SOLVING HIGHER-ORDER FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY

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Abstract
Higher-order fuzzy differential equations with initial value conditions are considered. We apply the new results to the particular case of second-order fuzzy linear differential equation.

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1. PRELIMINARIES

Denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis satisfying the following properties:

(i) $\forall u \in \mathbb{R}_F$, $u$ is normal, i.e. there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$;
(ii) $\forall u \in \mathbb{R}_F$, $u$ is fuzzy convex set;
(iii) $\forall u \in \mathbb{R}_F$, $u$ is upper semicontinuous on $\mathbb{R}$;
(iv) $cl\{s \in \mathbb{R}|u(s) > 0\}$ is compact, where $cl$ denotes the closure of a subset.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers.

The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}$, $D(u, v) = \sup_{\alpha \in [0,1]} \max\{|u^\alpha - v^\alpha|, |u^{\alpha} - v^{\alpha}|\}$.

2. GENERALIZED DIFFERENTIABILITY

In [1] a more general definition of derivative for fuzzy-number-valued functions is introduced. Using this differentiability concept we have the following definition.
Definition 2.1. Let $F : I \to \mathbb{R}_F$ and $m, n = 1, 2$. We say that $F$ is $(m)$-differentiable on $I$ if $F$ is differentiable as in Definition 5(m)[1] and its derivative is denoted by $D^1_m F$. Also, if $D^1_m F$ is $(n)$-differentiable as a fuzzy function, $D^2_{m,n} F$ denotes its second derivative on $I$.

Theorem 2.1. Let $D^1_1 F : I \to \mathbb{R}_F$ and $D^1_2 F : I \to \mathbb{R}_F$ be fuzzy functions.

(i) If $D^1_1 F$ is $(1)$-differentiable, then $f'_\alpha$ and $g'_\alpha$ are differentiable functions and $[D^1_{1,1} F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

(ii) If $D^1_2 F$ is $(2)$-differentiable, then $f'_\alpha$ and $g'_\alpha$ are differentiable functions and $[D^1_{2,2} F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.

(iii) If $D^1_2 F$ is $(1)$-differentiable, then $f'_\alpha$ and $g'_\alpha$ are differentiable functions and $[D^2_{2,1} F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.

(iv) If $D^1_2 F$ is $(2)$-differentiable, then $f'_\alpha$ and $g'_\alpha$ are differentiable functions and $[D^2_{2,2} F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

3. **Solving Fuzzy Differential Equations**

Consider the Cauchy problem for the second-order fuzzy differential equations

$$ y''(t) + ay'(t) + by(t) = \sigma(t), \quad y(0) = \gamma_0, \quad y'(0) = \gamma_1, \quad (1) $$

where $a, b \in \mathbb{R}$ and $\sigma(t)$ is a fuzzy function on some interval $I$. The interval $I$ can be $[0, A]$ for some $A > 0$ or $I = [0, \infty)$.

Our strategy of solving (1) is based on the choice of the derivative in the fuzzy differential equation. In order to solve (1) we have three steps: first we choose the type of derivative and change problem (1) to a system of ODE by using Theorem (2.2) and considering initial values. Second we solve the obtained ODE system. The final step is to find such a domain in which the solution and its derivatives have valid level sets. In view of these we propose the following definition.

Definition 3.1. Let $y : I \to \mathbb{R}_F$ be a fuzzy function such that $D^1_{m} y$ and $D^2_{m,n} y$ exist for $m, n = 1, 2$ on $I$. If $y, D^1_{m} y$ and $D^2_{m,n} y$ satisfy problem (1) we say that $y$ is a $(m,n)$-solution of problem (1).
We have the following alternatives for solving problem (1):

**1,1**-Solution: If we consider $y'(t)$ using the (1)-derivative and then its derivative, $y''(t)$ using the (1)-derivative we have

$$[y''(t)]^\alpha = [\widehat{y''}(t; \alpha), \overline{y''}(t; \alpha)]$$

and

$$[y'(t)]^\alpha = [\widehat{y'}(t; \alpha), \overline{y'}(t; \alpha)].$$

Now we proceed as follows:

(i) solve the differential system

\[
\begin{align*}
    y''(t; \alpha) + ay'(t; \alpha) + by(t; \alpha) &= \sigma(t; \alpha), \\
    \overline{y''}(t; \alpha) + a\overline{y'}(t; \alpha) + b\overline{y}(t; \alpha) &= \overline{\sigma}(t; \alpha), \\
    y(0) = \gamma_0, \overline{y}(0) = \overline{\gamma}_0, y'(0) = \gamma_1, \overline{y'}(0) = \overline{\gamma}_1,
\end{align*}
\]

for $y$ and $\overline{y}$.

(ii) ensure that $[\widehat{y}(t; \alpha), \overline{y}(t; \alpha)]$, $[\widehat{y'}(t; \alpha), \overline{y'}(t; \alpha)]$ and $[\widehat{y''}(t; \alpha), \overline{y''}(t; \alpha)]$ are valid level sets.

Other cases are similar to (1,1)-solution.

**Remark 3.1.** The solution of FDE (1) depends upon the selection of derivatives, in the first or second form.

**Example 3.1.** Consider the second order fuzzy initial value problem

$$y''(t) = \sigma_0, \quad y(0) = \gamma_0, \quad y'(0) = \gamma_1 \quad t \geq 0,$$

where $\sigma_0 = \gamma_0 = \gamma_1 = [\alpha - 1, 1 - \alpha]$. It possesses 4 different solutions.

In order to extend the results to Nth-order fuzzy differential equations, we can follow the proof of Theorem 2.2 to get the same results for derivatives of an arbitrary order $N$. We have at most $2^N$ solutions for a Nth-order fuzzy differential equation by choosing the different types of derivatives.

**References**
