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# TRANSVERSALITY CONDITIONS FOR INFINITE HORIZON OPTIMIZATION PROBLEMS: THREE ADDITIONAL ASSUMPTIONS

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**Abstract** We consider the transversality condition for the optimization problem: $\max\limits_{y(t),u(t)} \int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt$, subject to $\dot{y} = f\left(y\left(t\right),u\left(t\right),t\right)$. In economics, different forms of transversality conditions have been proposed to solve the problem. We show that the most general form of transversality condition can be derived under three additional assumptions following Chiang (1994)'s approach. We also reconsider the famous counterexamples of Halkin (1974) and Shell (1969), in the light of our transversality condition.

**Keywords:** transversality condition; dynamic optimization; infinite horizon.

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## 1. INTRODUCTION

This note studies the transversality conditions for the model

$$\begin{cases} \max\limits_{y(t),u(t)} \int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt \text{ subject to} \\ y\left(0\right) = y_0,\ \dot{y} = f\left(y\left(t\right),u\left(t\right),t\right), \forall t \geqslant 0,\ u\left(t\right) \in U, \end{cases} \tag{1}$$

where $v$ and $f$ are real-valued continuously differentiable functions. Letting $\lambda\left(t\right)$ be the co-state variable associated with the constraint in (1), the corresponding Hamiltonian is given by:

$$H\left(y\left(t\right),u\left(t\right),\lambda\left(t\right),t\right) \equiv v\left(y\left(t\right),u\left(t\right),t\right) + \lambda\left(t\right)f\left(y\left(t\right),u\left(t\right),t\right).$$

The following equation is generally referred to as the transversality condition to (1):

$$\lim_{T\to\infty}[\lambda(T)\,y(T)] = 0, \tag{2}$$

for any optimal path $\{y(t)\}$. By using an elementary perturbation argument, Michel (1982) provides a proof of the necessity of (2). However, the perturbation he considers is rather specific and his results are later generalized in Kamihigashi (2001). In this paper, we aim to provide a straightforward proof of this result, using Chiang (1994)'s approach, which is incorrect without certain assumptions. We show that three additional assumptions that are either implicitly assumed, or entirely overlooked, are required. We also investigate the famous counterexamples, Halkin (1974) and Shell (1969). They have been argued by Caputo (2005) as valid counterexamples, which would disqualify transversality condition in the form of $\lim_{T\to\infty}\lambda(T) = 0$ as a necessary condition. We show that both of them satisfy the three assumptions and hence our transverality condition, $\lim_{T\to\infty}\lambda(T)\,\Delta y(T) = 0$. However, since they have implicit fixed terminal states, $\lim_{T\to\infty}\lambda(T) = 0$ cannot be applied to the two examples.

## 2. TRANSVERSALITY CONDITIONS FOR FINITE HORIZON PROBLEMS (CHIANG, 1992)

Following Chiang (1992), we first consider the finite horizon version of (1):

$$\begin{cases} \max\int_0^T v\left(y\left(t\right),u\left(t\right),t\right)dt \text{ subject to} \\ y\left(0\right)=y_0,\ \dot{y}=f\left(y\left(t\right),u\left(t\right),t\right),\forall t\geqslant 0,\ u\left(t\right)\in U. \end{cases} \tag{1a}$$

We introduce a new functional

$$\begin{aligned} V &\equiv v+\int_0^T\left[\lambda\left(t\right)f\left(y\left(t\right),u\left(t\right),t\right)-\dot{y}\right]dt \\ &=\int_0^T\left\{v\left(y\left(t\right),u\left(t\right),t\right)+\left[\lambda\left(t\right)f\left(y\left(t\right),u\left(t\right),t\right)-\dot{y}\right]\right\}dt. \end{aligned} \tag{3}$$

As in Chiang (1992, pp. 177-181), when we vary $V$ infinitesimally by $\varepsilon$, from the standard variational methods, we have

$$\frac{dV}{d\varepsilon}=\int_0^T\left[\left(\frac{\partial H}{\partial y}+\dot{\lambda}\right)q\left(t\right)+\frac{\partial H}{\partial u}p\left(t\right)\right]dt+\left[H\right]_{t=T}\Delta T-\lambda\left(T\right)\Delta y\left(T\right)=0, \tag{4}$$

that leads to the transversality conditions for the following three cases, with $y\left(T\right)$, $\lambda\left(T\right)$ and $\left[H\right]_{t=T}$ denoting the optimal values

(i) *free terminal state* $(\Delta y\left(T\right)\neq 0)$ *and a fixed* $T(\Delta T=0)$

$$\lambda\left(T\right)=0; \tag{5}$$

(ii) *fixed terminal point* ($T$, $y_T$ given)

$$y\left(T\right)=y_T; \tag{6}$$

(iii) *fixed-end point* (*fixed terminal state* $(\Delta y\left(T\right)=0)$ *and a free* $T(\Delta T\neq 0)$)

$$\left[H\right]_{t=T}=0. \tag{7}$$

## 3. INFINITE HORIZON TRANSVERSALITY CONDITIONS AND THE THREE ASSUMPTIONS

We proceed to consider the infinite horizon case. As in most works, we assume that the objective functional converges for all admissible paths.

**Assumption (i).** $V=\int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt$ is finite.

The objective functional in (1) can then be restated as

$$V=\int_0^\infty v\left(y\left(t\right),u\left(t\right),t\right)dt=\lim_{T\to\infty}\int_0^T v\left(y\left(t\right),u\left(t\right),t\right)dt \tag{8}$$

. We have

$$V=\int_0^\infty(v+\lambda\left(f-\dot{y}\right))dt \tag{9}$$

$$=\lim_{T\to\infty}\int_0^T(v+\lambda\left(f-\dot{y}\right))dt$$

$$=\lim_{T\to\infty}\left(\int_0^T\left(v+\lambda f+\dot{y}\right)dt-\left[\lambda y\right]_0^T\right)$$

$$=\int_0^\infty\left(v+\lambda f+\dot{y}\right)dt-\left[\lambda y\right]_0^\infty$$

$$= \int_0^\infty \left(H + \dot{\lambda} f\right) dt - (\lambda(\infty) y(\infty)) + (\lambda(0) y(0)).$$

Note that the last equality of (9) will be invalid if either $\int_0^\infty \left(H + \dot{\lambda} y\right) dt$ or $(\lambda(\infty) y(\infty))$ is infinite. Therefore, we consider

**Assumption (ii).** Both $\int_0^\infty \left(H + \dot{\lambda} y\right) dt$ and $(\lambda(\infty) y(\infty))$ are finite.

Then (9) can be further stated as

$$\begin{aligned} V_\varepsilon &= \int_0^\infty \left[H(t, y^* + \varepsilon q, u^* + \varepsilon p, \lambda) + \dot{\lambda}(y^* + \varepsilon q)\right] dt - \\ &- \lambda(\infty)(y^*(\infty) + \varepsilon q(\infty)) + \lambda(0)(y^*(0) + \varepsilon q(0)). \end{aligned} \tag{10}$$

In general, $\frac{\partial}{\partial y}\left(\int_0^\infty v(x,y) dx\right) = \int_0^\infty \frac{\partial v(x,y)}{\partial y} dx$ only when $\lim\limits_{T\to\infty}\left[\int_0^T \frac{\partial v(x,y)}{\partial y} dx\right]$ converges uniformly for $y$ (Theorem 3.4 (p. 289), Lang, 1983). Hence, we impose

**Assumption (iii).** $\lim\limits_{T\to\infty} \int_0^T \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right) q + \frac{\partial H}{\partial u} p\right] dt$ converges uniformly for $y$.

When Assumptions (i)$\sim$(iii) are satisfied, setting

$$\frac{dV_\varepsilon}{d\varepsilon} = \underbrace{\int_0^\infty \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right) q(t) + \frac{\partial H}{\partial u} p(t)\right] dt}_{(I)} + \underbrace{\lim_{T\to\infty} H \Delta T}_{(II)} - \underbrace{\lim_{T\to\infty} \lambda(T) \Delta y(T)}_{(III)} = 0, \tag{11}$$

the first-order condition requires that each of the three component terms in (11) to be set equal to zero, respectively.

Again here we only consider the perturbation around the optimal values. This gives rise to *the general transversality conditions* for the infinite horizon problems:

$$\lim_{T\to\infty} H(T) \Delta T = 0 \text{ and } \lim_{T\to\infty} \lambda(T) \Delta y(T) = 0. \tag{12}$$

**Remark** $\lim_{T\to\infty} H(T) \Delta T = 0$ vanishes if the time horizon is assumed to be fixed at $\infty$.

Our general transversality condition is derived by directly following Chiang (1994)'s approach. It thus constitutes a straightforward proof of Kamihigashi (2001)'s fundamental results. Moreover, if the perturbation is to shift the entire optimal path downward by a small fixed proportion $\varepsilon \in [0, 1)$, as is in

Kamihigashi (2001), then $\Delta y(t) = \varepsilon y(t)$. Substituting $\varepsilon y(t)$ into the second equation in (12) leads to

$$\lim_{T\to\infty} \lambda(T)\,\varepsilon y(T) = \varepsilon \lim_{T\to\infty} \lambda(T)\,y(T) = 0. \tag{13}$$

Dividing both sides of (13) by $\varepsilon$, we then have (2).

Moreover, for problems with *a free terminal state*, as both the terminal time and the terminal state are not fixed ($\Delta T \neq 0$ and $\Delta y(T) \neq 0$), the transversality conditions comprise two conditions

$$\lim_{T\to\infty} [H(T)] = 0 \text{ and } \lim_{T\to\infty} \lambda(T) = 0 \tag{12$'$}$$

.

Note that the necessity of the second equation of (12') is derived by Michel (1982), who considers a specific perturbation that shifts the entire optimal path downward by a fixed value.

On the other hand, for problems with *a fixed terminal state* ($\Delta y(T) = 0$), the transversality condition is

$$\lim_{T\to\infty} [H(T)] = 0. \tag{12$''$}$$

Furthermore, from the definition of the Hamiltonian, we have

**Remarks. 1.**When $T \to \infty$, if the objective function converges to zero and the state equation is nonzero, then $\lim_{T\to\infty} [H(T)] = 0$ is equivalent to $\lim_{T\to\infty} \lambda(T) = 0$. Obviously, as the objective function for most discounted cases does approach zero when $T \to \infty$, either one of the two conditions in (12) can function as the transversality condition for such cases.

**2.** Sydster et al.'s necessary condition for an infinite horizon optimization problem with discounting (Theorem 9.11.2, 2005), can be readily derived from the Assumptions 2 and 3. Moreover, the "normal" transversality condition, $\lim_{T\to\infty} [\lambda(T) \cdot y(T)] = 0$, being redundant, remains valid.

**3.** Although economically intuitive, (2) can only be directly derived by variational approach that considers specific perturbations. However, it has to be noticed that there is no need to assume that the present value of the stock at the infinity should be zero.

**4.** For systems with steady-states, "inefficient overaccumulation of capital stock" does not necessarily imply that the present value of the stock approaches zero when $t$ approaches infinity, $\lim_{t\to\infty}[\lambda(t)\cdot y(t)]=0$, as argued in Weitzman (2003). It only requires that $\lim_{T\to\infty}\lambda(T)\,\Delta y(T)=0$, i.e., the variations in the values of the capital stock should approach zero.

Note that Chiang's (1992) derivation of the transversality conditions for the infinite horizon case is imprecise as the above three assumptions have not been explicitly stipulated, although it can be easily verified that for most discounted problems, Assumptions (i)∼(iii) are satisfied. Next, we consider the famous counterexamples of Halkin (1974) and Shell (1969).

## 4. APPLICATIONS

Several writers (Caputo (2005), for example) argue that there exist counterexamples that would disqualify transversality condition in the form of $\lim_{T\to\infty}\lambda(T)=0$ as a necessary condition. We shall demonstrate, however, these are not valid counterexamples, because they have implicit fixed terminal states.

### 4.1. HALKIN'S EXAMPLE (1974)

We first consider Halkin's example:

$$\begin{cases} \max\int_0^\infty (1-y)\,udt \text{ subject to} \\ y(0)=0,\ \dot{y}=(1-y)\,u,\ u(t)\in[0,1]\,. \end{cases} \tag{14}$$

From the equation of motion for $y$, we see that the definite solution is

$$y(t)=1-e^{-\int_0^t udt}. \tag{15}$$

Hence, problem (14) can be reformulated as

$$\max\int_0^\infty e^{-\int_0^t udt}udt \tag{16}$$

Let $\int_0^t udt=v(t)$, we see that $udt=dv$, and $\int_0^\infty udt=v(\infty)$. Now (16) can be restated as

$$\int_0^{v(\infty)} e^{-v}dv=-e^{-v}\Big|_0^{v(\infty)}=-e^{-v(\infty)}+1. \tag{17}$$

Obviously, (17) is maximized when $v(\infty) = \infty$, with the maximum being 1. In other words, the objective function in (14) is maximized if and only if $\int_0^\infty u(t)\,dt = \infty$.

Caputo (2005) incorrectly argues that Halkin's problem is a valid counterexample. He defines the control as $u(t) = \begin{cases} k \in [0,1] \;\; \forall t \in [0,\tau], \tau < \infty, \\ 0 \;\forall t \in (\tau, \infty), \end{cases}$ and argues that since the value of $\underset{t\to\infty}{} y(t)$ depends on $k$, the problem does not has a fixed terminal state. However, his choice of the control, although feasible, is not optimal as under such a case $\int_0^\infty u dt < \infty$.

Moreover, as $u(t) \in [0,1]$, we see that $e^{-\int_0^t u dt} \in (0,1]$. Consequently, $y(t) \in [0,1)$. It follows that

$$\underset{t\to\infty}{} y(t) = 1. \tag{18}$$

On the other hand, as

$$H = (1-y)\,u + \lambda\,(1-y)\,u = (1+\lambda)\,(1-y)\,u, \tag{19}$$

from the condition $\partial H/\partial u = 0$, we have $(1+\lambda)(1-y) = 0$, and the optimal co-state variable $\lambda^*(t) = -1$ inasmuch as $y$ is always less than 1. Therefore, $H = 0$. Note that $\underset{T\to\infty}{} \lambda^*(T) \neq 0$.

It is easy to verify that Assumption (i)$\sim$(iii) are satisfied for this problem. Moreover, from (18), we see the problem has an implicit fixed terminal state, (12") applies and $\underset{T\to\infty}{} \lambda(T) = 0$ does not apply.

## 5. THE SHELL PROBLEM (1969)

The Shell problem is a modified version of Ramsey's (1928) model, which maximizes the deviation from a "bliss level". Let $\bar{c}$ be the steady-state consumption level:

$$\begin{cases} \max \int_0^\infty (c(t) - \bar{c})\,dt \text{ subject to} \\ k(0) = k_0,\;\; \dot{k} = \phi(k(t)) - c(t) - (n+\delta)\,k, 0 \leqslant c(t) \leqslant \phi(k(t)) \;\; . \end{cases} \tag{20}$$

It is easy to verify that Assumption (i)$\sim$(iii) are satisfied. Hence, (12) applies. However, as pointed in Chiang (1992), the Shell problem contains an implicit fixed terminal state as $c \to \bar{c}$ when $t \to \infty$. Therefore, the transversality condition should be (12").

### Conclusions

As shown above, the dispute over the transversality condition for the infinite horizon problems, especially the interpretation concerning the "counterexamples", arises because the transversality conditions vary according to terminal states. Chiang correctly points out that both examples are not valid counterexamples as they all have implicitly fixed terminal states. However, a correct derivation of this result requires three additional assumptions to be explicitly stipulated, although it can be easily verified that Assumptions (i)$\sim$(iii) are satisfied for most discounted problems.

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