \times_{sr} q\Gamma_0 \mid$. Then $\mathcal{N}$-replique nX of the object X belongs to the subcategory $q\Gamma_0$.

Thus nX is a quasicomplete nuclear space. So it is semireflexive ([12] III 7.2. corollary 2, and also [12] IV 5.8 example 4).

3.16. Theorem. *Assume that $\mathcal{R} \in \mathbb{R}_b$, $\Gamma, \Gamma_1 \in \mathbb{R}_p$, $\Gamma \subset \Gamma_1$, and $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$, $g_1 : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_1$ are the reflector functors.*

1. If $\mathcal{R}$ is a c-reflective subcategory, then $g(\mathcal{R}) \subset \mathcal{R}$.

2. If $g(\mathcal{R}) \subset \mathcal{R}$, then $g_1(\mathcal{R}) \subset \mathcal{R}$.

Proof. 1. Let $X \in\mid \mathcal{R} \mid$, $g^X : X \longrightarrow gX$ be the Γ-replique of the object X, and $r^{gX} : gX \longrightarrow rgX$ the $\mathcal{R}$-replique of object gX. Then

$$r^{gX} g^X = r(g^X) \in \mathcal{M}_p,$$

since $r(\mathcal{M}_p) \subset \mathcal{M}_p$ for a c-reflective subcategory ([2], theorem 2.8). In the above equality $r(g^X) \in \mathcal{M}_p$, and the class $\mathcal{M}_p$ is the $\mathcal{E}pi$-cohereditary. So $r^{gX} \in \mathcal{M}_p \cap \mathcal{E}_u = \mathcal{I}so$.

$$\begin{tikzcd}
X \arrow[r, "g^X"] \arrow[d, equal] & gX \arrow[d, "r^{gX}"] \\
X{=}rX \arrow[r, "r(g^X)"] & rgX
\end{tikzcd}$$

Figure 2.6

2. Let $X \in\mid \mathcal{R} \mid$, and $g^X : X \longrightarrow gX$ and $g_1^X : X \longrightarrow g_1X$ be the respective repliques of the object X. Since $\Gamma \subset \Gamma_1$ it follows that

$$g^X = f g_1^X$$

for some morphism f.

$$\begin{tikzcd}
X \arrow[r, "g_1^X"] \arrow[dr, "g^X"'] & g_1X \arrow[d, dashed, "f"] \\
 & gX
\end{tikzcd}$$

Figure 2.7

Just like above we deduce that $f \in \mathcal{M}_p$. The hypothesis implies that $gX \in\mid \mathcal{R} \mid$ and $\mathcal{R}$ is a $\mathcal{E}_u$-reflective subcategory. So it is closed under $\mathcal{M}_p$-subobjects. It follows $g_1 X \in\mid \mathcal{R} \mid$.

3.17. Theorem. *Assume*

$$\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$$

where $\mathcal{R} \in \mathbb{R}_b$ *and* $\Gamma \in \mathbb{R}_p$. *If* $g(\mathcal{R}) \subset \mathcal{R}$, *then*

$$\mathcal{R} \subset \mathcal{B} = \mathcal{B}(\mathcal{L}).$$

Proof. Let X be an arbitrary object of the category $\mathcal{C}_2\mathcal{V}$, $r^X : X \longrightarrow rX$ and $g^{rX} : rX \longrightarrow grX$ - $\mathcal{R}$ and Γ-replique of the respective objects.

$$\begin{tikzcd}
X \arrow[r, "b^X"] \arrow[dr, "l^X"] \arrow[dd, "r^X"'] & bX \arrow[d, "p^X"] \arrow[ddl, dashed, "t"'] \\
 & lX \arrow[d, dashed, "f"] \\
rX \arrow[r, "g^{rX}"] & grX
\end{tikzcd}$$

Figure 2.8

Since $grX \in\mid \mathcal{R} \mid$ we deduce that $grX \in\mid \mathcal{L} \mid$.

Thus

$$fl^X = g^{rX} r^X \tag{1}$$

for some morphism f. Supposing that the $\mathcal{E}_u, \mathcal{M}_p$-factorization of the morphism l^X,

$$l^X = p^X b^X \tag{3}$$

holds, we deduce

$$g^{rX} r^X = f p^X b^X \tag{4}$$

where $b^X \in \mathcal{E}_u$, and $g^{rX} \in \mathcal{M}_p$, i.e. $b^X \perp g^{rX}$. Thus

$$r^X = tb^X, \tag{5}$$

for some morphism t,

$$g^{rX}t = fp^X. \tag{6}$$

The equality (4) indicates that $\mathcal{R} \subset \mathcal{B}$.

3.18. Conclusions. Returning to problems 3.5-3.8 we can make the following assertions.

1. The $\mathcal{L}$ elements of the lattice $\mathbb{R}_m$ can be presented as a semireflexive product

$$\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$$

with $\mathcal{R} \in \mathbb{R}_b$ and $\Gamma \in \mathbb{R}_p$ having the property $(\mathcal{SR})$ (Theorem 3.13).

2. $s\mathcal{R} = \mathcal{N} \times_{sr} q\Gamma_0$ and $\mathcal{N}$ is not a c-reflective subcategory.

3. Let $\mathcal{L} = \mathcal{R} \times_{sr} \Gamma$. Then neither the first nor the second factor is determined in a unique way (Theorem 3.13 and 3.15).

4. A partially answer is given to question 3.8 by Theorem 3.17.

2.19. Examples. The right product of two subcategories and following examples are examined in more detail in the article [2].

1. Since $(\mathcal{M}, \mathcal{S})$ is a pair of conjugated subcategories in the category $\mathcal{C}_2\mathcal{V}$ and $\Pi = \mathcal{S} \cap \Gamma_0$ we have ([2])

$$\mathcal{S} \times_{sr} \Gamma_0 = \mathcal{M} \times_d \Pi.$$

2. Let $q\Gamma_0$ be a subcategory of the quasicomplete spaces, and $s\mathcal{R}$ the subcategory of the semireflexive spaces [12]. Then

$$\mathcal{S} \times_{sr} (q\Gamma_0) = \mathcal{M} \times_d (\mathcal{S} \cap q\Gamma_0) = s\mathcal{R}.$$

3. The subcategory $\mathcal{S}c$ of Schwartz spaces is c-reflective. Let $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ for which $(\mathcal{K}, \mathcal{S}c)$ is a pair of conjugated subcategories. Then

$$\mathcal{S}c \times_{sr} \Gamma_0 = i\mathcal{R} = \mathcal{K} \times_d (\mathcal{S}c \cap \Gamma_0);$$

$i\mathcal{R}$ is a subcategory of the inductive semireflexive spaces ([4], theorem 1.5).

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