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SOLVING HIGHER-ORDER FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY

A. Khastan, F. Bahrami, K. Ivaz

Department of Applied Mathematics, University of Tabriz, Iran

khastan@gmail.com

Abstract Higher-order fuzzy differential equations with initial value conditions are considered. We apply the new results to the particular case of second-order fuzzy linear differential equation.

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1. PRELIMINARIES

Denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis satisfying the following properties:
(i) $\forall u \in \mathbb{R}_F$, u is normal, i.e. there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$;
(ii) $\forall u \in \mathbb{R}_F$, u is fuzzy convex set;
(iii) $\forall u \in \mathbb{R}_F$, u is upper semicontinuous on $\mathbb{R}$;
(iv) $cl\{s \in \mathbb{R} | u(s) > 0\}$ is compact, where cl denotes the closure of a subset.
Then $\mathbb{R}_F$ is called the space of fuzzy numbers.
The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$, $D(u, v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$.

2. GENERALIZED DIFFERENTIABILITY

In [1] a more general definition of derivative for fuzzy-number-valued functions is introduced. Using this differentiability concept we have the following definition.

Definition 2.1. *Let $F: I \to \mathbb{R}_F$ and $m, n = 1, 2$. We say that F is (m)-differentiable on I if F is differentiable as in Definition 5(m)[1] and its derivative is denoted by $D^1_m F$. Also, if $D^1_m F$ is (n)-differentiable as a fuzzy function, $D^2_{m,n} F$ denotes its second derivative on I.*

Theorem 2.1. *Let $D^1_1 F: I \to \mathbb{R}_F$ and $D^1_2 F: I \to \mathbb{R}_F$ be fuzzy functions.*
(i) If $D^1_1 F$ is (1)-differentiable, then f'_α and g'_α are differentiable functions and $[D^2_{1,1} F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.
(ii) If $D^1_1 F$ is (2)-differentiable, then f'_α and g'_α are differentiable functions and $[D^2_{1,2} F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.
(iii) If $D^1_2 F$ is (1)-differentiable, then f'_α and g'_α are differentiable functions and $[D^2_{2,1} F(t)]^\alpha = [g''_\alpha(t), f''_\alpha(t)]$.
(iv) If $D^1_2 F$ is (2)-differentiable, then f'_α and g'_α are differentiable functions and $[D^2_{2,2} F(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

3. SOLVING FUZZY DIFFERENTIAL EQUATIONS

Consider the Cauchy problem for the second-order fuzzy differential equations

$$y''(t) + ay'(t) + by(t) = \sigma(t), \quad y(0) = \gamma_0, \quad y'(0) = \gamma_1, \tag{1}$$

where $a, b \in \mathbb{R}$ and $\sigma(t)$ is a fuzzy function on some interval I. The interval I can be $[0, A]$ for some $A > 0$ or $I = [0, \infty)$.
Our strategy of solving (1) is based on the choice of the derivative in the fuzzy differential equation. In order to solve (1) we have three steps: first we choose the type of derivative and change problem (1) to a system of ODE by using Theorem (2.2) and considering initial values. Second we solve the obtained ODE system. The final step is to find such a domain in which the solution and its derivatives have valid level sets. In view of these we propose the following definition.

Definition 3.1. *Let $y: I \to \mathbb{R}_F$ be a fuzzy function such that $D^1_m y$ and $D^2_{m,n} y$ exist for $m, n = 1, 2$ on I. If $y, D^1_m y$ and $D^2_{m,n} y$ satisfy problem (1) we say that y is a (m,n)-solution of problem (1).*

We have the following alternatives for solving problem (1):

(1,1)-Solution: If we consider $y'(t)$ using the (1)-derivative and then its derivative, $y''(t)$ using the (1)-derivative we have

$$[y''(t)]^\alpha = [\underline{y}''(t;\alpha), \overline{y}''(t;\alpha)] \quad \textit{and} \quad [y'(t)]^\alpha = [\underline{y}'(t;\alpha), \overline{y}'(t;\alpha)].$$

Now we proceed as follows:

(i) solve the differential system

$$\begin{cases} \underline{y}''(t;\alpha) + a\underline{y}'(t;\alpha) + b\underline{y}(t;\alpha) = \underline{\sigma}(t;\alpha), \\ \overline{y}''(t;\alpha) + a\overline{y}'(t;\alpha) + b\overline{y}(t;\alpha) = \overline{\sigma}(t;\alpha), \\ \underline{y}(0) = \underline{\gamma_0}, \overline{y}(0) = \overline{\gamma_0}, \underline{y}'(0) = \underline{\gamma_1}, \overline{y}'(0) = \overline{\gamma_1}, \end{cases}$$

for $\underline{y}$ and $\overline{y}$.

(ii) ensure that $[\underline{y}(t;\alpha), \overline{y}(t;\alpha)]$, $[\underline{y}'(t;\alpha), \overline{y}'(t;\alpha)]$ and $[\underline{y}''(t;\alpha), \overline{y}''(t;\alpha)]$ are valid level sets.

Other cases are similar to (1,1)-solution.

Remark 3.1. *The solution of FDE (1) depends upon the selection of derivatives, in the first or second form.*

Example 3.1. *Consider the second order fuzzy initial value problem*

$$y''(t) = \sigma_0, \quad y(0) = \gamma_0, \quad y'(0) = \gamma_1 \qquad t \geqslant 0,$$

where $\sigma_0 = \gamma_0 = \gamma_1 = [\alpha - 1, 1 - \alpha]$. *It possesses 4 different solutions.*

In order to extend the results to Nth-order fuzzy differential equations, we can follow the proof of Theorem 2.2 to get the same results for derivatives of an arbitrary order N. We have at most 2^N solutions for a Nth-order fuzzy differential equation by choosing the different types of derivatives.

References

[1] B. Bede, S.G. Gal, *Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems. 151 (2005) 581-99.